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The energy criteria of instability in time-independent inelastic solids

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CONDITIONS sufficient for instability of deformation are examined for a class of incrementally nonlinear, time independent inelastic solids. Instability of a quasi-static deformation process (path) at varying loading is distinguished from a narrower concept of an unstable equilibrium state. The energy criterion is extended to deformation paths under general assumptions which ensure that the incremental boundary value problem can be given a variational formulation. For a discretized problem, fulfillment of the instability condition along a path is shown to imply either instability of the traversed equilibrium states in a dynamic sense or persisting possibility of quasi-static bifurcation at varying loading. For a continuum, instability at varying loading is interpreted as sensitivity of the incremental deformation to arbitrarily small perturbing forces.

1. Introduction

THE FOLLOWING question is investigated here: what processes of inelastic deformation cannot be practically realized in a physical system. A deformation process is thought to represent a theoretical quasi-static solution to an initial-boundary value problem formulated for an inelastic continuous body subject to *varying* loading. Real material properties and boundary conditions are assumed to be adequately modeled. The question posed above is related to the well known fact that there are simple and theoretically correct solutions, e.g. describing macroscopically uniform straining, which at sufficiently advanced deformations are no longer in accord with experimentally observed deformation modes. One of possible explanations is that this is a symptom of an instability of the solution beyond a certain critical stage.

Instability in inelastic solids at slowly varying loading may manifest itself in various ways, not necessarily involving observable dynamic effects. Familiar examples are buckling or necking as "geometric" instabilities and strain localization within shear bands as a "material" instability. According to the theory of uniqueness and stability in time-independent inelastic solids developed by HILL [3, 4], preceded by SHANLEY'S [23] pioneering observation of the essence of plastic column buckling, such phenomena need not be connected with loss of stability of *equilibrium* on the fundamental deformation path. Rather, they can be explained in many cases by appealing to the bifurcation theory or to sensitivity to initial imperfections (cf. e.g. [11, 16]).

Alternatively, the above mentioned phenomena can be consistently regarded as symptoms of instability provided that this term concerns not an equilibrium *state* (at constant loading) but rather a deformation *process* (or *path*—at varying loading). Traditionally, stability of a quasi-static deformation path is often identified with stability of equilibrium states traversed by the path. However, for path-dependent and incrementally nonlinear solids distinction must be made between these two kinds of instability. Stability of equilibrium means that the effects of disturbing influences become negligible when the strength of disturbance is vanishingly small *and* the value of the loading parameter is kept fixed. For path-dependent inelastic solids we generally cannot expect that after a transitory perturbed motion the initial equilibrium state will be restored exactly, even in an asymptotic sense. Now, if the final effect is very small but nonzero and if a quasi-static loading program is continued, there is no guarantee that the distance between the *subsequent* configurations along the perturbed and fundamental paths at the same loading levels will remain small; note that continuous dependence of this distance on initial data is necessarily broken e.g. at a bifurcation point. This observation alone indicates that stability of an equilibrium state and of a quasi-static process of inelastic deformation are essentially different concepts ([1]). The distinction is even more clear when the effect of application of small perturbing forces is studied under the assumption of a time-independent material. Depending on whether such forces are applied at constant loading or at varying loading, different branches of the incrementally nonlinear constitutive law can be activated. In the former case the straining "direction" is defined by the disturbance itself while in the latter, it is primarily determined by the fundamental velocity field; for elastic-plastic solids the "stiffness" in the latter case can be much smaller (cf. the reduced and tangent modulus "stiffness" of the Shanley column).

It can be concluded that the question of stability of a deformation process, and also its relation to bifurcation suggested by the above remarks, deserve more thorough studies. In particular, stability of post-bifurcation paths should be examined in order to establish which of them can have a practical meaning. In this paper, we will examine stability of an isothermal quasi-static process of rate-independent inelastic deformation under varying loading at presence of small disturbing influences ([2]).

Although the idea itself is not new (it could be traced back to Considère) and became used in particular problems, a respective general theory is lacking. To the author's knowledge, general stability conditions for such processes were, in effect, only postulated (cf. e.g. [2, 13, 18, 19, 1]). On the contrary, in the present paper certain conditions for instability are not only postulated but also related to a defined kind of instability of motion for a broad class of problems.

The general assumptions accepted here are essentially the same as in the bifurcation theory developed by HILL [4, 5, 6, 8]. It is assumed that the constitutive rate equations for the material do not contain a natural time and admit a potential, and that the incremental loading is conservative (possibly in an overall sense). Incremental nonlinearity of the material is otherwise arbitrary: for instance, elastoplastic models obeying the normality flow rule, with or without the yield-surface vertex effect, as well as so-called softening materials are included. The above assumptions ensure that any solution to the first-order rate boundary value problem will be a critical point of a velocity functional [4, 6]. On the basis of an energy interpretation of the functional, the author proposed [18, 19] the energy criterion of instability of a quasi-static deformation process. The condition of instability of equilibrium (cf. [9]) was included as a special case. The previous work [19] established a definite connection between the onset of instability of the fundamental deformation path, found from the second-order energy criterion, and the primary bifurcation point.

([1]) The scale of time and the meaning of a perturbed motion are also different: while in examining stability of equilibrium a natural time scale can be adopted and inertia effects should then be taken into account, for a quasi-static loading program the role of "time" is usually played by the loading parameter and it is then natural to neglect interia effects.

([2]) Disturbing influences (called also *disturbances*) should be carefully distinguished from geometric or material *imperfections*: the former perturb the motion of a given material system while the latter change the system itself.

However, the relation to general concepts of instability of motion and the meaning of predicted instability remained to be clarified.

Disturbing influences are often thought as perturbations of initial conditions. In reality, any material body is subject to perpetual disturbing influences *during* a deformation process; in the general theory of stability of motion such influences are called *persistent disturbances* (cf. e.g. [15]). They can be represented, for instance, by small perturbing forces acting independently of the loading program. If application of arbitrarily small perturbing forces in any finite interval of a loading parameter can cause finite deviations from a theoretical, unperturbed deformation path, then such a path may be regarded as being unstable and, consequently, practically unrealizable in a physical system.

The above concept of instability is, of course, not the only possible one: there is a vast literature on various theories of stability of motion, in particular, on Liapounov's theory where *initial* disturbances are considered. Moreover, the same solution can be stable or not depending on the chosen measures of the distance and disturbance: this is well known in the context of stability of equilibrium in elastic continua (cf. [14]). In absence of sufficiently strong reasons for assuming *a priori* a particular mathematical definition of stability or instability in inelastic continua, the following approach is adopted here: qualitative properties of certain post-critical solutions are first selected as *possible candidates* for conditions sufficient for instability, and then examined in detail in order to learn whether (or to what extent) they do imply instability in a physically acceptable sense. More specifically, the purpose of the present paper is to establish a connection between the energy criterion of path instability and the instability for persistent disturbances mentioned above.

That approach is new and will be shown to lead to the conclusions which are in accord with but are not derivable from Hill's theory of bifurcation and stability. Roughly speaking (details are given in subsequent sections), it will be demonstrated that if the basic velocity functional [4, 6] is not minimized along a solution path by the actual velocity field, then that deformation process is unstable in the sense of sensitivity of the incremental deformation to arbitrarily small perturbing forces. This in turn will provide additional and essential information, not given by Hill's theory, which post-bifurcation paths can be stable and thus practically realizable. Usually, the initial imperfection approach is adopted to obtain such information, and in simplest cases the predictions of both approaches will be similar. However, the present approach gives definite advantages since no comparable general criterion for incrementally nonlinear solids has been established yet via the considerations of initial imperfections. Therefore, a numerical analysis of development of various initial inhomogeneities for more complex examples, being avoided here, is still required in the usual approach to eliminate the possibility that the conclusions are only valid for particular imperfections. Moreover, from a physical point of view it is not clear why instabilities which are observed e.g. at large plastic strains should always have a form decided by *initial* imperfections. Of course, there are other advantages of the initial imperfection method as e.g. assessment of the effect of geometrical inaccuracies on the buckling loads.

A finite solid body with specified boundary conditions is considered here; instability of uniform deformation at the level of a material element will be examined in a separate paper. Sections 3 and 4 constitute the principal part of the present paper; for convenience of the reader, more important statements are distinguished in the text in italic and summarized in Sect. 6. The topic requires precise formulation of the assumptions and definitions; this is done in Sect. 2, along with certain extensions of Hill's results and their novel energy interpretation.

2. Preliminaries

2.1. Notation

Throughout the paper the standard symbolic notation is used. Vectors or tensors are denoted by boldface letters, and their Cartesian components are denoted by lower case Latin subscripts for which the summation convention is adopted. Contraction over two pairs of the subscripts is distinguished by a dot between two symbols, e.g. $A_{ij}B_{ji} = \mathbf{A}\cdot\mathbf{B}$, $C_{ijkl}B_{lk} = (\mathbf{C}\cdot\mathbf{B})_{ij}$, while $A_{ik}B_{kj} = (\mathbf{AB})_{ij}$, $p_iA_{ij} = (\mathbf{pA})_j$ or $p_iq_i = \mathbf{pq}$. The upper index T denotes a transpose, and $(A_{ij}A_{ij})^{1/2} = |\mathbf{A}|$.

In a reference configuration which is arbitrary but fixed, a continuous solid body, in general inhomogeneous, is assumed to occupy a spatial domain V bounded by a piecewise-regular surface S with a unit normal $\boldsymbol{\nu}$. $\boldsymbol{\xi}$ and $\mathbf{x}$ denote position vectors in the reference and current configuration, respectively, and $\mathbf{u} = \mathbf{x} - \boldsymbol{\xi}$ stands for displacement. A field defined on $\overline{V} = V \cup S$ is distinguished from its value at some $\boldsymbol{\xi}$ by a superimposed tilde. The symbol ∇ denotes the spatial gradient evaluated in the reference configuration. t stands for a scalar time-like parameter, called time for simplicity, identified with a natural time only when examining stability of an equilibrium state. A dot over a symbol denotes the rate of change with respect to t, understood as a material time derivative in the *right-hand* sense.

Finite strains and rotations are considered while their rates are assumed to be bounded. For the sake of conciseness, the formulae appearing in stability considerations are expressed in terms of rates of the deformation gradient $\mathbf{F} = \nabla\mathbf{x}$ and the (unsymmetric) nominal stress $\mathbf{N}$ (related to the symmetric Cauchy stress $\boldsymbol{\sigma}$ and Kirchhoff stress $\boldsymbol{\tau}$ by $\mathbf{FN} = \det(\mathbf{F})\boldsymbol{\sigma} = \boldsymbol{\tau}$). The formulae can be transformed in the routine way to other stress and strain measures when needed.

The first weak variation (Gateaux differential) of a functional J at its argument value $\tilde{\mathbf{v}}$ in the direction $\tilde{\mathbf{w}}$ is denoted by

$$(2.1) \qquad \delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) \equiv \frac{d}{d\gamma} J(\tilde{\mathbf{v}} + \gamma\tilde{\mathbf{w}})\,|_{\gamma=0}\,,$$

where $\tilde{\mathbf{w}} = \delta\tilde{\mathbf{v}}$ stands for an admissible variation of $\tilde{\mathbf{v}}$ and γ is a real variable.

In the following, $\mathbf{v} = \dot{\mathbf{x}}$ stands for velocity.

2.2. Constitutive rate equations

A general assumption is made that the mechanical properties of the material in isothermal deformation do not depend in any way on a natural time. Under restriction to piecewise smooth deformation paths, it is assumed that the stresses vary in time in a continuous and piecewise smooth manner. At the current state of a material element, a (homogeneous) constitutive relationship between the (right-hand) rates of stress and strain, no matter in which variables it is originally formulated, can be expressed as

$$(2.2) \qquad \dot{\mathbf{N}} = \dot{\mathbf{N}}(\dot{\mathbf{F}})\,.$$

The constitutive function $\dot{\mathbf{N}}(\cdot)$ itself is functionally dependent on the strain history prior to the current instant, however, this dependence (as well as a piecewise-smooth dependence on ξ) will not be indicated explicitly for simplicity. Following HILL [4], we assume that Eq. (2.2) admits a continuously differentiable potential $U(\dot{\mathbf{F}})$ (necessarily homogeneous of degree two), viz.

$$(2.3)\qquad \dot{\mathbf{N}} = \frac{\partial U}{\partial \dot{\mathbf{F}}^T} = \mathbf{C}(\dot{\mathbf{F}}) \cdot \dot{\mathbf{F}}, \quad U(\dot{\mathbf{F}}) = \frac{1}{2}\dot{\mathbf{N}}(\dot{\mathbf{F}}) \cdot \dot{\mathbf{F}}, \quad \mathbf{C} = \frac{\partial^2 U}{\partial \dot{\mathbf{F}}^T \partial \dot{\mathbf{F}}^T}.$$

The resulting dependence of the instantaneous moduli $C_{ijkl} = C_{klij}$ on the direction of strain rate is assumed to be piecewise smooth but may be arbitrarily nonlinear and discontinuous. Existence of the potential and symmetry of the moduli are preserved when Eq. (2.2) is expressed in terms of rates of any *work-conjugate* pair of stress and strain measures [8]. For elastic materials, this symmetry property is a consequence of existence of a strain energy potential. It is well known [3] that the form (2.3) is valid for conventional elastoplastic models (with a smooth yield surface and a linear relationship (2.2) for the loading branch) if and only if the normality flow rule holds in work-conjugate variables. Equation (2.3) is acceptable also at a yield surface vertex but then it is no longer a consequence of the normality flow rule alone. More generally, Eqs. (2.3) may be adopted as a constitutive restriction for solids other than elastic-plastic.

