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Physical Modeling of Magnetorheological Damper

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***Abstract* - The paper describes enhanced physical model of MR damper which takes into account the effects of blocking the flow between the chambers in case of low pressure difference and the compressibility of the fluid enclosed in each chamber. Combination of both effects is considered as the reason of generation of the characteristic shapes of force-velocity hysteresis loops. The subsequent sections of the paper contain derivation of the thermodynamic equations governing response of the damper and their implementation for two constitutive models of the magnetorheological fluid. Successful qualitative comparison against the experiment proves the correctness of applied assumptions and the relevance of the proposed model.**

I. INTRODUCTION

Among many applications of the magnetorheological fluids, one of the most promising are semi-active magnetorheological dampers [1]. The crucial issue for the optimal design and control of such devices is accurate mathematical modeling of their mechanical response under external excitation [2,3]. Two basic types of models of MR dampers widely considered in the literature are [4,5]: i) parametric phenomenological models typically based on Bouc-Wen model and ii) physical models typically utilizing Bingham plastic model of MR fluid and equations governing its flow through the orifice. Although a great amount of various parametric models is already developed and new modifications of these models are permanently proposed, it seems that a physical model which accurately clarifies all dissipative properties observed in the experiments is still missing [5]. This paper attempts to fill the gap in the literature by discussion of an enhanced physical model of the MR damper. The essence of the presented model is to combine the effect of blocking of the flow between the chambers in case of low pressure difference and the effect of compressibility of the MR fluid enclosed in each chamber. The conjunction of both effects influences dissipative characteristics of the damper and, in particular, it is the reason of generation of distinctive "z-shaped" force-velocity hysteresis loops.

The paper is aimed at derivation, implementation and analysis of the proposed model of MR damper. The first considered topic is detailed formulation of the mathematical model, which utilizes analytical model of viscous flow and fundamental laws of thermodynamics in order to obtain a convenient form of the equations governing balances of fluid volume and fluid energy. In turn, the second part is aimed at presentation of the governing equations and corresponding numerical results for two constitutive models of magnetorheological fluid involving different treatment of compressibility.

II. THERMODYNAMIC MODELLING OF MR DAMPER

As it is well known, modelling of the thermo-mechanical problems is based on three fundamental principles: balance of mass, balance of momentum and balance of energy, which are expressed as partial differential equations and supplemented by a proper constitutive relations. In classical models of two-chamber, hydraulic, pneumatic or magneto-rheological dampers subjected to a slow excitation, a reasonable assumption is homogeneity of parameters of the fluid in each chamber. In such situation the set of PDEs is solved only for the valve region in order to determine the mass flow rate through valve in terms of parameters of fluid in both chambers. The remaining part of the model is simplified to an initial-value problem involving:

- two ordinary differential equations governing the balance of fluid mass in each chamber,
- differential or algebraic equation governing the equilibrium of the piston,
- two ordinary differential equations governing the balance of fluid energy in each chamber.

In case of a damper subjected to kinematic excitation, the above problem is partially decoupled, i.e. the equations governing balance of mass and energy allow to determine thermodynamic parameters of gas without considering the equation of piston equilibrium. An additional important element of the model are constitutive equations defining the relation between parameters of the fluid (its pressure, temperature, mass, and volume). They are usually expressed either by algebraic equation of state or, alternatively, by definitions of the coefficients of compressibility and thermal expansion.

In the most straightforward approach, the analytical model of the flow and constitutive equations are introduced into mass and energy balances, which allows to define thermodynamic model of the damper subjected to kinematic excitation as set of four differential equations expressed in terms of pressures and temperatures of fluid in each chamber. The final form of governing equations and their coupling strongly depends on assumed model of the fluid flow and applied constitutive relations, especially fluid compressibility and its thermal expansion.

In this paper we will follow the above described classical methodology by using the constitutive model assuming small compressibility of the magneto-rheological fluid. Compressibility and thermal expansion of the fluid will be neglected at the

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stage of calculation of the flow rate of the fluid, however they will be taken into account while derivation of thermodynamic balances governing response of the fluid enclosed in each chamber of the absorber.

A. Model of the fluid flow

The first step of modelling is solution of the problem of the fluid flow through the valve and calculation of the corresponding volumetric flow rate of the fluid. This stage is conducted in a classical way and standard assumptions for the modelling of the magneto-rheological fluid are applied. The fluid is considered as fully incompressible, which reduces general flow equations to incompressibility equation, classical momentum equation and decoupled energy conservation equation. Since, the objective is calculation of fluid velocity **v** only two former equations are required:

$$\begin{gathered}\nabla \cdot \mathbf{v} = 0 \\ \rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}\end{gathered} \tag{1}$$

The magneto-rheological fluid is modelled as non-Newtonian viscous fluid described by Bingham constitutive equation where dependence between fluid stress $\boldsymbol{\sigma}$ and strain rate tensor $\mathbf{D}$ takes the form:

$$\begin{gathered}\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \\ |\boldsymbol{\tau}| \le \boldsymbol{\tau}_0 \;\rightarrow\; \mathbf{D} = 0 \\ |\boldsymbol{\tau}| > \boldsymbol{\tau}_0 \;\rightarrow\; \boldsymbol{\tau} = \pm\boldsymbol{\tau}_0 + 2\mu\mathbf{D}\end{gathered} \tag{2}$$

In above equations $\boldsymbol{\tau}_0$ indicates the limit value of the shear stress, which in general depends on the local value of applied magnetic field $\boldsymbol{\tau}_0 = \boldsymbol{\tau}_0(x, y)$. For the case of two-dimensional stationary flow in a channel with constant magnetic field the above problem can be solved fully analytically and the solution is well known in the literature. The essence of the method is assumption of a constant pressure gradient along the channel and calculation of distribution of shear stresses in fluid. Comparison of these stresses with a limit shear stress allows to distinguish two situations:

- for pressure gradient below a certain limit: the flow of fluid does not occur,
- for pressure gradient above this limit: the flow field can be divided into two external regions of viscous flow and internal region of the "plug flow" (flow with zero strain rate).

