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The short-wavelength instability of magnetically buoyant layer

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Abstract. We revisit the problem introduced by Gilman (1970) and Acheson (1979) of linear stability of a plane layer of compressible fluid permeated by a horizontal magnetic field of magnitude decreasing with height with respect to short-wavelength two-dimensional perturbations varying in the directions perpendicular to the applied field. We show, that in the limit of large horizontal wave numbers the perturbations become strongly localised in the vertical direction. The motiavtion for this study is of astrophysical nature and comes from the common belief, that the magnetic buoyancy effects produce short-wavelength instabilities in the solar tachocline. We analyse the solar tachocline parameter regime to speculate about the strength of the magnetic field at the base of the solar convective zone and the time scales of the field variations induced by the magnetic buoyancy instability on the Sun.

1. Introduction

The pioneering works on the magnetic buoyancy instability of Gilman (1970) and Acheson (1979) revealed some peculiarities of the short-wavelength limit, i.e. that the growth rate of the instability must necessarily vary with height in the buoyant layer. The solution to this difficulty lies in localisation of the eigenmodes, which in the limit of large horizontal wave number (corresponding to the direction perpendicular to the applied field) are strongly localised in the vertical direction and in the actual limit of $k \longrightarrow \infty$ there exist modes associated with different eigenvalues corresponding to every value of the vertical coordinate z within the layer. A detailed analysis of this problem is given in Mizerski *et al.* (2011). Similar behaviour has been found recently in the study of the inertial instability in geophysical flows by Griffiths (2008).

2. The short-wavelength limit

By choosing the layer depth d, the free fall time $\sqrt{(d/g)}$ and the free fall velocity $\sqrt{gd}$ as units of length, time and velocity respectively (where g is the acceleration of gravity, assumed constant) the set of hydro-magnetic equations governing the evolution of the velocity ($\mathbf{u}$), temperature (T), density (ρ) and magnetic field ($\mathbf{B}$) takes the following form

$$\rho\frac{\partial \mathbf{u}}{\partial t} = -\rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u} - \mathcal{P}\nabla p - \rho\hat{\mathbf{e}}_z + \Lambda\left(\nabla\times\mathbf{B}\right)\times\mathbf{B} - \mathcal{U}_\nu E^{-1}\rho\hat{\mathbf{e}}_z\times\mathbf{u} + \mathcal{U}_\nu\left[\nabla^2\mathbf{u} + \frac{1}{3}\nabla\left(\nabla\cdot\mathbf{u}\right)\right], \quad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{u} - \mathbf{B}\nabla\cdot\mathbf{u} + \mathcal{U}_\eta \nabla^2 \mathbf{B}, \tag{2}$$

$$\begin{aligned}\rho\frac{\partial T}{\partial t} &= -\rho\mathbf{u}\cdot\nabla T + \gamma\mathcal{U}_\kappa\nabla^2 T - \frac{\gamma-1}{\alpha}p\nabla\cdot\mathbf{u} + \frac{\gamma-1}{\alpha}\mathcal{U}_\eta\frac{\Lambda}{\mathcal{P}}(\nabla\times\mathbf{B})^2 \\ &\qquad + \frac{\gamma-1}{\alpha}\frac{\mathcal{U}_\nu}{\mathcal{P}}\left[2e_{ij}e_{ij} - \frac{2}{3}(\nabla\cdot\mathbf{u})^2\right],\end{aligned} \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0\,, \qquad p = \alpha\rho T, \tag{4}$$

where we have assumed that the dynamic viscosity $\nu\rho$ is constant, e_{ij} is the symmetric part of the velocity gradient tensor, and:

$$\mathcal{P} = \frac{p_s}{\rho_s g d}, \qquad \alpha = \frac{\rho_s R T_s}{p_s}, \tag{5}$$

$$\Lambda = \frac{B_s^2}{\mu_0\rho_s g d} = \beta^{-1}\mathcal{P}, \qquad E^{-1} = \frac{2\Omega d^2}{\nu}, \tag{6}$$

$$\mathcal{U}_\nu = \frac{\nu}{d\sqrt{gd}}, \qquad \mathcal{U}_\eta = \frac{\eta}{d\sqrt{gd}}, \qquad \mathcal{U}_\nu = \frac{\kappa}{d\sqrt{gd}}. \tag{7}$$