2.3. Boundary value problem and the fundamental deformation process

We are concerned with isothermal deformations of the time-independent solid body subject to a quasi-static loading program. Let $\lambda = \lambda(t)$ denote a scalar loading parameter which varies continuously in "time" t (infinitely slowly with respect to a natural time); it is convenient to keep the parameters t and λ as being distinct since t is always monotonically increasing while $\lambda(t)$ need not be a monotonic function. The standard boundary value problem is considered below; an extension to a more general case of configuration-dependent *conservative* loading will be discussed later in Sect. 5. Assume that on a given nonzero part S_u of S the displacements $\mathbf{u} = \bar{\mathbf{u}}(\xi, \lambda)$ are controlled while on a complementary part S_T the nominal surface tractions $\mathbf{T} = \bar{\mathbf{T}}(\xi, \lambda)$ (per unit reference area) are controlled. Nominal body forces $\mathbf{b} = \bar{\mathbf{b}}(\xi, \lambda)$ (per unit reference volume) are prescribed in V; note that λ need not be a load multiplier. The loading functions are assumed to be at least piecewise smooth with respect to their arguments and to exhibit no strong discontinuities other than across material surfaces or lines. An initial equilibrium state of the body at a certain value of λ is assumed to be known.

Suppose that a theoretical quasi-static response of the body to the above loading program is known, at least in principle. The respective deformation process is described by $\mathbf{x} = \chi^0(\xi, t)$, $\xi \in \bar{V}$, and called the fundamental process (path); the superscript 0 is used to distinguish the corresponding values of quantities of interest. We assume that the function χ^0 is continuous and at least piecewise continuously three times differentiable with respect to its arguments and, moreover, that the respective deformation gradient $\mathbf{F}^0$ varies continuously in time while the velocity $\mathbf{v}^0$ is a continuous function of place.

Let $\overline{\mathcal{W}}$ denote the linear space of admissible virtual velocity fields at any *fixed* λ (the fields from $\overline{\mathcal{W}}$ vanish on a given part S_u of the material surface S). Different regularity restrictions could be imposed at this moment; we assume that $\overline{\mathcal{W}}$ consists of all continuous and piecewise continuously twice differentiable vector fields on $\bar{V}$ which vanish on S_u. At any λ, $\mathcal{V} = \mathcal{V}(\lambda)$ denotes the class of kinematically admissible velocity fields $\tilde{\mathbf{v}}$ of the form $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}^0 + \tilde{\mathbf{w}}$, $\tilde{\mathbf{w}} \in \overline{\mathcal{W}}$ (so that $\mathbf{v} = \dot{\lambda}\dfrac{\partial \overline{\mathbf{u}}}{\partial \lambda}(\xi, \lambda)$ on S_u). Note that any functional defined on $\mathcal{V}$ can equivalently be regarded as a certain other functional defined on the *linear* space $\overline{\mathcal{W}}$. Since the zero element of $\overline{\mathcal{W}}$ will have to be excluded in a number of formulae, it will be convenient to consider also the class $\mathcal{W} = \{\tilde{\mathbf{w}} \in \overline{\mathcal{W}} : \tilde{\mathbf{w}} \neq \tilde{\mathbf{0}}\}$.

In absence of moving strong discontinuities in **N**, **b** and **T**, and when inertia forces are neglected, any solution $\tilde{\mathbf{v}} \in \mathcal{V}$ to the first-order rate boundary value problem satisfies the rate form of the virtual work principle, viz.

$$(2.4) \qquad \int_V \dot{\mathbf{N}}(\nabla \mathbf{v}) \cdot \nabla \mathbf{w}\, dV = \int_V \dot{\mathbf{b}}\mathbf{w}\, dV + \int_{S_T} \dot{\mathbf{T}}\mathbf{w}\, dS\,, \qquad \text{for every} \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}\,.$$

The variational equality (2.4) can be taken as a definition of a first-order solution; under the regularity restrictions imposed above, Eq. (2.4) is equivalent, by the well known transformation, to the standard local conditions of continuing equilibrium in V and on S_T. In the fundamental process, Eq. (2.4) is satisfied by $\tilde{\mathbf{v}}^0(t)$ at every t; it is recalled that the rates are understood in the right-hand sense.

2.4. Basic functionals

The following three functionals play an essential role in the analysis of uniqueness and stability (cf. [4]):

$$(2.5) \qquad J(\tilde{\mathbf{v}}) \equiv \int_V (U(\nabla \mathbf{v}) - \dot{\mathbf{b}}\mathbf{v})\, dV - \int_{S_T} \dot{\mathbf{T}}\mathbf{v}\, dS\,, \qquad \tilde{\mathbf{v}} \in \mathcal{V}\,,$$

$$(2.6) \qquad I^0(\tilde{\mathbf{w}}) \equiv \frac{1}{2}\int_V \nabla \mathbf{w} \cdot \mathbf{C}^0 \cdot \nabla \mathbf{w}\, dV\,, \qquad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}\,,$$

$$(2.7) \qquad I(\tilde{\mathbf{w}}) \equiv \int_V U(\nabla \mathbf{w})\, dV\,, \qquad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}\,.$$

The functionals are considered here at a certain stage of the fundamental deformation process. The quadratic functional I^0 is based on the so-called tangent moduli $\mathbf{C}^0 = \mathbf{C}(\nabla \mathbf{v}^0(\xi), \xi)$ corresponding to the fundamental continuation of deformation. The functional I^0 is well defined if the constitutive potential U is twice differentiable at $\nabla \mathbf{v}^0$ almost everywhere in V, i.e. except possibly in a region of zero volume only (on the elastic-plastic interface, for instance). This is tacitly assumed below whenever the functional I^0 appears in considerations.

A velocity field $\tilde{\mathbf{v}} \in \mathcal{V}$ is a solution to the rate boundary value problem if and only if it assigns to the functional (2.5) a stationary value [4], that is

$$(2.8) \qquad \delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) = 0 \quad \text{for every} \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}\,.$$

Suppose now that a fundamental first-order solution $\tilde{\mathbf{v}}^0$ is known, which is common case in practice. Uniqueness of $\tilde{\mathbf{v}}^0$ is then ensured when [8]

$$(2.9) \qquad \int_V (\dot{\mathbf{N}}(\nabla \mathbf{v}) - \dot{\mathbf{N}}^0) \cdot (\nabla \mathbf{v} - \nabla \mathbf{v}^0)\, dV > 0 \quad \text{for every} \quad \tilde{\mathbf{v}} \in \mathcal{V}\,, \quad \tilde{\mathbf{v}} \neq \tilde{\mathbf{v}}^0\,.$$

Under the stronger condition that inequality (2.9) is valid if $\tilde{\mathbf{v}}^0$ is replaced by *any* admissible field from $\mathcal{V}$, it was proved in [4, 8] that a velocity solution $\tilde{\mathbf{v}}^0$ strictly minimizes the value of $J(\tilde{\mathbf{v}})$ in $\mathcal{V}$, viz.

$$J(\tilde{\mathbf{v}}) > J(\tilde{\mathbf{v}}^0) \quad \text{for every} \quad \tilde{\mathbf{v}} \in \mathcal{V}, \quad \tilde{\mathbf{v}} \neq \tilde{\mathbf{v}}^0. \tag{2.10}$$

As an extension of this result, we shall prove the following proposition:

If $\tilde{\mathbf{v}}^0$ *is a solution to* Eq. (2.4) *then* (2.9) *implies* (2.10).

In proof, observe first that

$$\delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) = \int_V (\dot{\mathbf{N}}(\nabla \mathbf{v}) - \dot{\mathbf{N}}^0) \cdot \nabla \mathbf{w} \, dV, \tag{2.11}$$

on subtracting $\delta J(\tilde{\mathbf{v}}^0, \tilde{\mathbf{w}}) = 0$ from the expression for $\delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{w}})$. It follows that Eq. (2.9) is equivalent to

$$\delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{v}} - \tilde{\mathbf{v}}^0) > 0 \quad \text{for every} \quad \tilde{\mathbf{v}} \in \mathcal{V}, \quad \tilde{\mathbf{v}} \neq \tilde{\mathbf{v}}^0; \tag{2.12}$$

in passing, we note that (2.12) excludes (2.8) except for $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}^0$, implying uniqueness of the solution $\tilde{\mathbf{v}}^0$ on another route. From (A.8)(3) or (A.9) we obtain that if (2.12) holds then $J(\tilde{\mathbf{v}})$ is increasing along any straight "ray" in $\mathcal{V}$ with the origin at $\tilde{\mathbf{v}}^0$. This completes the proof of (2.10). It is worth mentioning that (2.12) does not imply convexity of the functional J.

Direct substitution shows that the uniqueness condition (2.9) holds if

$$I^0(\tilde{\mathbf{w}}) > 0 \quad \text{for every} \quad \tilde{\mathbf{w}} \in \mathcal{W}, \tag{2.13}$$

provided that I^0 is well defined and that the following constitutive inequality is satisfied for $\dot{\mathbf{F}}^0 \equiv \nabla \mathbf{v}^0$ [19]

$$(\dot{\mathbf{N}} - \dot{\mathbf{N}}^0) \cdot (\dot{\mathbf{F}} - \dot{\mathbf{F}}^0) \geq (\dot{\mathbf{F}} - \dot{\mathbf{F}}^0) \cdot \mathbf{C}(\dot{\mathbf{F}}^0) \cdot (\dot{\mathbf{F}} - \dot{\mathbf{F}}^0) \quad \text{for every } \dot{\mathbf{F}}. \tag{2.14}$$

This is an extension of the well known "comparison theorem" [4] based on the concept of relative convexity of the constitutive potential with respect to that for an incrementally linear comparison material. It is pointed out that (2.14) is generally a weaker restriction than the relative convexity property with respect to the "tangent" comparison material since $\dot{\mathbf{F}}^0$ in (2.14) is *fixed*. That straightforward modification of Hill's theory brings a definite advantage since (2.14) can be derived from micromechanical considerations [20].

Comparison with the standard theorem of the calculus of variations suggests that

$$I^0(\tilde{\mathbf{w}}) = \delta^2 J(\tilde{\mathbf{v}}^0, \tilde{\mathbf{w}}) \tag{2.15}$$

whenever I^0 is well defined, where

$$\delta^2 J(\tilde{\mathbf{v}}^0, \tilde{\mathbf{w}}) \equiv \frac{1}{2} \lim_{\gamma \to 0} (\delta J(\tilde{\mathbf{v}}^0 + \gamma \tilde{\mathbf{w}}, \tilde{\mathbf{w}}) - \delta J(\tilde{\mathbf{v}}^0, \tilde{\mathbf{w}}))/\gamma \tag{2.16}$$

is the second weak variation of J at $\tilde{\mathbf{v}}^0$ in the direction $\tilde{\mathbf{w}}$. However, the regularity assumptions of the theorem are not satisfied and the proof requires some modification. To prove (2.15) directly, substitute (2.11) into (2.16), which yields

$$\delta^2 J(\tilde{\mathbf{v}}^0, \tilde{\mathbf{w}}) = \frac{1}{2} \lim_{\gamma \to 0} \int_V \frac{1}{\gamma} (\dot{\mathbf{N}}(\nabla \mathbf{v}^0 + \gamma \nabla \mathbf{w}) - \dot{\mathbf{N}}(\nabla \mathbf{v}^0)) \cdot \nabla \mathbf{w} \, dV. \tag{2.17}$$

If the tangent moduli $\mathbf{C}^0$ are well defined almost everywhere in V then the integrand in (2.17) tends, as $\gamma \to 0$, to the integrand in (2.6) almost everywhere in V. On account of

(3) The letter "A" refers to formulae given in Appendix.

the estimate (A.1) the integrand in (2.17) for fixed $\tilde{\mathbf{w}}$ is uniformly bounded so that we can use the known theorem on the limiting passage under the integral sign. This completes the proof of (2.15).

2.5. Energy interpretation

The work of deformation in the body can be written down as

$$W = \int_V \int \mathbf{N} \cdot d\mathbf{F}\, dV\,, \tag{2.18}$$

where the stresses are determined pointwise by integration of the constitutive rate equations (2.2) along the deformation path. A potential energy of the loading device which applies nomial surface tractions $\overline{\mathbf{T}}$ and nomial body forces $\overline{\mathbf{b}}$ independently of the body configuration can be expressed as

$$\Omega = \Omega(\tilde{\mathbf{u}}, \lambda) = -\int_V \overline{\mathbf{b}}\mathbf{u}\, dV - \int_{S_T} \overline{\mathbf{T}}\mathbf{u}\, dS\,. \tag{2.19}$$

Introduce the *energy functional* [18, 19]

$$E = W + \Omega \tag{2.20}$$

defined for any kinematically admissible deformation process at varying or fixed λ. In general, E is a functional of the deformation history due to path-dependence of W. An increment of the value of E can be interpreted as the amount of energy which has to be supplied from external sources to the mechanical system consisting of the deformed body and the loading device in order to produce quasi-statistically a deformation increment, generally with the help of additional perturbing forces(4). It is emphasized that an increment of the value of $(-\Omega)$ is generally *not* equal to the work done by the controlled loads unless $\overline{\mathbf{b}}$ and $\overline{\mathbf{T}}$ are constant in time.