Obtained fluid velocities are functions of the pressure gradient and they depend on the location at the width of the channel. The volumetric flow rate of the fluid is calculated by integration of the fluid velocity over the cross section of the channel, while mass flow rate is obtained simply by additional multiplication by fluid density:

$$Q_V^f(\Delta p) = 2\int_{dA_1} v^{visc}(y)\,dA_1 + \int_{dA_2} v^{plug}\,dA_2 \quad \text{and} \quad Q_m^f = \rho Q_V^f \tag{3}$$

The exact formulae will not be quoted since they seem too detailed for the assumed high level of generality. Nevertheless, the important conclusion is that volumetric and mass flow rate of the fluid can be expressed analytically in terms of pressure gradient and that for a low value of pressure difference the flow does not occur. Although temperature of the gas can be computed from decoupled equation of energy balance, it is not required for further calculations.

B. Balance of fluid volume

Thermo-mechanical model of the processes arising in two chambers of MR damper will assume compressibility and thermal expansion of the magnetorheological fluid. The model will be derived in a slightly different manner than in classical approach since we will not directly use the equations governing the balance of mass. Instead, the model of the MR damper subjected to kinematic excitation will contain:

- two differential equations governing the balance of MR fluid volume for each chamber,
- two differential equations governing the balances of MR fluid energy for each chamber.

Although using the balance of volume instead the balance of mass may seem awkward, it will facilitate direct application of the exact definitions of fluid compressibility and thermal expansion coefficients, as well as, direct transformation into classical model assuming incompressibility of the fluid. In a proposed approach the volume of the fluid will be considered as thermodynamic potential of fluid pressure, temperature and mass:

$$V = V(p, T, m) \tag{4}$$

Thus, total differential of fluid volume can be expressed as a sum of partial derivatives with respect to subsequent thermodynamic parameters:

$$dV = \frac{\partial V}{\partial p}dp + \frac{\partial V}{\partial T}dT + \frac{\partial V}{\partial m}dm \tag{5a}$$

By performing differentiation over time and changing differentials into time derivatives we obtain:

$$\dot{V} = \frac{\partial V}{\partial p}\dot{p} + \frac{\partial V}{\partial T}\dot{T} + \frac{\partial V}{\partial m}\dot{m} \tag{5b}$$

The above equation indicates that total change of fluid volume (e.g. calculated on the basis of shaft displacement) equals to sum of changes of volume caused by change of particular thermodynamic parameters. The equation is useful for determination of fluid parameters when derivatives of volume with respect pressure, temperature and mass can be calculated, i.e. when equation of state is known. The equation of state can be expressed either in the in the form of algebraic equation:

$$f(p,T,m,V)=0 \tag{6}$$

or by definitions of coefficients of fluid compressibility, thermal expansion and, introduced by authors, "mass expansion":

$$\beta=-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \quad , \quad \alpha=\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \quad \text{and} \quad \gamma=\frac{1}{V}\left(\frac{\partial V}{\partial m}\right)_{p,T}=\frac{1}{m} \tag{7}$$

Let us note that the last quantity is, in fact, independent on equation of state and it is introduced only for the formal reasons and coherent formulation of the selected equations. By using the above definitions, the equation of volume balance can be rewritten to the form:

$$\dot{V}+\beta V\dot{p}-\alpha V\dot{T}=\gamma V\dot{m} \tag{8a}$$

Noticing that derivative of the volume over mass is the inverse of density and identifying the term at the right hand side as volumetric inflow rate of the fluid we obtain:

$$\dot{V}+\beta V\dot{p}-\alpha V\dot{T}=\frac{\dot{m}}{\rho} \quad \text{and} \quad \dot{V}+\beta V\dot{p}-\alpha V\dot{T}=Q_V \tag{8b,c}$$

The above equation can be directly applied to magnetorheological fluid enclosed in both chambers of the damper when the values or explicit formulae defining coefficients of compressibility and thermal expansion in terms of temperature and pressure are known and when volumetric flow rate of the fluid is determined. In such case the equations governing the balance of volume of MR fluid enclosed in both chambers take the form:

$$\begin{aligned} &\dot{V}_1+\beta_1 V_1\dot{p}_1-\alpha_1 V_1\dot{T}_1=Q_V^1 \\ &\dot{V}_2+\beta_1 V_2\dot{p}_2-\alpha_2 V_2\dot{T}_2=Q_V^2 \end{aligned} \tag{9}$$

The above described method will be further called a *global approach* to modelling of magnetorheological fluid since it requires global characteristics of the MR fluid and global model of the MR fluid flow. Let us note that due to the fact that considered fluid is compressible the volumetric flow rates to both chambers are not equal since they depend on local density of the fluid. Thus, volumetric inflow rate has to be determined for a compressible viscous fluid, which is more difficult than for the classical incompressible model and, moreover, often can not be done analytically. In turn, direct application of the incompressible flow model leads to a contradiction due to violation of the balance of mass.

Further, we will propose an alternative approach for considering compressibility of the magnetorheological fluid - the so called *decomposition approach*. We will assume that fluid is composed of two fluids of a different mechanical properties and constitutive equations:

- a classical viscous fluid with zero compressibility and thermal expansion constituting major part of the considered magnetorheological medium (the primary viscous fluid “f”),
- an inviscid fluid characterized by relatively large compressibility and thermal expansion coefficients being a minor part of the considered magnetorheological medium (the secondary compressible fluid “c”).

At first, the above approach is justified in case of modelling MR dampers filled with fluid containing gas bubbles. Secondly, it can be considered as the alternative method of modelling compressibility, where the constitutive equations are not defined for the entire medium but, instead, separately for its particular components. Finally, the method will allow for the application of the classical model of incompressible viscous flow without obtaining the contradiction in the balance of mass.

