On computation of sensitivities of multi-layer shells using elements with additional parameters

Krzysztof Wiśniewski[1] **and Przemysław Panasz**[1 *]

[1] *Department of Computational Science, Institute of Fundamental Technological Research, Polish Academy of Sciences*
ul. Pawinskiego 5B, 02-106 Warsaw, Poland
e-mail: kwisn@ippt.gov.pl, ppanasz@ippt.gov.pl

Abstract

We consider finite rotation shells based on the generalized Reissner kinematics, with an additional scalar parameter describing a position of the reference surface. Such kinematics is used to model composites with non-symmetric layer stacking sequence (LSS). Besides, we assume that the element formulation includes additional local parameters such as these appearing in the enhanced strain elements and in the mixed enhanced/non-enhanced elements based on the Hellinger-Reissner or Hu-Washizu functional.

The Design Sensitivity Analysis (DSA) provides derivatives for the design optimization but can serve also as a stand-alone tool which allows to verify robustness of a design to inaccuracies caused by manufacturing and assembling as being particularly dangerous for shells. In this paper, we describe the method to compute the design derivatives of displacements and rotations for the aforementioned kinematics and element's formulation. Besides, formulae for the design derivatives of shell strains and shell stress and couple resultants for composites are presented.

Keywords: 4-node shell element with 6 dofs/node, Assumed Strain method, Design Sensitivity Analysis, multi-layer composites

1. Characteristics of shell

The First Order Shell Deformation (FOSD) theory is based on the Reissner kinematics with an additional parameter z_0, which describes a position of the reference surface. The position vector of an arbitrary point of a shell is defined as

$$\mathbf{x}(\zeta) = \mathbf{x}_0 + (z - z_0)\,\mathbf{Q}_0\mathbf{t}_3, \qquad z, z_0 \in [0, h], \tag{1}$$

where $\mathbf{x}_0$ is the position of the reference surface, $\mathbf{t}_3$ is the shell director and h is the initial shell thickness. Besides, $\mathbf{Q}_0 \in SO(3)$ is a rotation tensor, parameterized by the canonical rotation vector $\boldsymbol{\psi}$ as follows

$$\mathbf{Q}_0(\boldsymbol{\psi}) \doteq \mathbf{I} + \frac{\sin\omega}{\omega}\,\tilde{\boldsymbol{\psi}} + \frac{1-\cos\omega}{\omega^2}\,\tilde{\boldsymbol{\psi}}^2, \tag{2}$$

where $\omega = \|\boldsymbol{\psi}\| = \sqrt{\boldsymbol{\psi}\cdot\boldsymbol{\psi}} \geq 0$ and $\tilde{\boldsymbol{\psi}} \doteq \boldsymbol{\psi} \times \mathbf{I}$. The Green strain tensor is approximated linearly over the thickness, i.e. $\mathbf{E}(\zeta) \approx \boldsymbol{\varepsilon} + (z - z_0)\boldsymbol{\kappa}$.

The effective (substitute) constitutive matrices for a multi-layer composite shell are obtained by an integration over the shell thickness accounting for the shift of the reference surface from the middle position, see eq. (1). The effective transverse shear stiffness is obtained as an inverse of the effective transverse shear flexibility matrix obtained for cylindrical bending in two perpendicular directions. This procedure was developed gradually in [Whitney, Pagano, 1970], [Chow, 1971], [Whitney, 1973], [Rohwer, 1988], [Vlachoutsis, 1992] and [Rolfes, Rohwer, 1997], see e.g. [3], and is suitable also for composites with non-symmetric LSS. Layers are assumed to be orthotropic and in plane stress.

2. DSA for shell elements with additional parameters

The geometric and material data for a layer of a composite, such as a thickness, orthotropic material constants and an orientation angle of orthotropy, are used as design parameters. We denote a design parameter by h.

Assume that the discrete governing functional F depends on two sets of variables: the nodal displacements $\mathbf{u}_I$ and the elemental parameters $\mathbf{q}$. For kinematically non-linear problems, the stationarity condition of $F(\mathbf{u}_I, \mathbf{q})$ yields a set of equilibrium equations,

$$\mathbf{r}_u \doteq \frac{\partial F(\mathbf{u}_I, \mathbf{q})}{\partial \mathbf{u}_I} = \mathbf{0}, \qquad \mathbf{r}_q \doteq \frac{\partial F(\mathbf{u}_I, \mathbf{q})}{\partial \mathbf{q}} = \mathbf{0}, \tag{3}$$

which can be linearized in a standard way and solved using the Newton method. For shells, $\mathbf{u}_I$ denotes nodal displacements and nodal rotational degrees of freedom.