By appealing to the virtual work principle it can be shown that the body is in equilibrium if and only if $\dot{E}(\tilde{\mathbf{v}})$, the first time derivative of the expression (2.20), has the same value for all kinematically admissible velocity fields from $\mathcal{V}$. The value of the second time derivative $\ddot{E}$ of E, when evaluated at an equilibrium state, does not depend on accelerations and differs from $2J(\tilde{\mathbf{v}})$ only by a state-dependent constant, that is [18, 19](5).

$$\frac{1}{2}\ddot{E}(\tilde{\mathbf{v}}_1) - \frac{1}{2}\ddot{E}(\tilde{\mathbf{v}}_2) = J(\tilde{\mathbf{v}}_1) - J(\tilde{\mathbf{v}}_2) \quad \text{for every} \quad \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2 \in \mathcal{V}\,. \tag{2.21}$$

The results quoted in the preceding subsection can now be given an energy interpretation, simply by replacing J by $\ddot{E}$ in the formulae (2.8), (2.10), (2.12), (2.15). Explicitly, the stationarity principle (2.8) valid for any solution $\tilde{\mathbf{v}}$ becomes

$$\delta \ddot{E}(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) = 0 \quad \text{for every} \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}\,. \tag{2.22}$$

The functional I^0 which appears in the uniqueness condition (2.13) and in the path insta-

(4) It is possible to consider an enlarged system consisting of the mechanical system placed in an ideal heat reservoir. In a quasi-static deformation process, an increment of E can be identified with an increment of the total internal energy of the enlarged system.

(5) There is a misprint in the proof given in [19]: in Eq. (18) there should be $\Delta\ddot{W}$ instead of $\ddot{W}$.

bility condition (2.30) below, has the following interpretation

$$I^{0}(\tilde{\mathbf{w}}) = \frac{1}{2}\delta^2 \ddot{E}(\tilde{\mathbf{v}}^{0}, \tilde{\mathbf{w}}), \quad \tilde{\mathbf{w}} \in \mathcal{W}. \tag{2.23}$$

The minimum property (2.10), implied by (2.9), is equivalent to

$$\ddot{E}(\tilde{\mathbf{v}}) > \ddot{E}(\tilde{\mathbf{v}}^{0}) \quad \text{for every} \quad \tilde{\mathbf{v}} \in \mathcal{V}, \quad \tilde{\mathbf{v}} \neq \tilde{\mathbf{v}}^{0}. \tag{2.24}$$

It is of interest to recall here the work interpretation of $J(\tilde{\mathbf{v}})$ which, unlike (2.21), is restricted to velocity *solutions*. If $\tilde{\mathbf{v}}$ satisfies (2.4) then we have the formula [5]

$$J(\tilde{\mathbf{v}}) = \frac{1}{2}\Big(\int_{S_u} \dot{\mathbf{T}}\dot{\bar{\mathbf{u}}}\, dS - \int_V \dot{\bar{\mathbf{b}}}\mathbf{v}\, dV - \int_{S_T} \dot{\bar{\mathbf{T}}}\mathbf{v}\, dS \Big), \tag{2.25}$$

where $\dot{\mathbf{T}} = \boldsymbol{\nu}\dot{\mathbf{N}}(\nabla \mathbf{v})$ is the reaction-rate on S_u. When multiplied by the square of a small time increment, this is the second-order work done on controlled displacements minus the second-order work done by controlled loads.

If λ is fixed then an increment of E along any kinematically admissible path is equal to the work of deformation minus the work of the given loads and, for a deformation process starting from an equilibrium state with a velocity field $\tilde{\mathbf{w}}$, we have [3]

$$\frac{1}{2}\ddot{E}(\tilde{\mathbf{w}}) = I(\tilde{\mathbf{w}}) \quad \text{at} \quad \lambda = \text{const}, \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}. \tag{2.26}$$

This is in agreement with (2.21) since at constant λ the functional J reduces to I. Contrary to that for the functionals J and I^{0}, the energy interpretation of I is well known.

2.6. Energy criteria of instability

As an object of the stability analysis we take a segment of the fundamental process χ^{0} in some interval of t for a given loading program, generally at varying λ. In other words, we wish to investigate stability of a quasi-static motion of *a system* consisting of the deformed body and the loading device. Instability of equilibrium is regarded as a particular case of instability of the fundamental deformation process when λ is fixed. More precisely, at any deformation stage (at $t = t_1$, say), the loading program may be stopped so that λ is kept fixed while the parameter t is increasing further up to $t = t_2$, say; the loading program can then be continued with the shift in "time" t by $t_2 - t_1$. Under the assumption of quasi-static and rate-independent deformations, the fundamental deformation process is then described by $\mathbf{x} = \hat{\chi}^{0}(\xi, t) = \chi^{0}(\xi, \hat{t})$ where $\hat{t} = t$, $\hat{t} = t_1$ or $\hat{t} = t - (t_2 - t_1)$ for $t < t_1$, $t_1 \le t \le t_2$ or $t > t_2$, respectively. The distinction between the processes χ^{0} and $\hat{\chi}^{0}$ is purely formal and they are thus identified with each other. Instability of the equilibrium state $\chi^{0}(\xi, t_1)$ is now identified with instability of the process $\hat{\chi}^{0}$ on an interval $[t_1, t_2)$ which represents an infinite interval of a natural time. If we assume that a process χ^{0} stable on some interval of t must be stable also on any subinterval, then instability of an equilibrium state (at $t = t_1$, say) implies instability (on any open interval containing t_1) of the process χ^{0} going through this state.

Without assuming any particular definition of stability, we define now the concept of "instability in the energy sense". An equilibrium state is said to be unstable in the energy sense if

$$(2.27) \qquad \ddot{E}(\tilde{\mathbf{w}}) < 0 \quad \text{for some} \quad \tilde{\mathbf{w}} \in \mathcal{W} \quad \text{at} \quad \lambda = \text{const}.$$

This condition is equivalent to that given in [9] as a *part* of a set of sufficiency conditions for instability of equilibrium in a dynamic sense. Although (2.27) means that a spontaneous departure from the equilibrium state is energetically possible, no convincing general argument has been given yet, to the author's knowledge, that (2.27) *alone* is sufficient for instability of equilibrium in a dynamic sense.

The fundamental *process* χ^0 is said to be unstable in the energy sense at some stage of the deformation if the respective velocity solution $\tilde{\mathbf{v}}^0$ does not minimize the value of $\ddot{E}$ in $\mathcal{V}$, that is, if

$$(2.28) \qquad \ddot{E}(\tilde{\mathbf{v}}) < \ddot{E}(\tilde{\mathbf{v}}^0) \quad \text{for some} \quad \tilde{\mathbf{v}} \in \mathcal{V}.$$

Alternatively, as a condition necessary for stability of the fundamental process in the energy sense, we take

$$(2.29) \qquad \ddot{E}(\tilde{\mathbf{v}}) \geq \ddot{E}(\tilde{\mathbf{v}}^0) \quad \text{for every} \quad \tilde{\mathbf{v}} \in \mathcal{V}.$$

This condition was proposed by the author [18, 19] as an intuitive criterion for predicting instability of a plastic deformation process([6]). In the special cases when $\lambda = \text{const}$ or when U is quadratic, (2.28) reduces to (2.27). Since at an equilibrium state the value of $\dot{E}(\tilde{\mathbf{v}})$ is independent of $\tilde{\mathbf{v}}$, the condition (2.28) (or (2.29)) can be written down in an equivalent incremental form as $\Delta E < \Delta E^0$ (or $\Delta E \geq \Delta E^0$), where ΔE^0 and $\Delta E \equiv \dot{E}\Delta t + \frac{1}{2}\ddot{E}(\Delta t)^2$ are increments of E evaluated to *second order* along the fundamental path and along any kinematically admissible branching path, respectively.

On the basis of the identity (2.21) and with the help of the constitutive inequality (2.14), the onset of instability found from this energy criterion has been shown [19] to coincide with the primary bifurcation point for a class of plasticity problems. We recall briefly that (2.28) (but not (2.27)) is implied by

$$(2.30) \qquad I^0(\tilde{\mathbf{w}}) < 0 \quad \text{for some} \quad \tilde{\mathbf{w}} \in \mathcal{W},$$

on account of (2.23). In typical cases examined so far in the literature, (2.30) is met along the *primary* deformation path immediately beyond the instant of primary bifurcation. In turn, by comparison with the minimum property (2.24), the inequality (2.29) holds in the uniqueness range (2.9). If (2.14) is valid then (2.29) holds exactly as long as $I^0(\tilde{\mathbf{w}})$ is nonnegative on $\mathcal{W}$. This follows from the implication: (2.13) $\Rightarrow$ (2.9) $\Rightarrow$ (2.24) which remains true when ($>$) is replaced in each of the inequalities by ($\geq$).

By appealing to homogeneity and continuity of $U(\dot{\mathbf{F}})$ it can be shown that (2.27) implies (2.28) [19] but the converse need not be true. This is in accord with the relation between instability of equilibrium and instability of a deformation process, and allows to treat (2.28) as a single energy criterion applicable to a variety of problems of inelastic instability. Below we will examine the question whether fulfillment of the condition (2.28) (or of (2.27) in particular) is connected with an instability in a physically acceptable sense.

([6]) The criterion was initially formulated [18] as a consequence of a more stringent postulate of stability of a deformation process at presence of persistent disturbances for which an energy measure was adopted. Later [19], the criterion in an incremental form was introduced as an independent hypothesis.

3. Discretized problems

In this section, admissible variations of velocities (or of displacements) are restricted to the form

$$\mathbf{w}(\xi) = \sum_{K=1}^{N} w_K \boldsymbol{\beta}_K(\xi), \tag{3.1}$$

where $\boldsymbol{\beta}_K$ are given "shape functions" from the class $\mathcal{W}$ and w_K are arbitrary scalar coefficients. Their number N is finite so that the fields (3.1) form a finite-dimensional subspace $\overline{\mathcal{W}}^N$ of $\overline{\mathcal{W}}$; it will be convenient to define also a set $\mathcal{W}^N = \{\tilde{\mathbf{w}} \in \overline{\mathcal{W}}^N : \tilde{\mathbf{w}} \neq \tilde{\mathbf{0}}\}$. The definition of a solution is accordingly changed, following the standard discretization procedure.

3.1. Instability of equilibrium

Throughout this subsection λ is kept constant, t is identified with a natural time, and inertia forces are taken into account. Let $\Delta\mathbf{N}(t) = \mathbf{N}(t) - \mathbf{N}^0$ denote a stress increment along a deformation path starting from an equilibrium state. By subtracting the virtual work expressions in the initial and current instants, the discretized equations of motion can be written in the form

$$\int_V (\Delta\mathbf{N} \cdot \nabla\mathbf{w} + \bar{\varrho}\mathbf{a}\mathbf{w})\, dV = 0 \quad \text{for every} \quad \mathbf{w} \in \overline{\mathcal{W}}^N, \tag{3.2}$$

where $\bar{\varrho}$ is the material density in the reference configuration and $\mathbf{a} = \ddot{\mathbf{u}}$ denote accelerations.

We will examine the possibility of departure from an equilibrium configuration $\tilde{\mathbf{u}}^0$ along a *direct* path of deformation which, by definition, can be approximated by a straight path

$$\tilde{\mathbf{u}}(t) = \tilde{\mathbf{u}}^0 + A(t)\tilde{\mathbf{w}}, \qquad A(0) = 0, \qquad \dot{A} \geq 0, \tag{3.3}$$

for sufficiently small values of A and for some fixed $\tilde{\mathbf{w}} \in \mathcal{W}^N$. Along straight (direct) paths, the direction of velocity gradient remains (approximately) constant and the rates in (2.2) can be replaced by small increments of $\mathbf{N}$ and $\mathbf{F}$ with a negligible error. Hence, for a path (3.3) and small A we obtain

$$\Delta\mathbf{N}(t) = A(t)\dot{\mathbf{N}}_{\mathrm{eq}}(\nabla\mathbf{w}), \tag{3.4}$$

where $\dot{\mathbf{N}}_{\mathrm{eq}}(\cdot)$ denotes the constitutive function at the considered equilibrium state. The approximation involved in (3.4) is similar to that in the linear theory of elastic stability, however, the distinction is that in path-dependent inelastic solids (3.4) need not be a valid approximation for circuitous paths corresponding to variable $\tilde{\mathbf{w}}$.

