Arch. Mech., **62**, 5, pp. 343–378, Warszawa 2010

Fundamentals of multiphysics modelling of piezo-poro-elastic structures

T. G. ZIELIŃSKI

Institute of Fundamental Technological Research
Polish Academy of Sciences
ul. Pawińskiego 5B
02-106 Warszawa, Poland
e-mail: tzielins@ippt.gov.pl

The paper discusses theoretical fundamentals necessary for accurate vibroacoustical modeling of structures or composites made up of poroelastic, elastic, and (active) piezoelectric materials, immersed in an acoustic medium (e.g. air). An accurate modeling of such hybrid active-passive vibroacoustic attenuators (absorbers or insulators) requires a multiphysics approach involving the finite element method to cope with complex geometries. Such fully-coupled, multiphysics model is given in this paper. To this end, first, the accurate PDE-based models of all the involved single-physics problems are recalled and, since a mutual interaction of these various problems is of the uttermost importance, the relevant couplings are thoroughly investigated and taken into account in the modeling. Eventually, the Galerkin finite element model is developed. This model should serve to develop designs of active composite vibroacoustic attenuators made up of porous foams with passive and active solid implants, or hybrid liners and panels made up of a core or layers of porous materials fixed to elastic faceplates with piezoelectric actuators, and coupled to air-gaps. A widespread design of such smart mufflers is still an open topic and should be addressed with accurate predictive tools based on the model proposed in the present paper. The model is accurate in the framework of kinematical and constitutive (material) linearity of behaviour. This is, however, the very case of the vibroacoustic application of elasto-poroelastic panels or composites, where the structural vibrations are induced by acoustic waves. The developed fully-coupled FE model is finally used to solve a generic two-dimensional example and some issues concerning finite element approximation and convergence are also discussed.

Key words: poroelasticity, piezoelectricity, multiphysics modeling, active vibroacoustics.

1. Introduction

1.1. Hybrid active-passive vibroacoustical attenuator

There are methods to cope with the unwanted vibroacoustical behaviour of structures which have already become classic. In the active structural acoustic control, the vibrations of noise radiating surfaces (plates, beams, shells) are actively controlled to reduce the generation of low-frequency noise [1]. These classic solutions have drawbacks and practical limitations and a hybrid approach has been recently proposed [2–7], relevant especially for barriers limiting the transmission of acoustic waves and, in general, for attenuators and dissipative materials for noise insulation and absorption. In such applications porous liners and multilayered panels (usually, with a core of porous material and thin elastic faceplates) are widely used but since they are passive, their efficiency is limited only to high and medium frequencies. The smart hybrid approach is also termed the *hybrid active-passive approach* [5, 7, 8] since it proposes an active control as a remedy for the lack of performance at low frequency, while in the high and medium-frequency range an excellent passive acoustic absorption should be guaranteed thanks to the inherited absorbing properties of well-chosen porous components.

Investigations concerning such smart hybrid approach started several years ago in the USA [2, 3] and France [4]. Recently new solutions have been developed in France by GALLAND and her collaborators [5–11]. They have started to study active sandwich panels with a core of poroelastic material. The panels are *active* in the sense that piezoelectric patches are added to their elastic faceplates which behave as a secondary vibrational source, interfering with the low-frequency disturbance propagating in the panel. Very recently, the active-passive concept of smart foams combining the passive dissipation capability of porous material in the medium and high frequency ranges and the active absorption ability of piezoelectric actuator (PVDF) in the low frequency range, has been investigated extensively by LEROY *et al.* [12–14].

Now, another even more innovative concept is being proposed: an active composite vibroacoustical attenuator made of porous layers (foams, etc.) with some solid implants (inclusions) – some of them may be passive (e.g., small distributed masses), the others are active (e.g., piezoelectric elements: patches of piezoceramic PZT, pieces of PVDF foil, piezo-fibres, etc.).

The widespread design of such smart mufflers (composites, liners, panels) is still an open topic and should be addressed with accurate predictive tools. Moreover, very often, an interaction of the mufflers with air-gaps or a fragment of the surrounding acoustical medium (the air) should also be taken into account in the modelling. The present paper intends to provide a complete theoretical basis necessary for the development of such tools using the finite element method which allows for modelling of complex geometries. Finally, the developed fully-coupled FE model will be used to solve a generic two-dimensional example of the aforementioned problems. In the context of this example, some convergence issues will be also discussed.

1.2. Physical problems involved, relevant theories, and general assumptions

Accurate modelling of active elasto-poroelastic noise attenuators (liners, panels or composites) means a multiphysics approach involving the finite element method to cope with a complex geometry. To this end the following theories will be used (relevant to the physical problems involved):

- the Biot's theory of poroelasticity – to model the vibroacoustic transmission and passive dissipation of acoustic waves in porous layers,
- the linear acoustics – to model the propagation of acoustic waves in the surrounding air and in air-gaps,
- the linear elasticity – to model the vibrations of elastic faceplates (and implants),
- the theory of piezoelectricity – to model the piezo-actuators for active control of low-frequency vibrations.

Moreover, a mutual interaction of all these various physical problems is of the uttermost importance, so the relevant couplings must be thoroughly investigated and taken into account in modelling.

The considered vibroacoustic application allows for using perfectly linear theories, so the superposition principle holds and may be effectively used. Consequently, the frequency analysis may be used as an efficient and sufficient tool for design and testing of the active liners, panels or composites. Therefore, apart from Eqs. (2.1) and (2.2) below, all other expressions (for all the problems involved) will be formulated for the case of harmonic oscillations with angular frequency $\omega = 2\pi f$, where f is the frequency of oscillations.

The equations of poroelasticity presented below assume no body forces acting on the poroelastic material. Consequently, the problems of elasticity and piezoelectricity are considered with zero body forces. Moreover, in the piezoelectricity problem there is no body electric charge applied. These assumptions comply with the modeling requirements of hybrid piezo-elasto-poroelastic noise attenuators.

Two sets of subscripts will be used, namely: $i, j, k, l \in \{1, 2, 3\}$, to denote vector and tensor components in the three-dimensional system of reference and $m, n \in \{1, \ldots N_{\rm dof}\}$, to number the degrees of freedom of a discrete model ($N_{\rm dof}$ is the total number of degrees of freedom). The summation convention is in use for both the types of subscripts. The (invariant) differentiation symbol is used which, in the Cartesian coordinate system, simply reads: $(.)_{|i} = \partial(.)/\partial x_i$. For brevity, symbols $d\Omega$ and $d\Gamma$ are skipped in all the integrals presented below since it is obvious that we integrate on the specified domain or boundary. Furthermore, the following notation rule for the symbol of variation (or test function) is used: $\delta(v\,w) = v\,\delta w + w\,\delta v$, where u and v are two dependent variables (fields) and δv and δw – their admissible variations.

2. Biot's theory of poroelasticity

2.1. Isotropic poroelasticity and the two formulations

Certainly, a vital component of a vibroacoustic panel, liner or composite is a layer (or layers) of porous material. Porous materials are quite often modeled in acoustics by using the so-called fluid-equivalent approach (see, e.g., [15]). This is acceptable for most of the porous media in some applications, especially in higher frequency range, where the vibrations of skeleton can be completely neglected; then, the so-called models of porous materials with rigid frame are valid. There are, however, many applications where the contribution of elastic frame vibrations is very significant, particularly in lower frequencies. This is relevant to sandwich panels with a poroelastic core (layer) and certainly it is the very case of porous composite noise absorbers. Here, the simple fluid-equivalent modelling is no longer valid and instead, a more complicated theory should be used since it is necessary to take into consideration the vibrations of elastic skeleton and their coupling to the wave propagating in the fluid in pores. There are two main theories which permit an adequate and thorough description of such problems: the Biot's theory of poroelasticity [15–17] and the so-called theory of porous media [18]; the latter one has been essentially established quite recently and is more general. An excellent work by de Boer [18] provides a current state of the theory of porous media, offering also highlights in the historical development and a comparison to the Biot's poroelasticity. The Biot's theory [16, 17] allows for modelling of materials made up of solid elastic skeleton (matrix, frame), with the pores filled up with a compressible fluid. Without doubt, within the framework of the geometrically- and physically-linear theory, it gives good results for a wide range of practical problems – in particular for the dynamic ones. A large number of applications have been worked out using this theory: starting from the acoustics [15, 19] (and vibroacoustics) up to the bio- and geomechanics [20]. In this theory, a *biphasic* approach is applied where the so-called *solid phase* is used to describe the behaviour of elastic skeleton, while the so-called *fluid phase* pertains to the fluid in the pores. Both the phases are, in fact, coupled homogenized continua of the "smeared" skeleton and pore-fluid. This homogenization (or rather, averaging) uses the concept of Average Volume Element (AVE) and it works very well if the shortest length of waves propagating in a porous medium is significantly greater than the characteristic dimension of pores.

In practical applications, the most frequently used is the Biot's *isotropic* theory of poroelasticity [15–17, 19]. In this approach, both the phases are isotropic. Moreover, the fluid is modeled as perfect (i.e., inviscid), though viscous forces are taken into account but only when modelling the interaction between the fluid and the frame. Two formulations of Biot's isotropic poroelasticity may be distinguished:

- the classical *displacement formulation* proposed by Biot, where the unknowns are the solid and fluid-phase displacements, which yields 6 degrees of freedom in every node of a three-dimensional numerical model;
- the *mixed displacement-pressure formulation*, where the dependent variables are the solid-phase displacements and the fluid-phase pressure; therefore, there are only 4 degrees of freedom in a three-dimensional model.

The second formulation is valid only for harmonic motion and it was presented by ATALLA *et al.* [21]. DEBERGUE *et al.* [22] discussed a very important subject of the boundary and interface-coupling conditions for this formulation.

2.2. The classical displacement formulation

As mentioned above, in the classical formulation [15–17] a state of poroelastic medium is unequivocally described by the displacements of solid, $\boldsymbol{u} = \{u_i\}$, and fluid phase, $\boldsymbol{U} = \{U_i\}$. Therefore, this is often referred to as the displacement-displacement, or the $\{\boldsymbol{u}, \boldsymbol{U}\}$ formulation. Biot's equations for a local dynamic equilibrium state of poroelastic medium, link partial stress tensors associated with the skeleton particle (σ^{s}_{ij}) and the macroscopic fluid particle (σ^{f}_{ij}) with the solid and fluid macroscopic displacements

$$\sigma^{\mathrm{s}}_{ij|j} = \varrho_{\mathrm{ss}}\,\ddot{u}_i + \varrho_{\mathrm{sf}}\,\ddot{U}_i + \tilde{b}\,(\dot{U}_i - \dot{u}_i), \tag{2.1}$$

$$\sigma^{\mathrm{f}}_{ij|j} = \varrho_{\mathrm{ff}}\,\ddot{U}_i + \varrho_{\mathrm{sf}}\,\ddot{u}_i + \tilde{b}\,(\dot{u}_i - \dot{U}_i), \tag{2.2}$$

where $\tilde{b}$ is the *viscous drag coefficient* and ϱ_{ss}, ϱ_{ff}, ϱ_{sf} are the so-called *effective densities*. First of these *equilibrium equations* refers to the solid phase, and the second one to the fluid phase; nevertheless, it is easy to notice that both the equations are strongly coupled by the inertial and viscous coupling terms: the viscous drag coefficient pertains to the traction between the interstitial fluid and the solid skeleton (the fluid by itself is inviscid, i.e., in the sense that there are no viscous forces between the fluid particles), whereas the last of the effective densities, ϱ_{sf}, is the so-called *mass coupling coefficient* responsible for consideration of the inertial interaction forces which occur between the solid skeleton and the fluid. The effective densities are expressed as follows:

$$\varrho_{\mathrm{ss}} = (1-\phi)\,\varrho_{\mathrm{s}} - \varrho_{\mathrm{sf}}, \qquad \varrho_{\mathrm{ff}} = \phi\,\varrho_{\mathrm{f}} - \varrho_{\mathrm{sf}}, \qquad \varrho_{\mathrm{sf}} = -(\alpha_\infty - 1)\,\phi\,\varrho_{\mathrm{f}}. \tag{2.3}$$

They depend on the porosity, ϕ, the tortuosity of pores, α_∞, the density of the material of skeleton, ϱ_{s}, and the density of saturating fluid, ϱ_{f}.

Consider now the case of harmonic motion (with the angular frequency ω). Then, all time-dependent quantities can be presented using the co-called complex notation involving a term $\exp(\mathrm{j}\,\omega\,t)$ (where $\mathrm{j}=\sqrt{-1}$ is the imaginary unit). This exponential term is reduced from the equations and all the relevant quantities are represented in these equations in the form of their (frequency-dependent) complex amplitudes. Remembering this, the equilibrium equations (2.1) and (2.2) read as follows:

$$\sigma^{\mathrm{s}}_{ij|j}+\omega^2\,\tilde{\varrho}_{\mathrm{ss}}\,u_i+\omega^2\,\tilde{\varrho}_{\mathrm{sf}}\,U_i=0, \tag{2.4}$$

$$\sigma^{\mathrm{f}}_{ij|j}+\omega^2\,\tilde{\varrho}_{\mathrm{ff}}\,U_i+\omega^2\,\tilde{\varrho}_{\mathrm{sf}}\,u_i=0, \tag{2.5}$$

where the so-called *frequency-dependent effective densities* are introduced:

$$\tilde{\varrho}_{\mathrm{ss}}=\varrho_{\mathrm{ss}}+\frac{\tilde{b}}{\mathrm{j}\,\omega},\qquad \tilde{\varrho}_{\mathrm{ff}}=\varrho_{\mathrm{ff}}+\frac{\tilde{b}}{\mathrm{j}\,\omega},\qquad \tilde{\varrho}_{\mathrm{sf}}=\varrho_{\mathrm{sf}}-\frac{\tilde{b}}{\mathrm{j}\,\omega}. \tag{2.6}$$

As a matter of fact, these densities are responsible not only for the inertia of solid- or fluid-phase particles but also for the combined inertial and viscous coupling (interaction) of both phases.

