**CONSTITUTIVE DSA FOR HUBER-MISES PLASTICITY**

**Derivative w.r.t. material parameters.**

In the paper we consider the constitutive model consisting of Hooke’s elasticity and Huber-Mises plasticity with nonlinear isotropic hardening of the saturation type, \( \kappa(\varepsilon^p) = \sigma_{00} + \kappa_1 \varepsilon^p + (\sigma_{01} - \sigma_{00})(1 - e^{-a \varepsilon^p}) \), and a linear kinematic hardening function \( H_s(\varepsilon^p) = H_0 + H' \varepsilon^p \). Hence, the set of material parameters is \( m = \{ E, \nu, \sigma_{00}, \kappa_1, \sigma_{01}, a \} \).

For a design parameter \( h \) and material parameters \( m \) depending on \( h \), i.e. \( m = m(h) \), we have \( D_h(\cdot) = D_m(\cdot) D_h m \), where \( D_h m \) is specified additionally, and it suffices to consider derivatives w.r.t. \( m \), only. For some \( A(m,B(m)) \), the total design derivative of \( A \) with respect to \( m \) is written as \( D_m A = \frac{\partial A}{\partial m} + A_B \frac{\partial B}{\partial m} B \), where \( \frac{\partial A}{\partial m} \) is the explicit derivative.

**Update of design derivatives of state variables.**

Let us denote by \( s \) the state variables which are updated incrementally, stored and retrieved as constitutive history data, \( s = \{ \Sigma, e^p, \alpha^p, e^p, \varepsilon_3, \psi_R \} \), where \( \psi_R \) are rotational parameters of the constitutive algorithm. The algorithm for \( s_{n+1} \) can be written as \( s_{n+1} = s_{n+1}(m, s_n(m), q_{n+1}(m), q_n(m)) \), and upon differentiation w.r.t. \( m \) we obtain the update formula

\[
D_m s_{n+1} = \frac{\partial s_{n+1}}{\partial m} D_m s_n + \frac{\partial s_{n+1}}{\partial q_{n+1}} D_m q_{n+1} + \frac{\partial s_{n+1}}{\partial q_n} D_m q_n.
\]

The derivatives, \( D_m s_n \) and \( D_m q_n \) are known as they are history variables, and \( D_m q_{n+1} \) in explicit dynamics is predicted. The derivatives, \( \frac{\partial s_{n+1}}{\partial m}, \frac{\partial s_{n+1}}{\partial q_n}, \frac{\partial s_{n+1}}{\partial q_{n+1}}, \frac{\partial s_{n+1}}{\partial q_{n+1}}, \frac{\partial s_{n+1}}{\partial q_n} \), must be explicitly calculated. This update includes the design derivative of back-rotated Kirchhoff stress \( \Sigma \) which belongs to \( s \).

**Design derivative of stress update algorithm for finite rotation shells.**

The algorithmic approach to plasticity for finite-rotation shells follows this of [1], i.e. the constitutive equations are written for the back-rotated Kirchhoff stress \( \Sigma \), and equations analogous to those for the small deformation case are used. The constitutive model is the 3D Huber-Mises material with nonlinear isotropic/linear kinematic hardening. The plane stress constraints are directly incorporated into constitutive equations, as in [3] or [2], i.e. the deviatoric stress is parameterized by the plane stress components. The resulting yield surface is ellipsoidal, and not a radial return algorithm but a return map algorithm, with a constitutive Newton loop, must be used. This model is applied to shell laminas.

The algorithm to calculate the 2nd Piola-Kirchhoff stress \( \Sigma \) can be written as \( \Sigma_{n+1} = \Sigma_{n+1}(m, s_n(m), q_{n+1}(m), q_n(m)) \) and upon differentiation w.r.t. \( m \) we obtain

\[
D_m \Sigma_{n+1} = \frac{\partial \Sigma_{n+1}}{\partial m} D_m s_n + \frac{\partial \Sigma_{n+1}}{\partial q_{n+1}} D_m q_{n+1} + \frac{\partial \Sigma_{n+1}}{\partial q_n} D_m q_n,
\]

where \( \{ D_m s_n, D_m q_{n+1}, D_m q_n \} \) are known, and the derivatives, \( \{ \frac{\partial \Sigma_{n+1}}{\partial m}, \frac{\partial \Sigma_{n+1}}{\partial s_n}, \frac{\partial \Sigma_{n+1}}{\partial q_{n+1}}, \frac{\partial \Sigma_{n+1}}{\partial q_n} \} \) can be explicitly calculated.