In the above E is the Ekman number, $\gamma = c_p/c_v$ is the ratio of specific heats, $R = c_p - c_v$ is the gas constant, μ_0 is the magnetic permeability of vacuum, ν, η and κ are the kinematic viscosity, magnetic diffusivity and thermal diffusivity respectively, p_s, ρ_s, T_s, B_s are the scales of pressure, density, temperature and the magnetic field respectively and $\beta = \mu_0 p_s/B_s^2$ is the plasma β, i.e. the ratio of the gas pressure to the magnetic pressure. The quantities $\mathcal{U}_i$ denote ratios of a certain velocity scale associated with the subscript 'i' with the free fall velocity. We study the linear stability of the following basic, stationary state

$$\mathbf{B}_0 = B_0(z)\,\hat{\mathbf{e}}_x = [1 - \Gamma + \Gamma z]\,\hat{\mathbf{e}}_x, \tag{8}$$

$$\Gamma = 1 - \frac{B_0^{bottom}}{B_0^{top}} < 0, \tag{9}$$

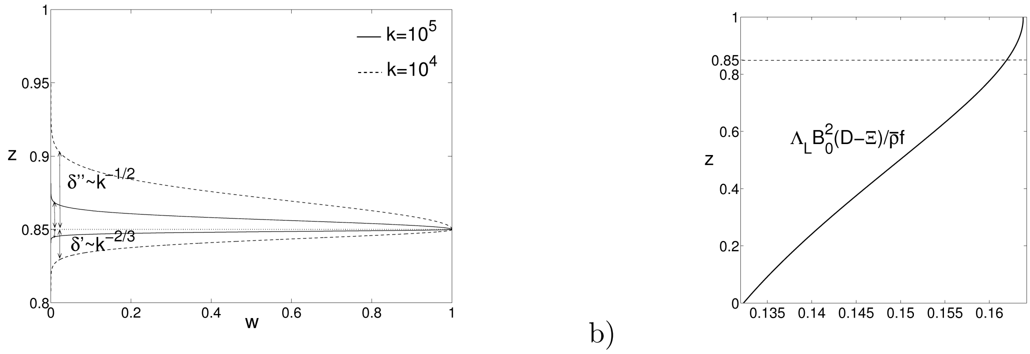
where Γ is the negative field gradient which makes the fluid layer magnetically buoyant and B_0^{top} and B_0^{bottom} are the magnitudes of the field at the top and bottom of the tachocline respectively. The value of B_0^{top} is used to non-dimensionalise the magnetic field and hence this is the quantity B_s used in the definition of Λ. For simplicity we will also assume that the fluid is isothermal, i.e. $T = 1$, and the dissipative effects will be neglected (the former assumption will be relaxed in the next section, where we study the solar tachocline limit) thus

$$\rho_0 = (\rho_B - d_1)\,\mathrm{e}^{-\kappa z} + d_1 + d_2 z, \tag{10}$$

where $\kappa = 1/\alpha\mathcal{P}$ and

$$d_1 = \Lambda\Gamma^2\left(\frac{1}{\kappa} - \frac{1-\Gamma}{\Gamma}\right), \qquad d_2 = -\Lambda\Gamma^2, \tag{11}$$

with ρ_B being the values of density at the bottom. Furthermore we neglect the Coriolis force (i.e. we set $\mathcal{U}_\Omega \ll 1$) and assume that the perturbations are two-dimensional and vary in the



a) b)