The partial solid and fluid stress tensors are linearly related to the partial strain tensors prevailing in the skeleton and the interstitial fluid. This is given by the following linear and isotropic *constitutive equations* of the Biot's theory of poroelasticity (where linear kinematic relations have already been used to replace the strain tensors with the gradients of displacements):

$$\sigma^{\mathrm{s}}_{ij}=\mu_{\mathrm{s}}\,(u_{i|j}+u_{j|i})+\big(\tilde{\lambda}_{\mathrm{s}}\,u_{k|k}+\tilde{\lambda}_{\mathrm{sf}}\,U_{k|k}\big)\delta_{ij}, \tag{2.7}$$

$$\sigma^{\mathrm{f}}_{ij}=\big(\tilde{\lambda}_{\mathrm{f}}\,U_{k|k}+\tilde{\lambda}_{\mathrm{sf}}\,u_{k|k}\big)\delta_{ij}. \tag{2.8}$$

(Here and below, δ_{ij} is the Kronecker's delta symbol.) One may clearly see that in this modelling both the phases are isotropic. Four material constants are involved here, namely μ_{s}, $\tilde{\lambda}_{\mathrm{s}}$, $\tilde{\lambda}_{\mathrm{f}}$, and $\tilde{\lambda}_{\mathrm{sf}}$. The first two are similar to the two Lamé coefficients of isotropic elasticity. Moreover, μ_{s} is the shear modulus of the poroelastic material and consequently, the shear modulus of the frame, since the fluid does not contribute to the shear restoring force. The three dilatational constants, $\tilde{\lambda}_{\mathrm{s}}$, $\tilde{\lambda}_{\mathrm{f}}$ and $\tilde{\lambda}_{\mathrm{sf}}$ are frequency-dependent and are functions of K_{b}, K_{s}, and $\tilde{K}_{\mathrm{a}}$ ($\tilde{\lambda}_{\mathrm{s}}$ depends also on μ_{s}), where: K_{b} is the bulk modulus of the frame at constant pressure in the fluid, K_{s} is the bulk modulus of the elastic solid from which the frame is made, and $\tilde{K}_{\mathrm{a}}$ is the effective bulk modulus of fluid in porous material. The adequate exact formulae to compute the poroelastic material constants can be found in [15]. Notice here that only one material constant, namely, the *constitutive coupling coefficient*, $\tilde{\lambda}_{\mathrm{sf}}$, is responsible for a multiphysics coupling occurring between the constitutive equations of both phases. However, the Reader should be reminded of the visco-inertial coupling present in the equations of equilibrium; thus, the interaction of solid skeleton with the fluid in the pores is very well represented in this biphasic approach.

The equations of equilibrium (2.1)–(2.2), or (2.4)–(2.5) in case of harmonic motion, form with the constitutive relations (2.7)-(2.8) the *displacement formulation* of linear, isotropic poroelasticity. Finally, total quantities are defined for this biphasic model, namely, the *total stress tensor* as a simple sum of the partial, i.e. phasic, stress tensors:

$$\sigma^{\mathrm{t}}_{ij} = \sigma^{\mathrm{s}}_{ij} + \sigma^{\mathrm{f}}_{ij} = \mu_{\mathrm{s}}\,(u_{i|j} + u_{j|i}) + \big[(\tilde{\lambda}_{\mathrm{s}} + \tilde{\lambda}_{\mathrm{sf}})u_{k|k} + (\tilde{\lambda}_{\mathrm{f}} + \tilde{\lambda}_{\mathrm{sf}})U_{k|k}\big]\delta_{ij}, \tag{2.9}$$

and the *total displacement vector* of poroelastic medium which reads

$$u^{\mathrm{t}}_i = (1-\phi)\,u_i + \phi\,U_i, \tag{2.10}$$

where the porosity-dependent contributions of the displacements of both phases are involved.

2.3. The mixed displacement-pressure formulation

Finite element models based on the displacement formulation of Biot's poroelasticity have been used to predict the acoustical and structural behavior of porous multilayer structures [23–26]. Since these models, while accurate, lead to large frequency-dependent matrices for three-dimensional problems, ATALLA *et al.* proposed in [21] a novel exact mixed displacement-pressure formulation derived directly from the Biot's poroelasticity equations. The boundary conditions for this formulation were extensively discussed in [22].

The mixed formulation uses the fact that the fluid-phase stress tensor can be expressed as $\sigma^{\mathrm{f}}_{ij} = -\phi\,p\,\delta_{ij}$, where p is the pressure of fluid in the pores. Basing on this relation, some mathematical manipulations applied to the harmonic case allow to get rid of the fluid-phase displacements, U_i, introducing instead a new unknown field of pressure in the pores, p. These manipulations are presented in the Appendix. Eventually, the harmonic equilibrium for the solid phase can expressed as

$$\sigma^{\mathrm{ss}}_{ij|j} + \omega^2\tilde{\varrho}\,u_i + \phi\left(\frac{\tilde{\varrho}_{\mathrm{sf}}}{\tilde{\varrho}_{\mathrm{ff}}} - \frac{\tilde{\lambda}_{\mathrm{sf}}}{\tilde{\lambda}_{\mathrm{f}}}\right)p_{|i} = 0 \qquad \text{where} \qquad \tilde{\varrho} = \tilde{\varrho}_{\mathrm{ss}} - \frac{\tilde{\varrho}^2_{\mathrm{sf}}}{\tilde{\varrho}_{\mathrm{ff}}}. \tag{2.11}$$

Here, a new stress tensor is introduced (which depends only on the solid-phase displacements)

$$\sigma^{\mathrm{ss}}_{ij} = \mu_{\mathrm{s}}\,(u_{i|j} + u_{j|i}) + \tilde{\lambda}_{\mathrm{ss}}\,u_{k|k}\,\delta_{ij} \qquad \text{where} \qquad \tilde{\lambda}_{\mathrm{ss}} = \tilde{\lambda}_{\mathrm{s}} - \frac{\tilde{\lambda}^2_{\mathrm{sf}}}{\tilde{\lambda}_{\mathrm{f}}}. \tag{2.12}$$

The fluid-phase equation is transformed to the following form

$$\frac{\phi^2}{\omega^2\,\tilde{\varrho}_{\rm ff}}\,p_{|ii} + \frac{\phi^2}{\tilde{\lambda}_{\rm f}}\,p - \phi\left(\frac{\tilde{\varrho}_{\rm sf}}{\tilde{\varrho}_{\rm ff}} - \frac{\tilde{\lambda}_{\rm sf}}{\tilde{\lambda}_{\rm f}}\right)u_{i|i} = 0, \tag{2.13}$$

where the last term couples this equation with the solid-phase equation (2.11). Equations (2.11) and (2.13) together with the constitutive relation (2.12) constitute the *mixed displacement-pressure formulation* of harmonic isotropic poroelasticity, where the primary dependent variables are solid-phase displacements, u_i, and pressure in the pores, p.

3. Weak forms of poroelasticity, elasticity, piezoelectricity, and acoustics

3.1. Weak form of the mixed formulation of poroelasticity

The weak integral form of the mixed formulation of Biot's poroelasticity was presented by ATALLA *et al.* [21]. An enhanced version of this weak formulation was proposed in [27]. Here, the enhanced version will be used since the enhancement allows to handle easily some boundary and interface-coupling conditions. This matter was extensively discussed in [27, 28]. Finite element models based on the enhanced weak form of the mixed poroelasticity problem involving coupling to elastic and acoustic media are presented in [27, 29, 28]. In [29] the convergence of model using hierarchical elements was investigated.

Let $\Omega_{\rm p}$ be a domain of poroelastic material and $\Gamma_{\rm p}$ – its boundary, with n_i being the components of the vector normal to the boundary and pointing outside the domain. The harmonic poroelasticity problem can be described in this domain by the mixed formulation equations (2.11) and (2.13). Both sides of these equations are multiplied by the so-called test (or weighting) functions, δu_i and δp, respectively for the solid phase equations and the fluid phase equation, and then integrated in the whole domain $\Omega_{\rm p}$ and summed up to one integral equation. Integration by parts of some of the terms and using the divergence theorem, yields the weak form for the harmonic poroelasticity problem, valid for any arbitrary yet admissible virtual displacements, δu_i, and pressure, δp. This form can be expressed as follows:

$$-\int\limits_{\Omega_{\rm p}}\sigma^{\rm ss}_{ij}\,\delta u_{i|j} + \int\limits_{\Omega_{\rm p}}\omega^2\,\tilde{\varrho}\,u_i\,\delta u_i - \int\limits_{\Omega_{\rm p}}\frac{\phi^2}{\omega^2\,\tilde{\varrho}_{\rm ff}}\,p_{|i}\,\delta p_{|i} + \int\limits_{\Omega_{\rm p}}\frac{\phi^2}{\tilde{\lambda}_{\rm f}}\,p\,\delta p + \int\limits_{\Omega_{\rm p}}\phi\left(1+\frac{\tilde{\varrho}_{\rm sf}}{\tilde{\varrho}_{\rm ff}}\right)\delta(p_{|i}\,u_i)$$
$$+\int\limits_{\Omega_{\rm p}}\phi\left(1+\frac{\tilde{\lambda}_{\rm sf}}{\tilde{\lambda}_{\rm f}}\right)\delta(p\,u_{i|i}) + \int\limits_{\Gamma_{\rm p}}\sigma^{\rm t}_{ij}\,n_j\,\delta u_i + \int\limits_{\Gamma_{\rm p}}\phi\,(U_i - u_i)\,n_i\,\delta p = 0. \tag{3.1}$$

Here, $\sigma_{ij}^{\rm ss} = \sigma_{ij}^{\rm ss}(\boldsymbol{u})$ is a function of solid phase displacements according to Eq. (2.12). The total stresses, $\sigma_{ij}^{\rm t} = \sigma_{ij}^{\rm t}(\boldsymbol{u}, p)$, and the fluid phase displacements, $U_i = U_i(\boldsymbol{u}, p)$, may also be formally considered as functions of solid phase displacements and fluid phase pressure (see Eqs. (A.8) and (A.3) in the Appendix), but they appear only in the boundary integrals and will be reduced or replaced by specific, prescribed values when considering the boundary or interface coupling conditions. These integrals of (Neumann) boundary conditions are in the last line of Eq. (3.1) while the second line contains the coupling terms. Thanks to the proposed weak formulation, the boundary and interface conditions are naturally handled for rigid piston displacements and when coupling to elastic medium or to a layer of another poroelastic material. They are also adequately simple when imposing a pressure field and in the case of coupling to an acoustic medium. Since the issue of boundary and coupling interface conditions is not a simple one in case of a double-phase modeling, this matter will be extensively discussed later on in Sections 4 and 5.

3.2. Weak form for an elastic solid

The theory of (linear) elasticity and the derivation of the weak form used by FEM and other variational methods can be found in many textbooks (e.g., [30]). Below, the weak integral of the principle of virtual work for a harmonic elastic-body system is given (without derivation), and the natural and essential boundary conditions are briefly discussed.

Let $\Omega_{\rm e}$ be an elastic solid domain with mass density $\varrho_{\rm e}$ and boundary $\Gamma_{\rm e}$, and let $n_i^{\rm e}$ be the components of the vector normal to the boundary and pointing outside the domain. Assuming zero body forces and the case of harmonic oscillations, the weak variational form of the problem of elasticity expressing the principle of virtual work reads

$$(3.2) \qquad -\int\limits_{\Omega_{\rm e}} \sigma_{ij}^{\rm e}\, \delta u_{i|j}^{\rm e} + \int\limits_{\Omega_{\rm e}} \omega^2 \varrho_{\rm e}\, u_i^{\rm e}\, \delta u_i^{\rm e} + \int\limits_{\Gamma_{\rm e}} \sigma_{ij}^{\rm e}\, n_j^{\rm e}\, \delta u_i^{\rm e} = 0,$$

where $u_i^{\rm e}$ are the elastic solid displacements and $\delta u_i^{\rm e}$ are their arbitral yet admissible variations; the elastic stress tensor $\sigma_{ij}^{\rm e} = \sigma_{ij}^{\rm e}(\boldsymbol{u}^{\rm e})$ substitutes here a linear function of elastic displacements. Generally, in the anisotropic case it equals

$$(3.3) \qquad \sigma_{ij}^{\rm e} = C_{ijkl}^{\rm e}\, \frac{u_{k|l}^{\rm e} + u_{l|k}^{\rm e}}{2},$$

where $C_{ijkl}^{\rm e}$ is the fourth-order tensor of linear elasticity. One may notice that the linear kinematic relations between the elastic strain tensor and the elastic displacements, $\varepsilon_{ij}^{\rm e} = \frac{1}{2}(u_{i|j}^{\rm e} + u_{j|i}^{\rm e})$, have been already used in (3.3). In the case of isotropy, the constitutive equation can be expressed as follows:

$$\sigma^{\mathrm{e}}_{ij} = \mu_{\mathrm{e}}\,(u^{\mathrm{e}}_{i|j} + u^{\mathrm{e}}_{j|i}) + \lambda_{\mathrm{e}}\,u^{\mathrm{e}}_{k|k}\,\delta_{ij}, \tag{3.4}$$

where the well-known Lamé coefficients appear, that is: the shear modulus, $\mu_{\mathrm{e}}\dfrac{E_{\mathrm{e}}}{2(1+\nu_{\mathrm{e}})}$, and the dilatational constant, $\lambda_{\mathrm{e}} = \dfrac{\nu_{\mathrm{e}}\,E_{\mathrm{e}}}{(1+\nu_{\mathrm{e}})(1-2\nu_{\mathrm{e}})}$, related to the material's Young modulus, E_{e}, and Poisson's coefficient, ν_{e}.

