Analysis of delamination and damage growth in joined bi-layer systems

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HIGHLIGHTS

• The paper presents analytical solution of the delamination process in a bi-layer plate using the cohesive zone model.
• The effect of elastic moduli and softening moduli on the delamination process has been clarified.
• It was shown that the limit load value is related to fracture energy and stiffness moduli of adhering plates.
• The scale effect is analytically expressed.

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ABSTRACT

The analysis of mechanical response of joined bi- or multi-layer systems is a typical problem for both geomechanics and composite technology. The elastic or visco-elastic layers interact through joining interfaces transferring stress state between layers and assuring structure integrity. The typical damage modes are related to progressive delamination at bonding interfaces, affected by distributed layer cracking. The present work is aimed to provide an analytical study of the stress state in a bi-layer system and of the progressive delamination process. The cohesive zone model is applied to simulate the interface response with shear stresses related to displacement discontinuities and to the specific fracture energies in shear mode. The following specific issues are discussed: delamination mode growth with the related critical and post-critical response of evaluation length of the process zone, scale effect of the critical stress. The analysis results can be applied to clarify the effect of material parameters on the damage process and to discuss experimental testing of epoxy joined ceramic elements, with specification of the connection strength, related to both the critical interface stress and the specific fracture energy.

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1. Introduction

Coating layers are usually deposited on boundary surfaces of structural elements in order to improve their mechanical response, such as wear, corrosion or fatigue resistance. Due to mismatch of the thermo-mechanical properties of film–substrate systems the residual stresses are generated both in the initial and loaded states. The damage and failure in layered structures or coatings on substrate constitutes the most important class of problems for this type of composite materials. In most cases the delamination at interface and layer cracking are the main modes of degradation of structure performance. An extensive research has been conducted in the last period in this area, with application of different approaches generally based on the fracture mechanics models or on Cohesive Zone Model (CZM) approach. In the case of surface cracking the small cracks can nucleate from a surface defect but for low load they do not channel through the film. Thus, as a result, small cracks remain stable
and can be tolerable for many applications. However, for higher stresses within the coating one observes a channeling process with a network of cracks surrounding islands of the intact film. Upon reaching coating/substrate interface several failure mechanisms can follow: cracks enter the substrate material and stabilize at a certain depth. They can also deviate and propagate along the coating/substrate interface resulting in a subsequent debonding of the protective film. The substrate spalling is another intriguing phenomenon: the crack enters the substrate and selects a path at a certain depth parallel to the interface. This type of failure is common for brittle substrates. The cracks arrested on the coating/substrate interface have been observed, also when cracks tend to interface under perpendicularly applied axial strain. Then new cracks may appear during loading with initially increased crack density to stabilize at a fixed value, unaffected by further loading. This mode of failure is known as a segmentation or multiple cracking of thin films. The fundamental mode of failure, when the load causes the separation of the layers by generated interface cracks, is delamination directly associated with the fracture toughness.

The fracture mechanics models are usually based on Linear Elastic Fracture Mechanics (LEFM) and energy criteria, namely on the potential energy release rate as the generalized driving force specifying progressive crack growth at its critical value. In Ref. the energy release rate has been derived in a closed form by applying the variational approach in crack analysis of coated and multilayer systems. In the analysis of segmentation cracking, assumed the constant shear stress value at the interface, corresponding to ductile response. The similar elastic–perfectly plastic model for the interface was applied by to predict the crack spacing within asphalt pavements. A considerable efforts have been devoted to develop fracture mechanics of interfacial cracking. provided an extensive study of the singular stress regime and of cracking modes along the interface combined with layer cracking. As the compliance discontinuity occurs, it affects the driving force and induces the local shear stresses at the crack tip. Due to the elastic mismatch the interface or near interface crack experiences both tensile and shear modes (Modes I and II) even if the remote loading corresponds to Mode I. Generally, the crack growth near or at the interface has been shown to depend on the relative strength of coating and substrate or the strength of interfacial bond relative to the strength of adjacent layers. Numerous studies have also been devoted to the fatigue failure of coated layers and correlation between the ability of cracks to traverse an interface undeflected and the bond strength of the interface. Suress and Sugimara successfully predicted the crack growth toward the interface in ductile-bimaterial systems by accounting for the constraint of plastic slip mechanisms ahead of the crack tip and adapted this approach to the analysis for crack growth in a system with brittle coatings.

The LEFM solutions for interface cracks exhibit singular oscillatory behavior of stress field in the crack tip zone, generating some difficulties in crack growth modeling. An useful alternative to LEFM is the shear lag (SL) or the cohesive zone model (CZM). The shear lag approach has been frequently applied in the analysis of stress transfer through the interfacial shear stress in composite or bonded materials. The interface tractions are then assumed to be related to displacement jumps. The analysis of rivet connected elements using this method was initiated by in 1938 and later by in a simple one dimensional model of analysis of stress transfer between a matrix and a fiber assuming the linear elastic relation between shear stress and tangential slip. The subsequent treatments by and were devoted to composite analysis assuming elastic or elastic–plastic interaction models. The cohesive zone models (CZM) first proposed by and were addressed to analyze the localized damage or plastic flow in the front of crack tip. Next they have been developed to become an effective numerical tool for the analysis of crack initiation and growth, also for study of interfacial fracture of composites. In the literature, numerous proposals for the non-linear interaction between interface tractions and displacement jumps have been presented for the cohesive elements, such as trapezoidal, perfectly plastic, polynomial and exponential rules. However, the most common and useful is the bilinear model, accounting for elastic and softening deformation stages preceding final failure. The cohesive zone models implemented in FEM have been widely exploited for specification of critical loads of bonded joints. The interface problems and delamination failure were numerically analyzed in Ref. The comparative analysis of several cohesive models in predicting the onset of cracking and failure loads was presented in Ref.

The analytical solutions for the cohesive zone models have been presented in several papers. The analytical treatment is important for the analysis of effect of material and geometric parameters on failure loads and their modes. The analytical solution of beam debonding form rigid substrate under normal load was presented by . The problem of fiber pull-out for the assumed cohesive model was analytically treated by and . The delamination process under compressive normal load was treated by and next by and indicating differing tangential deformation response due to coupling of delamination and sliding friction effects. The numerical treatment of cracking of surface coating layer in flexural mode was experimentally and numerically treated by et al. Crack patterns in thin layers under temperature loading were studied analytically by and presenting stress analysis and applying the total energy minimization method for prediction of crack pattern. The analytical solutions for specific cases of delamination process in a bi-material structure were presented by et al. and provided the insight into the initiation and growth of the damage...
process zone, crack initiation and growth length, leading to ultimate failure of the interface. The length evolution of the process zone along the interface is analyzed and compared with the predictions based on the fracture energy and critical stress concepts. Two cases are considered: first, the interface delamination for substrate plate loaded inducing delamination of the coating plate, next the lap joint failure process due to growth of delamination zones from two interface ends.

In Section 2 the problem formulation, governing equations and general solutions presenting displacement, and stress fields in both plates for different combination of elastic, softening and failure stress regimes are presented. The limiting cases for perfectly ductile or brittle response of bonding interface and plate are also considered. In Sections 3 and 4, the specific cases are considered when the increasing load is imposed. The analysis of progressive delamination is then presented. The analytical expression and diagrams illustrate the effect of geometric and material parameters on the damage mode in the bi-material system.

2. Problem formulation

Consider two plates $A$ and $B$ bonded along the interface $I$ and loaded uniaxially at two opposite edges (Fig. 1). The thickness of plates is $2h_A$ and $2h_B$, where $2h_B$ corresponding to the coating layer is usually much smaller than the substrate plate thickness $2h_A$. The uniform in-plane loading induces tensile or compressive stress in plates $A$ and $B$, bending and peel stresses are neglected. The elastic analysis of double lap joints by Goglio and Rosetto $^{52}$ with account for bending moments and peel stress indicates that this low value stress is compressive and does not affect the shear failure. The bonding interface is assumed to transfer shear stresses between two plates and its response is expressed in terms of displacement discontinuity of two plates. Denote the uniaxial stresses (uniform across the thickness) in plates by $\sigma_A = \sigma_A(x)$ and $\sigma_B = \sigma_B(x)$ and by $\tau_1 = \tau_1(x)$ the interface shear stress. The following equilibrium equations can be written for the resulting forces $T_A(x) = 2h_A \sigma_A$ and $T_B(x) = 2h_B \sigma_B$

$$\frac{dT_A}{dx} = \tau_1, \quad \frac{dT_B}{dx} = -\tau_1,$$

or

$$2h_A \frac{d\sigma_A}{dx} = \tau_1, \quad 2h_B \frac{d\sigma_B}{dx} = -\tau_1,$$

where $x$ denotes the position along the plate length. The displacement fields in plates are denoted by $u_A = u_A(x)$ and $u_B = u_B(x)$, the corresponding strains are

$$\varepsilon_A = \frac{du_A}{dx}, \quad \varepsilon_B = \frac{du_B}{dx}.$$  (2)

The substrate plate is assumed to follow linear elastic response $\sigma_B = E_B \varepsilon_B$ and its Young modulus is $E_B$. The mechanical response of the coating plate $A$ and of bonding interface is presented in (Fig. 2).

