STRING-BEAM UNDER MOVING INERTIAL LOAD

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Abstract

The paper deals with the original analytical-numerical approach to the Bernoulli-Euler beam with additional tensile effect under a moving inertial load. The authors applied the 2nd kind Lagrange equation to derive a motion differential equation of the problem. The moving mass can travel through the string-beam with a whole range constant speed, also overcritical. The analytical solution requires a numerical calculation in the last stage and is called a semi-analytical one.
Keywords: moving mass, inertial load, string, beam

Introduction

The problem of bridge spans under a moving inertial load has existed since the beginning of the railways development. The turning point in the literature devoted to moving loads was established by two historical publications [1, 2]. These analytical papers were elaborated with significant mathematical simplifications. The authors considered the complex acceleration of the moving mass. Its geometrical interpretation was presented by Renaudot [3]. Although the number of publications on the moving mass problem exceeded thousand items, still we do not have its detailed and fully analytical solutions. The approach given by Smith [4] seems to be a positive exception. He considered, however, the massless string only. There exist numerous review papers [5, 6, 7, 8] which discuss problems presented in hundreds of other publications. For a long time the main stream of works treated the problem in an analytical-numerical way [9, 10, 11, 12] or strictly numerically [13, 14, 15].

Together with increasing velocity of trains, the influence of the wave phenomenon is rising as well. Dynamic effects are generated by the load of train current collectors, travelling through the power supply cable of the overhead contact line. In this paper, we consider a cable as the string-beam model, since it has a certain flexible stiffness. The Bernoulli-Euler beam with additional tensile effect comprises this phenomenon (Fig. 1). In the paper the differential equation of the motion of a string-beam is derived from the Lagrange equation of the 2nd kind.

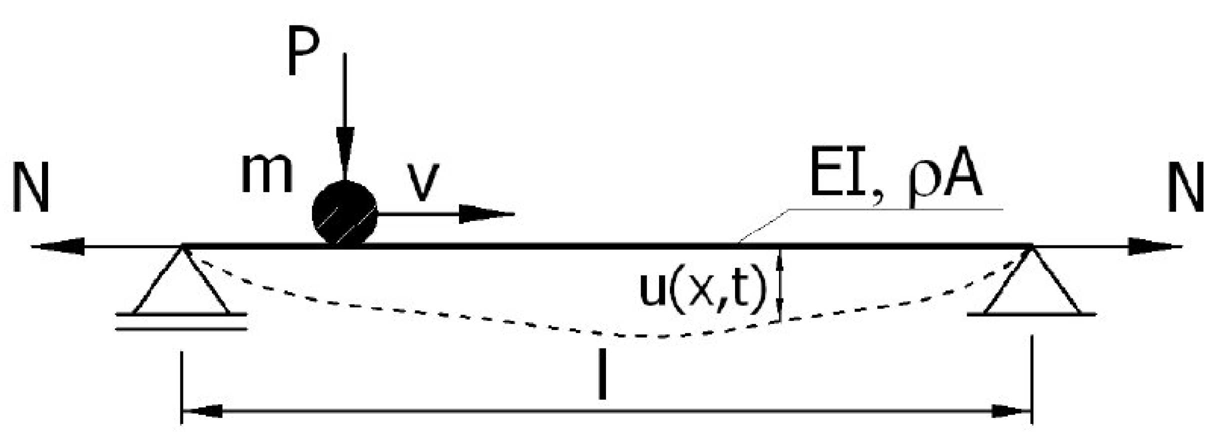

Figure 1: A string-beam under the moving mass.