Boundary conditions. Two kinds of boundary conditions will be discussed here, namely Neumann's and Dirichlet's, although they may be combined into the third specific type, the so-called Robin (or generalized Dirichlet) boundary condition. For the sake of brevity, the latter type will not be considered; remember only that, in practice, the well-known technique of Lagrange multipliers is usually involved when applying it. The Neumann (or natural) boundary conditions describe the case when forces $\hat{t}^{\mathrm{e}}_i$ are applied on a boundary, that is,

$$\sigma^{\mathrm{e}}_{ij}\,n^{\mathrm{e}}_j = \hat{t}^{\mathrm{e}}_i \qquad \text{on } \Gamma^t_{\mathrm{e}}, \tag{3.5}$$

whereas the displacements $\hat{u}^{\mathrm{e}}_i$ are prescribed by the Dirichlet (or essential) boundary conditions

$$u^{\mathrm{e}}_i = \hat{u}^{\mathrm{e}}_i \qquad \text{on } \Gamma^u_{\mathrm{e}}. \tag{3.6}$$

According to these conditions, the boundary is divided into two (directionally disjoint) parts, i.e., $\Gamma_{\mathrm{e}} = \Gamma^t_{\mathrm{e}} \cup \Gamma^u_{\mathrm{e}}$. There is an essential difference between the two kinds of conditions. The displacement constraints form the kinematic requirements for the trial functions while the imposed forces appear in the weak form; thus, the boundary integral, that is, the last left-hand-side term of Eq. (3.2), equals

$$\mathcal{BI}_{\mathrm{e}} = \int\limits_{\Gamma_{\mathrm{e}}} \sigma^{\mathrm{e}}_{ij}\,n^{\mathrm{e}}_j\,\delta u^{\mathrm{e}}_i = \int\limits_{\Gamma^t_{\mathrm{e}}} \hat{t}^{\mathrm{e}}_i\,\delta u^{\mathrm{e}}_i. \tag{3.7}$$

Here, the property $\delta u^{\mathrm{e}}_i = 0$ on Γ^u_{e} has been used.

3.3. Weak form of piezoelectricity

The theory of piezoelectricity is extensively discussed, for example, in [31, 32]. More or less brief recapitulations of the linear theory of piezoelectricity may be found also in many papers and books on active vibration control and piezoelectric actuators and sensors (e.g., [1, 33, 34]). A very good survey of the advances and trends in finite element modeling of piezoelectricity was presented by BENJEDDOU [35]. In this paper the basic theoretical considerations and equations of linear piezoelectricity as well as the variational piezoelectric equations are also given.

Piezoelectric elements (actuators and sensors) of the proposed active composites, liners and panels are to be modeled using the linear theory. It is adequate enough and, moreover, it is a very accurate model when comparing it to some frequently used approximations (as a matter of fact, the so-called thermal analogy approach is usually an acceptable approximation when modeling piezo-actuators). Here, a variational form of linear piezoelectricity will be presented as being the most used one for piezoelectric finite element formulations. This form should be regarded as the sum of the conventional principle of virtual mechanical displacements and the principle of virtual electric potential.

Let $\Omega_{\rm pz}$ be a domain of piezoelectric material, $\varrho_{\rm pz}$ its mass density, and $\Gamma_{\rm pz}$ its boundary. The unit boundary-normal vector, $n_i^{\rm pz}$, points outside the domain. The dependent variables of piezoelectric medium are the mechanical displacements, $u_i^{\rm pz}$, and electric potential, $V^{\rm pz}$. The case of harmonic oscillations (with the angular frequency ω) with no mechanical body forces and electric body charge is considered. Then, for arbitrary yet admissible virtual displacements, $\delta u_i^{\rm pz}$, and virtual electric potential, $\delta V^{\rm pz}$, the variational formulation of the piezoelectricity problem can be given as

$$(3.8)\qquad -\int\limits_{\Omega_{\rm pz}} \sigma_{ij}^{\rm pz}\,\delta u_{i|j}^{\rm pz} + \int\limits_{\Omega_{\rm pz}} \omega^2 \varrho_{\rm pz}\, u_i^{\rm pz}\,\delta u_i^{\rm pz} + \int\limits_{\Gamma_{\rm pz}} \sigma_{ij}^{\rm pz}\, n_j^{\rm pz}\,\delta u_i^{\rm pz} - \int\limits_{\Omega_{\rm pz}} D_i^{\rm pz}\,\delta V_{|i}^{\rm pz} + \int\limits_{\Gamma_{\rm pz}} D_i^{\rm pz}\, n_i^{\rm pz}\,\delta V^{\rm pz} = 0,$$

where $\sigma_{ij}^{\rm pz} = \sigma_{ij}^{\rm pz}(\boldsymbol{u}^{\rm pz}, V^{\rm pz})$ and $D_i^{\rm pz} = D_i^{\rm pz}(\boldsymbol{u}^{\rm pz}, V^{\rm pz})$ are expressions of mechanical displacements and electric potential. Obviously, from the physical point of view they represent the mechanical stress tensor and the electric displacement vector, respectively. As a matter of fact, these expressions are the so-called *stress-charge form* of the constitutive relations of piezoelectricity – they are given below for the case of linear anisotropic piezoelectricity:

$$(3.9)\qquad \sigma_{ij}^{\rm pz} = C_{ijkl}^{\rm pz}\,\frac{u_{k|l}^{\rm pz} + u_{l|k}^{\rm pz}}{2} - e_{kij}^{\rm pz}\,V_{|k}^{\rm pz}, \qquad D_i^{\rm pz} = e_{ikl}^{\rm pz}\,\frac{u_{k|l}^{\rm pz} + u_{l|k}^{\rm pz}}{2} + \epsilon_{ik}^{\rm pz}\,V_{|k}^{\rm pz}.$$

Here, $C_{ijkl}^{\rm pz}$, $e_{ikl}^{\rm pz}$, and $\epsilon_{ik}^{\rm pz}$ denote (the components of) the fourth-order tensor of elastic material constants, the third-order tensor of piezoelectric material constants, and the second-order tensor of dielectric material constants, respectively.

These three tensors of material constants characterize completely any piezoelectric material, i.e., its elastic, piezoelectric, and dielectric properties. Only one of these tensors is responsible for the piezoelectric effects. Therefore, piezoelectricity can be viewed as a multiphysics problem, where in one domain of a piezoelectric medium the problems of elasticity and electricity are coupled by the piezoelectric material constants present in (additional) coupling terms in the constitutive relations. One should notice that the (linear) kinematic relations, $\varepsilon_{ij}^{\text{pz}} = (u_{k|l}^{\text{pz}} + u_{k|l}^{\text{pz}})/2$, linking mechanical strain ($\varepsilon_{ij}^{\text{pz}}$) and displacements ($u_i^{\text{pz}}$), and the Maxwell's law for electrostatics, $E_i^{\text{pz}} = -V_{|i}^{\text{pz}}$, relating the electric field (E_i^{pz}) with its potential (V^{pz}), have been explicitly used in Eqs. (3.9).

Boundary conditions. In piezoelectricity the boundary conditions are divided into two groups – there are mechanical conditions (referring to the elasticity problem) and electrical conditions (referring to the electricity). Consequently, the boundary of piezoelectric domain can be subdivided as follows:

$$\Gamma_{\text{pz}} = \Gamma_{\text{pz}}^{t} \cup \Gamma_{\text{pz}}^{u} \qquad \text{and} \qquad \Gamma_{\text{pz}} = \Gamma_{\text{pz}}^{Q} \cup \Gamma_{\text{pz}}^{V}. \tag{3.10}$$

The parts belonging to the same group of subdivision are disjoint and both subdivisions are completely independent. Here, Γ_{pz}^{t} and Γ_{pz}^{Q} pertain to the Neumann conditions for surface-applied mechanical forces and electric charge, respectively, while Γ_{pz}^{u} and Γ_{pz}^{V} refer to the Dirichlet conditions on imposed mechanical displacements and electric potential, respectively. The third possibility of Robin boundary condition is skipped; however, it would involve another parts – one in the mechanical and one in the electric subdivision of the boundary.

First, consider the mechanical boundary conditions. The forces, $\hat{t}_i^{\text{pz}}$, applied to a boundary are expressed by the Neumann (or natural) condition

$$\sigma_{ij}^{\text{pz}}\, n_j^{\text{pz}} = \hat{t}_i^{\text{pz}} \qquad \text{on } \Gamma_{\text{pz}}^{t}, \tag{3.11}$$

whereas the imposed displacements, $\hat{u}_i^{\text{pz}}$, will appear in the Dirichlet (i.e., essential) boundary condition

$$u_i^{\text{pz}} = \hat{u}_i^{\text{pz}} \qquad \text{on } \Gamma_{\text{pz}}^{u}. \tag{3.12}$$

The Dirichlet condition must be *a priori* explicitly met by the trial functions while the Neumann condition (3.11) is used for the mechanical boundary integral, that is, the third term in Eq. (3.8), which equals

$$\mathcal{BI}_{\text{pz}}^{\text{mech}} = \int\limits_{\Gamma_{\text{pz}}} \sigma_{ij}^{\text{pz}}\, n_j^{\text{pz}}\, \delta u_i^{\text{pz}} = \int\limits_{\Gamma_{\text{pz}}^{t}} \hat{t}_i^{\text{pz}}\, \delta u_i^{\text{pz}}, \tag{3.13}$$

since $\delta u_i^{\text{pz}} = 0$ on Γ_{pz}^{u}.

The electric boundary condition of the Neumann kind serves for a surface electric charge $\hat{Q}^{\rm pz}$ applied to a boundary

$$-D_i^{\rm pz}\, n_i^{\rm pz} = \hat{Q}^{\rm pz} \qquad \text{on } \Gamma_{\rm pz}^{Q}, \tag{3.14}$$

whereas the Dirichlet condition allows to prescribe the electric potential $\hat{V}^{\rm pz}$ on a boundary

$$V^{\rm pz} = \hat{V}^{\rm pz} \qquad \text{on } \Gamma_{\rm pz}^{V}. \tag{3.15}$$

The electric boundary integral, that is the last term in Eq. (3.8), equals

$$\mathcal{BI}_{\rm pz}^{\rm elec} = \int\limits_{\Gamma_{\rm pz}} D_i^{\rm pz}\, n_i^{\rm pz}\, \delta V^{\rm pz} = -\int\limits_{\Gamma_{\rm pz}^{Q}} \hat{Q}^{\rm pz}\, \delta V^{\rm pz}. \tag{3.16}$$

Here, the Neumann condition for electric charge (3.14) has been used together with the condition for voltage variation, $\delta V^{\rm pz} = 0$ on $\Gamma_{\rm pz}^{V}$.

By summing up the mechanical and electrical boundary integrals (3.13) and (3.16), the following total mechanical-electric boundary integral results:

$$\mathcal{BI}_{\rm pz} = \mathcal{BI}_{\rm pz}^{\rm mech} + \mathcal{BI}_{\rm pz}^{\rm elec} = \int\limits_{\Gamma_{\rm pz}^{t}} \hat{t}_i^{\rm pz}\, \delta u_i^{\rm pz} - \int\limits_{\Gamma_{\rm pz}^{Q}} \hat{Q}^{\rm pz}\, \delta V^{\rm pz}. \tag{3.17}$$

3.4. Weak form for an acoustic medium

Classical acoustic media are homogeneous inviscid fluids where compressional acoustic waves propagate with velocity being the material property of the medium, termed the speed of sound. The classical linear time-harmonic acoustics is governed by the Helmholtz equation. The derivation of this equation may be found in many textbooks, e.g., in [36]. Finite (and infinite) element methods for time-harmonic acoustics are reviewed in [37, 38]. Below, the weak integral form (used by FEM) of harmonic acoustics is given and the relevant natural and essential boundary conditions are briefly discussed.

Let $\Omega_{\rm a}$ be an acoustic medium domain and $\Gamma_{\rm a}$ its boundary, with $n_i^{\rm a}$ being the components of unit normal vector pointing outside the domain. The dependent variable of acoustical medium is the acoustic pressure, $p^{\rm a}$. For harmonic motion with the angular frequency ω, the following weak form should be used:

$$-\int\limits_{\Omega_{\rm a}} \frac{1}{\omega^2 \varrho_{\rm a}}\, p_{|i}^{\rm a}\, \delta p_{|i}^{\rm a} + \int\limits_{\Omega_{\rm a}} \frac{1}{K_{\rm a}}\, p^{\rm a} \delta p^{\rm a} + \int\limits_{\Gamma_{\rm a}} \frac{1}{\omega^2 \varrho_{\rm a}}\, p_{|i}^{\rm a}\, n_i^{\rm a}\, \delta p^{\rm a} = 0, \tag{3.18}$$

where $\varrho_{\rm a}$ and $K_{\rm a}$ are the acoustic medium mass density and the bulk modulus, respectively. In the case of fluids, usually, the given data is how fast a sound wave propagates in the medium. Therefore, the bulk modulus can always be replaced by $K_{\mathrm{a}} = \varrho_{\mathrm{a}}\, c_{\mathrm{a}}^2$ where c_{a} is the speed of sound. However, in the case of fluid-equivalent models of porous materials (with rigid frame) one often prefers to use the bulk modulus which – together with the (now, effective) density – is a frequency-dependent quantity, i.e.: $K_{\mathrm{a}} = \tilde{K}_{\mathrm{a}}(\omega)$ and $\varrho_{\mathrm{a}} = \tilde{\varrho}_{\mathrm{a}}(\omega)$. Knowing the acoustic pressure one can always determine the (complex amplitudes of) displacements, velocities and accelerations of fluid particle using the following formulae:

$$(3.19)\qquad u_i^{\mathrm{a}} = \frac{1}{\omega^2 \varrho_{\mathrm{a}}}\, p_{|i}^{\mathrm{a}}, \qquad v_i^{\mathrm{a}} = \mathrm{j}\,\omega\, u_i^{\mathrm{a}} = -\frac{1}{\mathrm{j}\,\omega\,\varrho_{\mathrm{a}}}\, p_{|i}^{\mathrm{a}}, \qquad a_i^{\mathrm{a}} = -\omega^2\, u_i^{\mathrm{a}} = -\frac{1}{\varrho_{\mathrm{a}}}\, p_{|i}^{\mathrm{a}}.$$

Boundary conditions. Two kinds of boundary conditions will be considered: the Neumann condition when a rigid piston of known acceleration, $\hat{a}_i^{\mathrm{a}}$, is imposed on a boundary, and the Dirichlet condition when a value of acoustic pressure, $\hat{p}^{\mathrm{a}}$, is prescribed. In the harmonic case: $\hat{a}_i^{\mathrm{a}} = -\omega^2\, \hat{u}_i^{\mathrm{a}}$ with $\hat{u}_i^{\mathrm{a}}$ being the (complex) amplitude of displacements, and the Neumann condition reads

$$(3.20)\qquad \frac{1}{\omega^2\,\varrho_{\mathrm{a}}}\, p_{|i}^{\mathrm{a}} = \hat{u}_i^{\mathrm{a}} \qquad \text{on } \Gamma_{\mathrm{a}}^{u}.$$

The Dirichlet boundary condition simply states that

$$(3.21)\qquad p^{\mathrm{a}} = \hat{p}^{\mathrm{a}} \qquad \text{on } \Gamma_{\mathrm{a}}^{p}.$$

Like in the case of poroelastic, elastic, and piezoelectric media, the third (i.e., Robin's) kind of boundary conditions is skipped in the present discussion.