The coating layer $A$ is assumed to deform initially, as a linear elastic material and after reaching the critical stress value $\sigma^C_A$, the damage induced softening response follows until full failure. We can write

$$\sigma_A = \begin{cases} E_A \varepsilon_A & \text{for } \varepsilon_A \leq \frac{\sigma^C_A}{E_A} = \varepsilon^C_A \\ \sigma^C_A - E_A (\varepsilon_A - \varepsilon^C_A) & \text{for } \varepsilon_A > \varepsilon^C_A \\ 0 & \text{for } \varepsilon_A \geq \varepsilon^f_A. \end{cases}$$  (3)

The similar interface response presented in (Fig. 2(a)) can be specified in terms of displacement discontinuity $u_I = u_A - u_B$. The shear strain in the bonding layer of
The critical stress, \( \sigma \), with immediate stress drops to zero value after reaching \( E \). Thefully ductile response of the layer A occurs. On the other hand, when \( E_{\text{AS}} = \infty \), the perfectly brittle response with immediate stress drops to zero value after reaching the critical stress, \( \sigma_c \) and the strain \( \varepsilon_c \). The similar ductile or brittle responses of the bonding interface can be described. The critical failure strain \( \varepsilon_f \) or the critical displacement discontinuity \( u_A \) at the state of total failure now are

\[
\varepsilon_f = \frac{1}{E_A} + \frac{1}{E_{\text{AS}}}, \quad u_A = \frac{1}{K_{\text{eff}}} + \frac{1}{K_{\text{eff}}^C}.
\]

The total specific energy release due to coating layer or interface failure is

\[
G_A = \frac{1}{2} \sigma_A^C \varepsilon_f, \quad G_1 = \frac{1}{2} \varepsilon_f \tau_{f1}^C = \frac{1}{2} (\tau_{f1}^C)^2 \left( \frac{1}{K_{\text{eff}}} + \frac{1}{K_{\text{eff}}^C} \right).
\]

Accounting for both layer and interface damage, different interactive regimes can be generated, as illustrated in Fig. 2. Assuming first two layers as elastic, only interface damage growth and delamination process can be analyzed. The present constitutive model assumptions follow the previous more general formulation of interface damage and failure model for mixed mode conditions in Ref. \(^{45,46}\).
where, $C_3$, $\tilde{C}_3$, $C_4$, $\tilde{C}_4$ are constants to be specified from the boundary and continuity conditions. However, requiring $u_i = u_A - u_B$, the constants can be reduced, as follows: $\tilde{C}_4 = C_4$ and $\tilde{C}_3 = -E_B/E_A C_3$. Then,  

$$
\begin{align*}
\tau_i &= K_i^{ef}(C_1 \sinh[\lambda_s x] + C_2 \cosh[\lambda_s x]), \\
T_A &= \frac{K_i^{ef}}{\lambda_e}(C_1 \cosh[\lambda_e x] + C_2 \sinh[\lambda_e x]) + 2h_A C_3, \\
T_B &= -\frac{K_i^{ef}}{\lambda_e}(C_1 \cosh[\lambda_e x] + C_2 \sinh[\lambda_e x]) \\
&\quad + 2h_B E_B C_3, \\
u_A &= \frac{K_i^{ef} E_A}{2h_B E_B \lambda_e^2} (C_1 \sinh[\lambda_e x] + C_2 \cosh[\lambda_e x]) \\
&\quad + \frac{C_3}{E_A} x + C_4, \\
u_B &= -\frac{K_i^{ef} E_A}{2h_B E_B \lambda_e^2} (C_1 \sinh[\lambda_e x] + C_2 \cosh[\lambda_e x]) \\
&\quad + \frac{C_3}{E_A} x + C_4.
\end{align*}
\tag{12}
$$

2.2. Elastic plates-softening interface (E-S-E)

Assume first that plates are in the elastic state (E) and the interface in the softening state (S). Then we have  

$$
\begin{align*}
\tau_i = \tau_i^C - K_i^{ef}(u_i - u_i^C) &= \left(1 + \frac{K_i^{ef}}{K_i^{ef}}\right) \tau_i^C - K_i^{ef} u_i \\
&= A - K_i^{ef} u_i
\end{align*}
\tag{13}
$$

where the interface softening is beginning at $\tau_i^C$ value, as presented in Fig. 2(b). Then the equilibrium equations take the form

$$
\begin{align*}
2h_A E_A \frac{d^2 u_A}{dx^2} &= A - K_i^{ef} u_i, \\
2h_B E_B \frac{d^2 u_B}{dx^2} &= -A + K_i^{ef} u_i, \tag{14}
\end{align*}
$$

and the field $u_i(x)$ is specified from the equation

$$
\frac{d^2 u_i}{dx^2} + \lambda_i^2 u_i = F, \tag{15}
$$

where

$$
\begin{align*}
A &= \left(1 + \frac{K_s^{ef}}{K_i^{ef}}\right) \tau_i^C, \\
\lambda_i^2 &= K_i^{ef} \left(\frac{1}{2h_A E_A} + \frac{1}{2h_B E_B}\right) > 0, \\
F &= \left(\frac{1}{2h_A E_A} + \frac{1}{2h_B E_B}\right) \left(1 + \frac{K_i^{ef}}{K_i^{ef}}\right) \tau_i^C \\
&= \frac{\lambda_i^2}{K_s^{ef}} \left(1 + \frac{K_i^{ef}}{K_i^{ef}}\right) \tau_i^C.
\end{align*}
\tag{16}
$$

The integral equation (15) is of the form

$$
\begin{align*}
u_i &= D_1 \sin[\lambda_s x] + D_2 \cos[\lambda_s x] + \frac{F}{\lambda_i^2}, \tag{17}
\end{align*}
$$

The stress and displacement fields are now expressed as follows
\[ \tau_1 = -K_1^{ef}(D_1 \sin[\lambda_s x] + D_2 \cos[\lambda_s x]), \]
\[ T_A = K_2^{ef}(D_1 \cos[\lambda_s x] - D_2 \sin[\lambda_s x]) + 2D_3 h_A, \]
\[ T_B = K_2^{ef}(-D_1 \cos[\lambda_s x] + D_2 \sin[\lambda_s x]) - 2D_3 h_B, \]
\[ u_A = \frac{K_2^{ef}}{2h_A E_A \lambda_s^2} (D_1 \sin[\lambda_s x] + D_2 \cos[\lambda_s x]) \]
\[ + \frac{D_3}{E_A} x + D_4, \]
\[ u_B = \frac{K_2^{ef}}{2h_B E_B \lambda_s^2} (-D_1 \sin[\lambda_s x] - D_2 \cos[\lambda_s x]) \]
\[ - \frac{D_3}{E_B} x + D_4, \]
where, \( D_1, D_2, D_3, D_4, \tilde{D}_3, \tilde{D}_4 \) are constants. However, requiring \( u_i = u_A - u_B \), the constants can be reduced, as follows: \( D_4 = D_4 - F/\lambda_s^2 \) and \( D_3 = -E_B/E_A D_3 \). Then,
\[ \tau_1 = -K_1^{ef}(D_1 \sin[\lambda_s x] + D_2 \cos[\lambda_s x]). \]
\[ T_A = K_2^{ef}(D_1 \cos[\lambda_s x] - D_2 \sin[\lambda_s x]) + 2D_3 h_3, \]
\[ T_B = K_2^{ef}(-D_1 \cos[\lambda_s x] + D_2 \sin[\lambda_s x]) \]
\[ + 2h_B E_B / E_A D_3, \]
\[ u_A = \frac{K_2^{ef}}{2h_A E_A \lambda_s^2} (D_1 \sin[\lambda_s x] - D_2 \cos[\lambda_s x]) \]
\[ + \frac{D_3}{E_A} x + D_4, \]
\[ u_B = \frac{K_2^{ef}}{2h_B E_B \lambda_s^2} (-D_1 \sin[\lambda_s x] - D_2 \cos[\lambda_s x]) \]
\[ + \frac{D_3}{E_B} x + D_4 - F/\lambda_s^2. \]

The proper continuity conditions should be stated to match different stress and displacement regimes. The continuity of forces \( [T_A] = [T_B] = 0 \) at the transition point require that \( [u_A] = [u_B] = 0 \). These conditions can also be expressed as follows
\[ [u_i] = [u_B] = 0, \quad [T_A] = 0, \quad [u_A] = 0. \]
In fact, for the bi-layer structure loaded axially at its end, the condition \( T_A + T_B = T \) occurs, where \( T \) is the resultant boundary load. Thus, it suffices to specify \( u_i \), next force and displacement in one layer, say \( A \). Then force and displacement in the second layer \( B \) are determined in terms of \( u_i \), force \( T_A \) and displacement \( u_A \).

The case
\[ u_i = \tau_i = 0 \]
corresponds to uniform force distribution within both layers. As then \( \varepsilon_A = \varepsilon_B, T_A + T_B = T \), we have
\[ T_A = \frac{\rho}{1+\rho} T, \quad T_B = \frac{1}{1+\rho} T, \quad \rho = \frac{2h_A E_A}{2h_B E_B} \]  
and the forces \( T_A \) and \( T_B \) are specified in terms of the boundary load \( T \) and the layer stiffness ratio \( \rho \). The state (22) will be called the uniform stress state. It is important state for a multi-layer system, as the local stress state affected by boundary conditions or imperfections (layer cracks, debonded spots) tends to the uniform state (22) for increasing distance from the stress perturbation source.