1. The 2nd kind Lagrange equation

The motion equation of the string-beam under a moving mass m coupled with a force P can be written as follows

$$EI\,\frac{\partial^4 u(x,t)}{\partial x^4} - N\,\frac{\partial^2 u(x,t)}{\partial x^2} + \rho A\,\frac{\partial^2 u(x,t)}{\partial t^2} = \delta(x-vt)\,P - \delta(x-vt)\,m\,\frac{\partial^2 u(vt,t)}{\partial t^2}\,, \tag{1}$$

where EI is the beam stiffness, N is a tensile force and ρA is a linear mass density. Taking into account beam terms, we impose four boundary conditions

$$u(0,t) = 0\,, \qquad u(l,t) = 0\,, \qquad \left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=0} = 0\,, \qquad \left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=l} = 0 \tag{2}$$

and two initial conditions

$$u(x,0) = 0\,, \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0\,. \tag{3}$$

The kinetic energy of a string-beam and a travelling mass is described by

$$E_k = \frac{1}{2}\,\rho A \int_0^l \left[\frac{\partial u(x,t)}{\partial t}\right]^2 \mathrm{d}x + \frac{1}{2}\,m\left[\frac{\partial u(vt,t)}{\partial t}\right]^2\,. \tag{4}$$

The potential energy of a string-beam and a moving force is

$$E_p = \frac{1}{2}\,N \int_0^l \left[\frac{\partial u(x,t)}{\partial x}\right]^2 \mathrm{d}x + \frac{1}{2}\,EI \int_0^l \left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]^2 \mathrm{d}x - P\,u(vt,t)\,. \tag{5}$$

In order to separate variables, the displacement can be written in a form of the infinite series and then integrals in space x in Eqns. (4) and (5) can be computed

$$u(x,t) = \sum_{i=1}^{\infty} U_i(x)\xi_i(t)\,. \tag{6}$$

According to (6) the displacement under a moving load has the following form

$$u(vt,t) = \sum_{i=1}^{\infty} U_i(vt)\xi_i(t)\,. \tag{7}$$

The velocity of the displacement is determined by a chain rule

$$\frac{\partial u(vt,t)}{\partial t} = v\left.\sum_{i=1}^{\infty} U_i'(x)\xi_i(t)\right|_{x=vt} + \left.\sum_{i=1}^{\infty} U_i(x)\dot{\xi}_i(t)\right|_{x=vt}\,. \tag{8}$$

It is the function of general coordinates as well as their velocities

$$\frac{\partial u(vt,t)}{\partial t} = f\left(\xi_i, \dot{\xi}_i\right)\,. \tag{9}$$

After calculation of required derivatives of (6) with respect to t and x, the kinetic and potential energy can be written in the following forms

$$\begin{aligned} E_k &= \frac{1}{2}\rho A \sum_{i,j=1}^{\infty} \dot{\xi}_i(t)\dot{\xi}_j(t) \int_0^l U_i(x)U_j(x)\mathrm{d}x + \frac{1}{2}m\left[\frac{\partial u(vt,t)}{\partial t}\right]^2 , \qquad (10) \\ E_p &= \frac{1}{2}N \sum_{i,j=1}^{\infty} \xi_i(t)\xi_j(t) \int_0^l U_i'(x)U_j'(x)\mathrm{d}x + \\ &+\frac{1}{2}EI \sum_{i,j=1}^{\infty} \xi_i(t)\xi_j(t) \int_0^l U_i''(x)U_j''(x)\mathrm{d}x - P\sum_{i=1}^{\infty} U_i(vt)\xi_i(t) . \qquad (11) \end{aligned}$$

We assume orthogonal functions (12) which fulfil boundary conditions (2)

$$U_i(x) = \sin\frac{i\pi x}{l} . \qquad (12)$$

The orthogonality of functions $U_i(x)$ allows us to write

$$\int_0^l U_i(x)U_j(x)\mathrm{d}x = \begin{cases} \frac{1}{2}l & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} . \qquad (13)$$

With respect to (6), (12) and (13) the kinetic energy of the hole system is given by

$$E_k = \frac{1}{4}\rho A l \sum_{i=1}^{\infty} \dot{\xi}_i^2(t) + \frac{1}{2}m\left[\frac{\partial u(vt,t)}{\partial t}\right]^2 . \qquad (14)$$