The important assumption is that incompressible viscous fluid constitutes a major part of the considered medium. The volume of the secondary compressible fluid is assumed to be a small fraction k of the initial volume of the viscous fluid at initial conditions, which results in the corresponding relations between their volumes and masses:

$$V_c^0=kV_f^0 \quad \text{and} \quad m_c^0=k\frac{\rho_c^0}{\rho_f}m_f^0 \tag{10a,b}$$

Moreover, it is assumed that the main equation governing the flow of the medium is equation of incompressible viscous flow and mass flow rate of compressible fluid is proportional to mass flow rate of the viscous fluid. As a result the initial ratio between masses of the fluids (10b) is conserved i.e. the mass of the compressible fluid enclosed in each chamber is proportional to mass of viscous fluid during the entire process. In turn, the ratio of volumes is affected by change of density of compressible fluid caused by change of its pressure and temperature:

$$m_c=k\frac{\rho_c^0}{\rho_f}m_f \quad \text{and} \quad V_c=k\frac{\rho_c^0}{\rho_c}V_f \tag{10c,d}$$

In proposed approach the total volume the considered magnetorheological fluid V is a sum of volume of the primary viscous fluid V_f and volume of the secondary compressible fluid V_c:

$$V = V_f + V_c \quad \text{where:} \quad V_f = V_f(m_f) \quad \text{and} \quad V_c = V_c(p, T, m_c) \tag{11}$$

Substituting the above relations to general equation of volume balance (5) yields:

$$\dot{V} = \frac{\partial V_f}{\partial m_f}\dot{m}_f + \frac{\partial V_c}{\partial p}\dot{p} + \frac{\partial V_c}{\partial T}\dot{T} + \frac{\partial V_c}{\partial m_c}\dot{m}_c \tag{12}$$

By grouping terms related to fluid transfer (involving mass derivative) and by using definitions of the coefficients of compressibility, thermal expansion and mass expansion we obtain the governing equation in the form:

$$\dot{V} + \beta_c V_c \dot{p} - \alpha_c V_c \dot{T} = \gamma_f V_f \dot{m}_f + \gamma_c V_c \dot{m}_c \tag{13a}$$

Similarly as in previous case by rewriting the terms indicating fluid inflow we get:

$$\dot{V} + \beta_c V_c \dot{p} - \alpha_c V_c \dot{T} = \frac{\dot{m}_f}{\rho_f} + \frac{\dot{m}_c}{\rho_c} \quad \text{and} \quad \dot{V} + \beta_c V_c \dot{p} - \alpha_c V_c \dot{T} = Q_V^f + Q_V^c \tag{13b,c}$$

Right hand side of the above equations clearly indicates that total volumetric inflow of the magneto-rheological fluid is a sum of the inflow of incompressible viscous fluid and the inflow of compressible fluid. Alternatively, the volumetric flow rate of the compressible fluid can be expressed in terms of volumetric flow rate of the viscous fluid. Using the chain rule of differentiation for the time derivative of mass in the last term of (12) allows to express it in terms of $\dot{V}_f$ and to obtain:

$$\dot{V} + \beta_c V_c \dot{p} - \alpha_c V_c \dot{T} = Q_V^f\left[1 + \frac{\partial V_c}{\partial V_f}\right] \tag{14a}$$

Finally, by using definition of the mass of the compressible fluid (10c) we get two versions of the final form of equations governing the balance of fluid volume:

$$\dot{V} + \beta_c V_c \dot{p} - \alpha_c V_c \dot{T} = Q_V^f\left(1 + k\frac{\gamma_c V_c}{\gamma_c^0 V_c^0}\right) \quad \text{and} \quad \dot{V} + \beta_c V_c \dot{p} - \alpha_c V_c \dot{T} = Q_V^f\left[1 + k\frac{\rho_c^0}{\rho_c}\right] \tag{14b,c}$$

Now the right hand sides are expressed in terms of volumetric flow rate of viscous fluid and parameters of the compressible fluid. The first equation is formally more strict since it is expressed exclusively in terms quantities $\beta_c, \alpha_c, \gamma_c$ resulting from the constitutive equations. In turn, the second equation more clearly reveals dependence of the volumetric flow rate on density of the compressible fluid. The set of equations governing the balance of volume for two fluid chambers reads:

$$\begin{aligned} \dot{V}_1 + \beta_1^c V_1^c \dot{p}_1 - \alpha_1^c V_1^c \dot{T}_1 &= Q_V^f\left[1 + k\frac{\rho_c^0}{\rho_c^1}\right] \\ \dot{V}_2 + \beta_2^c V_2^c \dot{p}_2 - \alpha_2^c V_2^c \dot{T}_2 &= -Q_V^f\left[1 + k\frac{\rho_c^0}{\rho_c^2}\right] \end{aligned} \tag{15}$$

Let us note that for the primary viscous fluid the volumetric and mass flow rates are equal for both chambers and they differ only by a sign. In turn, for the secondary compressible fluid the volumetric mass flow rates are different, however mass flow rates (obtained by multiplication by fluid density) are equal. Thus, total balance of fluid mass is satisfied and application of incompressible flow model does not lead to a contradiction.

Concluding the section concerning the balance of fluid volume, it can be stated that equations (9) are convenient for application when global constitutive relations of the magnetorheological medium are known and when the volumetric flow rates are determined by using the model of *compressible viscous flow*. In turn, equations (15) are suitable when various models of compressibility and thermal expansion are considered and when the volumetric flow rates are determined on the basis of *incompressible viscous flow* model. The equations 9 and 15 are sufficient for calculation of fluid pressures and determination of the damper response only in a special case when coefficients of thermal expansion are equal to zero.