Since $\mathcal{W}^N$ is finite-dimensional and U depends continuously on $\dot{\mathbf{F}}$, the functional (2.7) is continuous in $\mathcal{W}^N$ and, when constrained to a level set of the function

$$\Phi(\tilde{\mathbf{w}}) \equiv \frac{1}{2} \int_V \bar{\varrho}|\mathbf{w}|^2\, dV, \tag{3.5}$$

attains a minimum at some $\mathbf{w}^* \in \mathcal{W}^N$. By the method of Lagrangian multipliers, there is a number μ such that

$$(3.6) \qquad \delta I(\tilde{\mathbf{w}}^*, \tilde{\mathbf{w}}) = \mu\, \delta\Phi(\tilde{\mathbf{w}}^*, \tilde{\mathbf{w}}) \quad \text{for every} \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}^N .$$

At the equilibrium state, this reads

$$(3.7) \qquad \int\limits_V \dot{\mathbf{N}}_{\mathrm{eq}}(\nabla \mathbf{w}^*) \cdot \nabla \mathbf{w}\, dV = \mu \int\limits_V \varrho \mathbf{w}^* \mathbf{w}\, dV , \quad \text{for every} \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}^N .$$

On multiplying both sides of (3.7) by A and using (3.4) we obtain that the stresses along the deformation path (3.3) satisfy (within the approximations involved in (3.4)) the discretized equations of motion (3.2) provided in (3.3) we take $\tilde{\mathbf{w}} = \tilde{\mathbf{w}}^*$ and $A(t)$ such that

$$(3.8) \qquad \ddot{A} = -\mu A .$$

Clearly, the sign of μ decides whether there is a tendency to decrease or to increase the speed of departure from equilibrium.

By homogeneity of degree two of I and Φ, substitution of $\tilde{\mathbf{w}} = \tilde{\mathbf{w}}^*$ into (3.6) yields $I(\tilde{\mathbf{w}}^*) = \mu\Phi(\tilde{\mathbf{w}}^*)$ and

$$(3.9) \qquad \mu = \min_{\tilde{\mathbf{w}} \in \mathcal{W}^N} \frac{I(\tilde{\mathbf{w}})}{\Phi(\tilde{\mathbf{w}})} .$$

Suppose that the equilibrium state is unstable in the energy sense (2.27), with $\mathcal{W}$ replaced by $\mathcal{W}^N$. From (2.26) we obtain that $I(\tilde{\mathbf{w}})$ takes negative values for some $\tilde{\mathbf{w}} \in \mathcal{W}^N$ so that $\mu < 0$. As an admissible function $A(t)$ in (3.3) satisfying (3.8) we can thus take

$$(3.10) \qquad A(t) = (\varepsilon/\kappa)\, \mathrm{sh}(\kappa t) , \quad \kappa = (-\mu)^{1/2} , \quad \varepsilon > 0$$

which corresponds to a free inertial motion starting from the equilibrium state at $t = 0$ with initial velocities $\varepsilon \mathbf{w}^*$. Evidently, a given small finite distance from the initial equilibrium configuration is exceeded for t sufficiently large no matter how small ε is. We have thus proved the following theorem on instability in the first approximation:

If an equilibrium state of a discretized system is unstable in the energy sense (2.27) *(with $\mathcal{W}$ replaced by $\mathcal{W}^N$) then it is also unstable in the dynamic sense for vanishingly small initial disturbances, under the approximation* (3.4)([7]).

An alternative to (3.10) is $A(t) = (\varepsilon/\kappa)(\cosh(\kappa t) - 1)$ which corresponds to another disturbance: initial velocities at $t = 0$ are not perturbed but rather a system of perturbing forces is applied. For that function $A(t)$ we have $\ddot{A} = -\mu A + \varepsilon\kappa$, and by an analogous argument as above we obtain that the perturbing forces corresponding to the path (3.3) with $\tilde{\mathbf{w}} = \tilde{\mathbf{w}}^*$ are constant in time and are vanishingly small when ε tends to zero. For $t \geq \tau$, say, we can take $A(t) = (\varepsilon/\kappa)(\cosh(\kappa t) - \cosh(\kappa(t - \tau)))$ which corresponds to a further motion free of perturbing forces since (3.8) is again satisfied. Hence, under the same approximations as above, instability of equilibrium in the energy sense (2.27) can be given the following interpretation: *if* (2.27) *holds then a finite distance from the equilibrium configuration can be reached in a dynamic motion caused by arbitrarily small perturbing forces.*

The method used here to demonstrate dynamic instability of equilibrium of an inelastic time-independent system is an extension of the well known "kinetic" method for linear

([7]) It appears likely that under reasonable regularity restrictions on the variation of the constitutive response $\dot{\mathbf{N}}(\dot{\mathbf{F}})$ along a smooth deformation path, a dynamic path with initial velocities $\varepsilon\tilde{\mathbf{w}}^*$ at $\mu < 0$ will be sufficiently close to the straight path in order to demonstrate instability of a discretized system without any constitutive approximations.

elastic systems where μ is defined as a square of the lowest natural frequency of vibrations. Of course, for inelastic solids μ defined by (3.9) has no longer this special interpretation. At the present generality, the above proof of instability seems to be new.

The limiting case when I is just positive semi-definite ($\mu = 0$) corresponds, under the approximation (3.4), also to dynamic instability of the considered equilibrium configuration which can be left with velocities $\varepsilon\tilde{\mathbf{w}}^*$ constant in time([8]).

Hence, under the approximation (3.4), a departure from equilibrium in a direction $\tilde{\mathbf{w}}^*$ is dynamically possible at vanishingly small disturbances unless

$$I(\tilde{\mathbf{w}}) > 0 \quad \text{for every} \quad \tilde{\mathbf{w}} \in \mathcal{W}^N . \tag{3.11}$$

On the other hand, (3.11) excludes such a departure in any direction from $\mathcal{W}^N$ by ensuring an insufficiency of energy [3, 4], as can be seen from the identity (2.26). The condition (3.11) is thus necessary and sufficient for a kind of stability which may be called *directional stability of equilibrium* (of a discretized system). It must be pointed out that due to path-dependence of the material, (3.11) does not guarantee stability for arbitrarily circuitous paths unless further assumptions are introduced (cf. [3, 4, 9, 17]).

3.2. Instability of a deformation process

Consider now the fundamental process χ^0 of quasi-static deformation at varying λ. Admissible velocity fields in the discretized problem have the form $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}^0 + \tilde{\mathbf{w}}$ where $\tilde{\mathbf{w}} \in \overline{\mathcal{W}}^N$; the class of such fields $\tilde{\mathbf{v}}$ is denoted by $\mathcal{V}^N$. A velocity field $\tilde{\mathbf{v}} \in \mathcal{V}^N$ is called a solution to the discretized (quasi-static) rate-problem if (2.4) holds for every $\tilde{\mathbf{w}} \in \overline{\mathcal{W}}^N$, and not necessarily for every $\tilde{\mathbf{w}} \in \overline{\mathcal{W}}$. In this subsection the fundamental velocity field $\tilde{\mathbf{v}}^0$ needs to be a solution in this weakened sense only. In analogy to (2.8), any solution $\bar{\mathbf{v}}$ to the discretized rate-problem is characterized by the stationarity property

$$\delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) = 0 \quad \text{for every} \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}^N . \tag{3.12}$$

Since U is a continuously differentiable function of $\dot{\mathbf{F}}$, J is differentiable in $\mathcal{V}^N$ and it follows that a minimizer of J in $\mathcal{V}^N$ is automatically a solution. In general, J may be unbounded from below so that an absolute minimum need not be attained. However, this cannot happen if the equilibrium state at which the discretized rate-problem is formulated is directionally stable in the sense of (3.11). As shown in Subsect. A.3 of Appendix we have the following

THEOREM 1. *If (3.11) holds then the discretized rate-problem has a solution which assigns to the functional J its absolute minimum value in $\mathcal{V}^N$.*

This existence theorem provides an interpretation for instability of the fundamental deformation process in the energy sense (2.28) when equilibrium states in the process are (directionally) stable. In the range of validity of (2.24), the solution guaranteed by the above theorem coincides with $\tilde{\mathbf{v}}^0$. However, if (2.28) holds true (with $\mathcal{V}$ replaced by $\mathcal{V}^N$), or equivalently, if

$$J(\tilde{\mathbf{v}}) < J(\tilde{\mathbf{v}}^0) \quad \text{for some} \quad \tilde{\mathbf{v}} \in \mathcal{V}^N , \tag{3.13}$$

([8]) When higher-order terms are taken into account then, depending on the problem, we can expect either instability of equilibrium (at constant λ) or path bifurcation at varying λ (with $\dot{\lambda} = 0$ initially), in analogy to the critical state of an elastic conservative system.

then the solution guaranteed by the Theorem 1 must be distinct from the fundamental solution $\tilde{\mathbf{v}}^0$. Hence, we have the following Corollary:

At any stage of deformation of a discretized system, if (3.11) *holds and the fundamental process is unstable in the energy sense* (2.28) *(with $\mathcal{V}$ replaced by $\mathcal{V}^N$), then there is a velocity solution different from $\tilde{\mathbf{v}}^0$.*

Evidently, the solution in velocities is nonunique also when $\tilde{\mathbf{v}}^0$ does minimize J in $\mathcal{V}^N$ but the minimum is not strict.

If the fundamental moduli $\mathbf{C}^0$ corresponding to the solution $\tilde{\mathbf{v}}^0$ are well defined almost everywhere in V then, by (2.15), a necessary condition for $\tilde{\mathbf{v}}^0$ to be a minimizer of J in $\mathcal{V}^N$ is that

$$I^0(\tilde{\mathbf{w}}) \geq 0 \quad \text{for every} \quad \tilde{\mathbf{w}} \in \mathcal{W}^N , \tag{3.14}$$

or equivalently, that the tangent stiffness matrix is positive definite or at least positive semi-definite (cf. [21]). Hence, we have proved the following theorem:

For a discretized system, there is a bifurcation in velocities at every point on the fundamental path along which (3.11) *holds and* (3.14) *does not.*

This generalizes the well known observation [23, 12, 10] that the incremental response of a straight plastic column under increasing compressive loading is nonunique between the tangent modulus load (above which (3.14) fails) and the reduced modulus load (at which (3.11) fails).

Contrary to the bifurcation at the typical critical instant at which I^0 is just positive semi-definite, in the range (3.13) the secondary solution $\tilde{\mathbf{v}}^*$ which minimizes $J(\tilde{\mathbf{v}})$ in $\mathcal{V}^N$ must correspond to a moduli field different from the fundamental moduli field $\tilde{\mathbf{C}}^0$. For, if the moduli fields were the same, then straightforward transformations with the help of symmetry of the moduli would yield $J(\tilde{\mathbf{v}}^*) - J(\tilde{\mathbf{v}}^0) = I^0(\tilde{\mathbf{v}}^* - \tilde{\mathbf{v}}^0)$. This would contradict (3.13) since from (2.11) and (3.12) we would have $I^0(\tilde{\mathbf{v}}^* - \tilde{\mathbf{v}}^0) = \delta J(\tilde{\mathbf{v}}^*, \tilde{\mathbf{v}}^* - \tilde{\mathbf{v}}^0) = 0$.

We may assume that there exists a quasi-static deformation path initiated with the displacement rate field $\tilde{\mathbf{v}}^*$ (for at least one minimizer if the minimum is not strict). Since $\tilde{\mathbf{v}}^* \neq \tilde{\mathbf{v}}^0$, the bifurcating path deviates from the fundamental one on a finite distance after some time increment, without the need of any disturbances if we neglect inaccuracy of the solution due to discretization of the problem. If this were only possible at an isolated instant then this would not suffice to conclude about instability of the fundamental post-bifurcation branch. However, as shown above, if (3.11) and (3.13) hold along a segment of the fundamental path then the bifurcation is possible at *every* point of the segment. This must be interpreted as an instability of the fundamental deformation process. For, if the velocity solutions at a bifurcation point have comparable chances to determine the actual continuation of the deformation then any segment of the considered fundamental path, no matter how short, has zero probability to be followed due to existence of infinitely many alternatives. Moreover, the secondary continuations of the deformation are energetically preferable, according to (2.21), the incremental form of (2.28) and the interpretation of an increment of E. From a physical point of view, this provides an additional argument for rejecting the fundamental solution (9).

(9) Recently, BAZANT [1] considered a possibility of deriving a criterion of path stability from thermodynamic considerations. Not going into a discussion on thermodynamic aspects of stability, we mention only that within the class of problems considered here the final criterion of choice of a stable path obtained by BAZANT (Eq. (35) in [1]) represents, on account of (2.25) and (2.22), a particular case of the energy criterion (2.29) (cf. Eq. (17) in [18]).

To summarize, if a discretized problem is considered then the energy inequality (2.28) with $\mathcal{V}$ replaced by $\mathcal{V}^N$, equivalent to (3.13), can be accepted as a sufficient condition for instability of the fundamental deformation process in the following sense. If (3.13) is satisfied when (3.11) fails (typically beyond a limit point) then *at any fixed* λ the current equilibrium state is unstable in the dynamic sense discussed. If (3.13) is satisfied in some interval of λ when (3.11) still holds (typically beyond a primary bifurcation point if (3.14) fails) then the fundamental deformation path can be left quasi-statically *starting from any* λ, even in absence of disturbances. This confirms the previous supposition that fulfillment of (2.29) is necessary for path stability.