Now, using the Neumann condition (3.20) and the condition for pressure variation, $\delta p^{\mathrm{a}} = 0$ on Γ_{a}^{p}, the boundary integral, that is, the last term in Eq. (3.18) can be written as follows:

$$(3.22)\qquad \mathcal{BI}_{\mathrm{a}} = \int\limits_{\Gamma_{\mathrm{a}}} \frac{1}{\omega^2\,\varrho_{\mathrm{a}}}\, p_{|i}^{\mathrm{a}}\, n_i^{\mathrm{a}}\, \delta p^{\mathrm{a}} = \int\limits_{\Gamma_{\mathrm{a}}^{u}} u_i^{\mathrm{a}}\, n_i^{\mathrm{a}}\, \delta p^{\mathrm{a}}.$$

4. Boundary conditions for poroelastic medium

4.1. The boundary integral

The boundary integral in the weak variational form of the mixed formulation of poroelasticity (3.1) has the following form:

$$(4.1)\qquad \mathcal{BI}_{\mathrm{p}} = \int\limits_{\Gamma_{\mathrm{p}}} \sigma_{ij}^{\mathrm{t}}\, n_j\, \delta u_i + \int\limits_{\Gamma_{\mathrm{p}}} \phi\,(U_i - u_i)\, n_i\, \delta p.$$

Here, two types of boundary conditions that may occur at the boundary of poroelastic medium will be discussed. Although some other conditions might be formally applied, these two are the most representative and important in practice. In other words, skipped will be, for example, the mixed conditions which prescribe in the same point of the boundary, different fields to both phases.

4.2. Imposed displacement field

A displacement field, $\hat{u}_i$, applied on a boundary of poroelastic medium describes, for example, the case of a piston in motion acting on the surface of the medium. Here, it is assumed that the solid skeleton is fixed to the surface of the piston while the fluid obviously cannot penetrate into the piston. Therefore,

$$u_i = \hat{u}_i, \qquad (U_i - u_i)\, n_i = 0. \tag{4.2}$$

The first condition expresses the continuity between the imposed displacement vector and the solid-phase displacement vector. The second equation expresses the continuity of normal displacements between the solid phase and the fluid phase. Using these conditions and the fact that the variations of the known solid displacements are zero ($\delta u_i = 0$), the boundary integral reduces to zero [27]:

$$\mathcal{BI}_{\mathrm{p}} = 0. \tag{4.3}$$

Notice that this result holds also when the poroelastic medium is not glued but only adherent to the rigid piston, provided that there is no friction or any imposed tangential forces at the interface between the piston and the poroelastic medium. In that case, the second boundary condition of formulae (4.2) holds (and so the second term of the boundary integral (4.1) disappears) whereas, instead of the three equations of the solid displacement condition $u_i = \hat{u}_i$ $(i = 1, 2, 3)$, there is one requirement for the *normal* solid displacement: $u_i\, n_i = \hat{u}$ (where $\hat{u}$ is the prescribed normal displacement of piston), and two additional requirements about the total stress vector $(\sigma^{\mathrm{t}}_{ij}\, n_j)$, which state that the components tangential to the surface of piston are zero. This assumption, together with the fact that the variation of the prescribed normal component of solid displacement must be zero ($\delta u_i\, n_i = 0$), make the first term of the boundary integral (4.1) vanish, and so the result (4.3) is valid. Remember, however, that this result cannot be used if the friction occurs between the piston and the poroelastic medium, or if any tangential forces are imposed. In this latter (rather academic) case the prescribed tangential forces would appear in the boundary integral. The case of friction can be important in practice and will yield a nonlinear boundary condition.

4.3. Imposed pressure field

A harmonic pressure field of amplitude $\hat{p}$ is imposed on the boundary of poroelastic domain what means that it affects at the same time the fluid in the pores and the solid skeleton. Therefore, the following boundary conditions must be met:

$$p = \hat{p}, \qquad \sigma^{\mathrm{t}}_{ij}\, n_j = -\hat{p}\, n_i. \tag{4.4}$$

The first condition is of Dirichlet type and must be applied explicitly. It describes the continuity of pressure in the fluid. It means also that the pressure variation is zero ($\delta p = 0$) at the boundary. The second condition expresses the continuity of the total normal stress. All this, when used for Eq. (4.1), leads to the following boundary integral [27, 28]:

$$\mathcal{BI}_{\mathrm{p}} = -\int\limits_{\Gamma_{\mathrm{p}}} \hat{p}\, n_i\, \delta u_i. \tag{4.5}$$

Now, consider an important case when there is no pressure (nor any displacement field) applied on the boundary of a poroelastic medium. In spite of appearances, this is not identical with, but can only be approximated by the case when the pressure at the boundary is kept at zero ($\hat{p} = 0$). Then, the boundary integral vanishes: $\mathcal{BI}_{\mathrm{p}} = 0$, and only the Dirichlet boundary condition, $p = 0$, must be applied.

5. Interface coupling conditions for poroelastic and other media

5.1. Poroelastic-poroelastic coupling

To begin with, consider the coupling conditions between two different poroelastic media (domains) fixed one to another. The superscripts 1 and 2 (put in parenthesis) denote which domain the superscripted quantity belongs to. Let $\Gamma_{(1)\text{-}(2)}$ be an interface between the two media and let $n_i^{(1)}$ be the components of the unit vector normal to the interface and pointing outside the medium 1 (and into the medium 2), while $n_i^{(2)}$ are the components of the unit normal vector pointing outside the medium 2 (into the medium 1), which means that at every point of the interface: $n_i^{(2)} = -n_i^{(1)}$. The coupling integral terms (given at the interface) are a combination of the boundary integrals resulting from the weak variational forms (3.1) obtained for both poroelastic domains, that is:

$$\begin{aligned}\mathcal{CI}_{(1)\text{-}(2)} = \int\limits_{\Gamma_{(1)\text{-}(2)}} \sigma^{\mathrm{t}(1)}_{ij}\, n_j^{(1)}\, \delta u_i^{(1)} + \int\limits_{\Gamma_{(1)\text{-}(2)}} \phi_{(1)} \big(U_i^{(1)} - u_i^{(1)}\big)\, n_i^{(1)}\, \delta p^{(1)} \\ + \int\limits_{\Gamma_{(1)\text{-}(2)}} \sigma^{\mathrm{t}(2)}_{ij}\, n_j^{(2)}\, \delta u_i^{(2)} + \int\limits_{\Gamma_{(1)\text{-}(2)}} \phi_{(2)} \big(U_i^{(2)} - u_i^{(2)}\big)\, n_i^{(2)}\, \delta p^{(2)}.\end{aligned} \tag{5.1}$$

It will be demonstrated that this coupling integral (resulting from the weak form (3.1) of the mixed formulation of harmonic poroelasticity) equals zero, what means that the coupling conditions are *naturally handled* [27, 28] at the interface between two domains made of poroelastic materials.

At the interface between two poroelastic media, the following coupling conditions must be met:

$$\begin{aligned} &\sigma^{\mathrm{t}(1)}_{ij}\, n^{(1)}_j = \sigma^{\mathrm{t}(2)}\, n^{(1)}_j, \qquad \phi_{(1)}(U^{(1)}_i - u^{(1)}_i)\, n^{(1)}_i = \phi_{(2)}(U^{(2)}_i - u^{(2)}_i)\, n^{(1)}_i, \\ &u^{(1)}_i = u^{(2)}_i, \qquad p^{(1)} = p^{(2)}. \end{aligned} \tag{5.2}$$

The first condition ensures the continuity of total stresses while the second one ensures the continuity of the relative mass flux across the interface. The two last conditions express the continuity of the solid-phase displacements and of the pressure of pore-fluids, respectively. This also entails that the appropriate variations are the same (i.e., $\delta u^{(1)}_i = \delta u^{(2)}_i$ and $\delta p^{(1)} = \delta p^{(2)}$). Now, applying the coupling conditions for Eq. (5.1) and taking into account that $n^{(2)}_i = -n^{(1)}_i$, it is easy to obtain the following result:

$$\mathcal{CI}_{(1)\text{-}(2)} = 0, \tag{5.3}$$

which means that the coupling conditions between two poroelastic media are naturally handled indeed [27, 28].

5.2. Poroelastic-elastic coupling

Let $\Gamma_{\text{p-e}}$ be an interface between poroelastic and elastic media. Let n_i be the components of the unit vector normal to the interface and pointing outside the poroelastic domain into the elastic one. The coupling integral combines the boundary integral terms resulting from both – poroelastic and elastic – weak forms (Eqs.(3.1) and (3.2), respectively):

$$\mathcal{CI}_{\text{p-e}} = \int\limits_{\Gamma_{\text{p-e}}} \sigma^{\mathrm{t}}_{ij}\, n_j\, \delta u_i + \int\limits_{\Gamma_{\text{p-e}}} \phi\,(U_i - u_i)\, n_i\, \delta p + \int\limits_{\Gamma_{\text{p-e}}} \sigma^{\mathrm{e}}_{ij}\, n^{\mathrm{e}}_j\, \delta u^{\mathrm{e}}_i, \tag{5.4}$$

where $n^{\mathrm{e}}_i = -n_i$ are the components of the unit normal vector pointing outside the elastic domain (and into the poroelastic medium). Now, the following coupling conditions must be met at the interface:

$$\sigma^{\mathrm{t}}_{ij}\, n_j = \sigma^{\mathrm{e}}_{ij}\, n_j, \qquad (U_i - u_i)\, n_i = 0, \qquad u_i = u^{\mathrm{e}}_i. \tag{5.5}$$

The first condition states the continuity of total stress tensor, the second one expresses that there is no mass flux across the interface, and the last one assumes the continuity of the solid displacements. The last condition involves also the equality of the variations of displacements, $\delta u_i = \delta u_i^{\mathrm{e}}$. Now, applying the coupling conditions for the coupling integral (5.4) results in

$$\mathcal{CI}_{\text{p-e}} = 0. \tag{5.6}$$

This is similar to the result obtained for coupling between two poroelastic domains: the coupling between poroelastic and elastic media is also *naturally handled* [27, 28].

5.3. Poroelastic-acoustic coupling

Now, the coupling between poroelastic and acoustic media will be discussed. Let $\Gamma_{\text{p-e}}$ be an interface between a poroelastic material and an acoustic medium, with n_i being the components of the unit vector normal to the interface and pointing outside the poroelastic domain (and into the acoustic medium), whereas n_i^{a} are the components of the similar unit normal vector pointing in the opposite direction; therefore, in every point of the interface $n_i^{\mathrm{a}} = -n_i$. The coupling integral is a combination of the boundary integral terms from the poroelastic weak form (3.1) and the acoustic weak form (3.18):

$$\mathcal{CI}_{\text{p-a}} = \int\limits_{\Gamma_{\text{p-a}}} \sigma_{ij}^{\mathrm{t}}\, n_j\, \delta u_i + \int\limits_{\Gamma_{\text{p-a}}} \phi\,(U_i - u_i)\, n_i\, \delta p + \int\limits_{\Gamma_{\text{p-a}}} \frac{1}{\omega^2 \varrho_{\mathrm{a}}}\, p_{|i}^{\mathrm{a}}\, n_i^{\mathrm{a}}\, \delta p^{\mathrm{a}}. \tag{5.7}$$

The coupling conditions between the two media express the continuity of (total) stresses, (total) normal displacements, and pressure – respectively:

$$\sigma_{ij}^{\mathrm{t}}\, n_j = -p\, n_i, \qquad \frac{1}{\omega^2 \varrho_{\mathrm{a}}}\, p_{|i}^{\mathrm{a}}\, n_i^{\mathrm{a}} = u_i^{\mathrm{t}}\, n_i^{\mathrm{a}}, \qquad p = p^{\mathrm{a}}. \tag{5.8}$$

Now, using these conditions and the expression for the total displacements of poroelastic medium, $u_i^{\mathrm{t}} = (1-\phi)\, u_i + \phi\, U_i$, the coupling integral (5.7) [27, 28] simplifies to

$$\begin{aligned} \mathcal{CI}_{\text{p-a}} &= -\int\limits_{\Gamma_{\text{p-a}}} p\, n_i\, \delta u_i + \int\limits_{\Gamma_{\text{p-a}}} \phi\,(U_i - u_i)\, n_i\, \delta p - \int\limits_{\Gamma_{\text{p-a}}} \big[(1-\phi)\, u_i + \phi\, U_i\big] n_i\, \delta p \\ &= -\int\limits_{\Gamma_{\text{p-a}}} \big(p\, n_i\, \delta u_i + u_i\, n_i\, \delta p\big) = -\int\limits_{\Gamma_{\text{p-a}}} \delta(p\, u_i\, n_i). \end{aligned} \tag{5.9}$$

5.4. Acoustic-elastic coupling

The coupling integral on an interface $\Gamma_{\text{a-e}}$ between the elastic and acoustic subdomains reads as follows:

$$\mathcal{CI}_{\text{a-e}} = \int\limits_{\Gamma_{\text{a-e}}} \frac{1}{\omega^2 \varrho_{\text{a}}} p^{\text{a}}_{|i}\, n^{\text{a}}_i\, \delta p^{\text{a}} + \int\limits_{\Gamma_{\text{a-e}}} \sigma^{\text{e}}_{ij}\, n^{\text{e}}_j\, \delta u^{\text{e}}_i. \tag{5.10}$$

On the interface, the conditions of continuity of the displacements normal to the interface and normal stresses must be satisfied, that is,

$$\frac{1}{\omega^2 \varrho_{\text{a}}} p^{\text{a}}_{|i}\, n^{\text{a}}_i = u^{\text{e}}_i\, n^{\text{a}}_i, \qquad \sigma^{\text{e}}_{ij}\, n^{\text{e}}_j = -p^{\text{a}}\, n^{\text{e}}_i. \tag{5.11}$$

These conditions are used for the integral and since on the interface the two unit normal vectors are in the opposite direction one to another, i.e., $n^{\text{a}}_i = -n^{\text{e}}_i$, the interface coupling integral (5.10) simplifies to

$$\mathcal{CI}_{\text{a-e}} = \int\limits_{\Gamma_{\text{a-e}}} \left(p^{\text{a}}\, n^{\text{a}}_i\, \delta u^{\text{e}}_i + u^{\text{e}}_i\, n^{\text{a}}_i\, \delta p^{\text{a}} \right) = \int\limits_{\Gamma_{\text{a-e}}} \delta\left(p^{\text{a}}\, u^{\text{e}}_i\, n^{\text{a}}_i \right). \tag{5.12}$$

Obviously, this result is also valid and, moreover, complete in the case of a piezoelectric medium in contact with an acoustic one, since the interface coupling occurs explicitly only between the acoustic problem and its mechanical (i.e., elastic) counterpart in the piezoelectric subdomain. To be formal, one should only change $\Gamma_{\text{a-e}}$ to $\Gamma_{\text{a-pz}}$, and u^{e}_i to u^{pz}_i in the formulae given above.