The remaining issue is determining the relation between thermodynamic coefficients for each component fluid and global coefficients for the considered compressible viscous medium. For the compressibility coefficient such relation takes the form:

$$\beta = -\frac{1}{V_f + V_c}\left(\frac{\partial V_c}{\partial p}\right)_T = \frac{V_c}{V_f + V_c}\beta_c = k\frac{\rho_c^0}{\rho_c}\left[1 + k\frac{\rho_c^0}{\rho_c}\right]^{-1}\beta_c \tag{16a}$$

Similarly, for thermal expansion coefficient and mass expansion coefficient we obtain:

$$\alpha = \frac{1}{V_f + V_c}\left(\frac{\partial V_c}{\partial T}\right)_p = \frac{V_c}{V_f + V_c}\alpha_c = k\frac{\rho_c^0}{\rho_c}\left[1 + k\frac{\rho_c^0}{\rho_c}\right]^{-1}\alpha_c \tag{16b}$$

Let us note that global coefficients for the entire medium depend on actual density, which is affected by fluid pressure and temperature. Even if coefficients defining the secondary compressible medium are assumed to be constant, the global coefficients depend on fluid pressure and temperature. Finally, for the coefficient of mass expansion we have:

$$\gamma = \frac{1}{V_f + V_c}\left(\frac{\partial V_f}{\partial m_f} + \frac{\partial V_c}{\partial m_c}\right)_{p,T} = \frac{V_f \gamma_f + V_c \gamma_c}{V_f + V_c} \qquad 16c$$

The above formulae eventually prove the equivalence of equations (9) and (15) and thus they prove the correctness of the proposed approach.

C. Balance of fluid energy

The second group of the governing equations concerns the balance of energy of the magnetorheological fluid enclosed in each chamber of the damper. General thermodynamic equation of energy balance combines energy transferred to the fluid in the form of heat δQ and submitted enthalpy dH with internal energy of the fluid dU and work done by fluid δW :

$$\delta Q + dH - dU - \delta W = 0 \qquad 17a$$

Hereafter, we will assume that the process is adiabatic and we will omit the term indicating energy transferred in the form of heat. Thus, the above equation of energy balance will be simplified to the form:

$$dH = dU + \delta W \qquad 17b$$

Further we will apply classical definitions of the differential of internal energy, the differential of enthalpy and the incremental work. The definition of change of internal energy can be derived by using fundamental thermodynamic relation:

$$dU = TdS - pdV \qquad 18a$$

By expressing dS in terms of change of temperature and volume, by using Maxwell relation to replace entropy derivative by pressure derivative and, finally, by applying the definition of heat capacity at constant volume C_V we obtain:

$$dU = C_V dT + \left[T\left(\frac{\partial p}{\partial T}\right)_V - p\right]dV \qquad 18b$$

Finally, the above formula can be expressed in terms of change of temperature and change of pressure by using general definitions of the coefficient of thermal expansion and coefficient of compressibility:

$$dU = \left[C_p - \alpha pV\right]dT + \left[\beta p - \alpha T\right]Vdp \qquad 18c$$

In turn, definition of the increase of enthalpy can be determined by using the classical relation:

$$dH = dU + d(pV) \qquad 19a$$

By using previously derived definition of the increase of internal energy (18b), the increase of enthalpy can be defined in terms of change of temperature and change of pressure:

$$dH = C_p dT + \left[V - T\left(\frac{\partial V}{\partial T}\right)_p\right]dp \qquad 19b$$

where C_p is the coefficient of heat capacity at constant pressure. Further, the above definition can be directly expressed in terms of coefficient of thermal expansion:

$$dH = C_p dT + \left[1 - \alpha T\right]Vdp \qquad 19c$$

Finally, the last component of the equation of energy balance is incremental work done by fluid, which can be directly expressed by fluid pressure and change of volume:

$$\delta W = pdV \qquad 20$$

Introducing definitions of internal energy, enthalpy and work (18c, 19c, 20) into equation of energy balance (17b), performing differentiation over time and identifying terms indicating volumetric flow rate through the valve leads to the most general form of the equation governing the energy balance:

$$\dot{m}c_p T_{(v)} + Q_V p_{(v)} - Q_V \alpha T_{(v)} p_{(v)} = \dot{m}\left(c_p T - \alpha p\frac{V}{m}T + \beta p\frac{V}{m}p - \alpha T\frac{V}{m}p\right) + mc_p\dot{T} - \alpha pV\dot{T} + \beta pV\dot{p} - \alpha TV\dot{p} + p\dot{V} \qquad 21a$$

In above equation index v indicates parameters of the fluid flowing through the valve which are not necessarily equal to parameters of the fluid in the considered chamber. Obviously, a complete model of the MR damper has to contain equations governing energy balance for the fluid enclosed in both chambers. The equation governing the chamber with fluid inflow has a general form analogous to eq. 21a. In turn, the equation governing the chamber with fluid outflow has a simplified form due to equality of parameters of fluid flowing through the valve and parameters of fluid in the chamber:

$$Q_V\, p = \dot{m}\left(-\alpha\, p \frac{V}{m} T + \beta\, p \frac{V}{m}\, p \right) + m c_p \dot{T} - \alpha\, pV\dot{T} + \beta\, pV\, \dot{p} - \alpha TV\, \dot{p} + p\dot{V} \qquad \text{21b}$$

Let us note that partial balance of energy for the chamber with fluid outflow can obtained by multiplication of the corresponding equation governing the balance of volume by the value of pressure:

$$Q_V\, p = p\dot{V} + p\beta V\dot{p} - p\alpha V\dot{T} \qquad \text{21c}$$

By subtracting the Eq. 21c from Eq. 21b we obtain the simplest possible form of the equation of the energy balance for the chamber with fluid outflow:

$$\dot{m}\left(-\alpha\, p \frac{V}{m} T + \beta\, p \frac{V}{m}\, p \right) + m c_p \dot{T} - \alpha TV\, \dot{p} = 0 \qquad \text{21d}$$

Although (21d) is much simpler than (21a) a typical periodic kinematic excitation causes that distinction between inflow and outflow chamber is only temporary and both equations have to be applied commutatively. Thus, more convenient option may be using a general form of energy balance for both chambers. Furthermore, summation of the equations governing the energy balance for the fluid in each chamber leads to global equation governing the energy balance for the entire fluid enclosed in the damper. Although considered flow is viscous and during the valve flow kinetic energy is transferred into heat, the total enthalpy of the transferred fluid remains constant. Consequently, adding of energy balances provides elimination of the enthalpy terms and resulting equation indicates the equivalence of increase of internal energy and work done on fluid. In a proposed model above described global equation of energy balance can replace energy balance for arbitrary selected chamber.