The term of the moving mass is not an integral, so we can't use the property of orthogonality. On this stage, the kinetic energy of the travelling load is left in the original form. According to (12) we have $U_i''(x) = -i^2\pi^2/l^2\, U_i(x)$. After integration by parts and taking into account the recent relation, the potential energy can be written in the form

$$\begin{aligned} E_p = &\frac{1}{2}N \sum_{i,j=1}^{\infty} \frac{i^2\pi^2}{l^2}\xi_i(t)\xi_j(t) \int_0^l U_i(x)U_j(x)\mathrm{d}x + \\ &+\frac{1}{2}EI \sum_{i,j=1}^{\infty} \frac{i^2 j^2\pi^4}{l^4}\xi_i(t)\xi_j(t) \int_0^l U_i(x)U_j(x)\mathrm{d}x - P\sum_{i=1}^{\infty} U_i(vt)\xi_i(t) . \end{aligned} \qquad (15)$$

Finally, the orthogonality of (13) allows us to write

$$E_p = \frac{1}{4}Nl\sum_{i=1}^{\infty}\frac{i^2\pi^2}{l^2}\xi_i^2(t) + \frac{1}{4}EIl\sum_{i=1}^{\infty}\frac{i^4\pi^4}{l^4}\xi_i^2(t) - P\sum_{i=1}^{\infty}\xi_i(t)\sin\frac{i\pi vt}{l} . \qquad (16)$$

The motion equation of the string-beam under a moving inertial load is obtained from the 2nd kind Lagrange equation, of which a general form is given by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_k}{\partial \dot{\xi}_i}\right) - \frac{\partial E_k}{\partial \xi_i} + \frac{\partial E_p}{\partial \xi_i} = 0\ . \tag{17}$$

This method results in the differential equation of variable coefficients (18).

$$\begin{aligned}
&\rho\, A\, \ddot{\xi}_i(t) + \frac{2m}{l}\sum_{j=1}^{\infty}\ddot{\xi}_j(t)\sin\frac{i\pi vt}{l}\sin\frac{j\pi vt}{l} + \frac{4m}{l}\sum_{j=1}^{\infty}\frac{j\pi v}{l}\dot{\xi}_j(t)\sin\frac{i\pi vt}{l}\cos\frac{j\pi vt}{l} + \\
&+ \quad N\frac{i^2\pi^2}{l^2}\xi_i(t) + EI\frac{i^4\pi^4}{l^4}\xi_i(t) - \frac{2m}{l}\sum_{j=1}^{\infty}\frac{j^2\pi^2v^2}{l^2}\xi_j(t)\sin\frac{i\pi vt}{l}\sin\frac{j\pi vt}{l} = \\
&= \quad \frac{2P}{l}\sin\frac{i\pi vt}{l}\ .
\end{aligned} \tag{18}$$

The equation (18) can not be easily solved and we must integrate it in a numerical way. We use the matrix notation here

$$\mathbf{M}\begin{bmatrix}\ddot{\xi}_1(t)\\ \ddot{\xi}_2(t)\\ \vdots\\ \ddot{\xi}_n(t)\end{bmatrix} + \mathbf{C}\begin{bmatrix}\dot{\xi}_1(t)\\ \dot{\xi}_2(t)\\ \vdots\\ \dot{\xi}_n(t)\end{bmatrix} + \mathbf{K}\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\\ \vdots\\ \xi_n(t)\end{bmatrix} = \mathbf{P} \tag{19}$$

which results in a short form $\mathbf{M}\ddot{\boldsymbol{\xi}} + \mathbf{C}\dot{\boldsymbol{\xi}} + \mathbf{K}\boldsymbol{\xi} = \mathbf{P}$, where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are square matrices for $i = j = 1, 2, ..., n$.