**DESIGN DERIVATIVE OF EXPLICIT DYNAMICS ALGORITHM**

The equations of dynamics for shells are complicated due to rotational inertia terms and parametrization of rotations (we use the canonical rotation vector). The design differentiation of the explicit dynamics algorithm yields, among others, the following equation

\[
D_m q_{n+1} = M^{-1}[ -D_m f_{n+1} - D_m c(\omega) - C D_m q_{n+1/2}],
\]

where \( q = \{ x_0, \psi_f \} \) and \( \psi_f \) is a canonical rotation vector. The design derivative of the internal force \( f \) is as follows

\[
D_m f = \frac{\partial}{\partial m} \int_V B^T S \, dV = \int_V \left( D_m B^T S + B^T D_m S \right) \, dV,
\]
where $S$ is the 2nd Piola-Kirchhoff stress, $E$ is the Green strain. The kinematical operator $B = \partial E / \partial q$ depends on $q$ for finite rotation shells, hence its design derivative is

$$D_m B = \frac{\partial B}{\partial q} \frac{dq}{dm} = B q D_m q,$$

(5)

where $B q = \partial^2 E / \partial q^2$ is a $(n_s \times n_{dof} \times n_{dof})$ matrix, and $n_s$ is a number of strain components.

**Design derivative of inertial term.**

The inertial term $c(\omega) = I_\omega \dot{\omega} \times (\Pi \omega)$ depends on $\psi_{n+1}$ and $\dot{\psi}_{n+1/2}$, because $\omega \approx T(\psi_{n+1}) \dot{\psi}_{n+1/2}$. Hence, the design derivative of it is computed as follows

$$D_h c(\omega) = \frac{\partial c}{\partial \psi_{n+1}} D_h \psi_{n+1} + \frac{\partial c}{\partial \dot{\psi}_{n+1/2}} D_h \dot{\psi}_{n+1/2}.$$

(6)

**NUMERICAL EXAMPLE: ELASTIC-PLASTIC COMPRESSION OF RECTANGULAR RAIL**

This example tests the DSA procedure in a realistic example of compression of a steel thin-walled rail, a quarter of which is shown in Fig. 1. To avoid contact modelling and enable clear conclusions regarding accuracy of the DSA, computations are performed for first few milliseconds only.

The load is a 270 kg mass with the initial velocity $v_3 = -7.7778$ m/s, applied at the rail end $x_3 = 370$ mm. The material data $E = 210000$, $\nu = 0.3$, $\sigma_y1 = 250$, $\sigma_y0 = 250$, $H' = 2000$, $a = 1$, $H_{kin} = 0$, thickness $h = 1.47$.

The shell element is a 4-node bilinear with 6 dofs/node, based on Reissner’s kinematics. The drilling rotation is included on use of the drilling RC-equation by the penalty method. The rotations are parameterized by a canonical rotation vector. The Green strain and 2nd Piola-Kirchhoff stress are used as a work-conjugate pair. For the transverse shear the ANS (Bathe, Dvorkin) approximation is used. A numerical integration with 2x2 Gauss points in lamina, and 5-point Simpson rule over thickness is used.

Design derivatives are computed analytically and by FD method; the derivatives of rotation at point P w.r.t. yield stress $\sigma_{y0}$ is given in Fig. 1b. As it can be seen, the finite-difference DSA results coincide with the analytical DSA results, which proves that, despite a substantial complexity of the algorithms, the analytical DSA equations are correct.

![Figure 1](image_url)

**Figure 1.** Elastic-plastic compression of rail: (a) geometry, (b) design derivatives of rotation at point P w.r.t. yield stress $\sigma_{y0}$.

**Acknowledgement.** This research was partially supported by: EC Growth Project G3RD-CT-2000-00276, and the Polish Committee for Scientific Research (KBN) Grant 7T11F01921.

**References**