3. Substrate plate loading

3.1. Elastic analysis

Consider now the case of the substrate plate \( B \) loaded at both ends by the force \( T_B(l) = T_B(-l) = T \) and the coating plate \( A \) with stress-free ends, \( T_A(l) = T_A(-l) = 0 \). (Fig. 4). The plates are bonded along the interface layer \( l \), whose thickness \( h \) is included in the definition of effective stiffness and softening layer moduli. As the boundary condition does not satisfy the uniform state condition (22), the interface shear stress \( \tau_1(x) \) is generated. Introducing the coordinate system \( x, y \) with its origin at the plate center and applying (12) with the boundary and symmetry conditions as
\[ T_A(l) = 0, \quad T_B(l) = T, \quad u_l(0) = 0 \]
we obtain
\[ C_1 = -\frac{\rho}{1+\rho} T \frac{\lambda_e}{K_1^{ef}} \frac{1}{\cosh[\lambda_e l]}, \quad C_2 = 0, \]
\[ C_3 = \frac{\rho}{1+\rho} \frac{T}{2h_A} \]
and the mechanical field is expressed as follows
\[ u_i^1(x) = -\frac{\rho}{1+\rho} T \frac{\lambda_e}{K_1^{ef}} \frac{\sinh[\lambda_e x]}{\cosh[\lambda_e l]}, \]
\[ \tau_i^1(x) = -\frac{\rho}{1+\rho} T \frac{\lambda_e}{K_1^{ef}} \frac{\sinh[\lambda_e x]}{\cosh[\lambda_e l]}, \]
\[ T_A^1(x) = \frac{1}{1+\rho} T \left( 1 - \frac{\cosh[\lambda_e l]}{\cosh[\lambda_e l]} \right), \]
\[ T_B^1(x) = \frac{1}{1+\rho} T \left( 1 + \frac{\cosh[\lambda_e x]}{\cosh[\lambda_e l]} \right). \]

Let us note that for increasing plate length, \( \cosh[\lambda_e l] \to \infty \) and the ratio \( T_A^1(0)/T_B^1(0) \) tends to \( \rho \), corresponding to the uniform stress regime. The displacement field for the boundary conditions \( u_i^1(0) = u_i^0(0) = 0 \) is obtained in the form
\[ u_i^0(x) = \int_0^x \frac{T_B^0(x)}{2h_B E_B} \mathrm{d}x \]
\[ = \frac{T}{2h_B E_B} \frac{1}{1+\rho} \left( x + \frac{\rho}{\lambda_e} \frac{\sinh[\lambda_e x]}{\cosh[\lambda_e l]} \right) \]
\[ u_i^1(x) = u_i^0(x) + u_i^0(x) \]
\[ = \frac{T}{2h_A E_A} \frac{\rho}{1+\rho} \left( x - \frac{\cosh[\lambda_e x]}{\cosh[\lambda_e l]} \right). \]

The interface damage initiation starts when \( \tau_i^0(l) = \tau_i^C \). The acting critical load then equals
dependence of critical load near the boundary increases and the value of the critical bonding layer to tension stiffness of plates, the shear stress varying values of stiffness parameter λ.

length the critical load tends to its asymptotic value which vanishes for λe → ∞. The diagram Tc − λe l specifies two domains of delamination and of no delamination.

3.2. Growth of process zone

For the elastic shear stress τi > |τe|, the symmetric damage process zones start to grow from two ends of plates. In view of symmetry, consider only the solution for 0 ≤ x ≤ l. Assume the elastic zone to exist in the central domain 0 ≤ x ≤ le and the damage process zone in the boundary domain le ≤ x ≤ l. Fig. 6(b), with its initial growing length dp = l − le. Fig 6(b) presents schematically the shear stress field in two zones.

Using formulae (12) and (19) the mechanical states in two zones are expressed as follows.

Elastic zone: 0 ≤ x ≤ le. The conditions τe(0) = 0 requires that the integration constant Ce = 0 and then we have

\[ τ^e_i(x) = \frac{K^e_i}{\lambda_e} C_1 \sinh[\lambda_e x], \]

\[ T^e_i(x) = \frac{K^e_i}{\lambda_e} C_1 \cosh[\lambda_e x] + \frac{\rho}{1 + \rho} T(l), \]

\[ T^e_i(x) = -\frac{K^e_i}{\lambda_e} C_1 \cosh[\lambda_e x] + \frac{1}{1 + \rho} T(l). \]

Damage process zone: le ≤ x ≤ l:

\[ u^p_i(x) = D_1 \sin[\lambda_e x] + D_2 \cos[\lambda_e x] + u^e_i, \]

\[ τ^p_i(x) = -\frac{K^e_i}{\lambda_e} (D_1 \sin[\lambda_e x] + D_2 \cos[\lambda_e x]), \]

\[ T^p_i(x) = \frac{K^e_i}{\lambda_s} (D_1 \cos[\lambda_s x] - D_2 \sin[\lambda_s x]) + \frac{\rho}{1 + \rho} T(l), \]

\[ T^p_i(x) = \frac{K^e_i}{\lambda_s} (-D_1 \cos[\lambda_s x] + D_2 \sin[\lambda_s x]) + \frac{1}{1 + \rho} T(l). \]

where \( u^e_i \) is defined by (5). Satisfying the conditions \( T^e_i(l) = 0, τ^e_i(l) = τ^p_i(l) = -τ^e_i \), the integration constants are specified, namely

\[ C_1 = -\frac{1}{K^e_i} \frac{\rho T(l)}{\lambda_s \cosh[\lambda_e l]}. \]

\[ D_1 = -\frac{1}{\lambda_s} \frac{\rho T(l)}{K^e_i} \cos[\lambda_e l] \] \[ \frac{\rho}{1 + \rho} T(l) \]

\[ D_2 = -\frac{1}{\lambda_s} \frac{\rho T(l)}{K^e_i} \sin[\lambda_e l] \]

Satisfying the continuity condition \( T^e_i(l) = 0 \), the relation is obtained specifying the length of the initial process zone \( dp = l - le \).

\[ \frac{T}{\tau^e_i} \frac{\rho}{1 + \rho} = \frac{\sin[\lambda_s dp]}{\lambda_s} + \frac{\cosh[\lambda_e l] \cos[\lambda_s dp]}{\lambda_e}. \]

Finally, the mechanical fields are in the elastic zone 0 ≤ x ≤ le:

\[ τ^e_i(x) = -\frac{\rho}{K^e_i} \frac{\rho T(l)}{\lambda_s \cosh[\lambda_e x]}. \]

\[ T^e_i(x) = -\frac{\rho}{K^e_i} \frac{\rho T(l)}{\lambda_s \sinh[\lambda_e x]} + \frac{\rho}{1 + \rho} T(l), \]

\[ T^e_i(x) = \frac{\rho}{K^e_i} \frac{\rho T(l)}{\lambda_s \sinh[\lambda_e x]} + \frac{1}{1 + \rho} T(l) \]

in the damage zone le ≤ x ≤ l:

\[ τ^p_i(x) = \frac{\rho T(l)}{\lambda_s \cosh[\lambda_s(l - le)]} \left[ \frac{\rho}{1 + \rho} T(l) \right] \]

\[ -\frac{\rho}{1 + \rho} T(l) \lambda_s \cos[\lambda_s l] \]

\[ -\tau^c_i \sin[\lambda_s l] \] \[ -\frac{\rho}{1 + \rho} T(l) \lambda_s \sin[\lambda_s l] \]

\[ +\frac{\rho}{1 + \rho} T(l) \lambda_s \cos[\lambda_s l] \]

\[ +\frac{\rho}{1 + \rho} T(l) \lambda_s \sin[\lambda_s l] \] \[ -\frac{\rho}{1 + \rho} T(l) \lambda_s \cos[\lambda_s l] \]

\[ +\frac{\rho}{1 + \rho} T(l) \lambda_s \sin[\lambda_s l] \] \[ -\frac{\rho}{1 + \rho} T(l) \lambda_s \cos[\lambda_s l] \]

\[ +\frac{\rho}{1 + \rho} T(l) \lambda_s \sin[\lambda_s l] \]
3.2.1. Growth of delamination zone

The full delamination starts when \( \tau_{1e}^{c}(l) = 0 \) and next the full delamination zone \( d = d_{1} - d_{p} = l - l_{d} \) grows from both ends, Fig. 6(b). From (32), setting \( \tau_{1e}^{p}(l) = 0 \), we obtain the load at the onset of delamination

\[
T = \frac{\rho + 1}{\rho} \frac{\tau_{1e}^{c}}{\lambda_{s} \sin[\lambda_{e} l_{d}]} = \frac{1}{\rho} \frac{1}{\lambda_{e} \sin[\lambda_{e} l_{d}]}.
\]

where, the value of \( d_{p} \) is specified from (31).

In the subsequent stage there are the delamination and damage process zones of lengths \( d \) and \( d_{p} \) evolving during the loading process, with the total damaged zone \( d_{1} = d_{p} + d \). In the delaminated zone, the mechanical state is as follows

\[
l_{d} \leq x \leq l : T_{A} = 0, \quad T_{B} = T, \quad \tau_{1e} = 0.
\]

In the process zone: \( l_{e} \leq x \leq l_{d} \) the same relations as in Eq. (29) apply and the following boundary conditions are stated \( T_{B}^{p}(l_{d}) = 0, \quad \tau_{1e}^{p}(l_{d}) = 0 \) and the constants are

\[
D_{1} = -\frac{\rho}{1 + \rho} \frac{\lambda_{s}}{K_{t1}} \cos[\lambda_{e} l_{d}],
\]

\[
D_{2} = \frac{\rho}{1 + \rho} \frac{\lambda_{s}}{K_{t1}} \sin[\lambda_{e} l_{d}].
\]

Similarly, for elastic fields: \( 0 \leq x \leq l_{e} \) the condition \( \tau_{1e}^{p}(l_{e}) = -\tau_{1e}^{c} \) provides the constant \( C_{1} \), thus

\[
C_{1} = -\frac{\tau_{1e}^{c}}{K_{t1}} \frac{1}{\frac{1}{\rho} + \frac{\lambda_{e}}{\sin[\lambda_{e} l_{e}]}},
\]
Fig. 6. Progressive growth of delamination: interface model (a); substrate mechanical loading and shear stress field for initial growth of damage zone, (b); growth of delamination and motion process zone, (c).