It is worth pointing out that the instability condition (2.28) is applicable irrespectively of the character of the critical point itself (for instance, instability may be induced by a discontinuous drop of the incremental stiffness of the material, without singularity of the tangent stiffness matrix at the critical point). It can be applied also to a secondary post-bifurcation path since it is a matter of choice which path is regarded as fundamental. Using (2.29) as a necessary condition for path stability, PETRYK and THERMANN [21] have proposed a new computational method for crossing bifurcation points with automatic rejection of an unstable post-bifurcation branch.

4. Instability of continuous systems

The results obtained in the preceding section for a discretized problem cannot be automatically extended to infinite-dimensional continuous systems. An essential point in proving instability was that a minimum of the considered functionals was actually attained, which need not be the case when the functional space $\overline{\mathcal{W}}$ is infinite-dimensional.

Measures of the distance between two configurations of a continuous body are in general not equivalent to each other. Taking into account that rigid body displacements at fixed λ have been excluded, we will use the measure

$$\|\tilde{\mathbf{w}}\| \equiv \left(\int_V |\nabla \mathbf{w}|^2 \, dV \right)^{1/2} \tag{4.1}$$

which is a norm on the linear space $\overline{\mathcal{W}}$ (cf. also the estimates (A.5)).

4.1. Directional stability of equilibrium

If (2.27) holds and $\mathcal{W}^N$ approximates $\mathcal{W}$ sufficiently well then $\ddot{E}(\tilde{\mathbf{w}})$ takes negative values also in $\mathcal{W}^N$ so that μ defined by (3.9) is negative. Hence, instability of equilibrium in the dynamic sense discussed in the preceding section can be inferred from (2.27) for any sufficiently "fine" discretization of the problem. On this basis we may assume, as a hypothesis, that the condition (2.27) is sufficient for a dynamic instability of equilibrium also for a non-discretized continuum. This is confirmed when a minimum of $I(\tilde{\mathbf{w}})$ on a hypersurface $\Phi(\tilde{\mathbf{w}})$ = const in $\mathcal{W}$ is actually attained so that the proof from Sect. 3, with $\mathcal{W}^N$ replaced by $\mathcal{W}$, remains valid. A general proof that (2.27) implies dynamic instability of a continuum is lacking at present; note that the functional $I(\tilde{\mathbf{w}})$ may be unbounded from below on a hypersurface $\Phi(\tilde{\mathbf{w}})$ = const in an infinite-dimensional space.

For the discretized problem, fulfillment of the condition (3.11) ensured that the distance from equilibrium attainable on a straight (direct) path vanished with a vanishing input of the energy supplied by a disturbance. The same inequality in an infinite-dimensional space does not ensure this rigorously since $I(\tilde{\mathbf{w}})$ may approach zero in a non-trivial way without achieving it. A possible extension of (3.11) to the continuum problem is that $I(\mathbf{w})$ is *uniformly* positive, that is

$$I(\tilde{\mathbf{w}}) \geq a\|\tilde{\mathbf{w}}\|^2 \quad \text{for every} \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}, \tag{4.2}$$

where a is a positive parameter, independent of place, which may be arbitrarily small. Equilibrium states of a continuum are said to be directionally stable if (4.2) holds; in a finite-dimensional space the conditions in (3.11) and (4.2) are equivalent to each other. As already remarked in Sect. 3, without further assumptions (4.2) cannot be regarded as being sufficient for dynamic stability of equilibrium for arbitrarily circuitous paths.

The parameter a can be defined as

$$a = \inf_{\tilde{\mathbf{w}} \in \mathcal{W}} \frac{I(\tilde{\mathbf{w}})}{\|\tilde{\mathbf{w}}\|^2}. \tag{4.3}$$

From (A.3) with $\tilde{\mathbf{v}} = \tilde{\mathbf{0}}$ it follows that a has always a finite value. We can expect that in typical circumstances a will vary along a deformation path and differ from zero with the exception of isolated points between the ranges where $a > 0$ and $a < 0$, that is, between the ranges of validity of (4.2) and (2.27). This means that in practice the condition (4.2) will frequently represent a negation of (2.27), possibly except at a critical point itself. Note that if the infimum in (4.3) is equal to zero and is actually reached in $\mathcal{W}$, then we have an *eigenstate* [9, 7].

4.2. Instability of a deformation process

Consider a certain stage of the fundamental process. We shall assume below that (4.2) holds for some constant $a > 0$. Then it can be proved (see Subsect. A.2 of Appendix) that

$$J(\tilde{\mathbf{v}}) \to +\infty \quad \text{when} \quad \|\tilde{\mathbf{v}}\| \to \infty, \quad \tilde{\mathbf{v}} \in \mathcal{V} \tag{4.4}$$

and that $J(\tilde{\mathbf{v}})$ is bounded from below in $\mathcal{V}$.

Contrary to the finite-dimensional case, a minimum of J in $\mathcal{V}$ need not in general be attained, even if $\overline{\mathcal{W}}$ is enlarged to be a complete space (the Sobolev space, for instance). Therefore, we modify the argument from the preceding section and examine a minimizing sequence rather than a minimizer itself.

Let $\{\tilde{\mathbf{v}}_n\} \in \mathcal{V}$, $n = 1, 2, \ldots$, be a minimizing sequence, i.e. such that

$$J(\tilde{\mathbf{v}}_n) \to d = \inf_{\tilde{\mathbf{v}} \in \mathcal{V}} J(\tilde{\mathbf{v}}) \quad \text{as} \quad n \to \infty. \tag{4.5}$$

For instance, the minimizing sequence can be determined by the Ritz method and the velocity fields $\tilde{\mathbf{v}}_n$ can be the solutions to a sequence of discretized rate-problems. As proved in Subsect. A.4 of Appendix, the sequence has the following property:

$$|\delta J(\tilde{\mathbf{v}}_n, \tilde{\mathbf{w}})| \leq \varepsilon_n \|\tilde{\mathbf{w}}\| \quad \text{for every} \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}, \quad \text{where} \quad \varepsilon_n \to 0 \quad \text{as} \quad n \to \infty. \tag{4.6}$$

Since the first variation of J vanishes at a solution field, its convergence to zero according to (4.6) means that inaccuracy of satisfying the equations of continuing equilibrium by the velocity fields $\tilde{\mathbf{v}}_n$ tends in a sense to zero. To clarify this, observe first that each velocity field $\tilde{\mathbf{v}}$ from $\mathcal{V}$ can be treated as a solution to the system of *perturbed* equations of continuing equilibrium, viz.

$$\text{(4.7)}\qquad \begin{aligned} &\operatorname{Div}\dot{\mathbf{N}}^T + \dot{\bar{\mathbf{b}}} + \dot{\mathbf{b}}^* = \mathbf{0} \quad \text{in } V \setminus S_D, \qquad \boldsymbol{\nu}_D[\dot{\mathbf{N}}] + \dot{\mathbf{T}}^* = \mathbf{0} \quad \text{on } S_D, \\ &\boldsymbol{\nu}\dot{\mathbf{N}} = \dot{\bar{\mathbf{T}}} + \dot{\mathbf{T}}^* \quad \text{on } S_T, \end{aligned}$$

where the volume and surface densities $\dot{\mathbf{b}}^* = \dot{\mathbf{b}}^*(\tilde{\mathbf{v}})$, $\dot{\mathbf{T}}^* = \dot{\mathbf{T}}^*(\tilde{\mathbf{v}})$ of the rates of perturbing forces are uniquely defined (locally) for each $\tilde{\mathbf{v}}$ (for simplicity, their dependence on ξ is not indicated explicitly). S_D stands for a collection of the surfaces of discontinuity (with $\boldsymbol{\nu}_D$ being a unit normal in the reference configuration and $[\cdot]$ denoting the jump) of the moduli (e.g. across an elastic/plastic interface) or of the velocity gradient. The first variation of J, after transforming it by using the Green theorem and substituting (4.7), can be written down in the form

$$\text{(4.8)}\qquad \delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) = \int_V \dot{\mathbf{b}}^*(\tilde{\mathbf{v}})\,\mathbf{w}\,dV + \int_{S_T \cup S_D} \dot{\mathbf{T}}^*(\tilde{\mathbf{v}})\,\mathbf{w}\,dS.$$

Consider an arbitrary material subdomain $G \subseteq V$ with a piecewise regular boundary ∂G and denote by $\mathbf{P}^*(G)$ the total perturbing force acting on G, including the perturbing tractions acting over ∂G. Assuming that $S_D \cap \overline{G}$ does not change discontinuously in time and that $\mathbf{b}^*$ and $\mathbf{T}^*$ are initially zero, the rate of $\mathbf{P}^*(G)$ which results from $\dot{\mathbf{b}}^*(\tilde{\mathbf{v}})$ is expressed by

$$\text{(4.9)}\qquad \dot{\mathbf{P}}^*(G, \tilde{\mathbf{v}}) = \int_G \dot{\mathbf{b}}^*(\tilde{\mathbf{v}})\,dV + \int_{\overline{G}\cap(S_T \cup S_D)} \dot{\mathbf{T}}^*(\tilde{\mathbf{v}})\,dS.$$

In Subsect. A.5 of Appendix it is proved that for a sequence $(\tilde{\mathbf{v}}_n)$ of velocity fields satisfying (4.6), the rates $\dot{\mathbf{P}}^*(G, \tilde{\mathbf{v}}_n)$ of perturbing forces acting on any G satisfy the inequality

$$\text{(4.10)}\qquad |\dot{\mathbf{P}}^*(G, \tilde{\mathbf{v}}_n)| \le \varepsilon_n \sqrt{3}\,|\partial G|$$

and tend thus to zero as $n \to \infty$ provided the domain G is fixed or varies with n such that its boundary measure $|\partial G|$ is bounded.

Perturbations of continuing equilibrium characterized by (4.6) *may thus be regarded as vanishingly small when* $n \to \infty$. In typical circumstances expected in the uniqueness range (2.9), such perturbations can cause only vanishingly small changes of the velocity field. To show this, suppose that

$$\text{(4.11)}\qquad \delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{v}} - \tilde{\mathbf{v}}^0) \ge b\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}^0\|^2 \quad \text{for every} \quad \tilde{\mathbf{v}} \in \mathcal{V}$$

for some positive constant b. Strictly speaking, the condition (4.11) is stronger than (2.12), however, in usual circumstances the ranges of validity of (2.12) and (4.11) along a deformation path can be expected to coincide, with the exception of isolated instants (cf. the discussion of (4.2)). In particular, if the constitutive inequality (2.14) holds then (4.11) is ensured so long as $I^0(\tilde{\mathbf{w}}) \ge b\|\tilde{\mathbf{w}}\|^2$. On substituting $\tilde{\mathbf{v}}_n$ in place of $\tilde{\mathbf{v}}$ into (4.11) and using (4.6) we obtain that $\|\tilde{\mathbf{v}}_n - \tilde{\mathbf{v}}^0\| \le \varepsilon_n / b$. It follows that *if* (4.11) *holds then any sequence* $\{\tilde{\mathbf{v}}_n\}$ *satisfying* (4.6) *tends to* $\tilde{\mathbf{v}}^0$ *in the sense that*

$$\text{(4.12)}\qquad \|\tilde{\mathbf{v}}_n - \tilde{\mathbf{v}}^0\| \to 0 \quad \text{as} \quad n \to \infty.$$

The situation is different when the process is unstable in the energy sense (2.28). As shown above, if (4.2) holds then (4.6) and thus also (4.10) are true for any sequence $(\tilde{\mathbf{v}}_n)$ minimizing $J(\tilde{\mathbf{v}})$ in $\mathcal{V}$. On account of (2.21), for a minimizing sequence $(\tilde{\mathbf{v}}_n)$ satisfying

(4.5) we thus have $J(\tilde{\mathbf{v}}^0) - J(\tilde{\mathbf{v}}_n) > \varepsilon = \text{const} > 0$ for n sufficiently large. From (4.4) and (A.7) it follows that the distance (4.1) between the perturbed and fundamental velocity fields remains finite as the infimum of $J(\tilde{\mathbf{v}})$ is approached at $n \to \infty$, viz.

$$M_2 > \|\tilde{\mathbf{v}}_n - \tilde{\mathbf{v}}^0\| > M_1 \quad \text{for} \quad n > K \tag{4.13}$$

for some positive constants M_1, M_2 and K. Hence, we have proved the following instability theorem:

THEOREM 2. *If* (4.2) *holds and the fundamental process is unstable in the energy sense* (2.28) *then finite deviations from the fundamental velocity field, measured by* (4.1), *can be caused by vanishingly small perturbations of the equations of continuing equilibrium, in the sense of* (4.6) *and* (4.10)([10]).

In short but less precisely (see below), we may say that if (2.28) holds simultaneously with (4.2) then the fundamental process is unstable in the sense of sensitivity of the incremental deformation to arbitrarily small perturbing forces.

If it happens that the infimum value in (4.5) is reached then the rates of perturbing forces can be exactly zero. In that case the path instability in the sense of Theorem 2 in some interval of λ is associated with persisting nonuniqueness in velocities, as in the finite-dimensional case considered in the preceding section.