6. Galerkin finite element model of a coupled system of piezoelectric, elastic, poroelastic and acoustic media

6.1. Weak form of the coupled multiphysics system

Consider a coupled multiphysics system (see the diagram in Fig. 1) made up of some piezoelectric, elastic, poroelastic, and acoustic subdomains, useful for analysis of some complex active noise absorbers or insulators. The Galerkin method will be used to approach the problem by means of finite elements.

To this end, a weak form of the coupled system must be constructed. The weak form combines all the weak forms for the corresponding problems presented in Sec. 3. The discussion of coupling interface conditions in Sec. 5 has presented very important results, namely, that the coupling of two poroelastic domains, or a poroelastic domain to an elastic one, is naturally handled; that is, the interface coupling integrals are zero what results from the continuity of the fields of primary variables. Such result is also straightforwardly obtained for elastic

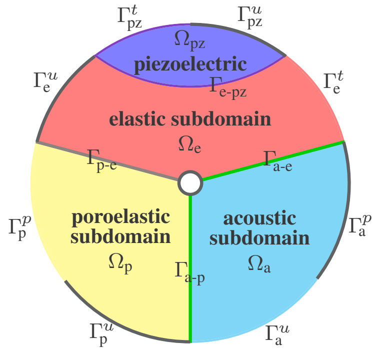

FIG. 1. Abstract configuration of a system made up of poroelastic, acoustic, elastic and piezoelectric media.

and piezoelectric domains. This is not the case of coupling to an acoustical domain. Therefore, define (for convenience) the following interface: $\Gamma_{\text{a-p,e,pz}} = \Gamma_{\text{a-p}} \cup \Gamma_{\text{a-e}} \cup \Gamma_{\text{a-pz}}$, which is a simple sum of all interfaces where the acoustic domain is coupled to the poroelastic, elastic, and piezoelectric domains. Now, the weak form of the coupled system reads

$$(6.1)\qquad \int\limits_{\Omega_{\text{pz}}} \big(-\sigma^{\text{pz}}_{ij}\,\delta u_{i|j} + \omega^2 \varrho_{\text{pz}}\, u_i\,\delta u_i + D^{\text{pz}}_i\,\delta V_{|i} \big) + \int\limits_{\Gamma^t_{\text{pz}}} \hat{t}^{\text{pz}}_i\,\delta u_i + \int\limits_{\Gamma^Q_{\text{pz}}} \hat{Q}^{\text{pz}}\,\delta V$$
$$+ \int\limits_{\Omega_{\text{e}}} \big(-\sigma^{\text{e}}_{ij}\,\delta u_{i|j} + \omega^2 \varrho_{\text{e}}\, u_i\,\delta u_i \big) + \int\limits_{\Gamma^t_{\text{e}}} \hat{t}^{\text{e}}_i\,\delta u_i + \int\limits_{\Omega_{\text{p}}} (\mathcal{P}) - \int\limits_{\Gamma^p_{\text{p}}} \hat{p}\, n_i\,\delta u_i$$
$$+ \int\limits_{\Omega_{\text{a}}} \left(-\frac{1}{\omega^2 \varrho_{\text{a}}}\, p_{|i}\,\delta p_{|i} + \frac{1}{K_{\text{a}}}\, p\,\delta p \right) + \int\limits_{\Gamma^u_{\text{a}}} \hat{u}^{\text{a}}_i\, n^{\text{a}}_i\,\delta p + \int\limits_{\Gamma_{\text{a-p,e,pz}}} n^{\text{a}}_i \big(\delta p\, u_i + p\,\delta u_i\big) = 0,$$

where $\mathcal{P}$ stands for the integrand of a poroelastic domain and equals

$$(6.2)\qquad \mathcal{P} = -\sigma^{\text{ss}}_{ij}\,\delta u_{i|j} + \omega^2 \tilde{\varrho}\, u_i\,\delta u_i - \frac{\phi^2}{\omega^2 \tilde{\varrho}_{\text{ff}}}\, p_{|i}\,\delta p_{|i} + \frac{\phi^2}{\tilde{\lambda}_{\text{f}}}\, p\,\delta p$$
$$+ \phi \left(1 + \frac{\tilde{\varrho}_{\text{sf}}}{\tilde{\varrho}_{\text{ff}}} \right) \big(\delta p_{|i}\, u_i + p_{|i}\,\delta u_i \big) + \phi \left(1 + \frac{\tilde{\lambda}_{\text{sf}}}{\tilde{\lambda}_{\text{f}}} \right) \big(\delta p\, u_{i|i} + p\,\delta u_{i|i} \big).$$

Here, u_i are the displacements of a piezoelectric or elastic solid, or of the solid-phase of poroelastic material, V is the electric potential in the piezoelectric domain and p is the pressure in the acoustic medium or in the pores of poroelastic medium. The variational equation (6.1) must be satisfied for *all* admissible variations (i.e., virtual or test functions) of primary variables: δu_i, δV, and δp. Furthermore:

$$(6.3)\qquad \sigma^{\rm pz}_{ij} = C^{\rm pz}_{ijkl}\,\frac{u_{k|l}+u_{l|k}}{2} + e^{\rm pz}_{kij}\,V_{|k}, \qquad D^{\rm pz}_{i} = e^{\rm pz}_{ikl}\,\frac{u_{k|l}+u_{l|k}}{2} - \epsilon^{\rm pz}_{ik}\,V_{|k},$$

$$(6.4)\qquad \sigma^{\rm ss}_{ij} = \mu_{\rm s}\,(u_{i|j}+u_{j|i}) + \tilde{\lambda}_{\rm ss}\,u_{k|k}\,\delta_{ij},$$

and

$$(6.5)\quad \sigma^{\rm e}_{ij} = \mu_{\rm e}\,(u_{i|j}+u_{j|i}) + \lambda_{\rm e}\,u_{k|k}\,\delta_{ij} \quad \text{or (in general)} \quad \sigma^{\rm e}_{ij} = C^{\rm e}_{ijkl}\,\frac{u_{k|l}+u_{l|k}}{2}.$$

Notice that the terms in the first line of the functional (6.1) of coupled multiphysics system refer to the piezoelectric subdomains, another two terms – to the elastic subdomains; then, there are two integrals pertaining to the poroelastic media, and the last line brings terms relevant for the acoustical medium, where the last integral describes the interface coupling to other media. Obviously, all material parameters involved in the functional are functions of position defined on the relevant subdomains. For elastic solids, the first formula in (6.5) refers to isotropic elastic materials while the second one – to elastic materials in general.

6.2. Galerkin finite element approximation

The discrete equations for a finite element model will be obtained from the functional (6.1) by using finite element interpolants for the trial and test functions, as stated by the Galerkin method. Remember that $i,j,k,l \in \{1,2,3\}$ are indices referring to the coordinates of the system of reference. Now, new subscripts are introduced for the degrees of freedom of a discrete model, namely: $m,n \in \{1,\dots N_{\rm dof}\}$, where $N_{\rm dof}$ is the total number of degrees of freedom. For simplicity and to avoid any inconsistency, the summation convention is in use also for these subscripts.

Let $\mathcal{N}^{u}_{im}$, $\mathcal{N}^{p}_{m}$, $\mathcal{N}^{V}_{m}$ be the interpolants, that is, the so-called *global shape functions* defined in the whole domain Ω; they are used to approximate the fields of displacements, pressure, and electric potential, respectively:

$$(6.6)\qquad u_i(\boldsymbol{x}) \approx \mathcal{N}^{u}_{im}(\boldsymbol{x})\,q_m, \qquad p(\boldsymbol{x}) \approx \mathcal{N}^{p}_{m}(\boldsymbol{x})\,q_m, \qquad V(\boldsymbol{x}) \approx \mathcal{N}^{V}_{m}(\boldsymbol{x})\,q_m,$$

where q_m are the *degrees of freedom* of discrete model, $(\boldsymbol{x}) \equiv (x_1,x_2,x_3)$. They form the *global vector of degrees-of-freedom*, $\mathbf{q}$, and can be divided into five groups of components as follows:

$$(6.7)\qquad \begin{aligned} q_m &\in \mathbf{q}^{u_i} && \text{if } q_m = u_i(\boldsymbol{x}) \quad (i=1,2,3),\\ q_m &\in \mathbf{q}^{p} && \text{if } q_m = p(\boldsymbol{x}),\\ q_m &\in \mathbf{q}^{V} && \text{if } q_m = V(\boldsymbol{x}). \end{aligned}$$

Here, $\mathbf{q}^{u_i}$ $(i=1,2,3)$, $\mathbf{q}^{p}$, and $\mathbf{q}^{V}$ are subvectors of the vector $\mathbf{q}$, corresponding to the three mechanical displacements, pressure, and electric potential, respectively.

The Galerkin method requires that the same shape functions are also used to approximate the corresponding test functions $\delta u_i(\boldsymbol{x})$, $\delta p(\boldsymbol{x})$, $\delta V(\boldsymbol{x})$, and using all these approximations for the functional (6.1) yields eventually the following system of algebraic equations:

$$\tilde{A}_{mn}\, q_n = F_m, \tag{6.8}$$

where the *governing matrix* and the *right-hand-side vector* can be presented as an assembled contributions of piezoelectric, elastic, poroelastic and acoustic subdomains, that is,

$$\tilde{A}_{mn} = \tilde{A}^{\mathrm{pz}}_{mn} + \tilde{A}^{\mathrm{e}}_{mn} + \tilde{A}^{\mathrm{p}}_{mn} + \tilde{A}^{\mathrm{a}}_{mn} + A^{\text{a-p,e,pz}}_{mn}, \tag{6.9}$$

$$F_m = F^{\mathrm{pz}}_m + F^{\mathrm{e}}_m + F^{\mathrm{p}}_m + F^{\mathrm{a}}_m. \tag{6.10}$$

The obtained matrices and vectors contributing to the global system of discrete equations are the results of integrating – over the relevant subdomains, boundaries and interfaces – the terms approximated by the known (i.e., assumed) shape functions. The relevant integrals defining the component matrices and vectors are presented below.

Notice that in the formula for the system governing matrix (6.9), there is also a contribution, $A^{\text{a-p,e,pz}}_{mn}$, resulting from coupling on the interface between the acoustic subdomain and the poroelastic and elastic (or piezoelectric) subdomains (the naturally-handled coupling between the poroelastic and elastic subdomains provides no contribution). It will be apparent further below that the system matrix and the first four component-matrices are frequency-dependent, while the interface-coupling matrix and the right-hand-side vector are not. Therefore, when carrying out frequency-analysis, these latter quantities (i.e., the boundary or interface terms resulting from the Neumann-type excitations or inter-subdomain coupling) are to be computed only once, and should be used then for any computational frequency.

The piezoelectric, elastic, poroelastic, and acoustic contribution matrices of Eq. (6.9) are composed from the following components:

$$\tilde{A}^{\mathrm{pz}}_{mn} = K^{\mathrm{pz}}_{mn} - \omega^2 M^{\mathrm{pz}}_{mn} + L^{\mathrm{pz}}_{mn} + B^{\mathrm{pz}}_{mn}, \tag{6.11}$$

$$\tilde{A}^{\mathrm{e}}_{mn} = K^{\mathrm{e}}_{mn} - \omega^2 M^{\mathrm{e}}_{mn}, \tag{6.12}$$

$$\tilde{A}^{\mathrm{p}}_{mn} = \tilde{K}^{\mathrm{p}}_{mn} - \omega^2 \tilde{M}^{\mathrm{p}}_{mn} + \frac{1}{\omega^2}\, \tilde{P}^{\mathrm{p}}_{mn} - \tilde{Q}^{\mathrm{p}}_{mn} - \tilde{R}^{\mathrm{p}}_{mn} - \tilde{S}^{\mathrm{p}}_{mn}, \tag{6.13}$$

$$\tilde{A}^{\mathrm{a}}_{mn} = \frac{1}{\omega^2}\, P^{\mathrm{a}}_{mn} - Q^{\mathrm{a}}_{mn}. \tag{6.14}$$

The frequency-dependence is explicitly shown in the above formulas. However, in the case of the poroelastic subdomain matrix (6.13), the frequency-dependence is also implicit because the component matrices depend on some frequency-dependent parameters of poroelastic material. Moreover, the component matrices of the acoustic subdomain matrix (6.14) can also be frequency-dependent, i.e., $P^{\mathrm{a}}_{mn} = \tilde{P}^{\mathrm{a}}_{mn}(\omega)$, $Q^{\mathrm{a}}_{mn} = \tilde{Q}^{\mathrm{a}}_{mn}(\omega)$; this happens when the acoustic medium is a fluid-equivalent model of a porous material with rigid frame rather than a simple fluid (like the air). Therefore, in the case of porous materials (both, with rigid and elastic frame), the component matrices for corresponding subdomains must be recalculated for every computational frequency, whereas in the case of piezoelectric or elastic media, or perfect fluids, the component matrices need to be calculated only once, and the corresponding subdomain matrices are then simply assembled for every computational frequency using Eqs. (6.11), (6.12), or (6.14), respectively.

