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On the null condition for nonlinearly elastic solids

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In Memory of Professor Henryk Zorski

Smooth solutions to the Cauchy problem for the equations of nonlinear elastodynamics exist typically only locally in time. However, under the assumption of small initial data and an additional restriction, the so-called *null condition*, global existence and uniqueness of a classical solution can be proved. In this paper, we examine this condition for the elastodynamic equations and study its connection with the property of genuine nonlinearity as well as its relation with the phenomenon of self-resonance of nonlinear elastic waves. Using a special structure of plane waves elastodynamics [13], we provide an alternative and simple formulation of the null condition. This condition is then evaluated for some examples of elastic constitutive laws in order to determine the nature of the restrictions that it imposes.

1. Introduction

The fact that the equations of elastodynamics belong to the class of *hyperbolic* systems can be expressed mathematically in a number of alternative ways. In particular, the equations are hyperbolic if the acoustic tensor is positive-definite. It is known that if a hyperbolic system is *nonlinear* then smooth solutions of initial-value or initial boundary-value problems for such a system do not usually exist globally in time. Singularities will develop, typically after a finite time, even when the initial or boundary data are smooth. Therefore, in general, we can expect only local results as far as the existence of smooth solutions for arbitrary data are concerned. A theorem on local well-posedness for quasi-linear hyperbolic systems with applications to nonlinear elastodynamics has been proved in Hughes *et al.* [1].

To study global-in-time well-posedness, one tries to extend the class of solutions by considering generalized, weak solutions. Most results on existence of weak solutions for the elastodynamic equations are restricted to problems with one space dimension (see, e.g., DiPerna [2]). In multi-dimensional space it is not clear in which classes of distributions or measures the generalized solution may exist globally in time (see, however, Demoulini *et al.* [3]).

On the other hand, the existence of smooth solutions to the initial-value problem has been proved in multi-dimensions even for an abstract general class of quasi-linear wave equations (see Klainerman [4]). This was achieved under the assumption of small initial data and the additional so-called *null condition*. Such a null condition was formulated for isotropic elastodynamics by Sideris [5, 6] (see also Agemi [7]) and used in the proof of the global existence of classical smooth solutions to the initial value problem for the isotropic elastodynamic equations with small initial data. Both assumptions, together, seem to be necessary for the global existence of smooth solutions of the elastodynamic equations in the general case since there are examples of blow-up of solution when either one of these conditions is violated (see Tahvildar-Zadeh [8] and John [9]).

Roughly speaking, the null condition provides restrictions on the quadratic nonlinearities. In particular, it excludes self interactions of plane waves. In Sec. 2, we summarize the equations of nonlinear elastodynamics and their specialization to the case of isotropic materials. An outline derivation of the null condition is provided in Sec. 2.1, while in Sec. 2.2 an alternative formulation of the condition is derived based on plane wave considerations. Specifically, we formulate the null condition for arbitrary plane wave elastodynamics using the special structure of these equations and then illustrate it in the case of the Murnaghan strain-energy function. In Sec. 2.3 the relationship between the null condition and plane wave self-interaction is highlighted. A general form of the null condition is given in Sec. 2.4 and again illustrated for the Murnaghan material. In Sec. 2.5, we examine the null condition for a class of elastic constitutive laws and discuss the restrictions imposed by the null condition on the material parameters. Some concluding remarks are contained in Sec. 3.

2. The equations of nonlinear elastodynamics

When written in Lagrangian form and in the absence of body forces, the equations of elastodynamics have the form

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathrm{Div}\mathbf{S}, \tag{2.1}$$

where ρ_0 is the constant density (in the reference configuration), $\mathbf{u}$ is the displacement vector, $\mathbf{S}$ is the first Piola–Kirchhoff stress tensor and Div denotes the divergence operator in the reference configuration.

These equations may also be written as a first-order system of partial differential equations, namely

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = \operatorname{Div} \mathbf{S}, \quad \frac{\partial \mathbf{F}}{\partial t} = \operatorname{Grad} \mathbf{v}, \tag{2.2}$$

where $\mathbf{v}$ is the velocity vector, $\mathbf{F}$ is the deformation gradient tensor and Grad denotes the gradient operator in the reference configuration.

In what follows we assume that the medium is hyperelastic. This means that there exists an energy density function $W = W(\mathbf{F})$, defined per unit volume in the reference configuration, such that

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}. \tag{2.3}$$

It is assumed that W is objective, so that it depends on $\mathbf{F}$ only through the right Cauchy–Green deformation tensor $\mathbf{C}$, which is defined as $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ (see, for example, OGDEN [10, 11])

Next, we define the *acoustic tensor* $\mathbf{Q}(\mathbf{N})$, which, for any vectors $\mathbf{m}$ (Eulerian) and $\mathbf{N}$ (Lagrangian), satisfies

$$\mathbf{m} \cdot [\mathbf{Q}(\mathbf{N})\mathbf{m}] = \boldsymbol{\mathcal{A}}[\mathbf{m} \otimes \mathbf{N}, \mathbf{m} \otimes \mathbf{N}] \equiv \mathcal{A}_{ijkl} m_i m_k N_j N_l,$$

where $\boldsymbol{\mathcal{A}} = \mathrm{D}^2_{\mathbf{F}} W(\mathbf{F})$ is the elasticity tensor, with components defined by

$$\mathcal{A}_{ijkl} = \frac{\partial^2 W(\mathbf{F})}{\partial F_{ij} \partial F_{kl}}, \tag{2.4}$$

and where $\mathrm{D}_{\mathbf{F}}$ represents $\partial/\partial \mathbf{F}$. In component form, we have

$$Q_{ik}(\mathbf{N}) = \mathcal{A}_{ijkl} N_j N_l = \frac{\partial^2 W(\mathbf{F})}{\partial F_{ij} \partial F_{kl}} N_j N_l.$$

For the considered hyperelastic material the acoustic tensor $\mathbf{Q}(\mathbf{N})$ is *symmetric*. We assume that the *strong ellipticity condition* is satisfied. This can be expressed as positive definiteness of the acoustic tensor, that is

$$\mathbf{m} \cdot [\mathbf{Q}(\mathbf{N})\mathbf{m}] > 0 \quad \forall \mathbf{m}, \mathbf{N} \neq \mathbf{0}.$$

The strong ellipticity condition ensures the hyperbolicity of the elastodynamic equations, which are given in the alternative forms (2.1) or (2.2).

In terms of the gradient of displacement $\operatorname{Grad} \mathbf{u}$, which we denote by $\mathbf{D}$, so that $\mathbf{F} = \mathbf{I} + \mathbf{D}$, where $\mathbf{I}$ is the identity tensor, the stress (2.3) is written

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{D}}, \tag{2.5}$$

and Eqs. (2.2) can be rewritten as

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = \operatorname{Div}\left(\frac{\partial W}{\partial \mathbf{D}}\right), \qquad \frac{\partial \mathbf{D}}{\partial t} = \operatorname{Grad}\mathbf{v}. \tag{2.6}$$

Now, by taking all the terms onto one side, we may write the system (2.6) in the form

$$\boldsymbol{\mathcal{L}}\mathbf{U} = \mathbf{0}, \tag{2.7}$$

where the vector $\mathbf{U}$ consists of the three components of the velocity vector $\mathbf{v}$ and the nine components of the displacement gradient tensor $\mathbf{D}$, written in an appropriate order, and $\boldsymbol{\mathcal{L}}$ is a first-order quasi-linear 12×12 matrix operator, which is hyperbolic under the assumption of positive definiteness of the corresponding acoustic tensor. The system (2.7) consists of 12 equations with 12 unknowns. For definiteness, we order the components of $\mathbf{U}$ so that

$$U_i = v_i, \quad i = 1,2,3, \quad (U_4, U_5, \ldots, U_{11}, U_{12}) = (D_{11}, D_{12}, \ldots, D_{32}, D_{33}).$$

The components of $\boldsymbol{\mathcal{L}}$ are then defined by

$$\begin{aligned}
L_{ij} &= -\rho_0 \delta_{ij} \frac{\partial}{\partial t}, \quad i,j \in \{1,2,3\},\\
L_{ij} &= \sum_{k=1}^{3} A_{ikj} \frac{\partial}{\partial X_k}, \quad i = 1,2,3,\ j = 4,5,\ldots,12,\\
L_{3+k\,1} = L_{6+k\,2} = L_{9+k\,3} &= \frac{\partial}{\partial X_k}, \quad k = 1,2,3,\\
L_{ii} &= -\frac{\partial}{\partial t}, \quad i = 4,5,\ldots,12,
\end{aligned}$$

with all other $L_{ij} = 0$, where we use the representation $A_{ikp} = \mathcal{A}_{ikjl}$, with $p = (4,5,\ldots,12)$ corresponding to $(jl) = (11,12,\ldots,33)$.