The mechanical fields are now expressed as follows in the elastic zone: $0 \leq x \leq l_e$:

\[
\begin{align*}
\tau^e_1(x) &= -\frac{\tau^e_1 \sinh[\lambda e x]}{\sinh[\lambda e l_e]}, \\
T^e_A(x) &= -\frac{\tau^e_1 \cosh[\lambda e x]}{\lambda e \sinh[\lambda e l_e]} + \frac{\rho}{1 + \rho} T, \\
T^e_B(x) &= \frac{\tau^e_1 \cosh[\lambda e x]}{\lambda e \sinh[\lambda e l_e]} + \frac{1}{1 + \rho} T
\end{align*}
\]

\hspace{2cm}

in the damage zone: $l_e \leq x \leq l_d$:

\[
\begin{align*}
\tau^p_1(x) &= \frac{\rho}{1 + \rho} T \lambda d \left[ \cos[\lambda d] \sin[\lambda x] - \sin[\lambda d] \cos[\lambda x] \right], \\
T^p_A(x) &= \frac{\rho}{1 + \rho} T \left[ 1 - \cos[\lambda d] \cos[\lambda x] - \sin[\lambda d] \sin[\lambda x] \right], \\
T^p_B(x) &= \frac{\rho}{1 + \rho} T \left[ 1 + \cos[\lambda d] \cos[\lambda x] + \sin[\lambda d] \sin[\lambda x] \right].
\end{align*}
\]

Requiring $\tau^p_1(l_e) = -\tau^e_1$ and continuity condition $[T_A] = 0$ at $x = l_e$, two equations specifying $l_e$ and $l_d$ are obtained
and the relation between the load of plate B and the elastic zone length is

\[
T = \frac{\rho + \frac{1}{\lambda_s} \frac{\tau_0^C}{\lambda_e}}{\rho \cos[\lambda_s \Delta p]} = \frac{\rho + \frac{1}{\lambda_s} \frac{\tau_0^C}{\lambda_e}}{\rho \sin[\lambda_s \Delta p]}.
\]

The displacement field of plate B is expressed for elastic, process and delaminated zones. Satisfying the boundary conditions \( u_B^A(0) = 0 \) and continuity conditions \( |u_B(l_e)| = 0 \) and \( |u_B(l_d)| = 0 \), we have Eq. (44) in Box II.

It is seen from (42) that the length \( d_p \) of the process zone decreases during the delamination process and vanishes for \( l_e \to 0 \). On the other hand, the applied load expressed by (43) increases and tends to infinity for \( l_e \to 0 \). This effect can easily be interpreted from the elastic solution and diagram in Fig. 5(c). In fact the required delamination load increases for short plate. Fig. 7 presents the progression of process zone length \( d_p \) versus total delamination affected portion of the interface \( l_d \). Initially \( d_p = d_i \) during the zone growth phase and next \( d_p \) decreases during the zone translation phase.

### 3.3. Specific case 1: elastic-perfectly brittle bond model

The elastic–brittle bonding model is frequently used to specify the lower bound on the delamination load. Now the process zone does not exist as there is an abrupt transition from the elastic shear stress value \( \tau_i^C \) to failure at fixed value of \( u_i = u_{lfr} \). Fig. 8. Following (25), we have for \( 0 < x < l_e \):

\[
\tau_i^C(x) = \frac{1}{1 + \rho} T\frac{\lambda_e \sin[\lambda_e x]}{\lambda_e \cosh[\lambda_e l_e]}
\]

and the displacement for the boundary condition \( u_B(0) = 0 \) is \( 0 < x < l_e \):

\[
\begin{align*}
\tau_A^P(x) & = \frac{1}{1 + \rho} T\left(1 - \frac{\cosh[\lambda_e x]}{\cosh[\lambda_e l_e]}\right), \\
\tau_B^P(x) & = \frac{1}{1 + \rho} T\left(1 + \frac{\cosh[\lambda_e x]}{\cosh[\lambda_e l_e]}\right),
\end{align*}
\]

Setting \( \tau_i^C(l_e) = -\tau_i^C \), it is obtained

\[
\tau_i^C = \frac{1 + \rho}{\rho} \frac{\tau_0^C}{\lambda_e} \tanh[\lambda_e l_e]
\]

and it is noted that \( u_B^A(l_e) = u_B^B(l_e) \) for \( l_e < l < I \):
3.4. Specific case 2: elastic-perfectly plastic bond model: $K_{eff}^t = 0$

Referring to the previous case, the growth of plastic zones from the plate ends induces higher load values in order to drive the delamination process. Fig. 9, namely

$$
\tau^C = \frac{1 + \rho}{\rho} \tau^C \left[ \frac{1}{\lambda_e} \coth[\lambda_e(l - d_i)] + d_i \right]
$$

and the displacement at the end of plate B is

$$
u^B_b(x) = u^e_b(l_e) = \frac{l_e}{1 + \rho} \frac{T}{2h_b E_b} + \frac{\tanh[\lambda_e l_e]}{2h_b E_b \lambda_e} \frac{T}{1 + \rho} - \tau^C \left( l_e - l \right).
$$

3.5. Specific case 3: rigid-softening interface layer: $\lambda_e \to \infty$

The analysis presented for the elastic-softening response can be particularized for several specific cases requiring more detailed discussion. Consider first the case when the elastic modulus $K_{eff}^t$ tends to infinity and the rigid-softening response occurs. This modulus is difficult to identify for the FEM analysis where high values are usually assumed. The softening rule is now expressed as follows

$$
\tau_1 = \tau^C_1 - K_{eff}^t u_1, \quad G_{II} = \frac{1}{2} \tau_C^2 u_{II} = \frac{1}{2} \frac{(\tau^C_1)^2}{K_{eff}^t}.
$$

As $u_1 = 0$ for $|\tau_1| < \tau^C_1$, then only homogeneous stress states are allowed for which $\tau_1 = u_1 = 0$. After load application the damage process zones are formed near the plates ends and in the central part the homogeneous stress state is generated, Fig. 10.

From the equilibrium equations for the damage growth phase

$$
\frac{dT_A}{dx} = 2h_a E_A \frac{d^2 u_A}{dx^2} = -\tau^C_1 - K_{eff}^t u_1
$$

$$
\frac{dT_B}{dx} = 2h_b E_B \frac{d^2 u_B}{dx^2} = \tau^C_1 + K_{eff}^t u_1.
$$

The following differential equation is obtained

$$
\frac{d^2 u_1}{dx^2} + \lambda_e^2 u_1 = -\left( \frac{1}{2h_a E_A} + \frac{1}{2h_b E_B} \right) \tau^C_1.
$$
where
\[ \lambda_s^2 = K_s^{ef}\left(\frac{1}{2h_A E_A} + \frac{1}{2h_B E_B}\right) = K_s^{ef} C_1 \]
and
\[ K_s^{ef} = \frac{(\tau_i^C)^2}{2G_{if}} \]

where, \( C_1 \) denotes the elastic layer compliance, expressed by the bracketed term of (56). Satisfying the continuity conditions \([u_l] = [u']_l = 0\) for \( x = l_e \) and the boundary conditions \( T_A(l) = 0 \) it is obtained

\[ u_l^p(x) = -\frac{\tau_i^C}{K_s^{ef}} (1 - \cos[\lambda_s(x - l_e)]) \]

\[ \tau_i^p(x) = -\frac{\tau_i^C}{K_s^{ef}} \cos[\lambda_s(x - l_e)] \]

\[ T_A^p(x) = \frac{\tau_i^C}{\lambda_s} (\sin[\lambda_s d] - \sin[\lambda_s(x - l_e)]) \]

\[ T_B^p(x) = T - \frac{\tau_i^C}{\lambda_s} (\sin[\lambda_s d] - \sin[\lambda_s(x - l_e)]) \]

where, \( d = l - l_e \). For \( x = l_e \), the values of \( T_A \) and \( T_B \) should be equal to the homogeneous state values, thus

\[ T_A = \frac{\tau_i^C}{\lambda_s} \sin[\lambda_s d] = \frac{\rho}{1 + \rho} T, \]

\[ \sin[\lambda_s d] = \frac{\lambda_s}{\tau_i^C} \frac{\rho}{1 + \rho} T \]

and the homogeneous stress state occurs for \(-l_e \leq x \leq l_e\).

The end of growth phase takes place when \( \tau_i^0(l) = 0 \), thus

\[ -\tau_i^C \cos[\lambda_s d_f] = 0, \quad \lambda_s d_f = \frac{\pi}{2} \quad \text{and} \quad d_f = \frac{\pi}{2} \frac{1}{\lambda_s}. \]  

(59)

It is seen that at the end of growth process the length of the process zone reaches its critical value \( d_f \) expressed by (59). Alternatively, it can also be presented in the form

\[ d_f = \frac{\pi}{2} K_s \frac{1}{\tau_i^C} \sqrt{2G_{if}} \frac{1}{C_1}. \]  

(60)

The critical value \( d_f \) depends on three parameters, with the most sensitive dependence on \( \tau_i^C \) and less sensitive dependence on \( G_{if} \) and \( C_1 \).