Let us define the vector of residuals, $\mathbf{r} \doteq [\mathbf{r}_u, \mathbf{r}_q]^T$ and re-write the equilibrium equation as follows: $\mathbf{r}(h, \mathbf{z}(h)) = \mathbf{0}$, where the state variable $\mathbf{z} \doteq [\mathbf{u}_I, \mathbf{q}]^T$ depends on the design parameter h. Differentiation of the equilibrium equation w.r.t. the design variable yields

$$\frac{D\mathbf{r}}{Dh} = \frac{\partial \mathbf{r}}{\partial h} + \frac{\partial \mathbf{r}}{\partial \mathbf{z}}\frac{d\mathbf{z}}{dh} = \mathbf{0}, \tag{4}$$

or

$$\begin{bmatrix} \frac{\partial \mathbf{r}_u}{\partial h} \\ \frac{\partial \mathbf{r}_q}{\partial h} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{L}^T & \mathbf{K}_{qq} \end{bmatrix} \begin{bmatrix} \frac{d\mathbf{u}_I}{dh} \\ \frac{d\mathbf{q}}{dh} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{5}$$

where the submatrices of the tangent operator are: $\mathbf{K} \doteq \partial\mathbf{r}_u/\partial\mathbf{u}_I$, $\mathbf{L} \doteq \partial\mathbf{r}_u/\partial\mathbf{q}$, $\mathbf{K}_{qq} \doteq \partial\mathbf{r}_q/\partial\mathbf{q}$. We note that the same tangent operator is used to solve the equilibrium equations (3) by the Newton method. From these equations we calculate the sensitivities $d\mathbf{q}/dh$ and $d\mathbf{u}_I/dh$ using

$$\frac{d\mathbf{q}}{dh} = -\mathbf{K}_{qq}^{-1}\left(\frac{\partial \mathbf{r}_q}{\partial h} + \mathbf{L}^T\Delta\mathbf{u}_I\right), \quad \mathbf{K}^*\,\frac{d\mathbf{u}_I}{dh} = -\left(\frac{\partial \mathbf{r}_u}{\partial h}\right)^*, \tag{6}$$

where the reduced stiffness matrix and the design derivative of the reduced residual are

$$\mathbf{K}^* \doteq \mathbf{K} - \underbrace{\mathbf{L}\mathbf{K}_{qq}^{-1}}_{\doteq\mathbf{A}^T}\mathbf{L}^T, \quad \left(\frac{\partial \mathbf{r}_u}{\partial h}\right)^* \doteq \frac{\partial \mathbf{r}_u}{\partial h} - \underbrace{\mathbf{L}\,\mathbf{K}_{qq}^{-1}}_{\doteq\mathbf{A}^T}\frac{\partial \mathbf{r}_q}{\partial h}, \tag{7}$$

*This research was partially supported by the Polish Ministry of Science and Higher Education under grant No. N501-290234.

for a symmetric non-singular $\mathbf{K}_{qq}$ and for $\mathbf{A} \doteq \mathbf{K}_{qq}^{-1}\mathbf{L}^T$ updated and stored using Scheme U2 of [1] p. 276. To perform the DSA, we use $\mathbf{K}^*$ and $\mathbf{A}$ from the last converged step of the static analysis. We calculated the explicit design derivatives of residuals $\partial \mathbf{r}_u/\partial h$ and $\partial \mathbf{r}_q/\partial h$ using direct differentiation.

3. Sensitivities of strains and stress/couple resultants

The shell strains $\{\boldsymbol{\varepsilon}, \boldsymbol{\kappa}\}$ and the shell stress and couple resultants $\{\mathbf{N}, \mathbf{M}\}$ (corresponding to the 2nd Piola-Kirchhoff stress) are the essential performance parameters and, therefore, we calculate their design derivatives. Differentiation of a performance $\boldsymbol{\Psi} = \boldsymbol{\Psi}(h, \mathbf{z}(h))$ w.r.t. the design variable h yields

$$\frac{D\boldsymbol{\Psi}}{Dh} = \frac{\partial \boldsymbol{\Psi}}{\partial h} + \frac{d\boldsymbol{\Psi}}{d\mathbf{z}}\frac{d\mathbf{z}}{dh}, \tag{8}$$

where $d\mathbf{z}/dh \doteq [d\mathbf{u}_I/dh,\ d\mathbf{q}/dh]^T$ was earlier computed from eq. (6) and is treated as known. For instance, let $\boldsymbol{\Psi} \doteq \mathbf{N}$, for which the effective constitutive equation is $\mathbf{N} = \mathbb{D}_0\boldsymbol{\varepsilon} + \mathbb{D}_1\boldsymbol{\kappa}$. Then, the derivatives are obtained as follows:

$$\frac{\partial \mathbf{N}}{\partial h} = \frac{\partial \mathbb{D}_0}{\partial h}\boldsymbol{\varepsilon} + \frac{\partial \mathbb{D}_1}{\partial h}\boldsymbol{\kappa}, \qquad \frac{\partial \mathbf{N}}{\partial \mathbf{z}} = \mathbb{D}_0\mathbf{B}_\varepsilon + \mathbb{D}_1\mathbf{B}_\kappa, \tag{9}$$

where $\partial\mathbb{D}_0/\partial h$ and $\partial\mathbb{D}_1/\partial h$ can be directly computed from the input data e.g. using the semi-analytical or analytical method. Besides, $\mathbf{B}_\varepsilon$ and $\mathbf{B}_\kappa$ are the shell strain-displacement tangent operators. Note that we have to differentiate also w.r.t. additional parameters $\mathbf{q}$ because $\mathbf{z}$ depends on $\mathbf{q}$. The design derivatives $D\mathbf{N}/Dh$ and $D\mathbf{M}/Dh$ are printed out and displayed in our program similarly as $\mathbf{N}$ and $\mathbf{M}$.

4. Numerical example

The panel consists of a composite skin, composite omega-shaped straight stringers and curved C-frames made of aluminium, see Fig. 1. The 8-layer composite has the symmetric LSS: $[45/\text{-}45/90/0]_s$. One curved boundary is clamped, all the other boundaries are supported in a way which will be explained during the presentation. The load consists of shear loads and compressive loads. The model was computed for various mesh densities, where $nl = 4$ denotes the densest mesh.

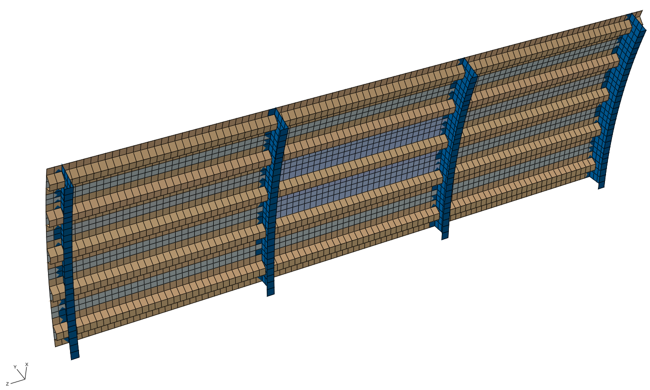

Figure 1: Geometry of panel.

The calculation were performed using our 4-node finite rotations enhanced strain element EADG5. The normal displacement at the center of panel is monitored. Nonlinear equilibrium curves were obtained using the arc-length method with the initial $\Delta F = 100$ and are shown in Fig. 2a. The DSA was performed along the solution path, for each converged configuration. The design derivative w.r.t. the frame thickness is shown in Fig. 2b. More DSA results for the panel and other benchmarks will be presented during the conference.

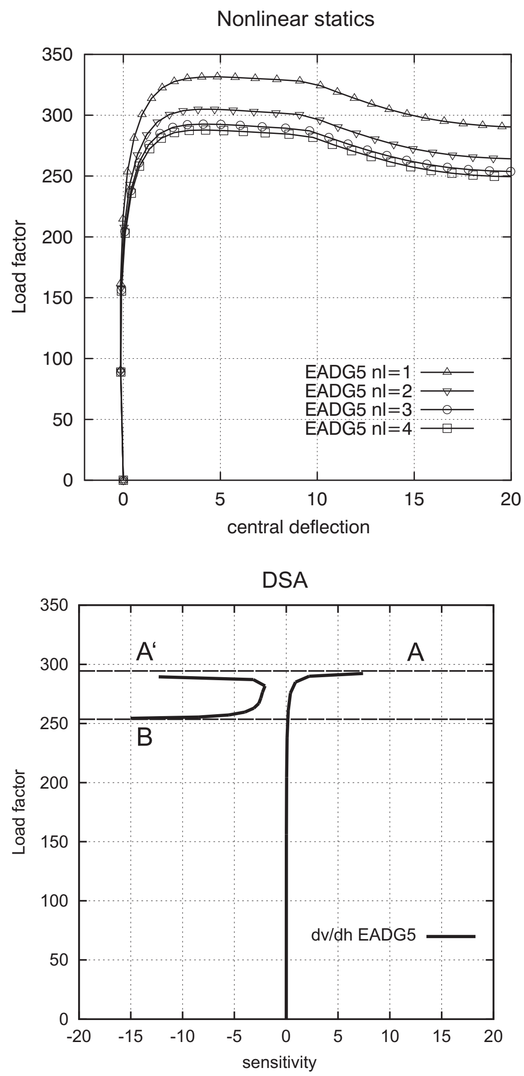

Figure 2: Normal displacements at the center of panel and its sensitivity w.r.t. frame thickness

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