To solve this system it is necessary to specify the strain-energy function. Here, we consider an isotropic material, for which the energy is a function of three independent deformation or strain invariants, which we denote by (I_1, I_2, I_3). We define these as the principal invariants of the Green strain tensor

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{D} + \mathbf{D}^T + \mathbf{D}^T\mathbf{D}\right).$$

Thus,

$$I_1 = \operatorname{tr}\mathbf{E}, \quad I_2 = \frac{1}{2}\left[(\operatorname{tr}\mathbf{E})^2 - \operatorname{tr}(\mathbf{E}^2)\right], \quad I_3 = \det\mathbf{E}. \tag{2.8}$$

It should be emphasized that the notation I_1, I_2, I_3 is frequently used for the principal invariants of $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ rather than $\mathbf{E}$, but here it is convenient to adopt the notation (2.8).

Because of the nonlinear dependence of the energy on the gradient of displacement, the system (2.1), equivalently (2.2), since it is quasi-linear and hyperbolic, does not in general have smooth solutions even for smooth initial or boundary data. Typically, shock waves form in a finite time. To avoid formation of singularities, SIDERIS [5] (see also AGEMI [7]) introduced the *null condition*. Under this condition the following theorem was proved.

THEOREM 1. *Assume that the* ***null condition*** *is satisfied. Then, there exists a positive constant* ε_0 *such that the initial value problem*

$$\boldsymbol{\mathcal{L}}\mathbf{U} = 0, \qquad \mathbf{U}|_{t=0} = \varepsilon\mathbf{U}_0$$

with $\mathbf{U}_0 \in \mathrm{C}_0^\infty(\mathbb{R}^3)$*, has a unique global-in-time* C^∞ *solution* $\mathbf{U}$ *for any* $\varepsilon \le \varepsilon_0$.

2.1. The null condition

We now briefly recall the ideas underlying the derivation of the null condition in the form provided by SIDERIS [5] and AGEMI [7]. This form of the null condition is expressed in terms of the derivatives of the strain-energy function with respect to the strain invariants. First, we write the equation of motion (2.1) as a second-order system with displacement as the dependent variable. This gives

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathrm{Div}\left(\frac{\partial W}{\partial \mathbf{D}}\right), \tag{2.9}$$

and in component form this can be expanded as

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \mathcal{A}_{ijkl} D_{kl,j} \equiv \mathcal{A}_{ijkl} u_{k,lj}, \tag{2.10}$$

where the coefficients $\mathcal{A}_{ijkl}$ are as defined in (2.4), but now written as

$$\mathcal{A}_{ijkl} = \frac{\partial^2 W}{\partial D_{ij} \partial D_{kl}}. \tag{2.11}$$

It suffices to consider explicitly only terms of first and second order in the components of $\mathbf{D}$ and its derivatives, and we therefore expand $\mathcal{A}_{ijkl}$ to the first order in the form

$$\mathcal{A}_{ijkl} = \mathcal{A}^0_{ijkl} + \mathcal{B}^0_{ijklmn} D_{mn}, \tag{2.12}$$

where the superscript ^{0} signifies evaluation in the reference configuration $\mathbf{D} = \mathbf{O}$ and

$$\mathcal{B}_{ijklmn} = \frac{\partial^3 W}{\partial D_{ij}\partial D_{kl}\partial D_{mn}} \tag{2.13}$$

are the components of the tensor $\boldsymbol{\mathcal{B}}$. To the considered order, Eq. (2.10) then takes the form

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} - \mathcal{A}^0_{ijkl} u_{k,jl} = \mathcal{B}^0_{ijklmn} D_{mn} D_{kl,j}, \tag{2.14}$$

wherein the linear terms are on the left-hand side and the nonlinear terms (truncated at second order) are on the right-hand side.

Now, since we are considering an isotropic material, we write $W = W(I_1, I_2, I_3)$ and it follows, using the component form of (2.5) and the formula (2.11), that

$$S_{ij} = \frac{\partial W}{\partial D_{ij}} = \sum_{p=1}^{3} W_p \frac{\partial I_p}{\partial D_{ij}}, \tag{2.15}$$

$$\mathcal{A}_{ijkl} = \sum_{p,\,q=1}^{3} W_{pq} \frac{\partial I_p}{\partial D_{ij}} \frac{\partial I_q}{\partial D_{kl}} + \sum_{p=1}^{3} W_p \frac{\partial^2 I_p}{\partial D_{ij}\partial D_{kl}}, \tag{2.16}$$

where $W_p = \partial W/\partial I_p, W_{pq} = \partial^2 W/\partial I_p \partial I_q$, and similarly, on use of (2.13),

$$\begin{aligned} \mathcal{B}_{ijklmn} = \sum_{p,\,q,\,r=1}^{3} & W_{pqr} \frac{\partial I_p}{\partial D_{ij}} \frac{\partial I_q}{\partial D_{kl}} \frac{\partial I_r}{\partial D_{mn}} \\ & + \sum_{p,\,q=1}^{3} W_{pq} \left(\frac{\partial I_p}{\partial D_{ij}} \frac{\partial^2 I_q}{\partial D_{kl}\partial D_{mn}} + \frac{\partial I_p}{\partial D_{kl}} \frac{\partial^2 I_q}{\partial D_{ij}\partial D_{mn}} \right. \\ & \left. + \frac{\partial I_p}{\partial D_{mn}} \frac{\partial^2 I_q}{\partial D_{ij}\partial D_{kl}} \right) + \sum_{p=1}^{3} W_p \frac{\partial^3 I_p}{\partial D_{ij}\partial D_{kl}\partial D_{mn}}, \end{aligned} \tag{2.17}$$

with $W_{pqr} = \partial^3 W/\partial I_p \partial I_q \partial I_r$.

We require the values of these expressions in the reference configuration, for which purpose we need to calculate the derivatives of I_p, $p = 1, 2, 3$, with respect to the components of $\mathbf{D}$. We first note that the invariants can be expanded in terms of $\mathbf{D}$ in the forms

$$I_1 = \operatorname{tr} \mathbf{D} + \frac{1}{2} \operatorname{tr} \left(\mathbf{D}^T \mathbf{D} \right), \tag{2.18}$$

$$I_2 = \frac{1}{4}\left[2(\operatorname{tr}\mathbf{D})^2 - \operatorname{tr}(\mathbf{D}^2) - \operatorname{tr}(\mathbf{D}\mathbf{D}^T)\right] + \frac{1}{2}\left[(\operatorname{tr}\mathbf{D})\operatorname{tr}(\mathbf{D}^T\mathbf{D}) - \operatorname{tr}(\mathbf{D}^2\mathbf{D}^T)\right], \tag{2.19}$$

$$I_3 = \frac{1}{12}\left[2(\operatorname{tr}\mathbf{D})^3 + \operatorname{tr}(\mathbf{D}^3)\right] - \frac{1}{4}\left[(\operatorname{tr}\mathbf{D})\operatorname{tr}(\mathbf{D}^2) + (\operatorname{tr}\mathbf{D})\operatorname{tr}(\mathbf{D}\mathbf{D}^T) - \operatorname{tr}(\mathbf{D}^2\mathbf{D}^T)\right], \tag{2.20}$$

where I_2 and I_3 have been truncated at the third order in $\mathbf{D}$. To the second order, the first derivatives of these invariants are

$$\frac{\partial I_1}{\partial \mathbf{D}} = \mathbf{I} + \mathbf{D}, \tag{2.21}$$

$$\frac{\partial I_2}{\partial \mathbf{D}} = (\operatorname{tr}\mathbf{D})\,\mathbf{I} - \frac{1}{2}\left(\mathbf{D} + \mathbf{D}^T\right) + \frac{1}{2}\operatorname{tr}\left(\mathbf{D}\mathbf{D}^T\right)\mathbf{I} + (\operatorname{tr}\mathbf{D})\,\mathbf{D} - \frac{1}{2}\left(\mathbf{D}\mathbf{D}^{\mathrm{T}} + \mathbf{D}^{\mathrm{T}}\mathbf{D} + \mathbf{D}^2\right), \tag{2.22}$$

$$\frac{\partial I_3}{\partial \mathbf{D}} = \frac{1}{2}(\operatorname{tr}\mathbf{D})\left[(\operatorname{tr}\mathbf{D})\mathbf{I} - \mathbf{D} - \mathbf{D}^T\right] + \frac{1}{4}\left(\mathbf{D}^T\mathbf{D}^T + \mathbf{D}\mathbf{D}^T + \mathbf{D}^T\mathbf{D} + \mathbf{D}^2\right) - \frac{1}{4}\left[\operatorname{tr}(\mathbf{D}^2) + \operatorname{tr}(\mathbf{D}\mathbf{D}^T)\right]\mathbf{I}. \tag{2.23}$$