When we calculate the value of general coordinates $\xi_i(t)$ for each i to n. Finally we can compute displacements of the string-beam $u(x,t)$

$$u(x,t) = \sum_{i=1}^{\infty}\xi_i(t)\sin\frac{i\pi x}{l} \tag{20}$$

Displacements given in the example below are dimensionless. They were calculated in relation to the static deflection u_0 of the string-beam loaded in the midpoint by the point force P: $u_0 = u_{0s}\, u_{0b}/(u_{0s}\, u_{0b})$. u_{0s} and u_{0b} are static deflections in the case of a string and a beam, respectively.

Example Let us assume the following data: $E = 1$, $I = 0.01$, $N = 1$, $\rho = 1$, $A = 1$, $P = -1$ and $m = 1$. We solve the problem for different speeds v of the moving load. The mass trajectory is depicted in Fig. 2. The simulation of the string-beam motion is depicted in Fig. 3.

2. Conclusions

In the paper the Bernoulli-Euler beam with additional tensile effect under moving inertial load was presented. The proposed semi-analytical approach can be applied

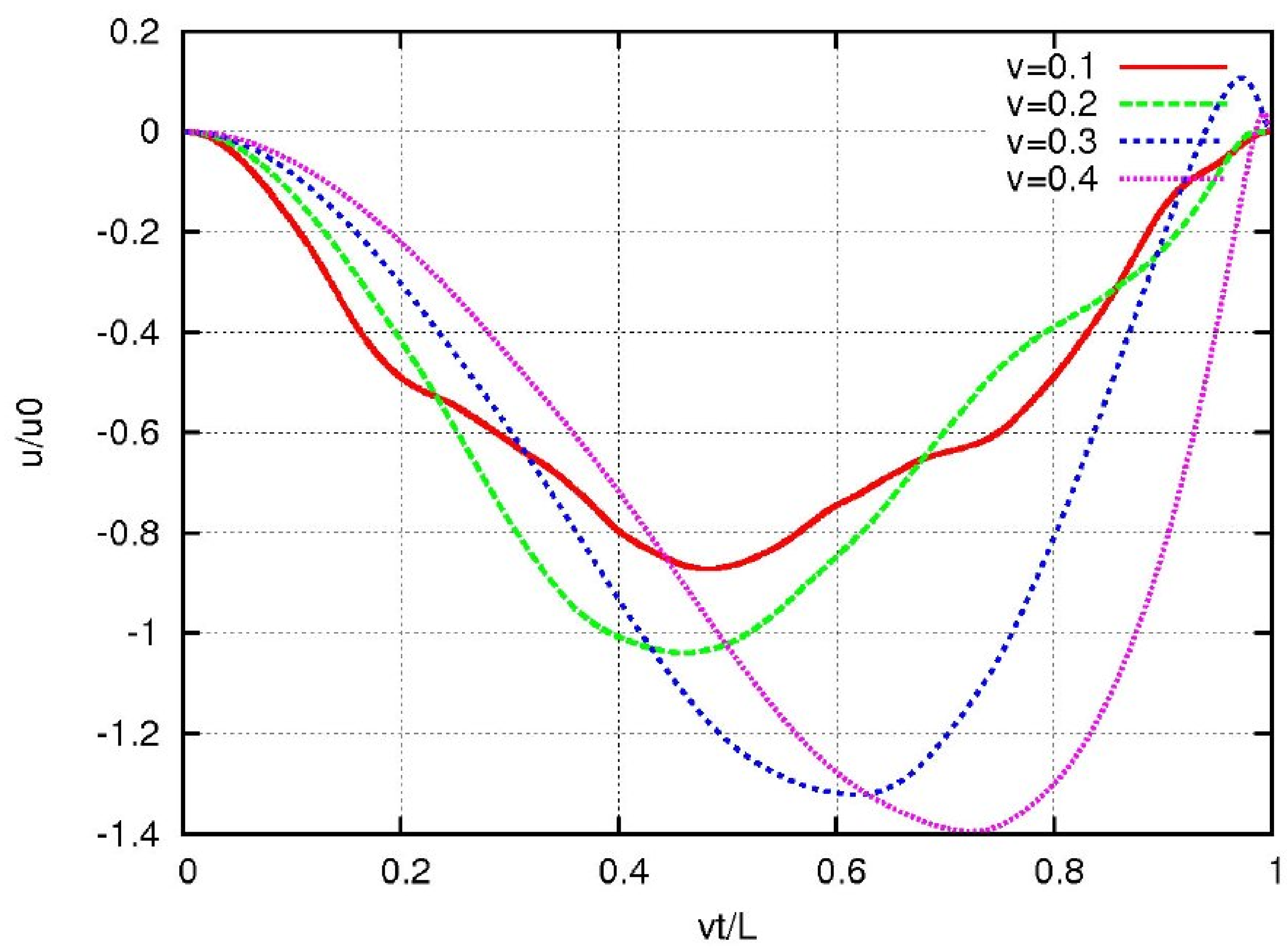

Figure 2: Mass trajectory for different speeds v.

in a hole range of the speed and for all points of the string-beam span. The accuracy of the solution (20) depends on the number of terms in the infinite series. The examined series of displacements is convergent, so it allows us to assume a limited number of terms in our example to $i = 130$.