The formulae and nomenclature (basing on some physical interpretations) for all the component matrices and vectors will be given below. Moreover, these submatrices and subvectors of the discrete system will be visualized in a diagram (see Fig. 2 on p. 368).

There are four subcomponents in the matrix (6.11) obtained for piezoelectric subdomain. They are: the *stiffness matrix* and the *mass matrix*,

$$K^{\mathrm{pz}}_{mn} = \frac{1}{2}\int\limits_{\Omega_{\mathrm{pz}}} C^{\mathrm{pz}}_{ijkl}\left(\mathcal{N}^{u}_{kn|l} + \mathcal{N}^{u}_{ln|k}\right)\mathcal{N}^{u}_{im|j}, \qquad M^{\mathrm{pz}}_{mn} = \int\limits_{\Omega_{\mathrm{pz}}} \varrho_{\mathrm{pz}}\,\mathcal{N}^{u}_{im}\,\mathcal{N}^{u}_{in}, \tag{6.15}$$

the *electric permittivity* matrix,

$$L^{\mathrm{pz}}_{mn} = \int\limits_{\Omega_{\mathrm{pz}}} \epsilon^{\mathrm{pz}}_{ik}\,\mathcal{N}^{V}_{m|i}\,\mathcal{N}^{V}_{n|k}, \tag{6.16}$$

and finally, the *piezoelectric coupling matrix*,

$$\begin{aligned} B^{\mathrm{pz}}_{mn} &= \int\limits_{\Omega_{\mathrm{pz}}} e^{\mathrm{pz}}_{kij}\,\mathcal{N}^{V}_{n|k}\,\mathcal{N}^{u}_{im|j} - \frac{1}{2}\int\limits_{\Omega_{\mathrm{pz}}} e^{\mathrm{pz}}_{ikl}\left(\mathcal{N}^{u}_{kn|l} + \mathcal{N}^{u}_{ln|k}\right)\mathcal{N}^{V}_{m|i} \\ &= \int\limits_{\Omega_{\mathrm{pz}}} e^{\mathrm{pz}}_{ikl}\left[\mathcal{N}^{V}_{n|i}\,\mathcal{N}^{u}_{km|l} - \frac{1}{2}\big(\mathcal{N}^{u}_{kn|l} + \mathcal{N}^{u}_{ln|k}\big)\mathcal{N}^{V}_{m|i}\right]. \end{aligned} \tag{6.17}$$

As shown in Eq. (6.13), six component matrices are distinguished for poroelastic subdomain, namely: the *stiffness matrix* of the skeleton in vacuo and the *mass matrix*,

$$
\begin{aligned}
\tilde{K}^{\mathrm{p}}_{mn} &= \int\limits_{\Omega_{\mathrm{p}}} \Big[\mu_{\mathrm{s}} \left(\mathcal{N}^{u}_{in|j} + \mathcal{N}^{u}_{jn|i}\right) \mathcal{N}^{u}_{im|j} + \tilde{\lambda}_{\mathrm{ss}} \mathcal{N}^{u}_{im|i} \mathcal{N}^{u}_{jn|j}\Big], \\
\tilde{M}^{\mathrm{p}}_{mn} &= \int\limits_{\Omega_{\mathrm{p}}} \tilde{\varrho} \mathcal{N}^{u}_{im} \mathcal{N}^{u}_{in},
\end{aligned}
\tag{6.18}
$$

the *kinetic* and *compressional energy matrices* of the fluid phase,

$$
\tilde{P}^{\mathrm{p}}_{mn} = \int\limits_{\Omega_{\mathrm{p}}} \frac{\phi^2}{\tilde{\varrho}_{\mathrm{ff}}} \mathcal{N}^{p}_{m|i} \mathcal{N}^{p}_{n|i}, \qquad \tilde{Q}^{\mathrm{p}}_{mn} = \int\limits_{\Omega_{\mathrm{p}}} \frac{\phi^2}{\tilde{\lambda}_{\mathrm{f}}} \mathcal{N}^{p}_{m} \mathcal{N}^{p}_{n},
\tag{6.19}
$$

and finally, the *matrix of visco-inertial (or kinetic) coupling* and the *matrix of elastic (or potential) coupling*, that is, respectively,

$$
\tilde{R}^{\mathrm{p}}_{mn} = \int\limits_{\Omega_{\mathrm{p}}} \phi \left(1 + \frac{\tilde{\varrho}_{\mathrm{sf}}}{\tilde{\varrho}_{\mathrm{ff}}}\right) \left(\mathcal{N}^{p}_{m|i} \mathcal{N}^{u}_{in} + \mathcal{N}^{p}_{n|i} \mathcal{N}^{u}_{im}\right),
\tag{6.20}
$$

$$
\tilde{S}^{\mathrm{p}}_{mn} = \int\limits_{\Omega_{\mathrm{p}}} \phi \left(1 + \frac{\tilde{\lambda}_{\mathrm{sf}}}{\tilde{\lambda}_{\mathrm{f}}}\right) \left(\mathcal{N}^{p}_{m} \mathcal{N}^{u}_{in|i} + \mathcal{N}^{p}_{n} \mathcal{N}^{u}_{im|i}\right).
\tag{6.21}
$$

These two coupling matrices can be treated together since they share the same degrees of freedom (as a matter of fact, they couple the displacement degrees of freedom with the pressure ones).

The elastic subdomain matrix (6.12) has two component matrices resulting from the stiffness and inertia of elastic medium. These *stiffness* and *mass matrices* read as follows:

$$
\begin{aligned}
K^{\mathrm{e}}_{mn} &= \int\limits_{\Omega_{\mathrm{e}}} \Big[\mu_{\mathrm{e}} \left(\mathcal{N}^{u}_{in|j} + \mathcal{N}^{u}_{jn|i}\right) \mathcal{N}^{u}_{im|j} + \lambda_{\mathrm{e}} \mathcal{N}^{u}_{im|i} \mathcal{N}^{u}_{jn|j}\Big], \\
M^{\mathrm{e}}_{mn} &= \int\limits_{\Omega_{\mathrm{e}}} \varrho_{\mathrm{e}} \mathcal{N}^{u}_{im} \mathcal{N}^{u}_{in}.
\end{aligned}
\tag{6.22}
$$

There are also two component matrices in the case of the acoustic subdomain matrix (6.14), namely, the *kinetic* and *compressional energy matrix*, respectively:

$$
P^{\mathrm{a}}_{mn} = \int\limits_{\Omega_{\mathrm{a}}} \frac{1}{\varrho_{\mathrm{a}}} \mathcal{N}^{p}_{m|i} \mathcal{N}^{p}_{n|i}, \qquad Q^{\mathrm{a}}_{mn} = \int\limits_{\Omega_{\mathrm{a}}} \frac{1}{K_{\mathrm{a}}} \mathcal{N}^{p}_{m} \mathcal{N}^{p}_{n}.
\tag{6.23}
$$

As it has already been mentioned, the acoustic medium contribution to the governing matrix (6.9) of the system (6.8) arises also from the interface coupling to the poroelastic and elastic (or piezoelectric) media. The relevant *interface coupling matrix* is computed as

$$A^{\text{a-p,e,pz}}_{mn} = A^{\text{a-p}}_{mn} + A^{\text{a-e}}_{mn} + A^{\text{a-pz}}_{mn} = -\int\limits_{\Gamma_{\text{a-p,e,pz}}} n^{\text{a}}_i \big(\mathcal{N}^p_m \mathcal{N}^u_{in} + \mathcal{N}^p_n \mathcal{N}^u_{im}\big). \tag{6.24}$$

Finally, the formulae for the component-vectors of the right-hand-side vector (6.10) of the system of equations (6.8) must be provided; they are:

$$\begin{gathered} F^{\text{pz}}_m = F^{\text{pz}t}_m + F^{\text{pz}Q}_m = \int\limits_{\Gamma^t_{\text{pz}}} \hat{t}^{\text{pz}}_i \mathcal{N}^u_{im} - \int\limits_{\Gamma^Q_{\text{pz}}} \hat{Q}^{\text{pz}} \mathcal{N}^V_m, \\ F^{\text{e}}_m = \int\limits_{\Gamma^t_{\text{e}}} \hat{t}^{\text{e}}_i \mathcal{N}^u_{im}, \qquad F^{\text{p}}_m = -\int\limits_{\Gamma^p_{\text{p}}} \hat{p}\, n_i \mathcal{N}^u_{im}, \qquad F^{\text{a}}_m = \int\limits_{\Gamma^u_{\text{a}}} \hat{u}^{\text{a}}_i n^{\text{a}}_i \mathcal{N}^p_m. \end{gathered} \tag{6.25}$$

These vectors arise from the Neumann boundary conditions of the piezoelectric, elastic, poroelastic, and acoustic subdomain, respectively. Notice that in case of elastic medium, an imposed pressure or traction results in the Neumann condition, whereas it is a prescribed displacement in case of an acoustic subdomain. As for the *biphasic* theory of poroelasticity, the mixed displacement-pressure formulation renders the imposed-pressure condition as a hybrid one, that is, essential for the fluid phase and natural for the solid one; the imposed-displacement condition is essential for the solid phase and naturally-handled by the fluid phase (thanks to the mentioned enhancement of the mixed formulation).

The linear algebraic system of equations (6.8) constitutes a discrete model of a multiphysics problem involving poroelastic, acoustic, elastic and piezoelectric media. Figure 1 (see Sec. 6.1) presents a schematic diagram of such a problem. The coupling interfaces as well as the boundaries for essential and natural conditions are presented. In the case of piezoelectric subdomain the boundary-division relevant to electrical conditions is skipped. The couplings of poroelastic or acoustic media to a piezoelectric material are similar to the couplings of these media to an elastic material and, for clearness, they are not presented.

Figure 2 shows a diagram of the system of algebraic equations where particular submatrices and subvectors are visualized. The system describes a discrete model of the multiphysics problem of coupled poroelastic, acoustic, elastic, and piezoelectric media. Different interface couplings are manifested by the intersections of the submatrices. Obviously, the sizes of submatrices are irrelevant since they depend on a particular problem. Thus, components for an acoustic subdomain may refer to a usually big region of air, yet in this region there is only one degree-of-freedom per node, and the mesh density can be usually very coarse (since the wavelength is comparatively very long); therefore, the relevant

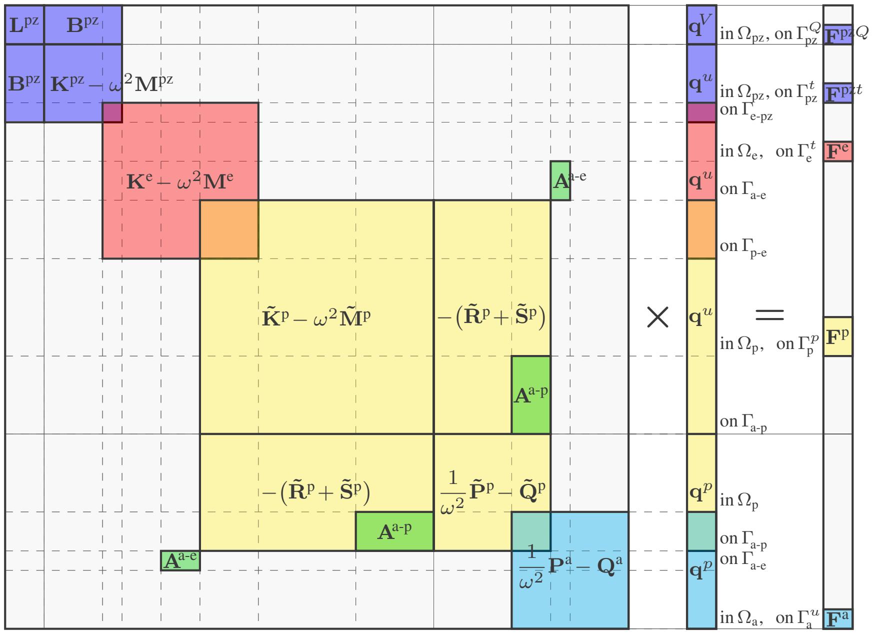

FIG. 2. Submatrices and subvectors of the global system of algebraic equations for a discrete model of piezo-elasto-poroealstic structure coupled to an acoustic medium.

contribution to the global matrix of coefficients may be rather limited. On the other hand, the piezoelectric actuators may occupy very small regions but they require 4 DOFs per node and dense meshes. Finally, the porous layers will generally occupy comparatively big regions and moreover, the poroelastic subdomains require 4 DOFs per node (for the mixed formulation in 3D) and rather dense meshes, and so their contribution to the global discrete system should be significant. Certainly, any contribution is also affected by the used approximation order. Generally speaking, for the acoustic subdomain linear shape functions can be used, whereas for the elastic or piezoelectric subdomains the second-order (quadratic) Lagrange polynomials should be preferred as shape functions for all the component fields of displacement, as well as for the scalar field of electric potential. It is important to emphasize that the usage of the second-order polynomial for the electric potential is quite important for accurate estimation of voltage amplitudes used in active control. A first-order interpolation would result in a linear through-thickness variation of the electric potential and that would neglect the induced potential and the electromechanical coupling would be partial. The poroelastic subdomain, which requires dense meshes and many DOFs per node, can be approximated with linear shape functions. Nevertheless, to prove convergence in the example below, two solutions will be presented: one with linear and the other with quadratic approximation.