Corresponding second- and third-order derivatives may also be written down but they are mostly quite lengthy so are not listed here. More specifically, in the reference configuration the first derivatives reduce simply to

$$\frac{\partial I_1}{\partial \mathbf{D}} = \mathbf{I}, \qquad \frac{\partial I_2}{\partial \mathbf{D}} = \frac{\partial I_3}{\partial \mathbf{D}} = \mathbf{O}, \tag{2.24}$$

and, in component form, the second derivatives are

$$\begin{aligned} \frac{\partial^2 I_1}{\partial D_{ij}\partial D_{kl}} &= \delta_{ik}\delta_{jl}, \\ \frac{\partial^2 I_2}{\partial D_{ij}\partial D_{kl}} &= \delta_{ij}\delta_{kl} - \frac{1}{2}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}), \\ \frac{\partial^2 I_3}{\partial D_{ij}\partial D_{kl}} &= 0, \end{aligned} \tag{2.25}$$

where δ_{ij} is the Kronecker delta, while the third derivative of I_1 vanishes and, in the reference configuration, the third derivatives of I_2 and I_3 are given by

$$\frac{\partial^3 I_2}{\partial D_{ij}\partial D_{kl}\partial D_{mn}} = \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{ij}\delta_{km}\delta_{ln} + \delta_{kl}\delta_{im}\delta_{jn} - \frac{1}{2}\big(\delta_{ik}\delta_{jm}\delta_{ln} + \delta_{ik}\delta_{lm}\delta_{jn} + \delta_{jl}\delta_{im}\delta_{kn} + \delta_{jl}\delta_{km}\delta_{in} + \delta_{il}\delta_{km}\delta_{jn} + \delta_{im}\delta_{jk}\delta_{ln}\big), \tag{2.26}$$

$$\begin{aligned}\frac{\partial^3 I_3}{\partial D_{ij}\partial D_{kl}\partial D_{mn}} &= \delta_{ij}\delta_{kl}\delta_{mn} \\ &- \frac{1}{2}\big(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{lm}\delta_{kn} + \delta_{kl}\delta_{im}\delta_{jn}\delta_{kl}\delta_{jm}\delta_{in} + \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{jk}\delta_{il}\delta_{mn}\big) \\ &+ \frac{1}{4}\big(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{jk}\delta_{lm}\delta_{in} + \delta_{jk}\delta_{im}\delta_{ln} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{jl}\delta_{km}\delta_{in} \\ &\qquad + \delta_{il}\delta_{km}\delta_{jn} + \delta_{jl}\delta_{im}\delta_{kn} + \delta_{ik}\delta_{lm}\delta_{jn}\big).\end{aligned} \tag{2.27}$$

In the reference configuration we assume that both $W = W^0$ and $\mathbf{S} = \mathbf{S}^0$ vanish, and since, on use of the above results in (2.15), $\mathbf{S}^0 = W_1^0\mathbf{I}$, we have

$$W^0 = 0, \quad W_1^0 = 0, \tag{2.28}$$

where again and in what follows the superscript ^{0} indicates evaluation in the reference configuration. Also, from (2.16), we obtain

$$\mathcal{A}^0_{ijkl} = \big(W^0_{11} + W^0_2\big)\,\delta_{ij}\delta_{kl} - \frac{1}{2}W^0_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

The corresponding expression for $\mathcal{B}^0_{ijklmn}$ may be obtained from (2.17) and (2.24)–(2.27) but is not given explicitly here.

The nonlinear terms are quite complicated and we do not list them all here since most are not relevant to the argument. Let us now set $\rho_0 c_l^2 \equiv W^0_{11}$ and $\rho_0 c_s^2 \equiv -\frac{1}{2}W^0_2$, where c_l and c_s are the speeds of the linearized longitudinal and shear waves, respectively. Noting that $\mathbf{D} = \operatorname{Grad}\mathbf{u}$, the equations of motion (2.9) can be written as

$$\mathbf{u}_{tt} - c_s^2\triangle\mathbf{u} - (c_l^2 - c_s^2)\nabla(\nabla\cdot\mathbf{u}) = N(\nabla\mathbf{u}, \nabla^2\mathbf{u}), \tag{2.29}$$

where N is a nonlinear function of its arguments and the operator ∇ has replaced Grad. On the left-hand side is the classical linear operator of elastodynamics, while on the right-hand side we have second-order and higher-order terms. Amongst the nonlinear terms only terms of the form $\nabla(\nabla\cdot\mathbf{u})^2$ interfere with the energy estimates needed in the proof of global existence of a smooth solution to elastodynamic Eqs. (2.1) (see SIDERIS [5] and AGEMI [7] for more details). We display this term explicitly and write

$$N(\nabla\mathbf{u},\nabla^2\mathbf{u}) = \frac{1}{2\rho_0}(3W_{11}^0 + W_{111}^0)\nabla(\nabla\cdot\mathbf{u})^2 + ...$$

The null condition is imposed to cancel this term.

Definition 1. *We say that the elastodynamic Eqs. (2.1) or (2.9) satisfy the null condition if*

$$(2.30) \qquad \left(3W_{11}^0 + W_{111}^0\right) \equiv \left(3\frac{\partial^2 W}{\partial I_1^2} + \frac{\partial^3 W}{\partial I_1^3}\right)\Bigg|_{I_1=I_2=I_3=0} = 0.$$

2.2. An alternative form of the null condition

In this and the following sections we formulate the null condition in two different ways. Both formulas are expressed with the use of plane waves. Therefore, we first reduce the nonlinear elastodynamic system (2.2) to a plane wave system. The null condition says that each of the nonlinear elastic plane waves is *not* genuinely nonlinear in the reference configuration. For another derivation of a null condition in terms of plane waves see also XIN [12].

2.2.1. Reduction to plane waves. In DOMAŃSKI and YOUNG [13] a general structure for plane wave elastodynamics for an arbitrary elastic solid was derived. Here we follow this derivation with the aim of providing alternative formulations of the null condition using plane waves. In this context the null condition asserts that no plane wave solution can be genuinely nonlinear or, equivalently, that quadratically nonlinear self-interaction of plane waves is forbidden.

Given an arbitrary fixed unit vector $\mathbf{N}$, we consider motions of the type

$$(2.31) \qquad \mathbf{x} = \boldsymbol{\pi}\mathbf{X} + \mathbf{u}(x,t),$$

where $\boldsymbol{\pi}$ is the gradient of a homogeneous background deformation (in particular $\boldsymbol{\pi}$ may be equal to $\mathbf{I}$), $\mathbf{u}$ is the superimposed displacement vector and

$$x = \mathbf{N}\cdot\mathbf{X}$$

is the projection of $\mathbf{X}$ on the direction $\mathbf{N}$. Since $\mathrm{Grad}\, x = \mathbf{N}$, the resulting deformation gradient is

$$\mathbf{F} = \boldsymbol{\pi} + \mathbf{d}\otimes\mathbf{N},$$

where $\mathbf{d} = \mathbf{u}_{,x}$ is the vector displacement gradient.