Let (4.2) and (2.28) hold along the fundamental deformation path in a certain interval of λ. As shown above, at each equilibrium state of the body within that interval there exists a sequence of perturbed velocity fields $\tilde{\mathbf{v}}_n$ satisfying (4.5) and (4.13) while the rates of perturbing forces tend to zero in the sense of (4.10). Consider a perturbed deformation path initiated with a displacement rate field $\tilde{\mathbf{v}}_n$; we assume that if the *rates* of perturbing forces are kept fixed along the path then variations of the velocity field in a small interval of λ along any such path can be neglected. According to (4.13), the measure (4.1) of the distance between body configurations in the fundamental and perturbed processes grows then in time with a *finite* rate and achieves a given small but finite value after a small time increment. This distance can be reached at perturbing forces as small as we please if n is taken sufficiently large, with the initial instant of their application being arbitrarily chosen from the considered interval of λ.

This is the final interpretation of the path instability implied by (2.28) when each equilibrium state separately is directionally stable. As remarked in the Introduction, that kind of instability corresponds to the instability of a process for so-called *persistent* disturbances in the absence of *initial* disturbances. However, it must be noted that the transition from the conclusion of Theorem 2 formulated in terms of the rates to the final statement for the *increments*, although it may seem intuitively obvious, is not mathematically rigorous in view of the additional assumption made. From a physical point of view, the conclusion about instability is strengthened by the fact that for n sufficiently large the perturbed continuations correspond to a smaller value of J and require thus incrementally less energy to be supplied to the system than the fundamental continuation. This follows from (2.21), (4.5), the incremental form of (2.28) and the interpretation of an increment of E.

([10]) In an equivalent formulation, if (4.2) and (2.28) hold simultaneously then there is $\varepsilon > 0$ such that for every $\gamma > 0$, however small, there is $\tilde{\mathbf{v}}_\gamma \in \mathcal{V}$ satisfying $|\delta J(\tilde{\mathbf{v}}_\gamma, \tilde{\mathbf{w}})| < \gamma\|\tilde{\mathbf{w}}\|$ for every $\tilde{\mathbf{w}} \in \mathcal{W}$ and $|\dot{\mathbf{P}}^*(G, \tilde{\mathbf{v}}_\gamma)| < \gamma|\partial G|$ for every G and such that $\|\tilde{\mathbf{v}}_\gamma - \tilde{\mathbf{v}}^0\| \geq \varepsilon$.

5. Extension to configuration-dependent conservative loading

In the cases of an elastic support or fluid-pressure loading, the nominal surface tractions **T** on S_T are not only functions of ξ and λ but depend also on the actual displacements or their surface gradients. For one- or two-dimensional idealizations of the solid body, generalized body forces dependent in a similar way on the displacement field are induced by lateral surface tractions; other examples can also be given. Suppose thus (cf. [6, 8]) that the incremental loading consists not only of a controllable part (distinguished by a bar), as it was assumed above, but also of a deformation-sensitive part, that is

$$\dot{\mathbf{T}} = \dot{\bar{\mathbf{T}}} + \mathbf{f}(\tilde{\mathbf{v}}), \qquad \dot{\mathbf{b}} = \dot{\bar{\mathbf{b}}} + \mathbf{g}(\tilde{\mathbf{v}}), \tag{5.1}$$

where $\mathbf{f}(\tilde{\mathbf{v}})$ and $\mathbf{g}(\tilde{\mathbf{v}})$ are linear homogeneous expressions in the velocity **v** and its gradient $\nabla\mathbf{v}$ at the considered material point. Coefficients in the expressions may depend piecewise continuously on ξ and sufficiently smoothly on λ and $\tilde{\mathbf{u}}$. We show below in outline (cf. [19]) that the energy criteria of instability can be extended to the case of configuration-dependent loading corresponding to (5.1) provided the loading is *conservative* in an overall sense (cf. [6, 22]). Namely, it is assumed that when λ is *fixed* then the total work done by the body forces $\mathbf{b} = \mathbf{b}(\xi, \lambda, \tilde{\mathbf{u}})$ and surface tractions $\mathbf{T} = \mathbf{T}(\xi, \lambda, \tilde{\mathbf{u}})$ in any virtual motion compatible with the kinematic constraints and leading from a configuration $\tilde{\mathbf{u}}^0$ to any sufficiently close configuration $\tilde{\mathbf{u}}$ is *path-independent*, viz.

$$\int_V \int_{\mathbf{u}^0}^{\mathbf{u}} \mathbf{b}\, d\mathbf{u}\, dV + \int_{S_T} \int_{\mathbf{u}^0}^{\mathbf{u}} \mathbf{T}\, d\mathbf{u}\, dS = \Omega(\tilde{\mathbf{u}}^0, \lambda) - \Omega(\tilde{\mathbf{u}}, \lambda), \qquad \lambda = \text{const.} \tag{5.2}$$

The functional $\Omega(\tilde{\mathbf{u}}, \lambda)$ is defined to within an additive function of λ which may be chosen arbitrarily. For a physically appropriate choice of that function, Ω can be identified with the potential energy of the loading device. When λ is *varying* then, in general, the left-hand expression in (5.2) is path-dependent and differs from the respective increment of $(-\Omega)$.

At a given body configuration $\tilde{\mathbf{u}}^0$ at certain λ, introduce the bilinear functional Q of velocities (or their variations) and the respective quadratic functional R, defined by

$$Q(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2) = \int_V \mathbf{g}(\tilde{\mathbf{v}}_1)\, \mathbf{v}_2\, dV + \int_{S_T} \mathbf{f}(\tilde{\mathbf{v}}_1)\, \mathbf{v}_2\, dS, \qquad R(\tilde{\mathbf{v}}) = -\frac{1}{2} Q(\tilde{\mathbf{v}}, \tilde{\mathbf{v}}). \tag{5.3}$$

The first variation of **T** and **b** at fixed λ can be identified with **f** and **g**, respectively. Path-independence of the left-hand expression in (5.2) requires (cf. e.g. [24]) that $Q(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}) = Q(\tilde{\mathbf{w}}, \tilde{\mathbf{w}}_1)$ for every $\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}} \in \overline{\mathcal{W}}$. Substitution of the difference of two fields from $\mathcal{V}$ in place of $\tilde{\mathbf{w}}_1$ shows that the value of the expression: $Q(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) - Q(\tilde{\mathbf{w}}, \tilde{\mathbf{v}})$ is independent of $\tilde{\mathbf{v}} \in \mathcal{V}$. Since **f** and **g** depend only locally on $\tilde{\mathbf{v}}$ this expression vanishes if $\mathbf{v} = 0$ in V wherever $\mathbf{w} \neq \mathbf{0}$. Consequently, it must vanish, by a suitable choice of $\tilde{\mathbf{v}}$, if $\mathbf{w} = \mathbf{0}$ in some neighborhood of S_u, and hence for every $\tilde{\mathbf{w}} \in \overline{\mathcal{W}}$ by the limit transition. It follows that (5.2) implies

$$Q(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2) = Q(\tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_1) \quad \text{for every} \quad \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2 \in \mathcal{V} \cup \overline{\mathcal{W}}; \tag{5.4}$$

this is an extension of HILL'S [6] "self-adjointness" condition for surface loading to the loading (5.1).

From (5.1), (5.3) and (5.4) we obtain the equality

$$\int_V \dot{\mathbf{b}}\mathbf{w}\,dV + \int_{S_T} \dot{\mathbf{T}}\mathbf{w}\,dS = \int_V \dot{\bar{\mathbf{b}}}\mathbf{w}\,dV + \int_{S_T} \dot{\bar{\mathbf{T}}}\mathbf{w}\,dS - \delta R(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) \tag{5.5}$$

valid for every $\tilde{\mathbf{v}} \in \mathcal{V}$ and $\tilde{\mathbf{w}} \in \overline{\mathcal{W}}$. By using this identity it can be shown (cf. [6, 19]) that all conclusions from the Sect. 2 remain valid when the basic functionals J, I^0, I are modified by adding to the right-hand expressions in (2.5) and (2.6), (2.7) the quadratic form $R(\tilde{\mathbf{v}})$ and $R(\tilde{\mathbf{w}})$, respectively. Accordingly, $R(\tilde{\mathbf{v}} - \tilde{\mathbf{v}}^0)$ and $R(\tilde{\mathbf{w}})$ have to be added to the integral expressions in (2.9) and (2.17), respectively, while $Q(\tilde{\mathbf{v}} - \tilde{\mathbf{v}}^0, \tilde{\mathbf{w}})$ has to be subtracted from the right-hand expression in (2.11). Instead of (2.19), by the trapezoid rule of quadrature we obtain the following second-order expression for the difference between the values of Ω in two body configurations close to each other:

$$\Omega(\tilde{\mathbf{u}}^0 + \tilde{\mathbf{w}}, \lambda) - \Omega(\tilde{\mathbf{u}}^0, \lambda) = -\int_V \bar{\mathbf{b}}\mathbf{w}\,dV - \int_{S_T} \bar{\mathbf{T}}\mathbf{w}\,dS + R(\tilde{\mathbf{w}}) + o(\|\tilde{\mathbf{w}}\|^2). \tag{5.6}$$

The other formulae in Sect. 2 remain unchanged, including the definition (2.20) of the energy functional and the basic identity (2.21) [19].

Similar modifications of the formulae in Sects. 3 and 4 are straightforward and hence need not be given in detail. By applying the Cauchy inequality and (A.5) to (5.3) it can be shown that the estimates in Appendix remain true (with modified constants). The theorems from Sects. 3 and 4 remain thus valid for the incremental loading (5.1) provided the loading is conservative in the sense of (5.2).

6. Conclusions

The principal results established above can be summarized as follows:

(i) If an equilibrium state of a discretized system is unstable in the energy sense (2.27) then it is also unstable in the dynamic sense, under the constitutive approximation (3.4).

(ii) If a process of quasi-static deformation of a discretized system is unstable in the energy sense (2.28) while the traversed equilibrium states are (directionally) stable then there is bifurcation in velocities at every such state, and the secondary solutions minimize the second-order increment of energy to be supplied to the system.

(iii) Under assumptions as in (ii) above but for a continuous system, the deformation process is unstable for persistent disturbances, in the sense of sensitivity of the incremental deformation to arbitrarily small perturbing forces.

More precise formulations have been given in preceding sections. The result (i) indicates that the familiar second-order energy inequality at constant loading (cf. (3.11) and (2.26)), usually interpreted as a sufficiency condition for stability of equilibrium, may be interpreted (with a sign $\geq$) as a condition necessary for stability in a dynamic sense. From (ii) we obtain that if the tangent stiffness matrix becomes indefinite (cf. (2.30)) along a deformation path without loss of stability of equilibrium, then a continuous nonuniqueness range is entered; moreover, the secondary solutions are energetically preferable. For continuous systems under analogous circumstances, the persisting possibility of path bifurcation is, according to (iii), replaced in general by the persisting possibility of switching to a secondary deformation path under action of arbitrarily small perturbing forces. Similarly to the case (ii), the secondary deformation modes are energetically preferable to the fundamental continuation.

Those statements and conclusions at the present generality are believed to be new. The obtained results lend support to the hypothesis that, for a broad class of problems, the theoretical deformation processes characterized by (2.28) are unstable in a physically meaningful sense and hence cannot be practically realized in a physical system. Certainly, there are many aspects of instability in inelastic solids which still remain to be examined; as remarked in the Introduction, investigation in that direction has been rather limited.

7. Illustration

To illustrate the meaning of the instability condition (2.28) on a possibly simple and well-known example, consider the classical Shanley model of an elastic-plastic column with a central two-flange hinge [23]. The model has two degrees of freedom: the rotation angle θ of rigid arms and the relative vertical displacement u of the end points (Fig. 1a); $\tilde{\mathbf{v}}$ can thus be identified with $(\dot{\theta}, \dot{u})$. We will examine stability of the fundamental deformation process $\theta \equiv 0$ when the vertical compressive load $P \equiv \lambda(t)$ is a given increasing function of time, so that $\Omega = -Pu$. At a certain stage of the fundamental process, the rate of the force carried by an elastoplastic flange (K), $K = 1$ or 2, is given by $\dot{P}_K = L(\dot{e}_K)\,\dot{e}_K$ (no sum), where $\dot{e}_K$ is the shortening rate of the flange (taken positive for compression). $L(\dot{e})$ is equal to L_p (current tangent modulus) or to L_e (elastic modulus) for $\dot{e} > 0$ (loading) or $\dot{e} < 0$ (unloading), respectively, with $L_e > L_p > 0$. At an unbuckled configuration with the current dimensions as shown in Fig. 1a we have $\dot{e}_1 = \dot{u} + \dot{\theta}$, $\dot{e}_2 = \dot{u} - \dot{\theta}$ while $\ddot{u} = \frac{1}{2}(\ddot{e}_1 + \ddot{e}_2) + 2l\dot{\theta}^2$ (all variables are treated as nondimensional). Elementary calculations show that at an unbuckled equilibrium state we have

$$J(\dot{\theta}, \dot{u}) = \frac{1}{2}\ddot{E}(\dot{\theta}, \dot{u}) + \frac{1}{2}\ddot{P}u = \frac{1}{2}L(\dot{e}_1)\dot{e}_1^2 + \frac{1}{2}L(\dot{e}_2)\dot{e}_2^2 - \dot{P}\dot{u} - P\dot{\theta}^2 l\,. \tag{7.1}$$