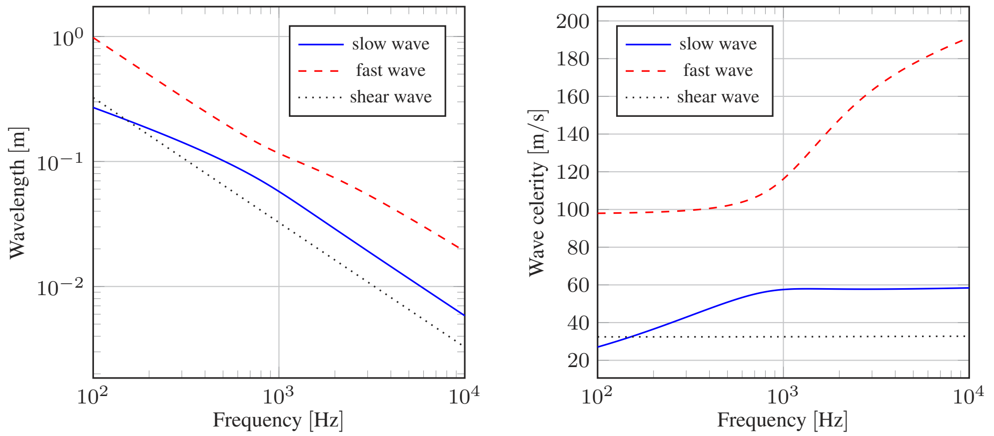

Fig. 3. The frequency-dependent wavelegths and wave velocities in the dispersive medium of poroelastic foam.

Obviously, different finite element discretization is required for different subdomains, and moreover, the discretization must depend on frequency. Different materials and media are supposed to interact in modelled problems and the wave propagation may change drastically between various subdomains, since the wavelengths are different. Furthermore, there are three waves that propagate with different speeds in poroelastic media: a fast compressional wave and a shear wave, both originating mainly from the elastic solid of skeleton, and a slow compressional fluid-borne wave. Moreover, the poroelastic medium is dispersive so that the velocities of (compressional) waves depend on frequency. This can be observed in Fig. 3, which shows the wavelengths and wave speeds for a poroelastic material used in the example below. Thus, generally speaking, the subdomains must be discretized into finite elements of sizes sufficiently small to satisfy the common requirement of several elements per wavelength, and this should be done for the shortest waves and so for the highest frequency of interest. It is obvious though, that the required size of elements will vary drastically for subdomains of various media and for some, let us say, "longer-wavelegth" subdomains, the elements in the vicinity of the interfaces with "shorter-wavelegth" subdomains may also be significantly smaller than the required size, in order to maintain the geometrical quality of the mesh. These requirements were fulfilled by the FE mesh used in the example below.

7. Numerical example

Figure 4 (left) presents a simple generic example for two-dimensional analysis of an active-passive acoustic panel. A simple slat-shaped panel is composed

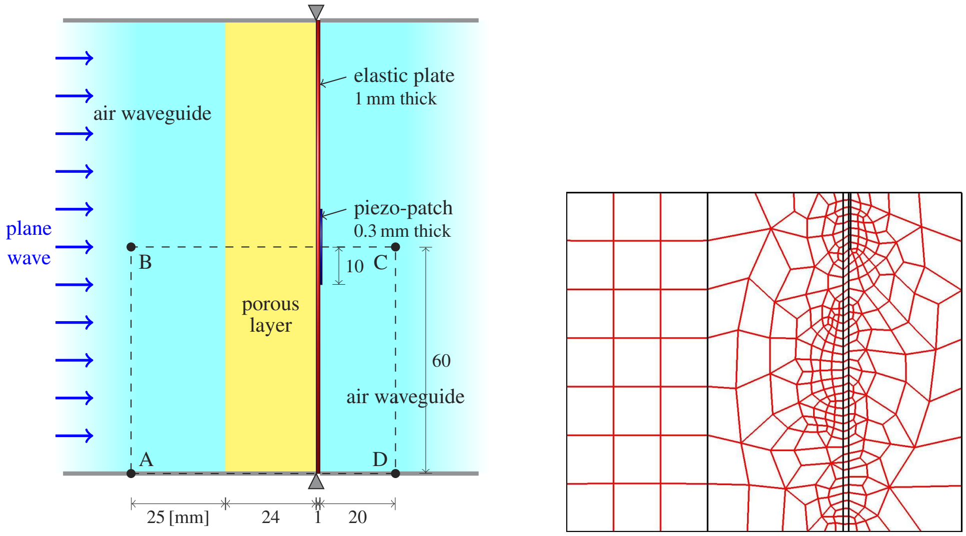

Fig. 4. A generic two-dimensional configuration of an active-passive acoustic panel and a finite element mesh for the modelled area ABCD.

of a single layer of highly-absorbing porous material fixed to an aluminium plate. The porous material is very light, its porosity being 99%. The frequency-dependent wavelengths and velocities for the three waves propagating in it have been presented in the previous section, in Fig. 3. The plate is 1 mm-thick and the thickness of porous layer is 24 mm, so that the total thickness of panel is 25 mm. The width of panel is 120 mm, whereas its length is considered to be significantly bigger, so that such slat panel could be modelled using two-dimensional (plane strain) approach. The panel is put into a 120 mm-wide slit of a waveguide filled with air. The plate of panel is simply-supported at both ends (at the walls) and the skeleton of porous layer can freely slide along the waveguide walls. (Such boundary conditions can be easily realized in practice.) The panel should allow an active approach: thus, in the centre of the free face of aluminium plate, a 0.3 mm-thick piezoelectric patch is glued; its width is 20 mm. A plane harmonic acoustic wave propagates in air of the waveguide, onto the panel. The amplitude of source pressure is $p_0 = 1$ Pa. Depending on its frequency, the wave can be partially reflected and absorbed by the panel, or transmitted through it.

By taking advantage of the symmetry, the problem can be modelled using the rectangular domain ABCD shown in Fig. 4, where a finite-element mesh of the modelled domain is also presented. Appropriate boundary conditions must be applied. The rigid-wall or sliding conditions are set on relevant parts of boundaries AD and BC (in the latter case, these conditions are valid because of the symmetry plane). On the boundary CD, the free radiation condition is set, or alternatively, the impedance boundary condition (using the characteristic impedance of air) can be set; as a matter of fact it was checked that both approaches give very similar results. Such conditions simulate for a finite domain (modelled with finite elements) the fact that there is no reflection at the relevant boundary and the wave can freely propagate outside the domain. Finally, a radiation condition with a plane incident pressure wave is set on the boundary AB. (Another approach, where simply $p = p_0$ is set on the boundary would result in the appearance of some additional cavity resonances.) The used radiation conditions are special boundary conditions, often called the non-reflecting boundary conditions (NRBC) [39], which ensure that no (or little) spurious wave reflection occurs from the boundary. For the case of this time-harmonic analysis, the second-order implementation of the Givoli and Neta's reformulation [40] of the Higdon conditions for plane waves is used.

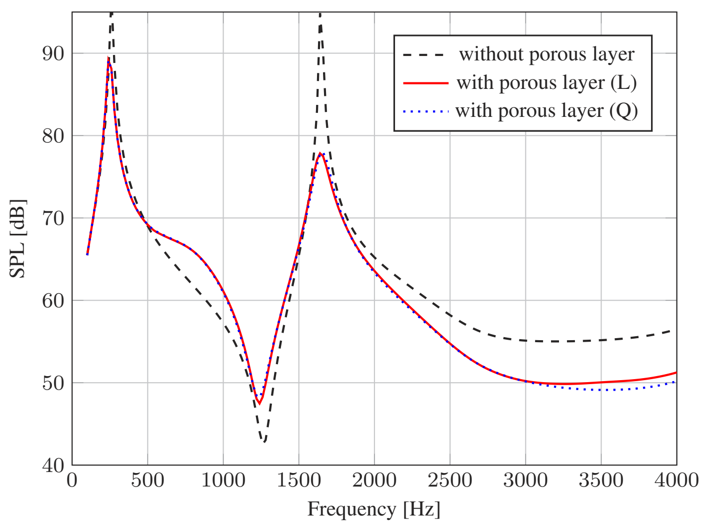

FIG. 5. SPL at point C for the passive panel without porous layer and with porous layer approximated with linear (L) or quadratic (Q) shape functions

Firstly, the behaviour of panel in passive state was analysed. The passive behaviour means that the piezoelectric patch was simply shunted (no voltage signal was sent to its electrodes), and the only excitation provided to the system was by the plane harmonic wave. The passive results are shown in Fig. 5 in the form of sound pressure level (SPL) curves calculated at point C for a wide range of frequencies. Remembering that the SPL of source is approx. 94 dB (for the pressure amplitude $p_0 = 1$ Pa), these curves illustrate the expected reduction of noise. Three curves are presented for the passive state; namely, two SPL curves obtained for the poro-elastic panel and an SPL curve obtained for the plate in absence of the porous layer (in the latter case, the subdomain of porous layer is modelled as air using the same mesh of elements). Notice that the two solutions computed for the system with porous layer are almost identical (Fig. 5), though the first solution, marked (L), was obtained when the poroelastic and acoustic domains were approximated with linear shape functions, whereas the second one, marked (Q), was computed for the approximation with quadratic shape functions. (The elastic and piezoelectric domains were in both cases approximated with quadratic shape functions.) Both solutions were calculated for the same FE mesh, so it can also be considered as an example of the P-convergence method of solution (where, in general, the FE mesh is unaltered and only the polynomial order of approximation is increased). The discrepancies between these two solutions are very small indeed, and moreover, appear only in the region of highest frequencies, which means that the assumed FE mesh seems to be sufficient for the linear approximation and for the quadratic shape functions, a coarser mesh could be used. Now, from the comparison of these solutions with the one obtained for the system without porous layer, it is clearly visible that for frequencies above 1.5 kHz the performance of the panel is better that that of the plate, and above 3 kHz this improvement reaches and exceeds 6 dB. The absorbing effect of a rather thin porous layer allows for significant reduction of vibro-acoustic transmission in higher frequencies. An important observation concerns two peaks that appear at approx. 240 Hz and 1700 Hz. They are the vibro-acoustic noise resulting from the first and second eigenmode vibrations of the plate, caused by the acoustic wave excitation of relevant frequency. It can be observed that the porous layer damps slightly the effect of noise emitted at plate eigenfrequencies; it is not so in absence of the porous layer (the inherent plate damping is comparatively very small and it wasn't considered in the model of elastic plate). Nevertheless, an active approach is necessary in order to reduce the vibro-acoustic transmission not only at these eigenfrequencies, but for lower frequencies in general.

Figure 6 shows some results of active analyses carried out for the panel excited by the plane acoustic wave with harmonic frequency of 240 Hz (which is approximately the first eigenfrequency of the plate) and a voltage signal of the same frequency applied to the electrodes of the piezoelectric patch in order to reduce the noisy vibrations. The generated/transmitted low-frequency noise was observed at point C for different amplitudes of the voltage signal in the range from 0 to 1 V. (It is worth to mention that a similar noise would be observable even farther from the plate since for the first eigenmode shape, there is no auto-cancellation of waves generated by the neighbouring parts of the plate and the total emitted wave quickly becomes plane in the waveguide.) The sound pressure level computed at point C for different voltage amplitudes is shown in Fig. 6: from this SPL curve it can be assessed that the necessary amplitude of the voltage signal is approximately 0.9 V. One should remember, however, that

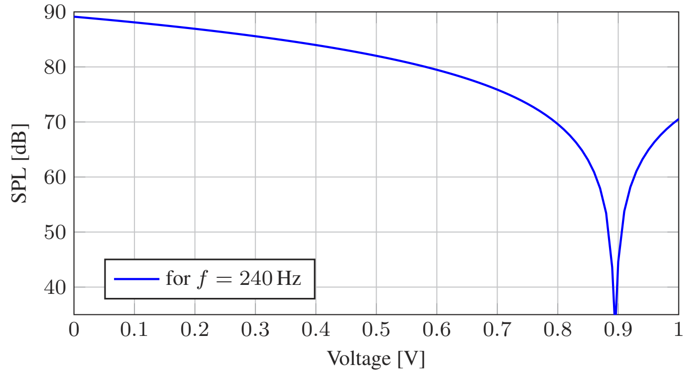

Fig. 6. SPL at point C at 240 Hz for different values of the voltage amplitude of the active signal.

in practice, the noisy vibrations are often induced by causes stronger than impinging acoustic waves, and would generally require higher voltages for active control. Finally, it was checked that such localization of piezo-patch actuator (in a node of the relevant eigenmode) does not allow to reduce the noisy vibrations at 1700 Hz.

8. Conclusions

The theoretical study presented in this paper constitutes a complete basis for the development of numerical tools, necessary for accurate multiphysics modeling of active vibroacoustical problems involving poroelastic, elastic, piezoelectric, and acoustic media. The discrete model derived according to the Galerkin method is ready for the implementation as a finite element code. The fully-coupled, multiphysics system has been developed in order to model hybrid vibroacoustical attenuators (absorbers, insulators) in the form of active-passive liners, panels, or composites. The system allows to use the advanced theory of poroelasticity to model porous media so that the mechanics of elastic skeleton is coupled with the acoustic wave propagation in the fluid in pores. The active elements can be modeled using the accurate (not reduced) model of piezoelectricity. Moreover, all the possible couplings on the interfaces between different media are also taken into account. Finally, the developed fully-coupled FE model is used to solve a generic two-dimensional example of the defined problems, and issues concerning finite element approximation and convergence are also discussed. Some, at least partial, experimental validation of the proposed system (since no porous media are involved) will be presented in [41] for a problem of active reduction of structure-borne noise for a thin aluminium plate, with actuators in the form of piezoelectric patches.

Acknowledgements

Financial support of the Foundation for Polish Science Team Programme co-financed by the EU European Regional Development Fund Operational Programme "Innovative Economy 2007-2013": Project "Smart Technologies for Safety Engineering – SMART and SAFE", No. TEAM/2008-1/4, and Project "Modern Material Technologies in Aerospace Industry", No. POIG.0101.02-00-015/08, is gratefully acknowledged.