Further we will consider the equation of energy balance in previously introduced *decomposition approach* where considered magnetorheological medium is assumed to be composed of primary incompressible viscous fluid and secondary compressible fluid. Again, the mass of the secondary compressible fluid is assumed to be proportional to the mass of the primary viscous fluid and it is defined by equations 10c-d. Derivation of the equations of energy balance for such compound fluid is straightforward i.e. all terms of eq. 17b are defined separately for both fluids and definitions of enthalpy and internal energy for incompressible viscous fluid are significantly simplified. General equation of energy balance takes the form:

$$\begin{aligned}&\left(\dot{m}_f\, c_p^f T_{(v)} + Q_V^f\, p_v\right) + \left(\dot{m}_c c_p^c T_{(v)} + Q_V^c\, p_{(v)} - Q_V^c \alpha_c T_{(v)}\, p_{(v)}\right) = \left(\dot{m}_f c_p^f T \; + m_f\, c_p^f \dot{T}\right) + \\ &\dot{m}_c\left(c_p^c T - \alpha_c\, p \frac{V_c}{m_c} T + \beta_c\, p \frac{V_c}{m_c}\, p - \alpha_c T \frac{V_c}{m_c}\, p \right) + \left(m_c\, c_p^c \dot{T} - \alpha_c\, pV_c \dot{T} + \beta_c\, pV_c\, \dot{p} - \alpha_c TV_c\, \dot{p}\right) + p\dot{V}_f + p\dot{V}_c\end{aligned} \qquad \text{22a}$$

In above equation the index v indicates the parameters of fluid flowing through the valve, while Q_V^f and Q_V^c indicate volumetric flow rates of viscous and compressible fluid, respectively. Simplification of the above equation for the case of chamber with fluid outflow is performed in similar manner as in case of homogeneous medium and it yields:

$$Q_V^f\, p_{(v)} + Q_V^c\, p_{(v)} = m_f\, c_p^f \dot{T} + \dot{m}_c\left(-\alpha_c\, p \frac{V_c}{m_c} T + \beta_c\, p \frac{V_c}{m_c}\, p \right) + \left(m_c\, c_p^c \dot{T} - \alpha_c\, pV_c \dot{T} + \beta_c\, pV_c\, \dot{p} - \alpha_c TV_c\, \dot{p}\right) + p\dot{V}_f + p\dot{V}_c \qquad \text{22b}$$

Moreover, equation of partial energy balance obtained by multiplication of the equation of volume balance by pressure reads:

$$Q_V^f\, p_{(v)} + Q_V^c\, p_{(v)} = \; p\dot{V}_f + p\dot{V}_c + p\beta_c V_c \dot{p} - p\alpha_c V_c \dot{T} \qquad \text{22c}$$

Reduction of the above equation from the equation describing energy balance for the outflow chamber causes elimination of selected terms related to viscous and compressible fluid:

$$m_c\, c_p^c \dot{T} + m_f\, c_p^f \dot{T} + \dot{m}_c\left(-\alpha_c\, p \frac{V_c}{m_c} T + \beta_c\, p \frac{V_c}{m_c}\, p \right) - \alpha_c TV_c\, \dot{p} = 0 \qquad \text{22d}$$

In all above equations the mass of the fluid can be also expressed in terms of coefficient gamma indicating mass expansion. Similarly as in case of homogenous medium, a complete model of the system had to contain two equations governing the balance of fluid energy: i) two equations in a general form (22a) or ii) one equation in general form (22a) and one equation in a simplified form (22d) or iii) combination of equation in a general form and sum of equations for two chambers.

Let us finally note that comparison of heat capacity coefficients arising in global approach and decomposition approach gives:

$$c_p = \frac{m_f c_p^f + m_c c_p^c}{m_f + m_c} = \left[1 + k \frac{\rho_c^0}{\rho_f}\right]^{-1} \left(c_p^f + k \frac{\rho_c^0}{\rho_f} c_p^c \right) \qquad \text{23}$$

The above formula indicates that scaling of heat capacity coefficient is performed with the use of mass of considered fluids, which is in agreement with its general definition. Let us remind here that coefficients of compressibility and thermal expansion were scaled by volume of the fluids. Comparing these results we notice that the quantities multiplied in balance equations by volume are scaled with the use of volume and, in turn, the quantities multiplied by mass are scaled by mass.

Concluding the proposed in this section approach for thermodynamic modeling of the magnetorheological dampers it can be stated that the model is always composed of analytical equation defining volumetric flow rate, two differential equations of volume balance and two differential equations of energy balance. In a *global approach*, used when global constitutive equations for the compressible magnetorheological fluid are known, the model contains balances of volume (Eq. 9), balances of energy (Eq. 21) and, moreover, the definitions of volumetric flow rate of compressible viscous medium has to be applied. Arising in the governing equations values of thermal expansion coefficient, compressibility coefficient and mass of the fluid can be determined from global constitutive equation. Eventually, the system of governing equations can be solved in order to find unknown pressures and temperatures of the fluid in both chambers of the damper.

In turn, in the proposed *decomposition approach* the constitutive equations are defined separately for the primary viscous fluid and secondary compressible fluid. The model of the system contains balances of mass (Eq. 15) and balances of energy (Eq. 22) and, moreover, standard equation defining volumetric flow rate of incompressible viscous fluid can be applied. Arising in the governing equations values of thermal expansion coefficient and compressibility coefficient can be determined from the constitutive equation of the compressible fluid, while its volume and mass can be determined as proportional to mass and volume of the viscous fluid (cf. Eq.10). Eventually, the system of equations can be solved in order to find unknown pressures and temperatures of the fluid similarly as in previous case.

III. EXAMPLE OF MODEL IMPLEMENTATION FOR TWO CONSTITUTIVE MODELS OF THE FLUID

In the current section the proposed methodology of thermodynamic modelling of magnetorheological dampers will be applied for two classical constitutive models of the secondary compressible medium, i.e.:

- the model of ideal gas with general pressure and temperature dependence,
- the model of volumetric linear elasticity with linear thermal expansion.