The progression phase of the process zone proceeds at the constant load

\[ T = \frac{1 + \rho}{\lambda_s} \frac{\tau_i^C}{\rho}, \]

and the increasing displacement \( u_l(\lambda) \), Fig. 1(b). We have

\[ l_p \leq x \leq l \quad \tau_i^0(x) = 0, \quad T_A = 0, \]

\[ T_B = T = \frac{\tau_i^C}{\lambda_s^{ef}} \frac{1 + \rho}{\rho}, \]

\[ l_e \leq x \leq l_p \quad \tau_i^p(x) = -\tau_i^C \cos[\lambda_s(x - l_e)], \quad (62) \]

\[ T_A^p(x) = \frac{\tau_i^C}{\lambda_s} (1 - \sin[\lambda_s(x - l_p)]), \]

\[ 0 \leq x \leq l_e : T_A = \frac{\rho}{1 + \rho} = \frac{\tau_i^C}{\lambda_s} \frac{1}{1 + \rho} \]

\[ T_B = \frac{\tau_i^C}{\lambda_s} \frac{1}{1 + \rho}. \]
The progression phase is determined when the damage process zones reach the symmetry axes at $x = 0$, Fig. 11. The final phase, named the degradation of process zone next proceeds. Assume $u_i$ to vary discontinuously at $x = 0$ with growing value $u_1$, at constant process zone length $d_f$, so that

$$u_i = -\frac{\tau_i^C - K_{ef}^u u_1}{K_{ef}^u} (1 - \cos[\lambda_s x]),$$

$$\tau_1 = -(\tau_i^C - K_{ef}^u u_1) \cos[\lambda_s x],$$

$$T = -\frac{\tau_i^C - K_{ef}^u u_1}{\lambda_s} (1 - \sin[\lambda_s x]), \quad T_B = T - T_A. \quad (63)$$

Final degradation occurs when $K_{ef}^u u_1 = \tau_i^C$, and $T_A = 0$, $T_B = T$. It is seen that for the rigid-softening model, the process of full delamination proceeds differently from that for elastic-softening model of bonding interface, where the continuing reduction of length of the process zone occurred at increasing plate loading.

Fig. 12 presents the dependence of the applied load $T$ or the length of delaminated zone. It is seen that for increasing value of $\lambda_v$, the $T - d_1$ diagram tends to that predicted by the rigid-softening model. Fig. 13 presents the $T - u_B(l)$ diagrams for several cases analyzed in the paper.
4. Lap-joint analysis

4.1. Elastic analysis

Consider the lap joint shown in Fig. 14 with the plate B attached to a fixed support at $x = 0$ and plate A axially loaded at its end $x = l$ by the force $T$ and with the free end at $x = 0$. The reference system $x, y$ is now put at the left plate end. Applying the elastic solution (12) with the boundary conditions $T_A(l) = T$, $T_A(0) = 0$, the static field can be determined.

From (12) it follows that

$$T(x) = T_A(x) + T_B(x) = C_3 2h_A \left( 1 + \frac{h_B E_B}{h_A E_A} \right)$$

$$= C_3 2h_A \left( 1 + \frac{1}{\rho} \right)$$

and

$$2h_A C_3 = \frac{\rho}{1 + \rho} T(l).$$

Satisfying the boundary conditions, the following formulae are obtained

$$T_A = \frac{T \rho \sinh[\lambda e(l - x)] - \sinh[\lambda e(l - x)] + \sinh[\lambda e x]}{1 + \rho \sinh[\lambda e l]},$$

$$T_B = \frac{T \sinh[\lambda e l] - \sinh[\lambda e x] + \rho \sinh[\lambda e(l - x)]}{1 + \rho \sinh[\lambda e l]}.$$ (65)

The stress $T_1$ at the plate ends $x = 0$ and $x = l$ is

$$T_1(l) = \frac{\lambda e T + \rho \cosh[\lambda e l]}{1 + \rho \sinh[\lambda e l]},$$

$$T_1(0) = \frac{\lambda e T + \rho \cosh[\lambda e l] + 1}{1 + \rho \sinh[\lambda e l]}.$$ (66)

and

$$\frac{T_1(0)}{T_1(l)} = \frac{1 + \rho \cosh[\lambda e l]}{\rho + \cosh[\lambda e l]}.$$ (67)

It is seen in view of the inequality $\cosh[\lambda e l] > 1$, that for $\rho < 1$ $T_1(l) > T_1(0)$, for $\rho = 1$ $T_1(l) = T_1(0)$, for $\rho > 1$ $T_1(l) < T_1(0)$.

Fig. 15 presents the distribution of the shear stress $T_1(x)$ and of the forces $T_A, T_B(x)$ for different values of $\lambda e$. It is seen that, similarly as in the previous case, for the increasing values of $\lambda e$, the shear stress in the end zones increases and the force distribution tends to its homogeneous state in the central point portion. The critical load value at the onset of damage process now is specified as follows

for $\rho < 1$ $T_1(l) = T_c^1,$

$$T_c^1 = \frac{\lambda e T_1(l)}{\rho + \cosh[\lambda e l]}.$$ (68)

for $\rho > 1$ $T_1(l) = T_c^1,$

$$T_c^1 = \frac{T_1(l)}{\rho + \cosh[\lambda e l]}.$$ (69)
Fig. 13. (a) Load–displacement response of the bi-layer structure for elastic-softening model and parameters: $l = 0.3$ m, $\rho = 11$, $\lambda_e = 21 \cdot m^{-1}$, $\lambda_s = 5 m^{-1}$, $h = 0.5 \cdot 10^{-3}$ m, $\tau^C = 2$ MPa. Here $A+B$ indicates elastic solution for joined plates, $B$ indicates elastic response for plate B only, I presents brittle interface response, Ila and IIb delamination with damage zone, III—perfectly plastic behavior; (b) Load–displacement diagrams for rigid-softening model.

Fig. 14. The interface model (a), and lap-joint mechanical loading (b).

Fig. 15. Elastic distributions of the shear stress $\tau(x)$. (a) the forces $T_A(x)$, $T_B(x)$, (b) the critical force initiating the delamination process versus length of specimen $l$ for different values of $\lambda_e$, (c).
4.2. Damage growth analysis for $\rho = 1$

For simplicity let us consider the case of symmetric delamination of lap joint for $\rho = 1$. As in the previous case, there are consecutive stages of delamination process, namely damage zone growth, delamination combined with motion of damage zone and finally degradation of damage zone. As for $\rho = 1$, there is symmetric distribution of the shear stress $\tau$ relative to the central axis, it is sufficient to provide the solution for $0 \leq x \leq l/2$ and next to extend it by applying the symmetry condition $\tau_1(x) = \tau_1(l - x)$.

4.2.1. Damage zone growth

The critical load value at the onset of damage growth is now specified by (69), thus

$$T_c = \frac{2\tau}{\lambda_e} \frac{\sinh[\lambda_e l]}{1 + \cosh[\lambda_e l]} = \frac{2\tau}{\lambda_e} \tanh \left[ \frac{\lambda_e l}{2} \right].$$  (70)

Assume that for the elastic shear stress $\tau_1(x) > \tau_c$ two damage process zones grow from both ends. Considering the growth process at the left end, assume existence of the damage zone of length $d_l$ and the elastic zone in the central portion. From (19) the stress state is obtained in the segment for $0 \leq x \leq d_l$ after satisfying the conditions $T_A = 0$ for $x = 0$ and $\tau_1 = \tau_c^l$ for $x = d_l$:

$$\tau_1^p(x) = -\lambda_s \frac{T \sin[\lambda_s(d_l - x)]}{\cos[\lambda_s d_l]} + \tau_c^l \frac{\cos[\lambda_s x]}{\cos[\lambda_s d_l]},$$

$$T_A^p(x) = \frac{T}{2} \left( 1 - \frac{\cos[\lambda_s(x - d_l)]}{\cos[\lambda_s d_l]} \right) + \frac{\tau_c^l \sin[\lambda_s x]}{\lambda_s \cos[\lambda_s d_l]}.  \quad (71)$$

The elastic state for $d_l \leq x \leq l/2$ is determined from the relations (12) after satisfying the conditions $\tau_1^p(d_l) = \tau_c^l$ and $\tau_1^p(l/2) = 0$. It is obtained

$$\tau_1^p(x) = \tau_c^l \frac{\cosh[\lambda_e(l/2 - x)]}{\cosh[\lambda_e(l/2 - d_l)]},$$

$$T_A^p(x) = \frac{T}{2} \left( 1 - \frac{\cosh[\lambda_e(l/2 - x)]}{\lambda_e \cosh[\lambda_e(l/2 - d_l)]} \right).  \quad (72)$$

From the condition of continuity of $T_A$ for $x = d_l$, the relation specifying the length of damage zone is obtained, thus

$$\frac{T}{2} \frac{1}{\cos[\lambda_s d_l]} = \tau_c^l \frac{\tan[\lambda_s d_l]}{\lambda_s} + \frac{\tau_c^l}{\lambda_e} \tan[\lambda_e(l/2 - d_l)].$$  \quad (73)