Appendix A. Derivation of the mixed displacement-pressure formulation of poroelasticity

Equations for the mixed formulation of poroelasticity will be derived here to show that it has the form of a classical coupled fluid-structure problem, involving the dynamic equations of the skeleton *in vacuo* and the equivalent fluid in the rigid skeleton limit. First, notice that the fluid-phase stress tensor can be expressed as

$$\sigma^{\mathrm{f}}_{ij} = -\phi\, p\, \delta_{ij} \tag{A.1}$$

where p is the pressure of fluid in the pores (it should not be mistaken for the pressure of fluid phase which equals ϕp). Using this relation for the constitutive equation of fluid phase (2.8) yields the following expressions:

$$p = -\frac{\tilde{\lambda}_{\mathrm{f}}}{\phi}\, U_{k|k} - \frac{\tilde{\lambda}_{\mathrm{sf}}}{\phi}\, u_{k|k}, \qquad \text{or} \qquad U_{k|k} = -\frac{\phi}{\tilde{\lambda}_{\mathrm{f}}}\, p - \frac{\tilde{\lambda}_{\mathrm{sf}}}{\tilde{\lambda}_{\mathrm{f}}}\, u_{k|k}. \tag{A.2}$$

They are valid for the general case since only a constitutive equation has been used. Now, however, the interest is restricted to the harmonic oscillations (with the angular frequency ω). In this case, by using Eq. (A.1) in the harmonic equilibrium equation of fluid phase (2.5), one can express the fluid-phase displacements as a function of the pressure in the pores and the solid-phase displacements:

$$U_i = \frac{\phi}{\omega^2 \tilde{\varrho}_{\mathrm{ff}}}\, p_{|i} - \frac{\tilde{\varrho}_{\mathrm{sf}}}{\tilde{\varrho}_{\mathrm{ff}}}\, u_i. \tag{A.3}$$

And this in turn can be used for Eq. (2.4); so now, the harmonic equilibrium for the solid phase can be expressed as follows:

$$\sigma^{\mathrm{ss}}_{ij|j} + \omega^2 \tilde{\varrho}\, u_i + \phi\left(\frac{\tilde{\varrho}_{\mathrm{sf}}}{\tilde{\varrho}_{\mathrm{ff}}} - \frac{\tilde{\lambda}_{\mathrm{sf}}}{\tilde{\lambda}_{\mathrm{f}}}\right) p_{|i} = 0 \qquad \text{where} \qquad \tilde{\varrho} = \tilde{\varrho}_{\mathrm{ss}} - \frac{\tilde{\varrho}^2_{\mathrm{sf}}}{\tilde{\varrho}_{\mathrm{ff}}}. \tag{A.4}$$

Here, a new stress tensor is introduced

$$\sigma^{\mathrm{ss}}_{ij} = \mu_{\mathrm{s}}\, (u_{i|j} + u_{j|i}) + \tilde{\lambda}_{\mathrm{ss}}\, u_{k|k}\, \delta_{ij} \qquad \text{where} \qquad \tilde{\lambda}_{\mathrm{ss}} = \tilde{\lambda}_{\mathrm{s}} - \frac{\tilde{\lambda}^2_{\mathrm{sf}}}{\tilde{\lambda}_{\mathrm{f}}}. \tag{A.5}$$

This tensor depends only on the solid-phase displacements and has an interesting physical interpretation: it is called the stress tensor of the skeleton *in vacuo* because it describes the stresses in the skeleton when there is no fluid in the pores or when at least the pressure of fluid is constant in the pores. This can be easily noticed when putting $p(\boldsymbol{x}) = \text{const.}$ in Eq. (A.4); then, the last term (which couples this equation with its fluid-phase counterpart) vanishes and so the remaining terms clearly describe the behaviour of the skeleton of poroelastic medium filled with a fluid under the same pressure everywhere. The new stress tensor is related to the solid phase stress tensor in the following way:

$$\sigma^{\mathrm{s}}_{ij} = \sigma^{\mathrm{ss}}_{ij} - \phi\,\frac{\tilde{\lambda}_{\mathrm{sf}}}{\tilde{\lambda}_{\mathrm{f}}}\,p\,\delta_{ij}. \tag{A.6}$$

Now, the fluid-phase displacements are to be eliminated from the fluid-phase harmonic equilibrium equations (2.5). To this end, Eq. (A.1) and the second formula from Eqs. (A.2) are used for Eq. (2.5) to obtain (after multiplication by $-\phi/\omega^2\tilde{\varrho}_{\mathrm{ff}}$)

$$\frac{\phi^2}{\omega^2\,\tilde{\varrho}_{\mathrm{ff}}}\,p_{|ii} + \frac{\phi^2}{\tilde{\lambda}_{\mathrm{f}}}\,p - \phi\left(\frac{\tilde{\varrho}_{\mathrm{sf}}}{\tilde{\varrho}_{\mathrm{ff}}} - \frac{\tilde{\lambda}_{\mathrm{sf}}}{\tilde{\lambda}_{\mathrm{f}}}\right)u_{i|i} = 0. \tag{A.7}$$

This equation pertains to the fluid phase but the last term couples it with the solid-phase equation (A.4). This term vanishes for the rigid body motion of the skeleton (that is, when $u_i = \text{const.}$). This means that the main terms describe the behaviour of the fluid when the skeleton is motionless or rigid. Notice also that the expression which stands by $u_{i|i}$ in the coupling term is similar to the one standing by $p_{|i}$ in the coupling term of the solid-phase equation (A.4). This feature is quite important when constructing the so-called weak variational formulation (and it justifies to present Eq. (A.7) in such a form), since it permits to simplify the handling of some coupling conditions at the interface between two different poroelastic media.

Equations (A.4), and (A.7) together with the constitutive relation (A.5), constitute the *mixed displacement-pressure formulation* of harmonic isotropic poroelasticity. For completeness, the total stresses and total displacements in terms of the fluid pressure and solid-phase displacements are given here:

$$\sigma^{\mathrm{t}}_{ij} = \sigma^{\mathrm{ss}}_{ij} - \phi\left(1 + \frac{\tilde{\lambda}_{\mathrm{sf}}}{\tilde{\lambda}_{\mathrm{f}}}\right)p\,\delta_{ij} = \mu_{\mathrm{s}}\,(u_{i|j} + u_{j|i}) + \left[\tilde{\lambda}_{\mathrm{ss}}\,u_{k|k} - \phi\left(1 + \frac{\tilde{\lambda}_{\mathrm{sf}}}{\tilde{\lambda}_{\mathrm{f}}}\right)p\right]\delta_{ij}, \tag{A.8}$$

$$u^{\mathrm{t}}_i = \left[1 - \phi\left(1 + \frac{\tilde{\varrho}_{\mathrm{sf}}}{\tilde{\varrho}_{\mathrm{ff}}}\right)\right]u_i + \frac{\phi^2}{\omega^2\,\tilde{\varrho}_{\mathrm{ff}}}\,p_{|i}. \tag{A.9}$$

References

1. C.R. Fuller, S.J. Elliott, P.A. Nelson, *Active Control of Vibration*, Academic Press, 1996.

2. C. Guigou, C.R. Fuller, *Control of aircraft interior broadband noise with foam-PVDF smart skin*, J. Sound Vib., **220**, 3, 541–557, 1999.

3. B.D. Johnson, C.R. Fuller, *Broadband control of plate radiation using a piezoelectric, double-amplifier active-skin and structural acoustic sensing*, J. Acoust. Soc. Am., **107**, 2, 876–884, February 2000.

4. O. Lacour, M.-A. Galland, D. Thenail, *Preliminary experiments on noise reduction in cavities using active impedance changes*, J. Sound Vib., **230**, 1, 69–99, 2000.

5. M.-A. Galland, B. Mazeaud, N. Sellen, *Hybrid passive/active absorbers for flow ducts*, Appl. Acoust. **66**, 691–708, 2005.

6. T.G. Zielinski, M.-A. Galland, M.N. Ichchou, *Active reduction of vibroacoustic transmission using elasto-poroelastic sandwich panels and piezoelectric materials*, [in:] Proceedings of Symposium on the Acoustics of Poro-Elastic Materials SAPEM'05, Lyon, 2005.

7. C. Batifol, T.G. Zielinski, M.-A. Galland, M.N. Ichchou, *Hybrid piezo-poroelastic sound package concept: numerical/experimental validations*, [in:] Conference Proceedings of ACTIVE 2006, 2006.

8. C. Batifol, T.G. Zielinski, M.N. Ichchou, M.-A. Galland, *A finite-element study of a piezoelectric/poroelastic sound package concept*, Smart Mater. Struct. **16**, 168–177, 2007.

9. T.G. Zielinski, M.-A. Galland, M.N. Ichchou, *Further modeling and new results of active noise reduction using elasto-poroelastic panels*, [in:] Conference Proceedings of ISMA2006, 2006.

10. N. Sellen, M. Cuesta, M.-A. Galland, *Noise reduction in a flow duct: Implementation of a hybrid passive/active solution*, J. Sound Vib., **297**, 492–511, 2006.

11. M.-A. Galland, J.-B. Dupont, *A new hybrid passive/active cell to realize a complex impedance boundary condition*, [in:] Conference Proceedings of Acoustics'08, 2008.

12. P. Leroy, A. Berry, N. Atalla, P. Herzog, *Numerical analysis of smart foam for acoustic absorption*, [in:] Conference Proceedings of 19th International Congress on Acoustics ICA2007, 2007.

13. P. Leroy, P. Herzog, A. Berry, N. Atalla, *Experimental assessment of the performance of a smart foam absorber*, [in:] Conference Proceedings of Acoustics'08, 2008.

14. P. Leroy, N. Atalla, A. Berry, P. Herzog, *Three-dimensional finite element modeling of smart foam*, J. Acoust. Soc. Am., **126**, 6, 2873–2885, 2009.

15. J.F. Allard, *Propagation of Sound in Porous Media. Modelling Sound. Absorbing Materials*, Elsevier, 1993.

16. M.A. Biot, *The theory of propagation of elastic waves in a fluid-saturated porous solid*, J. Acoust. Soc. Am., **28**, 2, 168–191, 1956.

17. M.A. Biot, *Mechanics of deformation and acoustic propagation in porous media*, J. Appl. Phys., **33**, 4, 1482–1498, April 1962.

18. R. DE BOER, *Theory of Porous Media: Highlights in Historical Development and Current State*, Springer, 1999.

19. K. WILMANSKI, *A few remarks on Biot's model and linear acoustics of poroelastic saturated materials*, Soil Dyn. Earthq. Eng., **26**, 509–536, 2006.

20. M. SHANZ, *Wave Propagation in Viscoelastic and Poroelastic Continua: A Boundary Element Approach*, Vol. 2, Lecture Notes in Applied and Computational Mechanics, Springer, 2001.

21. N. ATALLA, R. PANNETON, P. DEBERGUE, *A mixed displacement-pressure formulation for poroelastic materials*, J. Acoust. Soc. Am., **104**, 3, 1444–1452, September 1998.

22. P. DEBERGUE, R. PANNETON, N. ATALLA, *Boundary conditions for the weak formulation of the mixed (u, p) poroelasticity problem*, J. Acoust. Soc. Am., **106**, 5, 2383–2390, November 1999.

23. P. GÖRANSSON, *A 3-D, symmetric, finite element formulation of the Biot equations with application to acoustic wave propagation through an elastic porous medium*, Int. J. Numer. Meth. Eng., **41**, 167–192, 1998.

24. N.-E. HÖRLIN, M. NORDSTRÖM, P. GÖRANSSON, *A 3-D hierarchical FE formulation of Biot equations for elasto-acoustic modelling of porous media*, J. Sound Vib., **245**, 4, 633–652, 2001.

25. R. PANNETON, N. ATALLA, *An efficient finite element scheme for solving the three-dimensional poroelasticity problem in acoustics*, J. Acoust. Soc. Am., **101**, 6, 3287–3298, June 1997.

26. R. PANNETON, N. ATALLA, *Numerical prediction of sound transmission through finite multilayer systems with poroelastic materials*, J. Acoust. Soc. Am., **100**, 1, 346–354, July 1996.

27. N. ATALLA, M.A. HAMDI, R. PANNETON, *Enhanced weak integral formulation for the mixed (u, p) poroelastic equations*, J. Acoust. Soc. Am., **109**, 6, 3065–3068, June 2001.

28. S. RIGOBERT, F.C. SGARD, N. ATALLA, *A two-field hybrid formulation for multilayers involving poroelastic, acoustic, and elastic materials*, J. Acoust. Soc. Am., **115**, 6, 2786–2797, June 2004.

29. S. RIGOBERT, N. ATALLA, F.C. SGARD, *Investigation of the convergence of the mixed displacement-pressure formulation for three-dimensional poroelastic materials using hierarchical elements*, J. Acoust. Soc. Am., **114**, 5, 2607–2617, November 2003.

30. J.N. REDDY, *Energy Principles and Variational Methods in Applied Mechanics*, Wiley, 2002.

31. T. IKEDA, *Fundamentals of Piezoelectricity*, Oxford University Press, 1990.

32. G.A. MAUGIN, *Continuum Mechanics of Electromagnetic Solids*, Elsevier, 1988.

33. A. PREUMONT, *Vibration Control of Active Structures*, Springer, 2002.

34. J.N. REDDY, *On laminated composite plates with integrated sensors and actuators*, Eng. Struct., **21**, 568–593, 1999.

35. A. BENJEDDOU, *Advances in piezoelectric finite element modeling of adaptive structural elements: a survey*, Comput. Struct., **76**, 347–363, 2000.

36. D.T. Blackstock, *Fundamentals of Physical Acoustics*, Wiley, 2000.

37. I. Harari, *A survey of finite element methods for time-harmonic acoustics*, Comput. Methods Appl. Mech. Engrg., **195**, 1594–1607, 2006.

38. L.L. Thompson, *A review of finite-element methods for time-harmonic acoustics*, J. Acoust. Soc. Am., **119**, 3, 1315–1330, March 2006.

39. D. Givoli, *High-order local non-reflecting boundary conditions: a review*, Wave Motion, **39**, 319–326, 2004.

40. D. Givoli, B. Neta, *High-order non-reflecting boundary scheme for time-dependent waves*, J. Comput. Phys., **186**, 24–46, 2003.

41. T.G. Zielinski, *Multiphysics modeling and experimental validation of the active reduction of structure-borne noise*, J. Vib. Acoust., **132**, 6, pp. 14, 2010.

Received May 24, 2009; revised version June 9, 2010.