The graph of $J(\dot{\theta}, \dot{u})$ consists of four quadrics joined smoothly along the lines $\dot{\theta} = \pm\dot{u}$. The uniqueness range (2.9) corresponds to $P < P_t$, where $P_t = L_p/l$ is the tangent modulus load, and in this range the expression (7.1) is minimized by the fundamental continuation $\dot{\theta}^0 = 0$, $\dot{u}^0 = \dot{P}/2L_p$. If L_p decreases continuously along the fundamental path then the uniqueness range terminates at $P = P_t$; this is the primary bifurcation point at which an absolute minimum of $J(\dot{\theta}, \dot{u})$ is attained at any $\tilde{\mathbf{v}}$ from the "fan" of velocity solutions: $\dot{u} = \dot{u}^0$, $|\dot{\theta}| \leq \dot{u}^0$ (cf. Fig. 1a). However, if L_p and thus also P_t decrease discontinuously then the range $P > P_t$ need not be preceded by $P = P_t$. Within the range $P_t < P < P_r$, where $P_r = 2L_e L_p/l(L_e + L_p)$ is the reduced modulus load, the fundamental solution $(\dot{\theta}^0, \dot{u}^0)$ becomes a saddle point of J so that (2.28) is met while the condition (3.11) of stability of equilibrium is still satisfied. In accord with Theorem 1, along this segment of the fundamental path the absolute minimum value of J is attained at the secondary solution points (cf. Fig. 1b) which are easily found to be

$$\dot{\theta}^* = \pm\frac{\dot{P}}{2L_p}z\,, \qquad \dot{u}^* = \frac{\dot{P}}{2L_p}(1 + (P/P_t - 1)z)\,, \qquad \text{where} \qquad z \equiv \frac{1 - L_p/L_e}{2(1 - P/P_r)}\,. \tag{7.2}$$

As P is increasing, the column can start to buckle with these rates at *any* point along the segment, without any perturbing forces: this is the present interpretation of the path instability associated with (2.28). According to the energy interpretation of J, buckling initiated with $(\dot{u}^*, \dot{\theta}^*)$ requires incrementally less energy to be supplied to the *system* than the fundamental solution, i.e. $\Delta E^* < \Delta E^0$. If $P(t)$ is a gravitational force coming from a

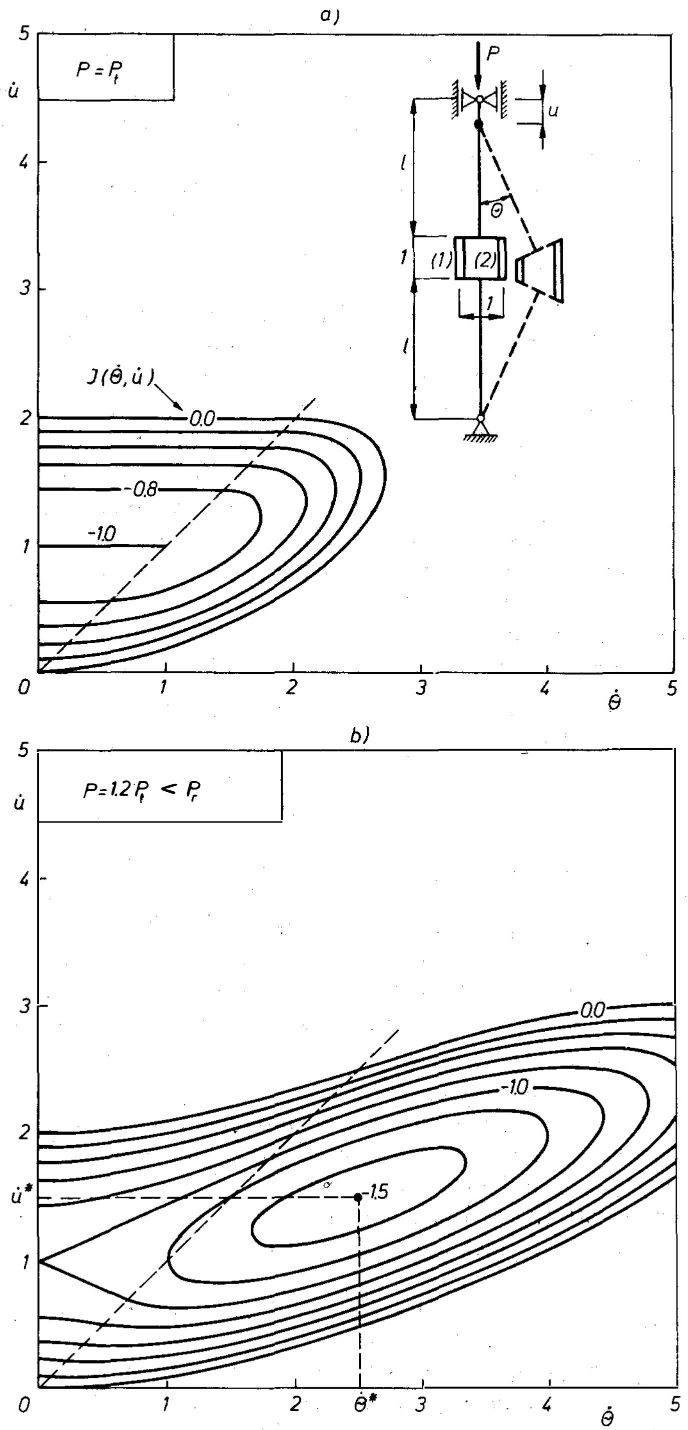

FIG. 1. Representation of the velocity functional (2.5) for the Shanley column (a) at the tangent modulus load, (b) between the tangent and reduced modulus loads. Numerical values correspond to $\dot P/2L_p = 1$, $L_p/L_e = 0.5$, $L_p = 1$. Contours of $J(\dot\theta, \dot u) > 0$ are not shown.

mass supplied with a prescribed rate then $\Delta E^* - \Delta E^0$ can be identified with the decrease of the work needed to transport the mass; note that $\dot{u}^* > \dot{u}^0$.

For $P > P_r$ the condition (2.28) is satisfied simultaneously with (2.27), and J is unbounded from below. In this range, instability of the fundamental process is interpreted as instability of the traversed equilibrium states in a dynamic sense.

As indicated in Sect. 3, the qualitative picture presented above can be extended to much more complicated finite-dimensional systems. This is illustrated elsewhere [21] on the example of necking under plane strain tension, calculated by using the finite element method.

The picture may change if a continuous system is considered: from one side, the absolute minimum of the functional J in the range where (2.28) and (4.2) hold need not be attained among ordinary functions, and from another side, the number of solution paths emanating from a bifurcation point may be even infinite. Both cases can be illustrated e.g. on the example of necking in a biaxially stretched, infinitely thin sheet of finite in-plane dimensions, by assuming specific boundary conditions. In the second case, the stability condition (2.29) can be applied to *secondary* post-bifurcation branches to select the incipient width of a necking band which is undetermined by the bifurcation theory alone; details will be given elsewhere.

Appendix

A.1. Certain estimates and identities

The function $\dot{\mathbf{N}}(\cdot)$ has been assumed to be positively homogeneous of degree one, continuous and piecewise smooth (outside $\dot{\mathbf{F}} = \mathbf{0}$). It follows that for any pair $\dot{\mathbf{F}}_1, \dot{\mathbf{F}}_2$ of velocity gradients we have the estimate

$$|\dot{\mathbf{N}}(\dot{\mathbf{F}}_1) - \dot{\mathbf{N}}(\dot{\mathbf{F}}_2)| \leq C|\dot{\mathbf{F}}_1 - \dot{\mathbf{F}}_2|, \tag{A.1}$$

where C is a positive constant, independent of place if it is taken as a supremum over V of the local constant. From (A.1) and the identity

$$U(\dot{\mathbf{F}}_1) - U(\dot{\mathbf{F}}_2) = \frac{1}{2}\dot{\mathbf{N}}(\dot{\mathbf{F}}_2) \cdot (\dot{\mathbf{F}}_1 - \dot{\mathbf{F}}_2) + \frac{1}{2}(\dot{\mathbf{N}}(\dot{\mathbf{F}}_1) - \dot{\mathbf{N}}(\dot{\mathbf{F}}_2)) \cdot \dot{\mathbf{F}}_1,$$

we obtain

$$|U(\dot{\mathbf{F}}_1) - U(\dot{\mathbf{F}}_2)| \leq \frac{1}{2}C|\dot{\mathbf{F}}_1 - \dot{\mathbf{F}}_2|(|\dot{\mathbf{F}}_1| + |\dot{\mathbf{F}}_2|). \tag{A.2}$$

Let the definition (2.7) of the functional I be formally extended to the class $\mathcal{V}$. For any pair $\tilde{\mathbf{v}}, \tilde{\mathbf{w}}$ of vector fields from $\mathcal{V} \cup \overline{\mathcal{W}}$, from (A.2), (4.1) and the Cauchy inequality for integrals we obtain the estimate

$$|I(\tilde{\mathbf{v}}) - I(\tilde{\mathbf{w}})| \leq \frac{1}{2}C\|\tilde{\mathbf{v}} - \tilde{\mathbf{w}}\|(\|\tilde{\mathbf{v}}\| + \|\tilde{\mathbf{w}}\|). \tag{A.3}$$

Similarly, for any fields $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2$ and $\tilde{\mathbf{w}}$, from (2.1), (2.5) and (A.1) it follows that

$$|\delta J(\tilde{\mathbf{v}}_1, \tilde{\mathbf{w}}) - \delta J(\tilde{\mathbf{v}}_2, \tilde{\mathbf{w}})| \leq C\|\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2\|\,\|\tilde{\mathbf{w}}\|. \tag{A.4}$$

Since $\tilde{\mathbf{w}} \in \overline{\mathcal{W}}$ means that $\mathbf{w} = 0$ on a finite part S_u of the boundary of V, it is well known that

$$\int_{S_T} \mathbf{w}\mathbf{w}\,dS \le k_1\|\tilde{\mathbf{w}}\|^2\,, \qquad \int_V \mathbf{w}\mathbf{w}\,dV \le k_2\|\tilde{\mathbf{w}}\|^2 \quad \text{for every} \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}\,, \tag{A.5}$$

where k_1 and k_2 are positive constants, dependent on V. Since $\dot{\bar{\mathbf{b}}}$ and $\dot{\bar{\mathbf{T}}}$ in (2.5) are bounded, after standard transformations with the use of the Cauchy inequality and (A.5), we obtain

$$\left| \int_V \dot{\bar{\mathbf{b}}}\mathbf{w}\,dV + \int_{S_T} \dot{\bar{\mathbf{T}}}\mathbf{w}\,dS \right| \le k_3\|\tilde{\mathbf{w}}\|\,, \qquad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}\,, \tag{A.6}$$

where k_3 is another positive constant.

Let $\tilde{\mathbf{v}}_1$, $\tilde{\mathbf{v}}_2$ be any pair of velocity fields from $\mathcal{V}$ so that $\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2 \in \mathcal{W}$. From (2.5), (A.6) and (A.3) we arrive at the estimate

$$|J(\tilde{\mathbf{v}}_1) - J(\tilde{\mathbf{v}}_2)| \le \|\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2\| \left(k_3 + \frac{1}{2}C(\|\tilde{\mathbf{v}}_1\| + \|\tilde{\mathbf{v}}_2\|) \right). \tag{A.7}$$

From the definition (2.1) it follows that at $\gamma > 0$ we have the identity

$$\frac{d}{d\gamma}J(\tilde{\mathbf{v}} + \gamma\tilde{\mathbf{w}}) = \delta J(\tilde{\mathbf{v}} + \gamma\tilde{\mathbf{w}}, \tilde{\mathbf{w}}) = \frac{1}{\gamma}\delta J(\tilde{\mathbf{v}} + \gamma\tilde{\mathbf{w}}, \gamma\tilde{\mathbf{w}}) \tag{A.8}$$

which yields

$$\begin{aligned} J(\tilde{\mathbf{v}} + \tilde{\mathbf{w}}) - J(\tilde{\mathbf{v}}) &= \int_0^1 \delta J(\tilde{\mathbf{v}} + \gamma\tilde{\mathbf{w}}, \tilde{\mathbf{w}})\,d\gamma \\ &= \delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) + \int_0^1 (\delta J(\tilde{\mathbf{v}} + \gamma\tilde{\mathbf{w}}, \tilde{\mathbf{w}}) - \delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}))\,d\gamma\,. \end{aligned} \tag{A.9}$$

From (A.4) we obtain the estimate

$$\int_0^1 |(\delta J(\tilde{\mathbf{v}} + \gamma\tilde{\mathbf{w}}, \tilde{\mathbf{w}}) - \delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}))|\,d\gamma \le \int_0^1 C\|\gamma\tilde{\mathbf{w}}\|\,\|\tilde{\mathbf{w}}\|\,d\gamma = \frac{1}{2}C\|\tilde{\mathbf{w}}\|^2\,.$$

On substituting it into (A.9), we arrive at:

$$|J(\tilde{\mathbf{v}} + \tilde{\mathbf{w}}) - J(\tilde{\mathbf{v}}) - \delta J(\tilde{\mathbf{v}}, \tilde{\mathbf{w}})| \le \frac{1}{2}C\|\tilde{\mathbf{w}}\|^2\,. \tag{A.10}$$

A.2. Proof of property (4.4)

Let $\tilde{\mathbf{v}}'$ be a fixed field from $\mathcal{V}$, $\tilde{\mathbf{w}}$ be any field from $\mathcal{W}$ and $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}' + \tilde{\mathbf{w}}$. From (A.3) and the triangle inequality it follows that

$$I(\tilde{\mathbf{v}}) \ge I(\tilde{\mathbf{w}}) - \frac{1}{2}C\|\tilde{\mathbf{v}}'\|(\|\tilde{\mathbf{v}}'\| + 2\|\tilde{\mathbf{w}}\|)\,. \tag{A.11}$$

On substituting (A.11) into (2.5), rearranging and using (A.6), we obtain

$$J(\tilde{\mathbf{v}}) \ge I(\tilde{\mathbf{w}}) - k'\|\tilde{\mathbf{w}}\| - c'\,, \tag{A.12}$$

where the numbers k' and c' depend on $\tilde{\mathbf{v}}'$ but are independent of $\tilde{\mathbf{w}}$. If (4.2) holds then (A.12) implies that

$$J(\tilde{\mathbf{v}}) \to +\infty \quad \text{as} \quad \|\tilde{\mathbf{w}}\| \to \infty \,. \tag{A.13}$$

$\tilde{\mathbf{v}}'$ is fixed so that $\|\tilde{\mathbf{w}}\| \to \infty$ is equivalent to $\|\tilde{\mathbf{v}}\| \to \infty$. Since every $\tilde{\mathbf{v}} \in \mathcal{V}$ can be expressed as $\tilde{\mathbf{v}}' + \tilde{\mathbf{w}}$, the property (4.4) has been proved.