Next, since the stress $\mathbf{S}$ now depends on x through $\mathbf{F}$ and $\mathbf{N}$ is independent of x, we calculate

$$\operatorname{Div}\mathbf{S} = (\mathbf{S}\,\mathbf{N})_{,x}. \tag{2.32}$$

Also,

$$\operatorname{Grad}\mathbf{v} = \mathbf{v}_{,x}\otimes\mathbf{N}, \qquad \mathbf{F}_{,t} = \mathbf{d}_{,t}\otimes\mathbf{N}. \tag{2.33}$$

We now define the (reduced) stress vector $\mathbf{s} = \mathbf{s}(\mathbf{d})$ by

$$\mathbf{s}\,(\mathbf{d}) \equiv \mathbf{S}\,(\mathbf{F})\,\mathbf{N} = \mathbf{S}\,(\boldsymbol{\pi} + \mathbf{d}\otimes\mathbf{N})\,\mathbf{N}. \tag{2.34}$$

Then, on use of (2.32)–(2.34) in the equations of motion, we obtain the plane wave system

$$\rho_0\mathbf{v}_{,t} = [\mathbf{s}(\mathbf{d})]_{,x}, \qquad \mathbf{d}_{,t} = \mathbf{v}_{,x}, \tag{2.35}$$

which is a 6×6 system in one space variable x and time t. Note that we omitted the explicit dependence of $\mathbf{s}$ on $\mathbf{N}$.

In parallel with the definition of the stress vector $\mathbf{s}$ we define the (reduced) energy $V = V(\mathbf{d})$ via

$$V(\mathbf{d}) \equiv W(\mathbf{F}) = W(\boldsymbol{\pi} + \mathbf{d}\otimes\mathbf{N}), \tag{2.36}$$

from which we calculate

$$\mathbf{s}\,(\mathbf{d}) = \mathrm{D}_{\mathbf{d}}V, \tag{2.37}$$

where $\mathrm{D}_{\mathbf{d}} = \partial/\partial\mathbf{d}$, so that (2.35) may be rewritten as

$$\rho_0\mathbf{v}_{,t} = (\mathrm{D}_{\mathbf{d}}V)_{,x}, \qquad \mathbf{d}_{,t} = \mathbf{v}_{,x}. \tag{2.38}$$

2.2.2. The one-dimensional quasi-linear system. We now restrict attention to the plane wave system (2.38), which, provided $V \in \mathrm{C}^2$, may be written in the quasi-linear form

$$\mathbf{w}_{,t} + \mathbf{A}(\mathbf{w},\mathbf{N})\mathbf{w}_{,x} = \mathbf{0}, \tag{2.39}$$

where

$$\mathbf{w} = \begin{bmatrix} \mathbf{v}(x,t) \\ \mathbf{d}(x,t) \end{bmatrix}, \qquad \mathbf{A}(\mathbf{w},\mathbf{N}) = -\begin{pmatrix} \mathbf{0} & \mathbf{B}(\mathbf{d},\mathbf{N}) \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \tag{2.40}$$

and, for convenience, we have normalized the density by setting $\rho_0 = 1$. Here $\mathbf{B}$ is the symmetric matrix

$$\mathbf{B}(\mathbf{d}, \mathbf{N}) \equiv \mathrm{D}_{\mathbf{d}}\mathbf{s} = \mathrm{D}^2_{\mathbf{d}} V, \tag{2.41}$$

with components

$$B_{ij} = \frac{\partial^2 V(\mathbf{d})}{\partial d_i \partial d_j}.$$

In fact, it is straightforward to show that $\mathbf{B}$ is precisely the acoustic tensor:

$$\mathbf{B} = \mathbf{Q}(\mathbf{N}), \qquad B_{ij} = Q_{ij}(\mathbf{N}) = \mathcal{A}_{ikjl} N_k N_l.$$

Thus, the assumption of strong ellipticity implies that the matrix $\mathbf{B}(\mathbf{d}, \mathbf{N})$ is positive definite. In particular, since $\mathbf{B}(\mathbf{d}, \mathbf{N})$ is symmetric and positive definite, $\mathbf{A}(\mathbf{w}, \mathbf{N})$ always has a full set of (real) eigenvectors, and the system (2.39) is therefore hyperbolic. We will often suppress the dependence of these quantities on the vector $\mathbf{N}$, which is a fixed but arbitrary unit vector.

2.2.3. Eigensystems for A and B. In order to find connections between the eigenvalues and eigenvectors of $\mathbf{A}$ and $\mathbf{B}$, we first prove a general Lemma.

Lemma 1. *Let $\boldsymbol{A}$ be a $2m \times 2m$ matrix of the form*

$$\boldsymbol{A} = -\begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{I}_m & \mathbf{0} \end{pmatrix}, \tag{2.42}$$

where $\mathbf{B}$ is a symmetric $m \times m$ matrix and $\mathbf{I}_m$ the identity $m \times m$ matrix. Then, for $i = 1, ..., m$, the eigensystem $\{\kappa_i, \mathbf{q}_i\}$ of the matrix $\mathbf{B}$ is related to the eigensystem $\{\lambda_j, \mathbf{r}_j\}$ of the matrix $\mathbf{A}$ through the connections

$$\lambda_{2i-1} = -\sqrt{\kappa_i} = -\lambda_{2i}, \tag{2.43}$$

$$\mathbf{r}_{2i-1} = \begin{bmatrix} \sqrt{\kappa_i}\mathbf{q}_i \\ \mathbf{q}_i \end{bmatrix}, \quad \mathbf{r}_{2i} = \begin{bmatrix} -\sqrt{\kappa_i}\mathbf{q}_i \\ \mathbf{q}_i \end{bmatrix}. \tag{2.44}$$

P r o o f. We have

$$\det\,(\lambda \mathbf{I}_{2m} - \mathbf{A}) = \det \begin{pmatrix} \lambda\mathbf{I}_m & \mathbf{B} \\ \mathbf{I}_m & \lambda\mathbf{I}_m \end{pmatrix} = \det\,(\lambda^2 \mathbf{I}_m - \mathbf{B}).$$

Thus, λ is an eigenvalue of the $\mathbf{A}$ if and only if $\kappa = \lambda^2$ is an eigenvalue of $\mathbf{B}$. Hence, in order to find the eigenvalues λ_j of the $2m \times 2m$ matrix $\mathbf{A}$, it suffices to find the eigenvalues κ_i of the $m \times m$ matrix $\mathbf{B}$ and then to obtain the eigenvalues of the matrix $\mathbf{A}$ through (2.43) for $i = 1, \ldots, m$.

Similarly for the corresponding eigenvectors. Assuming that $\mathbf{r}$ is the eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, we have

$$(\lambda \mathbf{I}_{2m} - \mathbf{A})\mathbf{r} = \mathbf{0}.$$

We separate the $2m$-component eigenvector $\mathbf{r}$ into two m-component vectors $\mathbf{p}$ and $\mathbf{q}$:

$$\mathbf{r} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix}.$$

This gives

$$(\lambda \mathbf{I}_{2m} - \mathbf{A})\mathbf{r} = \begin{pmatrix} \lambda \mathbf{I}_m & \mathbf{B} \\ \mathbf{I}_m & \lambda \mathbf{I}_m \end{pmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \mathbf{0},$$

and hence $\mathbf{p} = -\lambda \mathbf{q}$ and $\mathbf{q}$ is the eigenvector of the matrix $\mathbf{B}$ corresponding to the eigenvalue λ^2. Thus, in order to find an eigenvector $\mathbf{r}_j$ of the $(2m \times 2m)$ matrix $\mathbf{A}$ it is only necessary to find the corresponding eigenvector $\mathbf{q}_i$ of the smaller $(m \times m)$ matrix $\mathbf{B}$ and then to use the (2.44) with (2.43). Thanks to Lemma 1, we can reduce the calculation of the eigensystem for plane elastodynamic waves from a 6×6 system to a 3×3 system. For reference an outline calculation of the eigensystem for $\mathbf{A}(\mathbf{0})$ is provided in the Appendix.