Let us note that for $d_l = 0$ the critical load value (70) is obtained from (72). The full delamination process starts when $\tau_1^p(0) = 0$, and from (71), (72) the required load
value and the damage zone length are obtained

\[ T^0 = \frac{2\tau^c}{\lambda_s \sin[\lambda_s d_p]}, \]
\[ \tan[\lambda_s d_i] \tanh \left[ \lambda_e \left( \frac{l}{2} - d_i \right) \right] = \frac{\lambda_e}{\lambda_s}. \]  

\[ (74) \]

4.2.2. Delamination growth

Referring to Fig. 16, the delamination portion length is denoted by \( d_0 \) and the total damaged length by \( d_i \), with the damaged process zone of length \( d_p \). Now the conditions: \( \tau_1(l_0) = 0, \tau_1(d_i) = \tau_1^c \) and \( T_A(d_i) = 0 \) are applied, and for \( l_0 \leq x \leq d_i \) we have

\[ \tau_1(x) = \frac{\tau_1^c \sin[\lambda_s (x - l_0)]}{\sin[\lambda_s d_p]}, \]
\[ T_A(x) = \frac{T}{2} - \frac{\tau_1^c \cos[\lambda_s (x - l_0)]}{\lambda_s \sin[\lambda_s d_p]}. \]  

\[ (75) \]

In the elastic zone \( d_i \leq x \leq l/2 \) the stress state is presented as follows

\[ \tau_1(x) = \frac{\tau_1^c \cosh[\lambda_e (l/2 - x)]}{\cosh[\lambda_e (l/2 - d_i)]}, \]
\[ T_A(x) = \frac{T}{2} \frac{\tau_1^c \sinh[\lambda_e (l/2 - x)]}{\lambda_e \cosh[\lambda_e (l/2 - d_i)]}. \]  

\[ (76) \]

From the continuity condition of \( T_A \) for \( x = d_i \), the relation specifying the length of the process zone is obtained

\[ \cot[\lambda_s d_p] = \frac{\lambda_s}{\lambda_e} \tanh \left[ \lambda_e \left( \frac{l}{2} - d_i \right) \right]. \]  

\[ (77) \]

As two process zones meet at \( x = l/2 \), their length \( 2d_p \) then is

\[ \cos[\lambda_s d_p] = 0, \quad \lambda_s d_p = \pi/2, \quad D_p = 2d_p = \frac{\pi}{\lambda_s}. \]  

\[ (78) \]

which is a characteristic value for the bi-layer system. The corresponding load value is specified by \( (74) \) with \( d_i \) replaced by \( d_p \). As the length of process zone \( d_i \) increases, the load value decreases and its maximum corresponds to \( T^c \) or \( T^0 \).

Fig. 17 presents the progression of damage zone length \( d_i \) versus delamination part of the interface \( d_i \) for constant \( \lambda_e \) and different values of \( \lambda_s \). The first part of the delamination process is associated with exit zones from the plate ends and next with partial decohesion of the layers. Fig. 18 presents evolution of the driving force \( T \) for increasing length \( d_i \) of delamination affected interface portion. Fig. 19 presents load-displacement responses during the progressing delamination process for different interface models discussed in the paper. The final process zone degradation is discussed in detail in the next section.
4.2.3. Specific case 1: elastic-perfectly brittle bond model for $\rho = 1$: $K_{\text{eff}} = \infty$

Referring to Fig. 20, for $K_{\text{eff}} = \infty$, the damage process zone does not exist. Then the elastic stress field (76) occurs in the central part and the condition $T_A(d_l) = 0$ is valid. We obtain from (76)

$$T^C = \frac{2\tau^C}{\lambda_e} \tanh \left[ \frac{\lambda_e}{2} \left( \frac{1}{2} - d_l \right) \right]$$

(79)

and the load decreases during the whole process.

The displacement at the end of plate $A$ is

$$u^p_A(l) = \frac{\tau^C}{4} \lambda_e \left[ 4E_A h_A + d_i K_{\text{eff}} \left( 3d_l + \rho (2l - d_l) \right) \right] - 2K_{\text{eff}} K^C \left( 2d_l + l \rho \right) \tanh \left[ \frac{\lambda_e}{2} \left( d_l - \frac{1}{2} \right) \right]$$

(82)

**Box III.**

![Elastic–brittle interface model](image)

Fig. 20. Elastic–brittle interface model (a); shear stress field for growth of delamination in lap-joint system, (b).

4.2.4. Specific case 2: elastic-perfectly plastic model, $K_{\text{eff}} = 0$

As the plastic zone $\tau^p = \tau^C$ is developed in the portion $0 \leq x \leq l_d$ and $T(l_d) = \tau^C l_d$, Fig. 21, from (76) we obtain

$$T^C = 2\tau^C \left[ d_l + \frac{\tanh \left[ \lambda_e \left( \frac{1}{2} - d_l \right) \right]}{\lambda_e} \right]$$

(81)

and the displacement at the end of plate $A$ is given in **Box III.**

4.3. Special case 3: rigid-softening model

4.3.1. Growth of process zones

For $\lambda_e \to \infty$ two damage process zones are initiated after application of the increasing load $T$, Fig. 22. Denote the lengths of the right and left process zones by $d^p_{r}$ and $d^p_{l}$. Along the joint axis we have three segments $P_l - H - P_r$, with the homogeneous stress state in the central $H$ segment. The stress field in the process zone is expressed
in (19), thus

\[ u_I = D_1 \sin[\lambda_s x] + D_2 \cos[\lambda_s x] + \frac{\tau^c I}{K_s^{ef}}, \]

\[ \tau_I = -K_s^{ef}(D_1 \sin[\lambda_s x] + D_2 \cos[\lambda_s x]), \]

\[ T_A = \frac{K_s^{ef}}{\lambda_s} (D_1 \cos[\lambda_s x] - D_2 \sin[\lambda_s x]) + D_3, \]

\[ T_B = \frac{K_s^{ef}}{\lambda_s} (-D_1 \cos[\lambda_s x] + D_2 \sin[\lambda_s x]) - \frac{E_B}{E_A} D_3, \]

where, \( D_1, D_2, D_3 \) are the integration constants.

In order to calculate the displacement fields \( u_B(x) \) and \( u_A(x) \), we start from the supported left end of plate B and determine its displacement in the segment \( P_1 : 0 \leq x \leq d^I_p, \) thus

\[ u_B(x) = \int_0^x \frac{T_B(x)}{2h_B E_B} \, dx \]  \hspace{1cm} (84)  

and next the displacement \( u_A(x) \) is determined as \( u_A = u_I + u_B \). The integration constants of (83) are specified from the boundary conditions

\[ x = l_0 : \ u_I = u'_I = 0 \]
\[ x = l : \ T_A = T, \quad T_B = 0 \]
\[ x = d^I_p : \ u_I = u'_I = 0 \]
\[ x = 0 : \ T_A = 0, \quad T_B = T. \]  \hspace{1cm} (85)
In the left damage zone \( P_l, 0 \leq x \leq d_p^l \) we have

\[
\begin{align*}
\upsilon_l &= \frac{\tau_c}{K^s}(1 - \cos[\lambda_s(d_p^l - x)]), \\
\tau_l &= \frac{\tau_c}{\lambda_s} \cos[\lambda_s(d_p^l - x)], \\
T_A &= \frac{\tau_c}{\lambda_s} \left( \sin[\lambda_s(d_p^l - x)] - \cos[\lambda_s(d_p^l - x)] \right), \\
T_B &= T - \frac{\tau_c}{\lambda_s} \left( \sin[\lambda_s(d_p^l - x)] - \sin[\lambda_s(d_p^l - x)] \right),
\end{align*}
\]

and the displacement field takes the form

\[
\begin{align*}
\upsilon_B(x) &= \frac{1}{2h_B E_B} \left[ T x - \frac{\tau_c}{\lambda_s} \left( \sin[\lambda_s d_p^l] x \\
&\quad - \cos[\lambda_s(d_p^l - x)] + \cos[\lambda_s d_p^l] \right) \right] + \frac{\tau_c}{K^s}(1 - \cos[\lambda_s(d_p^l - x)]), \\
\upsilon_A(x) &= \frac{1}{2h_B E_B} \left[ T x - \frac{\tau_c}{\lambda_s} \left( \sin[\lambda_s d_p^l] x \\
&\quad - \cos[\lambda_s(d_p^l - x)] + \cos[\lambda_s d_p^l] \right) \right] + \frac{\tau_c}{K^s}(1 - \cos[\lambda_s(d_p^l - x)]).
\end{align*}
\]

In the homogeneous zone \( H, d_p^l \leq x \leq l \) we have

\[
\begin{align*}
\upsilon_l &= \upsilon_0 = 0, \\
T_A &= T, \\
T_B &= T - \frac{1}{1 + \rho},
\end{align*}
\]

and displacement field is

\[
\begin{align*}
\upsilon_B(d_p^l) &= \upsilon_0(d_p^l), \\
\upsilon_A(d_p^l) &= \upsilon_0(d_p^l), \\
\upsilon_B(x) &= \upsilon_0(x), \\
\upsilon_A(x) &= \upsilon_0(x).
\end{align*}
\]

In the right damage zone \( P_r, l \leq x \leq l \) we have

\[
\begin{align*}
\upsilon_l &= \frac{\tau_c}{K^s}(1 - \cos[\lambda_s(x - l)]) , \\
\tau_l &= \frac{\tau_c}{\lambda_s} \cos[\lambda_s(x - l)], \\
T_A &= T - \frac{\tau_c}{\lambda_s} \left( \sin[\lambda_s d_p^r] - \sin[\lambda_s(x - l)] \right), \\
T_B &= \frac{\tau_c}{\lambda_s} \left( \sin[\lambda_s d_p^r] - \sin[\lambda_s(x - l)] \right).
\end{align*}
\]