If we reduce the flexural stiffness of the system, we observe the discontinuity near the end support. It was broadly presented and proved in [11]. The discontinuity had appeared also in the case of the Timoshenko beam [12]. In the matrix form (19) we can use classical numeric methods for the integration of the final differential motion equation, for example Newmark method.

References

1. C.E. Inglis. *A Mathematical Treatise on Vibrations in Railway Bridges.* Cambridge University Press, 1934.
2. A. Schallenkamp. Schwingungen von Trägern bei bewegten Lasten. *Arch. of Appl. Mech. (Ingenieur Archiv)*, 8(3):182–198, June 1937.
3. Renaudot. Etude de l'influence des charges en mouvement sur la resistance, des ponts metallique a poutres droites. *Annales des Ponts et Chausses*, 1:145, 1861.
4. C.E. Smith. Motion of a stretched string carrying a moving mass particle. *J. Appl. Mech.*, 31(1):29–37, 1964.
5. J. Panovko. Historical outline of the theory of dynamic influence of moving load (in Russian). *Engineering Academy of Air Forces*, 17:8–38, 1948.
6. N.Z. Jakushev. Certain problems of dynamics of the beam under moving load (in Russian). *Publisher of the Kazan Univ.*, 12:199–220, 1974.
7. A.S. Dmitrijev. The analysis of solutions of problems with lateral oscillatory vibrations of various beam structures under the motion of non spring point load (in Russian). *Machine Dynamics Problems*, 24:18–28, 1985.