In each case the corresponding form of the governing equations will be derived, the values of vicarious global coefficients for the entire medium will be determined and, moreover, obtained numerical results will be presented.

A. Compressible fluid modelled as ideal gas

One of the most classical models in thermodynamics is the model of ideal gas described by well-known equation of state where pressure, volume and mass of the gas are linked by the relation:

$$pV = mRT \quad \text{or} \quad p = \rho RT \tag{24a}$$

with R being a gas constant. The above equation of state can be used in order to determine the values of compressibility coefficient, thermal expansion coefficient and mass expansion coefficient according to definitions (7):

$$\beta = \frac{1}{p}, \quad \alpha = \frac{1}{T} \quad \text{and} \quad \gamma = \frac{1}{m} \tag{24b}$$

In general, the constitutive equation (24a) and the formulae defining the values of the corresponding coefficients (24b) can be used alternatively. The knowledge of the constitutive equation allows to define mass and volume of the secondary compressible fluid in terms of volume of the primary viscous fluid according to eq. 10c and eq. 10d:

$$m_c = k\frac{p_0}{RT_0}V_f \quad \text{and} \quad V_c = k\frac{p_0}{p}\frac{T}{T_0}V_f \tag{25}$$

The above relations allow to rewrite general equations governing the balances of fluid volume in decomposition approach (Eq. 15) to the following form:

$$\begin{aligned}
\dot{V}_1 + k\left(\frac{p_0}{T_0}\frac{T_1}{p_1^{\,2}}\right)V_1^f\,\dot{p}_1 - k\left(\frac{p_0}{T_0}\frac{1}{p_1}\right)V_1^f\,\dot{T}_1 = Q_V^f\left[1 + k\frac{p_0}{T_0}\frac{T_1}{p_1}\right]\\
\dot{V}_2 + k\left(\frac{p_0}{T_0}\frac{T_2}{p_2^{\,2}}\right)V_2^f\,\dot{p}_2 - k\left(\frac{p_0}{T_0}\frac{1}{p_2}\right)V_2^f\,\dot{T}_2 = Q_V^f\left[1 + k\frac{p_0}{T_0}\frac{T_2}{p_2}\right]
\end{aligned} \tag{26}$$

As it was previously described the volumetric flow rate Q_V^f can be determined from the classical model of incompressible viscous flow described in Sec. II.A. Let us also note that the quantities which are not directly known are actual volumes of the viscous fluids V_i^f as well as actual total volumes of the magnetorheological medium $V_i = V_i^f + V_i^c$. In turn, the quantities which are directly defined during kinematic excitation, as the functions of displacement of the piston, are actual volumes of the fluid chambers V_i^{ch}. Thus, both unknown volumes can be determined from the set of equations:

$$\begin{aligned}
V_1^f + V_1^c(p_1, T_1, V_1^f) + V_1^g(m_1^0, p_1, T_1) = V_1^{ch}(u)\\
V_2^f + V_2^c(p_2, T_2, V_2^f) + V_2^g(m_2^0, p_2, T_2) = V_2^{ch}(u)
\end{aligned} \tag{27}$$

where V_i^g indicate volumes of additional gas cushions typically present in each chamber of the damper.

Finally, the equations governing the balance of volume (26) can be expressed exclusively in terms of applied kinematic excitation and unknown pressures and temperatures in the chambers. In a special case when the process is isothermal and, consequently, the thermal expansion coefficient equals to zero, the equations governing the balance of volume are expressed exclusively in terms of pressure of the fluid:

$$\begin{aligned} \dot{V}_1 + k\left(\frac{p_0}{{p_1}^2}\right)V_1^f\,\dot{p}_1 = Q_V^f\left[1 + k\frac{p_0}{p_1}\right] \\ \dot{V}_2 + k\left(\frac{p_0}{{p_2}^2}\right)V_2^f\,\dot{p}_2 = Q_V^f\left[1 + k\frac{p_0}{p_2}\right] \end{aligned} \qquad 28$$

and they are sufficient to determine mechanical response of the absorber. The final aspect of the considerations is calculation of global thermo-mechanical constants of the magnetorheological fluid. For the compressibility coefficient and thermal expansion coefficient we obtain:

$$\beta = \frac{V_c}{V_f + V_c}\beta_c = k\frac{p_0}{T_0}\frac{T}{p^2}\left[1 + k\frac{p_0}{T_0}\frac{T}{p}\right]^{-1}\beta_c = k\frac{p_0}{T_0}\frac{{\beta_c}^2}{\alpha_c}\left[1 + k\frac{p_0}{T_0}\frac{\beta_c}{\alpha_c}\right]^{-1} \qquad 29a$$

and

$$\alpha = \frac{V_c}{V_f + V_c}\alpha_c = k\frac{p_0}{T_0}\frac{1}{p}\left[1 + k\frac{p_0}{T_0}\frac{T}{p}\right]^{-1}\alpha_c = k\frac{p_0}{T_0}\beta_c\left[1 + k\frac{p_0}{T_0}\frac{\beta_c}{\alpha_c}\right]^{-1} \qquad 29b$$

Let us note that in proposed approach the global thermodynamic coefficients of the magnetorheological medium depend on its initial conditions, initial volumetric fraction of compressible fluid k and both local coefficients α_c and β_c. In addition, in case of simplified versions of the above formulae, when volume of the compressible fluid in the denominator is omitted, the global thermal expansion coefficient depends exclusively on the compressibility of the secondary fluid. Finally, for global coefficient of mass expansion we get the formula:

$$\gamma = \left[\gamma_f + k\frac{p_0}{p}\frac{T}{T_0}\gamma_c\right]\left[1 + k\frac{p_0}{p}\frac{T}{T_0}\right]^{-1} = \left[\gamma_f + k\frac{p_0}{T_0}\frac{\beta_c}{\alpha_c}\gamma_c\right]\left[1 + k\frac{p_0}{T_0}\frac{\beta_c}{\alpha_c}\right]^{-1} \qquad 29c$$

which indicates that it depends on all three thermodynamic coefficients of the secondary compressible fluid. The above equalities can be used to find global constitutive equation of the compressible magnetorheological fluid, which in considered case becomes fairly complicated.