Finally, we have

\[
\begin{align*}
\upsilon_B(l) &= \upsilon_B(d_p^r) + \frac{T}{2h_B E_B} \left( l - d_p^r \right), \\
\upsilon_A(l) &= \upsilon_B(l).
\end{align*}
\]

The force continuity conditions at the interfaces between the damage zones \( P_l, P_r \) and the homogeneous stress zone \( H \) provide two relations

\[
\begin{align*}
\upsilon_B(l) &= \upsilon_B(l_e) + \frac{1}{2h_B E_B} \left[ \frac{\tau_c}{\lambda_s} \left( \sin[\lambda_s d_p^l] x - l_e \right) \\
&\quad \cos[\lambda_s(x - l_e)] \right], \\
\upsilon_A(l) &= \upsilon_B(l_e), \\
&\quad + \frac{1}{2h_B E_B} \left[ \frac{\tau_c}{\lambda_s} \left( \sin[\lambda_s d_p^r] d_p^r + \cos[\lambda_s d_p^r] \right) \right] .
\end{align*}
\]

And finally

\[
\upsilon_A(l) = \upsilon_B(l) + \frac{\tau_c}{K^s}(1 - \cos[\lambda_s d_p^r]).
\]

Continuity condition at the interface between the damage zones \( d_p^l, d_p^r \) and the homogeneous stress zone \( H \) provide two relations:

\[
\begin{align*}
T_A(l_e) &= \frac{\rho}{1 + \rho} \Rightarrow T_A(l_e) = T - \frac{\tau_c}{\lambda_s} \sin[\lambda_s d_p^r] \\
&= \frac{\rho}{1 + \rho} T, \\
T_A(d_p^r) &= \frac{\tau_c}{\lambda_s} \sin[\lambda_s d_p^r] = \frac{\rho}{1 + \rho} T.
\end{align*}
\]

The lengths of process zones \( d_p^l \) and \( d_p^r \) are now specified by the formulae

\[
\begin{align*}
\begin{cases}
\lambda_s & \frac{1}{\tau_c} \frac{T}{1 + \rho} = \sin[\lambda_s d_p^l] \\
\lambda_s & \frac{\rho}{\tau_c} \frac{T}{1 + \rho} = \sin[\lambda_s d_p^r]
\end{cases} \Rightarrow \rho = \frac{\sin[\lambda_s d_p^l]}{\sin[\lambda_s d_p^r]}.
\end{align*}
\]

Thus, for \( \rho > 1 \) there is \( d_p^l > d_p^r \) and for \( \rho < 1 \) there is \( d_p^l < d_p^r \). In our analysis we consider the case \( \rho < 1 \) and \( d_p^l < d_p^r \). The initiation of full delamination occurs when

\[
\tau_1(l) = \frac{\tau_c}{\lambda_s} \cos[\lambda_s d_p^r] = 0.
\]

The length of the process zone then reaches its critical value

\[
\lambda_s d_p^r = \frac{\pi}{2}, \quad d_p^r = \frac{\pi}{2} \frac{1}{\lambda_s},
\]

and

\[
T = T_A = \frac{\tau_c}{\lambda_s} (1 + \rho).
\]

\[
\begin{align*}
\tau_1(0) &= \frac{\tau_c}{\lambda_s} \cos[\lambda_s d_p^r] = \tau_1 \sqrt{1 - \rho^2},
\end{align*}
\]

4.3.2. Process zone propagation

For \( \rho < 1 \), the full delamination starts to grow from the right end at constant force value expressed by (99) and for increasing displacement of plate A at \( x = l \). The left process zone remains stagnant since \( \tau_1(0) > 0 \) remains fixed. Referring to Fig. 23, the stress state is specified by
the same formulae (91) with replacement of \( I \) by \( l_p \). The displacement field in consecutive zones is expressed by the following formulae:

In left damage zone (\( P_L \)), \( 0 \leq x \leq d_p^l \):

\[
\begin{align*}
\tau_B(x) &= \frac{1}{2h_B E_B} \left[ Tx - \frac{\tau_1^C}{\lambda_s} \left( \sin[\lambda_s d_p^l] x \\
&- \frac{\cos[\lambda_s (d_p^l - x)]}{\lambda_s} \right) \right],
\end{align*}
\]

\( \tau_1(x) = \frac{\tau_1^C}{K_{st}} (1 - \cos[\lambda_s (d_p^l - x)]) \),  
\( u_A(x) = u_l(x) + u_B(x) \).

In the plate center zone (\( H \), \( d_p^l \leq x \leq l_c \)):

\[
\begin{align*}
\tau_B(d_p^l) &= \frac{1}{2h_B E_B} \left[ T d_p^l - \frac{\tau_1^C}{\lambda_s} \left( \sin[\lambda_s d_p^l] d_p^l \\
&- \frac{1}{\lambda_s} + \cos[\lambda_s d_p^l] \right) \right],
\end{align*}
\]

\( u_A^c(d_p^l) = u_B^c(d_p^l) \),  
\( u_A^c(x) = u_B^c(x) \),  
\( u_B^c(x) = u_B^c(d_p^l) + \frac{T}{2h_B E_B} \frac{x - d_p^l}{1 + \rho} \).

In right damage zone (\( P_R \), \( l_c \leq x \leq l_p \)):

\[
\begin{align*}
\tau_B(l_c) &= \frac{1}{2h_B E_B} \left[ \frac{\tau_1^C}{\lambda_s} \left( \sin[\lambda_s d_p^l] (x - l_c) \\
&+ \frac{\cos[\lambda_s (x - l_c)]}{\lambda_s} \right) \right],
\end{align*}
\]

\( u_B(l_c) = u_B^c(l_c) + \frac{T}{2h_B E_B} \frac{l_c - d_p^l}{1 + \rho} \),  
\( u_B(x) = u_B^c(l_c) + \frac{1}{2h_B E_B} \left[ \frac{\tau_1^C}{\lambda_s} \left( \sin[\lambda_s d_p^l] (x - l_c) \\
&+ \frac{\cos[\lambda_s (x - l_c)]}{\lambda_s} \right) \right] \),  
\( u_B(l_p) = u_B^c(l_c) + \frac{1}{2h_B E_B} \frac{\tau_1^C}{\lambda_s} d_p^l \),

and

\( u_A(l_p) = u_B(l_p) + \frac{\tau_1^C}{\lambda_s} \).

In the delaminated zone (\( D \), \( l_p \leq x \leq l \)):

\[
\begin{align*}
\tau_B(x) &= \frac{\tau_1^C}{K_{st}} (1 - \cos[\lambda_s (x - d_p^l)]) \\
&+ u_1 \cos[\lambda_s (x - d_p^l)],
\end{align*}
\]

where \( u_1 \) is the growing displacement discontinuity at \( x = d_p^l \). The stress state in the damage process zone is now expressed as follows for \( 0 \leq x \leq l_p \):

\[
\begin{align*}
\tau_1(x) &= (\tau_1^C - K_{st} u_1) \cos[\lambda_s (x - d_p^l)] ,
\end{align*}
\]

\( T_A(x) = \frac{\tau_1^C - K_{st}^e u_1}{\lambda_s} \left( \sin[\lambda_s d_p^l] - \sin[\lambda_s (d_p^l - x)] \right) \).  
\( T_B(x) = T - \frac{\tau_1^C - K_{st}^e u_1}{\lambda_s} \left( \sin[\lambda_s d_p^l] - \sin[\lambda_s (d_p^l - x)] \right) \).

Note that \( \sin[\lambda_s d_p^l] = 1 \) and \( \sin[\lambda_s (d_p^l - x)] = \rho \), so for \( x = l_p \) we have

\[
\begin{align*}
\tau_1(l_p) &= \frac{\tau_1^C}{K_{st}^e} ,
\end{align*}
\]

\( T_A(l_p) = \frac{\tau_1^C - K_{st}^e u_1}{\lambda_s} (1 + \rho) = T \),  
\( T_B(l_p) = 0 \).

The degradation process proceeds for the decreasing load and increasing \( u_1 \) until \( K_{st}^e u_1 = \tau_1^C \) and \( T_A = 0 \). The displacement field during the process zone degradation is

\[
\begin{align*}
\tau_B(x) &= \frac{1}{2h_B E_B} \left[ \frac{\tau_1^C - K_{st}^e u_1}{\lambda_s} \left( x + \frac{\cos[\lambda_s (d_p^l - x)]}{\lambda_s} \right) \\
&- \frac{\cos[\lambda_s d_p^l]}{\lambda_s} \right] ,
\end{align*}
\]

\( u_A(x) = u_B(x) + u_1(x) \)  

Fig. 23. The interface model (a), the lap-joint mechanical loading (b).

4.3.3. Degradation of process zones
When the left and right damage process zones join each other at \( x = d_p^l \), their total length equals \( l_p = d_p^l + d_p^r \). Fig. 24. In the consecutive damage process the length \( l_p \) is assumed to remain fixed and the displacement discontinuity \( u_1 \) to increase in the zone \( 0 \leq x \leq l_p \), thus
and the displacement of plate A at \(x = l_p\) equals

\[
u_A(l_p) = \frac{1}{2h_AE_A} \left[ \frac{\tau^C}{\lambda_s} (1 + \rho) + (\tau^C - K^C_{ef} u_1) \right] \times \left( - \frac{1 - \rho^2}{\lambda_s} \right).
\]

The inequality (117) specifies the length of joint for which the snap back response and uncontrolled dynamic failure occurs. On the other hand, for \(\chi_l < \chi_s\), the displacement controlled static failure process will proceed.