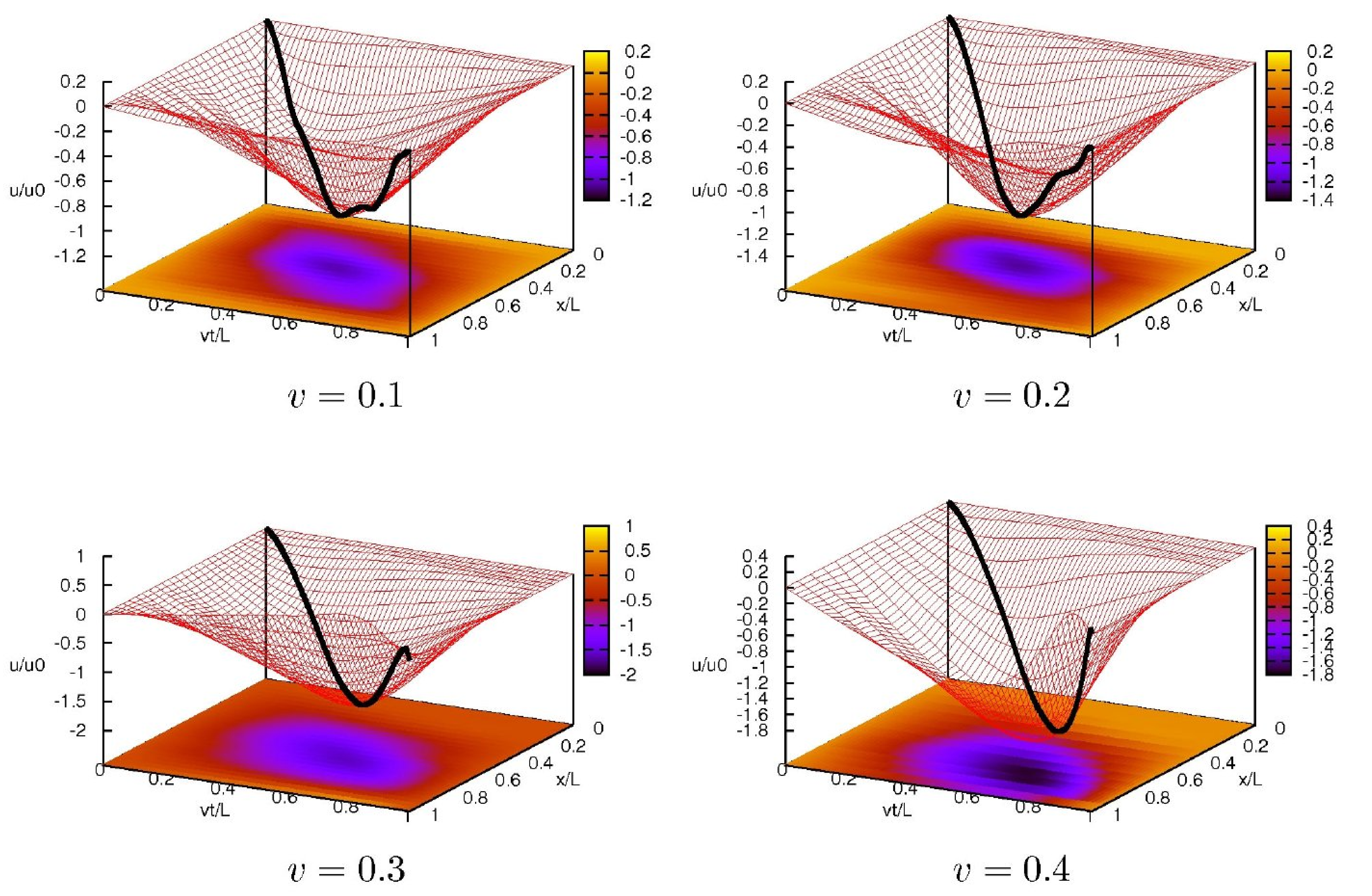

Figure 3: Simulation of the string-beam motion under the mass moving at v=0.1, 0.2, 0.3 and 0.4.

8. W. Szcześniak. Inercyjne obciążenia ruchome na belkach. Prace Naukowe, Politechnika Warszawska, budownictwo 112, 1990.
9. S. Sadiku and H.H.E. Leipholz. On the dynamics of elastic systems with moving concentrated masses. *Ing. Archiv*, 57:223–242, 1987.
10. E.C. Ting, J. Genin, and J.H. Ginsberg. A general algorithm for moving mass problems. *J. Sound Vib.*, 33(1):49–58, 1974.
11. B. Dyniewicz and C.I. Bajer. Paradox of the particle's trajectory moving on a string. *Arch. Appl. Mech.*, 2008. in print.
12. B. Dyniewicz and C.I. Bajer. Discontinuous trajectory of the mass particle moving on a string or a beam. *Machine Dyn. Probl.*, 31(3), 2007.
13. F.V. Filho. Finite element analysis of structures under moving loads. *The Shock and Vibration Digest*, 10(8):27–35, 1978.
14. J.R. Rieker, Y.-H. Lin, and M.W. Trethewey. Discretization considerations in moving load finite element beam models. *Finite Elements in Analysis and Design*, 21:129–144, 1996.
15. C.I. Bajer and B. Dyniewicz. Space-time approach to numerical analysis of a string with a moving mass. *Int. J. Numer. Meth. Engng.*, 2008. in print.