Remark 1. Denoting by $[\![\ ]\!]$ the largest integer that is less than or equal to a given number, we can express the correspondence between the eigensystem $\{\kappa_i, \mathbf{q}_i\}$ of $\mathbf{B}$ and the eigensystem $\{\lambda_j, \mathbf{r}_j\}$ of $\mathbf{A}$ in a more compact form. First let us notice that $i = [\![(j+1)/2]\!]$. With this notation we then have

$$\lambda_j = (-1)^{j+1}\sqrt{\kappa_i}, \qquad \mathbf{r}_j = \begin{bmatrix} (-1)^{j+1}\sqrt{\kappa_i}\,\mathbf{q}_i \\ \mathbf{q}_i \end{bmatrix}. \tag{2.45}$$

2.2.4. The null condition in terms of the eigensystem of B In order to formulate the null condition in a simpler form, we now prove the following Lemma.

Lemma 2. Let

$$\mathbf{A}(\mathbf{w}) = -\begin{pmatrix} \mathbf{0} & \mathbf{B}(\mathbf{d}) \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \tag{2.46}$$

be the matrix from the formula (2.40), with the argument $\mathbf{N}$ *omitted and with* $\mathbf{w} = [\mathbf{v}, \mathbf{d}]^T$. *Suppose that* $\kappa_i(\mathbf{d}) \neq 0$ *are the eigenvalues of* $\mathbf{B}(\mathbf{d})$, $\mathbf{q}_i(\mathbf{d})$ *the corresponding eigenvectors* $(i = 1, 2, 3)$, *and that* $\lambda_j(\mathbf{w})$ *are the eigenvalues of* $\mathbf{A}(\mathbf{w})$ *with* $\mathbf{r}_j(\mathbf{w})$ *the corresponding eigenvectors* $(j = 1, \ldots, 6)$. *Then*

$$(\mathrm{D}_{\mathbf{w}}\, \lambda_j(\mathbf{w})) \cdot \mathbf{r}_j = 0 \iff (\mathrm{D}_{\mathbf{d}}\, \kappa_i(\mathbf{d})) \cdot \mathbf{q}_i = 0. \tag{2.47}$$

P r o o f. Taking into account Lemma 1, we calculate

$$(2.48) \qquad (\mathrm{D}_{\mathbf{w}}\, \lambda_{2i-1}) \cdot \mathbf{r}_{2i-1} = (\mathrm{D}_{\mathbf{w}}\, (-\sqrt{\kappa_i})) \cdot \begin{bmatrix} \sqrt{\kappa_i}\mathbf{q}_i \\ \mathbf{q}_i \end{bmatrix}$$

$$= \left[\mathbf{0}, -\frac{1}{2\sqrt{\kappa_i}}(\mathrm{D}_{\mathbf{d}}\, \kappa_i)\right]^T \cdot \begin{bmatrix} \sqrt{\kappa_i}\mathbf{q}_i \\ \mathbf{q}_i \end{bmatrix} = -\frac{1}{2\sqrt{\kappa_i}}(\mathrm{D}_{\mathbf{d}}\, \kappa_i) \cdot \mathbf{q}_i.$$

Similarly,

$$(2.49) \qquad (\mathrm{D}_{\mathbf{w}}\, \lambda_{2i}) \cdot \mathbf{r}_{2i} = (\mathrm{D}_{\mathbf{w}}\, \sqrt{\kappa_i}) \cdot \begin{bmatrix} -\sqrt{\kappa_i}\mathbf{q}_i \\ \mathbf{q}_i \end{bmatrix}$$

$$= \left[\mathbf{0}, \frac{1}{2\sqrt{\kappa_i}}(\mathrm{D}_{\mathbf{d}}\, \kappa_i)\right]^T \cdot \begin{bmatrix} -\sqrt{\kappa_i}\mathbf{q}_i \\ \mathbf{q}_i \end{bmatrix} = \frac{1}{2\sqrt{\kappa_i}}(\mathrm{D}_{\mathbf{d}}\, \kappa_i) \cdot \mathbf{q}_i.$$

Hence we obtain Eq. (2.47).

We can now state that the *null condition for elastodynamics* can be formulated as

$$(2.50) \qquad (\mathrm{D}_{\mathbf{d}}\, \kappa_i \cdot \mathbf{q}_i)\big|_{\mathbf{d}\,=\,\mathbf{0}} = 0. \quad \text{for } i = 1, 2, 3.$$

2.2.5. Application to the Murnaghan material Here we illustrate the condition (2.50) for the Murnaghan isotropic elastic material, for which the strain-energy function may be written (see MURNAGHAN [14])

$$(2.51) \qquad W \;=\; \frac{1}{2}(\lambda_L + 2\mu_L)I_1^2 - 2\mu_L I_2 + \frac{1}{3}(l + 2m)I_1^3 - 2mI_1I_2 + nI_3,$$

where λ_L and μ_L are the Lamé elastic constants and l, m, n are the third-order Murnaghan elastic constants.

We express the energy function (2.51) in terms of the displacement gradient, truncating the resulting expression so that the stress tensor has only quadratically nonlinear terms. We then restrict attention to plane waves propagating in the $[1, 0, 0]$ direction. We obtain the plane wave system (2.39) with the matrix $\mathbf{B}(\mathbf{d})$ from (2.41) given by (see, also, DOMAŃSKI [15])

$$(2.52) \qquad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & 0 \\ B_{13} & 0 & B_{22} \end{pmatrix},$$

where

$$(2.53) \qquad B_{11} \;=\; \lambda_L + 2\mu_L + (3\alpha + 2l + m)d_1,$$

$$(2.54) \qquad B_{22} \;=\; \mu_L + \alpha d_1, \qquad B_{12} = \alpha d_2, \qquad B_{13} = \alpha d_3,$$

and $\alpha = \lambda_L + 2\mu_L + m$.

The eigenvalues of $\mathbf{B}$ are

$$\kappa_1 = \frac{1}{2}\left(B_{11} + B_{22} + \delta\right), \qquad \kappa_2 = \frac{1}{2}\left(B_{11} + B_{22} - \delta\right), \qquad \kappa_3 = B_{22}, \tag{2.55}$$

where

$$\delta = \sqrt{(B_{11} - B_{22})^2 + 4(B_{12}^2 + B_{13}^2)}. \tag{2.56}$$

The corresponding eigenvectors have components

$$\mathbf{q}_1 = \left[(B_{11} - B_{22} + \delta)/2\alpha, d_2, d_3\right], \tag{2.57}$$

$$\mathbf{q}_2 = \left[(B_{11} - B_{22} - \delta)/2\alpha, d_2, d_3\right], \tag{2.58}$$

$$\mathbf{q}_3 = \left[0, -d_3, d_2\right]. \tag{2.59}$$

Note that since $\mathbf{B}$ is symmetric it is not necessary to distinguish between its left and right eigenvectors, unlike the situation for $\mathbf{A}$.

Let us recall that from Lemma 1 we know that the pairs $\{\lambda_j, \mathbf{r}_j\}$ from the eigensystem of the whole matrix $\mathbf{A}$ are related to the pairs $\{\kappa_i, \mathbf{q}_i\}$ from the eigensystem of $\mathbf{B}$, specifically

$$\lambda_1 = -\sqrt{\kappa_1} = -\lambda_2, \qquad \lambda_3 = -\sqrt{\kappa_2} = -\lambda_4, \qquad \lambda_5 = -\sqrt{\kappa_3} = -\lambda_6,$$

$$\mathbf{r}_1 = \begin{bmatrix} -\sqrt{\kappa_1}\mathbf{q}_1 \\ \mathbf{q}_1 \end{bmatrix}, \qquad \mathbf{r}_3 = \begin{bmatrix} -\sqrt{\kappa_2}\mathbf{q}_2 \\ \mathbf{q}_2 \end{bmatrix}, \qquad \mathbf{r}_5 = \begin{bmatrix} -\sqrt{\kappa_3}\mathbf{q}_3 \\ \mathbf{q}_3 \end{bmatrix},$$

$$\mathbf{r}_2 = \begin{bmatrix} \sqrt{\kappa_1}\mathbf{q}_1 \\ \mathbf{q}_1 \end{bmatrix}, \qquad \mathbf{r}_4 = \begin{bmatrix} \sqrt{\kappa_2}\mathbf{q}_2 \\ \mathbf{q}_2 \end{bmatrix}, \qquad \mathbf{r}_6 = \begin{bmatrix} \sqrt{\kappa_3}\mathbf{q}_3 \\ \mathbf{q}_3 \end{bmatrix}.$$

Therefore, by taking into account the form of the eigenvectors $\mathbf{q}_i$, we conclude that the first pair $\{\kappa_1, \mathbf{q}_1\}$ corresponds to the pair of quasi-longitudinal waves characterized by the speeds $\pm\lambda_1$ and 'polarizations' $\mathbf{r}_1, \mathbf{r}_2$. The second pair $\{\kappa_2, \mathbf{q}_2\}$ corresponds to the quasi-shear waves propagating with the speeds $\pm\lambda_3$ and 'polarizations' $\mathbf{r}_3, \mathbf{r}_4$. Finally the third pair $\{\kappa_3, \mathbf{q}_3\}$ is related to a pair of pure shear waves whose speeds are $\pm\lambda_5$ and 'polarizations' $\mathbf{r}_5, \mathbf{r}_6$.