The derivation of the equation governing fluid energy balance is performed by introducing the values of thermo-mechanical coefficients into general equation (22a):

$$\left(\dot{m}_f\,c_p^f T_{(v)} + Q_V^f\,p_v\right) + \dot{m}_c c_p^c T_{(v)} = \left(\dot{m}_f c_p^f T + m_f\,c_p^f \dot{T}\right) + \left(\dot{m}_c c_p^c T - \dot{m}_c\,p\frac{V_c}{m_c}\right) + \left(m_c\,c_p^c \dot{T} - \frac{pV_c}{T}\dot{T}\right) + p\dot{V}_f + p\dot{V}_c \qquad 30a$$

By using the ideal gas law and the relation between constant volume and constant pressure heat capacity we recognize classical terms indicating enthalpy and internal energy of viscous fluid and ideal gas:

$$\left(\dot{m}_f\,c_p^f T_{(v)} + Q_V^f\,p_v\right) + \dot{m}_c c_p^c T_{(v)} = \left(\dot{m}_f c_p^f T + m_f\,c_p^f \dot{T}\right) + \left(\dot{m}_c c_V^c T + m_c\,c_V^c \dot{T}\right) + p\dot{V}_f + p\dot{V}_c \qquad 30b$$

Applying the standard simplifications obligatory for the outflow chamber (cf. general equation 22b or equation 30b) we obtain:

$$Q_V^f\,p + Q_V^c\,p = m_f\,c_p^f \dot{T} + m_c\,c_V^c \dot{T} + p\dot{V}_f + p\dot{V}_c \qquad 30c$$

In turn, equations of partial energy balance obtained by multiplying the equations of volume balances by pressures take the form (cf. eq. 22c):

$$Q_V^f\,p + Q_V^c\,p = p\dot{V}_f + p\dot{V}_c + V_c\,\dot{p} - mR\dot{T} \qquad 30d$$

and their subtracting from the energy equation for the outflow chamber gives:

$$m_f\,c_p^f \dot{T} + m_c\,c_p^c \dot{T} - V_c\,\dot{p} = 0 \qquad 30e$$

Finally, the general set of equations of energy balance for two chambers of the damper takes the form:

$$\begin{aligned} \left(\dot{m}_1^f\,c_p^f T_{(v)} + Q_V^f\,p_v\right) + \dot{m}_1^c c_p^c T_{(v)} = \left(\dot{m}_1^f\,c_p^f T_1 + m_1^f\,c_p^f \dot{T}_1\right) + \left(\dot{m}_1^c c_V^c T_1 + m_1^c\,c_V^c \dot{T}_1\right) + p_1\dot{V}_1^f + p_1\dot{V}_1^c \\ \left(\dot{m}_2^f\,c_p^f T_{(v)} + Q_V^f\,p_v\right) + \dot{m}_2^c c_p^c T_{(v)} = \left(\dot{m}_2^f\,c_p^f T_2 + m_2^f\,c_p^f \dot{T}_2\right) + \left(\dot{m}_2^c c_V^c T_2 + m_2^c\,c_V^c \dot{T}_2\right) + p_2\dot{V}_2^f + p_2\dot{V}_2^c \end{aligned} \qquad 31$$

The alternative for using two equations in a general form is using the simplified equation for the outflow chamber or using the sum of two above equations indicating global balance of energy of fluid enclosed in a considered system. The additional comment concerns the simplified equation of the energy balance (30e). Since the volume of the compressible fluid is assumed as relatively small, the value of the last term is also small and thus temperature in the outflow chamber is approximately constant during the entire process. In turn, the temperature of the fluid in inflow chamber increases as the result of work done on system by the shaft movement.

Basic results achieved by using the above introduced model, presented in Figure 1, reveal characteristic shapes of force-velocity hysteresis loops, which are not obtained from the classical non-parametric models of MR dampers.

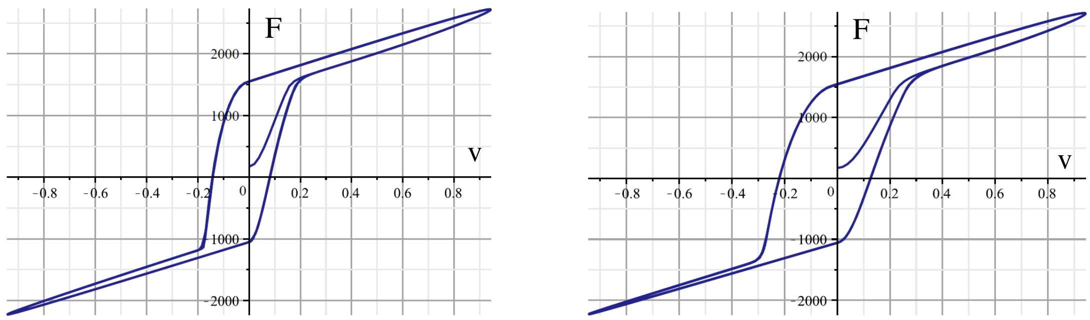

Figure 1. Force-velocity hysteresis loops obtained from the models with compressible fluid described by ideal gas law: a) k=0.2, b) k=0.5.