From (A.13) it follows that for $\|\tilde{\mathbf{w}}\| \geq M$ with M sufficiently large we have $J(\tilde{\mathbf{v}}) > J(\tilde{\mathbf{v}}')$. For $\|\tilde{\mathbf{w}}\| \leq M$, $J(\tilde{\mathbf{v}})$ is bounded on account of (A.7). Hence, $J(\tilde{\mathbf{v}})$ is bounded from below in $\mathcal{V}$.

A.3. Proof of Theorem 1

Let a be defined by (4.3) with $\mathcal{W}$ replaced by $\mathcal{W}^N$. Since $\mathcal{W}^N$ is finite-dimensional and I is homogeneous of degree two and continuous in $\mathcal{W}^N$, the infimum value is reached at some nonzero $\tilde{\mathbf{w}}^*$. From (3.11) it follows that $I(\tilde{\mathbf{w}}^*) > 0$ so that $a > 0$. By the argument given in A.2 (with $\mathcal{V}$ and $\mathcal{W}$ replaced by $\mathcal{V}^N$ and $\mathcal{W}^N$, respectively) we obtain that $J(\tilde{\mathbf{v}}) > J(\tilde{\mathbf{v}}')$ for $\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}'\| \geq M$, where $\tilde{\mathbf{v}}, \tilde{\mathbf{v}}' \in \mathcal{V}^N$, $\tilde{\mathbf{v}}'$ is fixed and M is a sufficiently large number. Since $\mathcal{V}^N$ is finite dimensional and J is continuous in $\mathcal{V}^N$, the absolute minimum of J in the ball $\|\tilde{\mathbf{v}} - \tilde{\mathbf{v}}'\| < M$, and thus in $\mathcal{V}^N$, is attained at some $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}^*$. Since J is Gateaux differentiable, (3.12) must be satisfied at the minimum point so that $\tilde{\mathbf{v}}^*$ is a solution to the discretized rate-problem. This completes the proof of Theorem 1.

A.4. Proof of property (4.6)

Consider a sequence $(\tilde{\mathbf{v}}_n)$ in $\mathcal{V}$ such that (4.5) holds, and a sequence of numbers γ_n. On substituting $\tilde{\mathbf{v}}_n$ and $\gamma_n\tilde{\mathbf{w}}$ into (A.10) in place of $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}$, respectively, and rearranging, we obtain

$$-\gamma_n \delta J(\tilde{\mathbf{v}}_n, \tilde{\mathbf{w}}) - \frac{1}{2} C \gamma_n^2 \|\tilde{\mathbf{w}}\|^2 \leq J(\tilde{\mathbf{v}}_n) - J(\tilde{\mathbf{v}}_n + \gamma_n \tilde{\mathbf{w}}) \,. \tag{A.14}$$

Let $\|\tilde{\mathbf{w}}\| = 1$ and $\gamma_n = -\delta J(\tilde{\mathbf{v}}_n, \tilde{\mathbf{w}})/C$. On taking into account that $J(\tilde{\mathbf{v}}_n + \gamma_n \tilde{\mathbf{w}}) \geq d$, we arrive at the result

$$(\delta J(\tilde{\mathbf{v}}^n, \tilde{\mathbf{w}}))^2 \leq 2C(J(\tilde{\mathbf{v}}_n) - d) \quad \text{for every} \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}}, \quad \|\tilde{\mathbf{w}}\| = 1 \,.$$

This can be rewritten as

$$|\delta J(\tilde{\mathbf{v}}^n, \tilde{\mathbf{w}})| \leq \varepsilon_n \|\tilde{\mathbf{w}}\| \quad \text{for every} \quad \tilde{\mathbf{w}} \in \overline{\mathcal{W}},$$

where

$$\varepsilon_n = (2C)^{1/2} (J(\tilde{\mathbf{v}}_n) - d)^{1/2} \,.$$

Since $J(\tilde{\mathbf{v}}_n)$ tends to d, $\varepsilon_n \to 0$ as $n \to \infty$. The property (4.6) has been proved.

A.5. Proof of property (4.10)

Consider a domain $G \subset V$ with a regular boundary ∂G. Let $h(\boldsymbol{\xi})$ denote the distance from a point $\boldsymbol{\xi}$ to the set G, i.e. $h(\boldsymbol{\xi}) = |\boldsymbol{\xi} - \boldsymbol{\xi}'|$, where $\boldsymbol{\xi}'$ is a point from $\overline{G}$ closest to $\boldsymbol{\xi}$. Denote by G_H a spatial set surrounding G and consisting of $\boldsymbol{\xi}$ such that $0 < h(\boldsymbol{\xi}) < H$.

We will consider the following trial scalar fields $\widetilde{w}$ defined on $\overline{V}$:

$$w(\xi) = \begin{cases} H & \text{if } \xi \in \overline{G}\,, \\ H - h(\xi) & \text{if } \xi \in G_H\,, \\ 0 & \text{if } h(\xi) \geq H\,. \end{cases}$$

Let $w_1 = w$ and $w_2 \equiv w_3 \equiv 0$. ∂G may contain a part of S_T but we assume that $\partial G \cap \overline{S_u} = \emptyset$. ∇h at typical $\xi \in G_H$ exists and is equal to a unit vector (normal to ∂G at ξ'). For H sufficiently small S_u does not intersect G_H and $\widetilde{\mathbf{w}}$ just defined belongs to $\overline{\mathcal{W}}$. We have then $w_{1,i}w_{1,i} = 1$ in G_H and $\|\widetilde{\mathbf{w}}\| = |G_H|$, where $|G_H|$ denotes the volume of G_H. On substituting (4.8) with that field $\widetilde{\mathbf{w}}$ into (4.6) and rearranging, we obtain

$$|\dot{P}_1^*(G, \widetilde{\mathbf{v}}_n)| \equiv \left| \int_G \dot{b}_1^*(\widetilde{\mathbf{v}}_n)\, dV + \int_{(S_T \cup S_D)\cap \overline{G}} \dot{T}_1^*(\widetilde{\mathbf{v}}_n)\, dS \right|$$
$$\leq \varepsilon_n |G_H|/H + \left| \int_{G_H} \dot{b}_1^*(\widetilde{\mathbf{v}}_n)(1 - h/H)\, dV + \int_{(S_T \cup S_D)\cap G_H} \dot{T}_1^*(\widetilde{\mathbf{v}}_n)(1 - h/H) dS \right|.$$

In the limit as $H \to 0$, the set G_H becomes vanishingly narrow and the integrals on the right-hand side vanish while $|G_H|/H \to |\partial G|$, the surface measure of the boundary ∂G. By repeating the above procedure for the remaining coordinates, we arrive at (4.10) for any regular domain $G \subset V$ such that $\partial G \cap \overline{S_u} = \emptyset$.

A domain $G \subset V$ with a piecewise regular boundary disjoint with $\overline{S_u}$ can be approximated by a sequence (G_k) of regular domains considered above. If this is done, as is always possible, in such a manner that $|\partial G \backslash \partial G_k| \to 0$ then in the limit of approximation we obtain (4.10). The same argument shows that (4.10) remains valid when $\partial G \cap \overline{S_u}$ is not empty but is of zero surface measure. If the common part of S_u and ∂G is of finite area then in the limit expression the rate of perturbing tractions acting on this common surface part cannot be taken into account. However, $\dot{\mathbf{T}}^* = 0$ on S_u (any reaction on S_u is statically admissible), so that (4.10) is finally valid for any domain $G \subseteq V$ with a piecewise regular boundary.

References

1. Z. P. BAZANT, *Stable states and stable paths of propagation of damage zones and interactive fractures*, in: Cracking and Damage, J. Mazars and Z. P. Bazant [Eds.], pp. 183–206, Elsevier, London and New York 1989.
2. D. C. DRUCKER, *A definition of stable inelastic material*, J. Appl. Mech., **26**, 101–106, 1959.
3. R. HILL, *A general theory of uniqueness and stability in elastic-plastic solids*, J. Mech. Phys. Solids, **6**, 236–249, 1958.
4. R. HILL, *Some basic principles in the mechanics of solids without a natural time*, J. Mech. Phys. Solids, **7**, 209–225, 1959.
5. R. HILL, *Bifurcation and uniqueness in non-linear mechanics of continua*, in: Problems of Continuum Mechanics, N.I. Muskhelishwili Anniversary Volume, pp. 155–164, S.I.A.M., Philadelphia 1961.
6. R. HILL, *Uniqueness criteria and extremum principles in self-adjoint problems of continuum mechanics*, J. Mech. Phys. Solids, **10**, 185–194, 1962.
7. R. HILL, *Eigenmodal deformations in elastic-plastic continua*, J. Mech. Phys. Solids, **15**, 371–386, 1967.
8. R. HILL, *Aspects of invariance in solid mechanics*, Adv. in Appl. Mech., **18**, 1–75, Acad. Press, New York 1978.
9. R. HILL and M. J. SEWELL, *A general theory of inelastic column failure-II*, J. Mech. Phys. Solids, **8**, 112–118, 1960.
10. R. HILL and M. J. SEWELL, *A general theory of inelastic column failure-III*, J. Mech. Phys. Solids, **10**, 285–300, 1962.

11. J. W. HUTCHINSON, *Plastic buckling*, Adv. in Appl. Mech., **14**, pp. 67–144, Academic Press, New York 1974.
12. T. VON KÁRMÁN, *Discussion on Shanley's paper: Inelastic column theory*, J. Aero. Sci., **14**, 267–268, 1947.
13. V. D. KLUSHNIKOV, *Stability of elastic-plastic systems* [in Russian], Nauka, Moscow 1980.
14. R. J. KNOPS and E. W. WILKES, *Theory of elastic stability*, Handbuch der Physik VI a/3, Springer, Berlin 1973.
15. N. N. KRASOVSKII, *Stability of motion*, Stanford Univ. Press, California 1963.
16. A. NEEDLEMAN and V. TVERGAARD, *Aspects of plastic postbuckling behavior*, in: Mechanics of Solids, The Rodney Hill 60th Anniversary Volume, H. G. Hopkins and M. J. Sewell [Eds.], pp. 453–497, Pergamon Press, Oxford 1982.
17. Q. S. NGUYEN and D. RADENKOVIC, *Stability of equilibrium in elastic plastic solids*, Springer Lecture Notes in Mathematics, vol. 503, pp. 403–414, 1975.
18. H. PETRYK, *A consistent energy approach to defining stability of plastic deformation processes*, in: Stability in the Mechanics of Continua, F. H. Schroeder [Ed.], Proc. IUTAM Symp., Nümbrecht 1981; pp. 262–272, Springer, Berlin, Heidelberg 1982.
19. H. PETRYK, *On energy criteria of plastic instability*, in: Plastic Instability, Proc. Considère Memorial, pp. 215–226, Ecole Nat. Ponts Chauss. Press, Paris 1985.
20. H. PETRYK, *On constitutive inequalities and bifurcation in elastic-plastic solids with a yield-surface vertex*, J. Mech. Phys. Solids, **37**, 265–291, 1989.
21. H. PETRYK and K. THERMANN, *On discretized plasticity problems with bifurcations*, Int. J. Solids Structures [in press].
22. M.J. SEWELL, *On configuration-dependent loading*, Arch. Rat. Mech. Anal., **23**, 327–351, 1967.
23. F.R. SHANLEY, *Inelastic column theory*, J. Aero. Sci., **14**, 261–267, 1947.
24. M. M. VAINBERG, *Variational methods for the study of nonlinear operators* [in Russian], G.I.T.T.L., Moscow 1956 (English transl.: Holden-Day, San Francisco 1964).

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