We now calculate the gradients of the eigenvalues with respect to the vector $\mathbf{d}$. We have, in component form,

$$\mathrm{D}_{\mathbf{d}}\kappa_1 = \frac{1}{2}\left[4\alpha + 2l + m + \delta_{,d_1}, \delta_{,d_2}, \delta_{,d_3}\right], \tag{2.60}$$

$$\mathrm{D}_{\mathbf{d}}\kappa_2 = \frac{1}{2}\left[4\alpha + 2l + m - \delta_{,d_1}, -\delta_{,d_2}, -\delta_{,d_3}\right], \tag{2.61}$$

$$\mathrm{D}_{\mathbf{d}}\kappa_3 = \left[\alpha, 0, 0\right], \tag{2.62}$$

and we note that

$$\mathrm{D}_{\mathbf{d}}\delta = \left[(B_{11} - B_{22})(2\alpha + 2l + m), 4\alpha^2 d_2, 4\alpha^2 d_3\right]/\delta. \tag{2.63}$$

We see immediately that

$$\mathrm{D}_{\mathbf{d}}\kappa_3 \cdot \mathbf{q}_3 = 0 \quad \text{for any} \quad \mathbf{d}, \tag{2.64}$$

and hence the pure shear waves propagating with speeds λ_5 and λ_6 are *globally linearly degenerate*.

At $\mathbf{d} = \mathbf{0}$, we have also

$$(\mathrm{D}_{\mathbf{d}}\kappa_2 \cdot \mathbf{q}_2)\big|_{\mathbf{d}=\mathbf{0}} = 0. \tag{2.65}$$

Thus, quasi-shear waves are *locally linearly degenerate*. It follows from (2.64) and (2.65) that the null condition provides no restrictions for shear and quasi–shear waves. On the other hand, it does impose restrictions on quasi-longitudinal waves. We have

$$(\mathrm{D}_{\mathbf{d}}\kappa_1 \cdot \mathbf{q}_1)\big|_{\mathbf{d}=\mathbf{0}} = (3\alpha + 2l + m)\,(\lambda_L + \mu_L)/\alpha. \tag{2.66}$$

It follows from (2.66) that the null condition (2.50) for the Murnaghan material is given by

$$3\alpha + 2l + m \equiv 3(\lambda_L + 2\mu_L) + 2(l + 2m) = 0.$$

Here we are assuming that $\lambda_L + 2\mu_L > 0$. Thus, if the null condition is to be satisfied, the material constants l and m must be such that $l + 2m$ is negative.

2.3. The null condition and wave self-interaction

Here we present the derivation of the null condition in yet another form, in terms of the nullity of wave self-interaction coefficients Γ^j_{jj}, which determine the magnitude of the interaction of the wave with itself. These coefficients are given by the formula

$$\Gamma^j_{jj} = \mathbf{l}_j \cdot (\mathrm{D}_{\mathbf{w}}\mathbf{A}\mathbf{r}_j)\,\mathbf{r}_j, \tag{2.67}$$

where $\mathbf{l}_j$ and $\mathbf{r}_j$ are left and right eigenvectors of the matrix $\mathbf{A}$ in (2.40); see DOMAŃSKI [15], p. 53, for details. Differentiation of the right-hand side of (2.67) in the direction $\mathbf{r}_j$ leads, after some algebra, to

$$\Gamma^j_{jj} = \mathrm{D}_{\mathbf{w}}\lambda_j \cdot \mathbf{r}_j.$$

Hence, the wave self-interaction coefficient is equal to the coefficient that determines the genuine nonlinearity.

IN DOMAŃSKI and YOUNG [13] general formulas for wave interaction coefficients were obtained for arbitrary elastic plane waves and expressed in terms of derivatives of the strain energy. Following [13] we re-derive these formulas here, with attention focused on the self-interaction coefficients. From (2.45) and the fact that we assume the normalization $\mathbf{l}_j \cdot \mathbf{r}_j = \delta_{ij}$, we obtain the left eigenvectors of $\mathbf{A}$ as

$$\mathbf{l}_j(\mathbf{w}) = \left[(-1)^{j+1} \frac{\mathbf{q}_i}{2\sqrt{\kappa_i}}, \quad \frac{\mathbf{q}_i}{2} \right]^T, \tag{2.68}$$

and we recall that $i = [\![(j+1)/2]\!]$.

We also have

$$\mathrm{D}_{\mathbf{w}}\mathbf{A}(\mathbf{r}) = -\begin{bmatrix} \mathbf{0} & \mathrm{D}_{\mathbf{d}}\mathbf{B}(\mathbf{q}) \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

for any vector with six components $\mathbf{r} = [\mathbf{p}, \mathbf{q}]^T$, where $\mathbf{q}$ is any vector with three components. Then, we find that the self-interaction coefficients are equal to

$$\Gamma^j_{jj} = \mathbf{l}_j \cdot (\mathrm{D}_{\mathbf{w}}\mathbf{A}\mathbf{r}_j)\,\mathbf{r}_j = \frac{(-1)^{j+1}}{2\sqrt{\kappa_i}}\,\mathbf{q}_i \cdot (\mathrm{D}_{\mathbf{d}}\mathbf{B}\,\mathbf{q}_i)\,\mathbf{q}_i. \tag{2.69}$$

Moreover, since, from (2.41), $\mathbf{B} = \mathrm{D}^2_{\mathbf{d}}V$, (2.69) becomes

$$(-1)^{j+1}\Gamma^j_{jj} = \frac{1}{2\sqrt{\kappa_i}}\,\mathrm{D}^3_{\mathbf{d}}V[\mathbf{q}_i, \mathbf{q}_i, \mathbf{q}_i], \tag{2.70}$$

where $\mathrm{D}^3_{\mathbf{d}}V$ is a symmetric trilinear form. Thus, another way of expressing the *null condition* is as

$$\mathrm{D}^3_{\mathbf{d}}V[\mathbf{q}_i, \mathbf{q}_i, \mathbf{q}_i] = 0 \quad \text{for} \quad i = 1, 2, 3. \tag{2.71}$$

In the next section we express the self-interaction coefficients in terms of the strain energy $W(\mathbf{E})$ in order to obtain yet another general form of the null condition.

2.4. A general form of the null condition

In this subsection we derive a general form of the null condition, formulated as a condition that excludes plane wave self-interactions. We derive this condition for an arbitrary hyperelastic medium and for any direction of plane wave propagation.

First, we note that since

$$\mathbf{B} = \mathbf{Q}(\mathbf{N}) \equiv \mathcal{A}_{ijkl}N_jN_l,$$

it follows that

$$\mathrm{D}_{\mathbf{d}}\mathbf{B} \equiv \mathrm{D}^3_{\mathbf{d}}V = \mathcal{B}_{ijklmn}N_jN_lN_n,$$

where $\mathcal{B}_{ijklmn}$ is defined by (2.13). The null condition (2.71) can therefore be written as

$$\mathcal{B}[\mathbf{q}_i \otimes \mathbf{N}, \mathbf{q}_i \otimes \mathbf{N}, \mathbf{q}_i \otimes \mathbf{N}] = 0.$$

We may also write this in terms of derivatives of W with respect to $\mathbf{E}$. The result is

$$\mathrm{D}^3_{\mathbf{E}}W[\mathbf{N}\otimes\mathbf{Q}_i, \mathbf{N}\otimes\mathbf{Q}_i, \mathbf{N}\otimes\mathbf{Q}_i] + 3\mathrm{D}^2_{\mathbf{E}}W[\mathbf{N}\otimes\mathbf{N}, \mathbf{N}\otimes\mathbf{Q}_i]\,\mathbf{q}_i\cdot\mathbf{q}_i = 0, \tag{2.72}$$

where $\mathbf{Q}_i = \mathbf{F}^T\mathbf{q}_i$.