B. Compressible fluid modelled by linear elasticity and linear thermal expansion

The second considered classical model of the secondary compressible medium is the model of linear volumetric elasticity with linear thermal expansion. In such model the increase of pressure with respect to a certain reference state depends linearly on the increase of volume and increase of temperature. The corresponding constitutive equation reads:

$$p - p_0 = \bar{k}(V_0 - V) + \bar{\bar{k}}(T - T_0) \tag{32a}$$

and it can be transformed to the form:

$$V = V_0 + C_1(p_0 - p) + C_2(T - T_0) \quad \text{where:} \quad C_1 = \bar{k}^{-1} \quad \text{and} \quad C_2 = \bar{k}^{-1}\bar{\bar{k}} \tag{32b}$$

Further, the above constitutive equation can be applied to determine the values of compressibility coefficient and thermal expansion coefficient according to formulae (7):

$$\beta_c = \frac{C_1}{V_0 + C_1(p_0 - p) + C_2(T - T_0)}, \quad \alpha_c = \frac{C_2}{V_0 + C_1(p_0 - p) + C_2(T - T_0)} \tag{32c}$$

Let us note that in considered case the explicit dependence between volume and mass of the fluid is not defined. However, by the analogy to previously analyzed model of ideal gas, the volume of the secondary compressible fluid will be assumed as proportional to actual volume of the primary viscous fluid:

$$V_c = \frac{kV_f}{V_0}\left[V_0 + C_1(p_0 - p) + C_2(T - T_0)\right] \tag{33}$$

For the sake of simplicity the thermal expansion of the fluid will be further neglected and the attention will be focused on its compressibility. Definition of the compressibility coefficient (32c) as well as definition of the volume of compressible fluid (33) allow to derive an explicit form of the equations governing the balance of fluid volume (cf. Eq.14):

$$\begin{aligned} \dot{V}_1 + \frac{kC_1}{V_1^0} V_1^f \dot{p}_1 &= Q_V^f \left[1 + k + \frac{kC_1}{V_1^0}(p_0 - p_1)\right] \\ \dot{V}_2 + \frac{kC_1}{V_2^0} V_2^f \dot{p}_2 &= -Q_V^f \left[1 + k + \frac{kC_1}{V_2^0}(p_0 - p_2)\right] \end{aligned} \tag{34}$$

Arising in the above equations volumes of the viscous fluid can be determined with the use of equations defining total volume of the fluid chamber being the sum of volume of primary viscous fluid, volume of the secondary compressible fluid and possible volume of gas cushions (vide Eq. 27).

The global value of the compressibility coefficient can be determined from equations (16) and it is equal:

$$\beta = \frac{V_c}{V_f + V_c}\beta_c = \frac{kC_1}{V_0 + k\left[V_0 + C_1(p_0 - p)\right]}, \tag{35}$$

Let us note that by neglecting the volume of the compressible fluid in the denominator (as being relatively small in comparison to volume of the viscous fluid) we obtain a constant value of the compressibility coefficient: $\beta = kC_1 / V_0$. In turn, using the definition of the fluid compressibility allows to define equation with unknown volume of the compressible fluid:

$$-\frac{1}{V_f + V_c(p)}\left(\frac{\partial V_c(p)}{\partial p}\right)_T = \beta\,,\;\; V_c(0) = V_c^0 \tag{36}$$

which has an analytical solution:

$$V_c(p) = V_f\left[e^{-\beta(p-p_0)} - 1\right] + V_c^0 e^{-\beta(p-p_0)} \tag{37}$$

By introducing simplified value of the compressibility coefficient and determined formula for the volume of the compressible fluid into equations 14 we obtain:

$$\begin{aligned} \dot{V}_1 + \beta V_1\ \dot{p}_1 &= Q_V^f e^{-\beta(p_1 - p_0)} \\ \dot{V}_2 + \beta V_2 \dot{p}_2 &= -Q_V^f e^{-\beta(p_2 - p_0)} \end{aligned} \tag{38}$$

The above system of equations (with multipliers on the r.h.s. equal to 1) is often used as a simplified balance of volume for compressible fluid. Nevertheless, presented derivation highlights two intrinsic aspects of the model. At first, derived equations can be treated as simplified version of the balance of volume assuming linear elastic model of compressible fluid. At second, the multipliers at the right hand side of Eq. 38 provide that volumetric flow rate Q_V^f can be determined from the model of incompressible viscous flow and it does not cause violation of the global balance of mass of the fluid.

Results obtained from the above simple model and presented in Figure 2 confirm the occurrence of the characteristic shapes of force-velocity hysteresis loops resembling the ones obtained from the model with ideal gas.

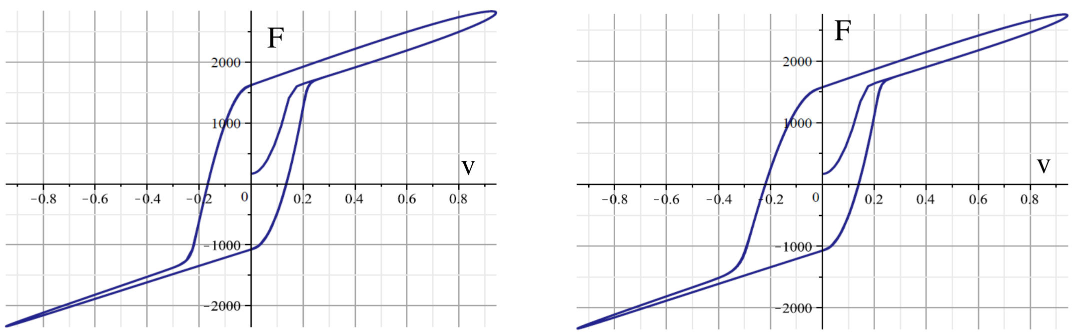

Figure 2. Force-velocity hysteresis loops obtained from the models involving compressible fluid described by linear elasticity.

IV. Final remarks

Presented enhanced physical model of MR damper considerably differs from the classical models since it takes into account the combination of the effects of blocking the flow between the chambers in case of low pressure difference and the compressibility of the fluid enclosed in each chamber. We had proved that taking into account both these phenomena is required to model dissipative characteristics of the damper. In further, not presented in the paper, part of the research the proposed model was preliminarily validated against the experiment where MR damper was subjected to kinematic excitation. Satisfactory agreement of numerical and experimental results, in particular the occurrence of characteristic "z-shaped" force-velocity hysteresis loops, proved the correctness of the applied assumptions and the relevance of the proposed model.

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