The inequality (117) specifies the length of joint for which the snap back response and uncontrolled dynamic failure occurs. On the other hand, for \(\chi_l < \chi_s\), the displacement controlled static failure process will proceed.

The load–displacement diagram is identical to that presented in Fig. 14(b). Note that

\[
l_p = d^f_p + d^t_p = \frac{\beta}{\lambda_s}, \quad \beta = \frac{\pi}{2} + \arcsin[\rho].
\]

Fig. 25 illustrates the load–displacement response for joints of different lengths. The analysis presented can be referred to laminated or joined composite structure and to geomechanical layer interaction of rock or soil strata systems. The specific cases treated analytically in the paper well illustrate the effect of material parameters \(K^C_{ef}, K^C_{ls}, \tau^C\) on damage growth and delamination process. The elastic response is controlled by the parameter \(\lambda_e\) defined by Eq. (9) as the square root of ratio of the effective stiffness moduli of bonding layer to the stiffness modulus of interacting plates. Similarly, the stress distribution in the damage process zone and its length depend essentially on the softening stiffness parameter \(\lambda_s\) defined by (16). The alternative forms of these parameters are

\[
\lambda_e = (K^C_{ef}C_p) = \left( \frac{K^C_{ef}}{S_p} \right) = \left( \frac{K^C_{ef}}{2h_AE_A \rho} \right)^{\frac{1}{2}}
\]

\[
\lambda_s = (K^C_{ls}C_p) = \left( \frac{K^C_{ls}}{S_p} \right) = \left( \frac{K^C_{ls}}{2h_AE_A \rho} \right)^{\frac{1}{2}}
\]

where, \(C_p\) denotes the plates compliance modulus and \(S_p = C_p^{-1}\) denotes the stiffness modulus. It was
demonstrated that $\lambda_c$ affects essentially the character of stress distribution. For large value of $\lambda_c$ the transition to the uniform stress state occurs along a very short distance from the stress inhomogeneity. For rigid softening model, we have $\lambda_c \rightarrow \infty$. The damage and delamination growth then differ essentially from that proceeding for the elastic-softening model. The damage process zones of the constant length $d_p = \frac{\tau_{\text{eff}}}{\tau_c}$ are generated at the plate ends and next translate along the contact interface at constant load, $T_l$. For the lap joint we have

$$T_l = \frac{\tau_{\text{eff}}}{\lambda_s} (1 + \rho) = \frac{\tau_c (1 + \rho)}{\left( K_{\text{eff}}^C \rho \right)^{\frac{1}{2}}} \left( 1 + \rho \right). \quad (119)$$

As $T_l$ represents the limit load, it is seen that its value does not depend on the critical stress $\tau_c$, but on the fracture energy and the plate compliance modulus. The scale effect is clearly demonstrated by specifying the average shear stress per unit length

$$\tau_{\text{eff}} = \frac{\tau_c}{\rho} = \frac{\tau_{\text{eff}} (1 + \rho)}{\tau_c} \left( \frac{1}{\lambda_s} \right) \quad (120)$$

which decreases with the length of plates.

The value of $\lambda_s$ is also very important for specifying the limit load. The solutions for cases $\lambda_s = 0$ and $\lambda_s = \infty$ clearly illustrate the load dependence on this parameter. The reference is made to experimental testing of strength of joined composite elements by Ferrarisi et al.$^{53}$ and Ventrella et al.$^{54}$ where the apparent shear strength of lap joints was determined, but the intrinsic parameters affecting adhesive joint thickness on the shear strength was examined in torsion, four point bending, single and double-lap tests. In fact, the varying joint strength may affect the elastic and softening moduli $K_{\text{eff}}^C$, $K_{\text{eff}}$ and parameters $\lambda_c, \lambda_s$ controlling the interface stress distribution. The apparent joint strength measured as the averaged shear stress referred to the joint area is expressed by (120) for the rigid-softening model. In view of (119) if depends on the square root of the ratio of fracture energy and plates compliance. It is also inversely proportional to the joint length, $l$, thus demonstrating the scale effect. In the present paper it was assumed that the joint length is larger than the process zone length, so the process zone translation and length evolution can occur. The finite element elastic analysis presented$^{53,54}$ provides some information on the shear and peel stress distributions, however it is not satisfactory to clarify the mode of progressive damage and failure.

The elastic-softening model response well illustrates the evolution of the damage process zone. In the first example of substrate plate loading, the damaged zone length $d_p$ decreases in the progressive delamination process. In the second example of lap joint the length of two zones $d_p$ increases before their final degradation stage.

Let us refer to the numerical finite element analysis presented in papers by Alfano and Crisfield,$^{55}$ Goyal et al.$^{56}$ and Turon et al.$^{57}$ In Ref. [55] the interface constitutive model was formulated assuming linear elastic and softening responses, with numerical analysis of the opening failure mode of two layers under transverse loads. In Ref. [56] the delamination process of precracked beams in opening, shear and mixed modes was studied assuming the model with an exponential softening rule. In Ref. [57] the material parameter selection and the specification of the cohesive zone length was discussed for a bilinear elastic-softening model, identical to that used in this paper. The illustrative numerical examples of the layered composite delamination were next presented. The effective elastic stiffness modulus $K_{\text{eff}}$ was recommended in Ref. [57] to be specified by the formula

$$h \tau_{\text{eff}} = \alpha E,$$  \hspace{1cm} (121)

where, $h$ and $E$ are the averaged thickness and elastic modulus of bonded plates and $\alpha \geq 50$ is the proposed value of stiffness parameter, assuring sufficiently high value of $K_{\text{eff}}$. The values of $K_{\text{eff}}$ based on literature data and resulting from (121) are in the range $K_{\text{eff}} = 4 \times 10^5 \div 10^8$ N mm$^{-1}$. The length of the process zone was assumed as constant in the numerical analysis and specified from the Irwin formula

$$d_{\text{eq}} = \frac{M E f}{\tau_{\text{eff}}} \quad (122)$$

where $M$ is the parameter varying in the range $0.2 \div 1.0$ depending on the geometric joint parameters. This formula can be now compared with the cohesive zone length $d_p$ specified by (98) for the rigid-softening model. In view of (118), we have

$$d_p = \frac{\pi}{2} \frac{1}{\lambda_s} = \frac{\pi}{2} \frac{1}{\left( K_{\text{eff}}^C \rho \right)^{0.5}} = \frac{\pi}{\tau_c} \frac{E h_A g_I}{\tau_{\text{eff}}} \frac{0.5}{1 + \rho} \quad (123)$$

and the value of $d_p$ specifies an upper bound on the length of process zone for an elastic-softening model.

Following the parameter data in Ref. [58] for a carbon fibre reinforced epoxy laminate

$$K_{\text{eff}}^C = 4 \times 10^5 \quad \text{N/mm}^2, \quad E = 150 \text{ GPa},$$

$$G_f = 0.352 \quad \text{N/mm}, \quad \tau_c = 60 \quad \text{N/mm}^2$$

and assuming the bonded plate thickness $h_A = h_B = 5$ mm, we obtain from (118)–(123)

$$\lambda_s = 82.5 \quad \text{m}, \quad \lambda_c = 730 \quad \text{cm}, \quad K_{\text{eff}} = 5113 \quad \text{N/mm}^3,$$

$$d_{\text{eq}} = 11.73 \quad \text{mm}, \quad d_p = 19.03 \quad \text{mm}.$$

The large value of $\lambda_c$ indicates that there is very fast transition from the inhomogeneous stress state in the process zone to the uniform stress state and the solution for the rigid-softening model well describes the damage process. Then the developed process zone length $d_p = \frac{\tau_{\text{eff}}}{\tau_c}$ remains constant during zone propagation. The similar conclusion was reached in the numerical analysis in Ref. [58], indicating that the constant length assumption of the process zone is legitimate and the maximal load at failure does not depend on the $\tau_{\text{eff}}$ value. Note that $d_p$ predicted by (123) is larger than $d_{\text{eq}}$ predicted by the Irwin formula, as it constitutes the upper bound on the cohesive zone length.

Referring to rock mechanics the mode of localized shear failure is fundamental in the compressive stress regime.
The delamination process of bi-layer system can occur when one layer deforms excessively relative to adjacent layers under increasing stress and induces large shear stress at the interface. The first example of substrate loading can then provide the insight into the delamination mode. The elastic stiffness of interface bond now depends on the normal pressure. Assume the following parameters following Morache et al.²⁸

\[
K_t^{\text{eff}} = 20 \cdot \frac{\text{MPa}}{\text{mm}}, \quad E = 70 \text{ GPa},
\]

\[
G_f = 0.07 \cdot \frac{\text{N}}{\text{mm}^2}, \quad \tau_f = 5 \text{ MPa}, \quad h = 50 \text{ mm}.
\]

Now the calculated model parameters and process zone lengths are

\[
\lambda_e = 2.39 \cdot \frac{1}{\text{m}}, \quad \lambda_5 = 7.14 \cdot \frac{1}{\text{m}}, \quad d_{cz} = 0.156 \text{ m},
\]

\[
d_p = 0.219 \text{ m}.
\]

The extended analysis of delamination process of a plate under lateral compressive stress with account for damage and sliding friction effect was presented by Białas and Mróz.²⁴,²⁵

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