To obtain this result we have used the formulas (given in index notation)

$$\frac{\partial E_{pq}}{\partial F_{ij}} = \frac{1}{2}(\delta_{jp}F_{iq} + \delta_{jp}F_{ip}),$$

and

$$(\mathrm{D}_{\mathbf{F}}W)_{ij} = F_{ip}(\mathrm{D}_{\mathbf{E}}W)_{pj}, \tag{2.73}$$

$$(\mathrm{D}^2_{\mathbf{F}}W)_{ijkl} = F_{ip}F_{kq}(\mathrm{D}^2_{\mathbf{E}}W)_{jplq} + (\mathrm{D}_{\mathbf{E}}W)_{jl}\delta_{ik}, \tag{2.74}$$

$$\begin{aligned}(\mathrm{D}^3_{\mathbf{F}}W)_{ijklmn} = F_{ip}F_{kq}F_{mr}(\mathrm{D}^3_{\mathbf{E}}W)_{jplqnr} + (\mathrm{D}^2_{\mathbf{E}}W)_{jnlp}F_{kp}\delta_{im} \\ + (\mathrm{D}^2_{\mathbf{E}}W)_{jpln}F_{ip}\delta_{km} + (\mathrm{D}^2_{\mathbf{E}}W)_{jlnp}F_{mp}\delta_{ik}.\end{aligned} \tag{2.75}$$

For an isotropic material, the acoustic tensor and the null condition can be calculated explicitly using (2.16) and (2.17), respectively. Alternatively, if W is treated as a function of $\mathbf{E}$ then we use the connections (2.74) and (2.75) together with the expansions

$$(\mathrm{D}_{\mathbf{E}}W)_{ij} = \sum_{p=1}^{3} W_p(\mathrm{D}_{\mathbf{E}}I_p)_{ij},$$

$$(\mathrm{D}^2_{\mathbf{E}}W)_{ijkl} = \sum_{p,q=1}^{3} W_{pq}(\mathrm{D}_{\mathbf{E}}I_p)_{ij}(\mathrm{D}_{\mathbf{E}}I_q)_{kl} + \sum_{p=1}^{3} W_p(\mathrm{D}^2_{\mathbf{E}}I_p)_{ijkl},$$

$$(\mathrm{D}^3_{\mathbf{E}}W)_{ijklmn} = \sum_{p,q,r=1}^{3} W_{pqr}(\mathrm{D}_{\mathbf{E}}I_p)_{ij}(\mathrm{D}_{\mathbf{E}}I_q)_{kl}(\mathrm{D}_{\mathbf{E}}I_r)_{mn}$$
$$+ \sum_{p,q=1}^{3} W_{pq}(\mathrm{D}_{\mathbf{E}}I_p)_{ij}(\mathrm{D}^2_{\mathbf{E}}I_q)_{klmn} + \sum_{p,q=1}^{3} W_{pq}(\mathrm{D}_{\mathbf{E}}I_p)_{kl}(\mathrm{D}^2_{\mathbf{E}}I_q)_{ijmn}$$
$$+ \sum_{p,q=1}^{3} W_{pq}(\mathrm{D}_{\mathbf{E}}I_p)_{mn}(\mathrm{D}^2_{\mathbf{E}}I_q)_{ijkl} + \sum_{p=1}^{3} W_p(\mathrm{D}^3_{\mathbf{E}}I_p)_{ijklmn}.$$

Additionally, we need the formulas

$$\mathrm{D}_{\mathbf{E}}I_1 = \mathbf{I}, \qquad \mathrm{D}_{\mathbf{E}}I_2 = I_1\mathbf{I} - \mathbf{E}, \qquad \mathrm{D}_{\mathbf{E}}I_3 = \mathbf{E}^2 - I_1\mathbf{E} + I_2\mathbf{I},$$
$$\mathrm{D}^2_{\mathbf{E}}I_1 = \mathbf{O}, \qquad \mathrm{D}^2_{\mathbf{E}}I_2 = \mathbf{I}\otimes\mathbf{I} - \boldsymbol{\mathcal{I}},$$

and

$$\frac{\partial^2 I_3}{\partial\mathbf{E}\partial\mathbf{E}} = 2\boldsymbol{\mathcal{J}}\mathbf{E} - (\mathbf{I}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{I}) + I_1(\mathbf{I}\otimes\mathbf{I} - \boldsymbol{\mathcal{I}}),$$

where, in index notation,

$$(\mathbf{I}\otimes\mathbf{E})_{ijkl} = \delta_{ij}E_{kl}, \quad \mathcal{I}_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right),$$

$$\mathcal{J}_{ijklmn} = \frac{1}{4}\left(\delta_{ik}\mathcal{I}_{jlmn} + \delta_{il}\mathcal{I}_{jkmn} + \delta_{im}\mathcal{I}_{kljn} + \delta_{in}\mathcal{I}_{kljm}\right),$$

and

$$\boldsymbol{\mathcal{J}}\mathbf{E} = \mathcal{J}_{ijklmn}E_{mn} = \frac{1}{2}\left(\delta_{ik}E_{jl} + \delta_{il}E_{jk} + \delta_{jl}E_{ik} + \delta_{jk}E_{il}\right).$$

Finally, the third derivatives of I_1 and I_2 vanish and, in index notation,

$$(\mathrm{D}^3_{\mathbf{E}}I_3)_{ijklmn} = 2\mathcal{J}_{ijklmn} + \delta_{ij}\delta_{kl}\delta_{mn} - (\delta_{ij}\mathcal{I}_{klmn} + \delta_{kl}\mathcal{I}_{ijmn} + \delta_{mn}\mathcal{I}_{ijkl}).$$

The resulting expressions for the acoustic tensor and the null condition are very lengthy in this form and are not therefore listed here in the general case.

2.4.1. General null condition for the Murnaghan material. The general formula (2.72) obtained for the null condition can also, because of the symmetry of $\mathbf{E}$, be expressed by replacing $\mathbf{N}\otimes\mathbf{Q}_i$ by $\mathbf{Q}_i\otimes\mathbf{N}$ or by its symmetric part. When evaluated in the reference configuration the null condition (2.72) becomes

$$\left[\left(2W^0_{111} + 3W^0_{12}\right)(\mathbf{q}_i\cdot\mathbf{n})^2 + 3\left(2W^0_{11} - W^0_{12}\right)(\mathbf{q}_i\cdot\mathbf{q}_i)\right](\mathbf{q}_i\cdot\mathbf{n}) = 0, \tag{2.76}$$

where we have now written $\mathbf{q}_i = \mathbf{Q}_i, \mathbf{n} = \mathbf{N}$. This is the general form of the null condition for an isotropic elastic material. It can be seen immediately that for transverse waves ($\mathbf{q}_i \cdot \mathbf{n} = 0$) no restriction is imposed by the null condition. For pure longitudinal waves with $\mathbf{q}_i = \mathbf{n}$ and $\mathbf{n} \cdot \mathbf{n} = 1$, on the other hand, the first factor in (2.76) yields the restriction $W^0_{111} + 3W^0_{11} = 0$, thus recovering the result (2.30).

For the Murnaghan material the first factor in (2.76) reduces the null condition to

$$(2l + m)(\mathbf{n} \cdot \mathbf{q}_i)^2 + 3\alpha(\mathbf{q}_i \cdot \mathbf{q}_i) = 0,$$

where we recall that $\alpha = \lambda_L + 2\mu_L + m$. For $\mathbf{q}_i = \mathbf{n}$ this becomes $2l + m + 3\alpha$, as obtained in Sec. 2.2.5.