Fig. 1 The leading order asymptotic solution of equation (12) in the limit of large wave numbers $k \gg 1$ under the assumptions that the function $\Lambda B_0^2 (D - \Xi) / \bar{\rho} f (z)$ is an increasing and concave function of 'z' at $z = z_0$, for internal layer at $z_0 = 0.85$ – (a), and the function defining the growth rates $\Lambda B_0^2 (D - \Xi) / \bar{\rho} f (z)$ – (b). The parameter values chosen for this plot are $\Gamma = -1.35$, $\Lambda = 0.2$, $1/\alpha\mathcal{P} = gd/RT_S = 0.8$. (after Mizerski *et al.* 2011)

directions perpendicular to the basic field $\mathbf{B}_0$. Under these assumptions the linear stability problem is governed by the following equation for the vertical perturbation velocity

$$\begin{aligned} \sigma^2 \bar{\rho} w \;=\; & \frac{k^2 \Lambda B_0^2}{\sigma^2 + k^2 f (z)} (D - \Xi) w + \frac{\sigma^2 \bar{\rho}}{\sigma^2 + k^2 f (z)} \left(D w + \frac{\mathrm{d}w}{\mathrm{d}z} \right) + \frac{\mathrm{d}}{\mathrm{d}z} \left\{ \frac{\sigma^2 \bar{\rho} f (z)}{\sigma^2 + k^2 f (z)} \frac{\mathrm{d}w}{\mathrm{d}z} \right. \\ & \left. - \frac{\sigma^2}{\sigma^2 + k^2 f (z)} \left[\Lambda B_0^2 (D - \Xi) - \bar{\rho} f (z) D \right] w \right\} \end{aligned} \tag{12}$$

where $D = \bar{\rho}^{-1} \mathrm{d}_z \bar{\rho}$, $\Xi = B_0^{-1} \mathrm{d}_z B_0 = B_0^{-1} \Gamma$ and $f (z) = \alpha\mathcal{P} + \Lambda B_0^2 / \bar{\rho}$. The growth rates σ of the perturbations are defined by the following function of 'z'

$$\sigma^2 = \frac{\Lambda B_0^2}{\bar{\rho} f (z)} (D - \Xi) . \tag{13}$$

The solution, obtained analytically by the use of the singular perturbation method is depicted on figure 1. For large wave numbers k the modes are localised at every z_0 in a region defined by $-k^{-2/3} \ll z - z_0 \ll k^{-1/2}$, if z_0 is not a local extremum of σ in which case the point z_0 is sourrounded by a layer of thickness $k^{-1/2}$ from both sides.

3. The solar tachocline parameter regime

We proceed to study the parameter regime corresponding to that of the solar tachocline

$$\alpha \sim \mathcal{P} \sim 1 , \qquad \mathcal{U}_\Omega \sim \Lambda \sim \beta^{-1}, \tag{14}$$

$$\mathcal{U}_\kappa \sim \beta^{-2} , \qquad \mathcal{U}_\nu \sim \mathcal{U}_\eta \sim \beta^{-3}. \tag{15}$$

and we now include the variations of temperature in the dynamics. The growth rates of the perturbations are purely real at the leading order and thus oscillations may appear only at higher orders. Therefore our asymptotic analysis, with the use of the multiple timescale method suggests that the timescale of oscillations induced by the magnetic buoyancy in the solar tachocline is likely to be of the order of years,

$$\beta \sqrt{\frac{d}{g}} \approx 2.1 \times 10^7 \, s \sim years, \tag{16}$$

which agrees with the typical cycle time in the Sun. Furthermore, following the idea introduced in Mizerski *et al.* (2011) we propose that locally, in the region of upwelling convective currents at the bottom of the solar convection zone, the magnetic field's intensity is increased due to the magnetic field lines being brought closer together by the convective flow. We find, that with stronger magnetic fields the magnetic buoyancy instability is much more vigorous, in other words if the field is $\beta^{1/2}$ times stronger, causing the local ratio of the gas pressure to the magnetic pressure to be of order unity, the growth rate is also $\beta^{1/2}$ times greater than in the weak field case. This suggests, that the magnetic field is more likely to be dragged by the flow out of the tachocline into the convection zone in the regions of upwelling convective currents, and what follows, can be then further advected towards the surface by the convective flow.

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