REMARK 2. For the St. Venant–Kirchhoff material, for which the strain-energy function is given by (see, e.g., CIARLET [16], p. 155)

$$W = \frac{1}{2}(\lambda_L + 2\mu_L)I_1^2 - 2\mu_L I_2, \tag{2.77}$$

the corresponding general null condition is found to be

$$(\lambda_L + 2\mu_L)(\mathbf{q}_i \cdot \mathbf{q}_i)(\mathbf{n} \cdot \mathbf{q}_i) = 0.$$

In particular, for the longitudinal wave with $\mathbf{q}_i = \mathbf{n}$, this yields

$$\lambda_L + 2\mu_L = 0,$$

which contradicts the usual assumptions adopted for the Lamé moduli, one of which is $\lambda_L + 2\mu_L > 0$. This follows from the strong ellipticity condition in the reference configuration, which we assume to hold.

2.5. Application to a class of strain-energy functions

In the literature there are many examples of isotropic strain-energy functions for *incompressible* elastic materials, but far fewer for *compressible* materials. Amongst the available compressible energy functions are the classes introduced by JIANG and OGDEN [17]. These include energy functions of the form

$$W = \bar{W}(\bar{I}_1, \bar{I}_3) = \bar{f}(\bar{I}_1)h_1(\bar{I}_3) + h_2(\bar{I}_3), \tag{2.78}$$

where the overbars indicate that the invariants are principal invariants of $\mathbf{C}$, not $\mathbf{E}$, with $\bar{f}$ a function of $\bar{I}_1$ and h_1 and h_2 functions of $\bar{I}_3$ that satisfy certain conditions which we do not specify here.

We note the connections

$$\bar{I}_1 = 3 + 2I_1, \qquad \bar{I}_3 = 1 + 2I_1 + 4I_2 + 8I_3, \tag{2.79}$$

which will be used below.

For simplicity of illustration we specialize (2.78) by setting $\bar{f}(\bar{I}_1) = f(I_1)$, $h_1 \equiv 1$, $h_2 = h$, so that it becomes

$$W = W(I_1, I_2, I_3) = f(I_1) + h(\bar{I}_3), \tag{2.80}$$

in which $(2.79)_2$ is required.

For the energy and the stress to vanish in the reference configuration we require, recalling (2.28),

$$W^0 = f(0) + h(1) = 0, \quad W_1^0 = f'(0) + 2h'(1) = 0, \tag{2.81}$$

while for compatibility with the classical theory we must have

$$4h'(1) = -2\mu_L = W_2^0, \quad W_{11}^0 = f''(0) + 4h''(1) = \lambda_L + 2\mu_L, \tag{2.82}$$

from which we deduce $f'(0) = \mu_L$.

We also calculate $W_{111}^0 = f'''(0) + 8h'''(1)$, so that the null condition (2.30) becomes

$$3(\lambda_L + 2\mu_L) + f'''(0) + 8h'''(1) = 0. \tag{2.83}$$

Jiang and Ogden [17] considered, in particular, functions of the form

$$\bar{f}(\bar{I}_1) = \frac{\mu_L}{k}\left(\frac{\bar{I}_1 - 1}{2}\right)^k \quad \longrightarrow \quad f(I_1) = \frac{\mu_L}{k}\left(I_1 + 1\right)^k, \tag{2.84}$$

where $k \geq 1/2$ is a material parameter, f satisfies

$$f(0) = \frac{\mu_L}{k}, \qquad f'(0) = \mu_L, \qquad f''(0) = (k-1)\mu_L, \tag{2.85}$$

and h is restricted according to

$$h(1) = -\frac{\mu_L}{k}, \qquad h'(1) = -\frac{1}{2}\mu_L, \qquad h''(1) = \lambda_L + (3-k)\mu_L. \tag{2.86}$$

For the considered specialization equation (2.83) becomes

$$3(\lambda_L + 2\mu_L) + \mu_L(k-1)(k-2) + 8h'''(1) = 0. \tag{2.87}$$

Subject to (2.86) there is considerable flexibility in the choice of the form of h, in particular in the value of $h'''(1)$. Thus, it is possible to find energy functions within the considered class for which the null condition is either satisfied or not satisfied, and this comment applies to wide ranges of other possible energy functions within different classes.

3. Concluding remarks

We have found expressions for the null condition in several different forms. These have been derived for an arbitrary hyperelastic medium and for any plane wave direction, irrespective of the use of the invariants or the assumption of isotropy. They can therefore be useful for application to anisotropic materials, although we do not claim that the null condition alone suffices to prove global existence and uniqueness in the general anisotropic case.

We have derived explicit restrictions imposed by the null condition for the Murnagham material in terms of the second- and third-order elastic constants (Lamé and Murnaghan constants). We have also noted that for the St. Venant–Kirchhoff material the null condition is too strong in that it contradicts the inequality $\lambda_L + 2\mu_L > 0$ and would therefore exclude self-interaction of longitudinal waves and *a fortiori* the propagation of longitudinal waves in the reference configuration. Finally, we have examined the form of the null condition for a special class of energy functions, within which particular energy functions can be constructed that either do or do not satisfy the null condition.

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Appendix

The eigensystem of matrix $\mathbf{A}(\mathbf{0})$

Let $\mathbf{N} = [1, 0, 0]$. The matrix $\mathbf{A}(\mathbf{0}, \mathbf{N}) = \mathbf{A}(\mathbf{0})$ is

$$\mathbf{A}(\mathbf{0}) = -\begin{pmatrix} \mathbf{0} & \mathbf{B}(\mathbf{0}) \\ \mathbf{I} & \mathbf{0} \end{pmatrix},$$

where

$$\mathbf{B}(\mathbf{0}) = -\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}$$

with

$$\alpha_1 = \lambda_L + 2\mu_L, \quad \alpha_2 = \mu_L.$$

The eigenvalues of $\mathbf{A}(\mathbf{0})$ have the form (recall that we have set $\rho_0 = 1$)

$$\lambda_1 = -\sqrt{\alpha_1} = -\lambda_2, \quad \lambda_3 = -\sqrt{\alpha_2} = -\lambda_4, \quad \lambda_5 = -\sqrt{\alpha_2} = -\lambda_6,$$

the right eigenvectors are

$$\mathbf{r}_1 = [-\lambda_1,\, 0,\, 0,\, 1,\, 0,\, 0], \quad \mathbf{r}_2 = [-\lambda_2,\, 0,\, 0,\, 1,\, 0,\, 0] \quad \text{(longitudinal)},$$
$$\mathbf{r}_3 = [0,\, -\lambda_3,\, 0,\, 0,\, 1,\, 0], \quad \mathbf{r}_4 = [0,\, -\lambda_4,\, 0,\, 0,\, 1,\, 0] \quad \text{(transverse)},$$
$$\mathbf{r}_5 = [0,\, 0,\, -\lambda_5,\, 0,\, 0,\, 1], \quad \mathbf{r}_6 = [0,\, 0,\, -\lambda_6,\, 0,\, 0,\, 1] \quad \text{(transverse)},$$

and the left eigenvectors

$$\mathbf{l}_1 = \frac{1}{2}[-\lambda_1^{-1},\, 0,\, 0,\, 1,\, 0,\, 0], \quad \mathbf{l}_2 = \frac{1}{2}[-\lambda_2^{-1},\, 0,\, 0,\, 1,\, 0,\, 0] \quad \text{(longitudinal)},$$
$$\mathbf{l}_3 = \frac{1}{2}[0,\, -\lambda_3^{-1},\, 0,\, 0,\, 1,\, 0], \quad \mathbf{l}_4 = \frac{1}{2}[0,\, -\lambda_4^{-1},\, 0,\, 0,\, 1,\, 0] \quad \text{(transverse)},$$
$$\mathbf{l}_5 = \frac{1}{2}[0,\, 0,\, -\lambda_5^{-1},\, 0,\, 0,\, 1], \quad \mathbf{l}_6 = \frac{1}{2}[0,\, 0,\, -\lambda_6^{-1},\, 0,\, 0,\, 1] \quad \text{(transverse)}.$$

We assume always that $\lambda_L + 2\mu_L > 0$ and $\mu_L > 0$.

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