QUANTIZATION OF TWO-DIMENSIONAL AFFINE BODIES WITH STABILIZED DILATATIONS

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Discussed are some quantization problems of two-dimensional affine bodies. Quantum dilatational motion is stabilized by some appropriately chosen model potentials. Isochoric part of the dynamics is geodetic, i.e. potential-free. Surprisingly enough, this is compatible with the existence of discrete spectrum (bounded quantum motion). The Sommerfeld polynomial method is used to perform the quantization of such problems.

Keywords: affinely-rigid body; two-polar decomposition; Sommerfeld polynomial method.

1. Introduction and motivation

The paper is a continuation of [7, 8] where we have discussed affine symmetry in classical and quantum mechanics of collective and internal modes.

Here we consider the two-dimensional situation on the quantum level [8], which is also of some physical interest. Obviously, it may have some direct physical applications when we deal with flat molecules or other structural elements. Besides, our two-dimensional models may be useful in the theory of surface phenomena. The general ideas of the model remain valid in the physical three-dimensional space and in "academic" considerations concerning the general n-dimensional models. The peculiarity of two-dimensional spaces (flat, spherical and hyperbolic ones) is that there exist then realistically looking potentials with which the system is integrable and may be solved in quadratures by the separation of variable method. This follows from the exceptional geometry of the groups $SO(2,\mathbb{R})$, $GL(2,\mathbb{R})$ among all $SO(n,\mathbb{R})$, $GL(n,\mathbb{R})$ with $n > 2$. The fundamental reason is that $SO(2,\mathbb{R})$ is commutative (and one-dimensional). The models in two-dimensional spaces are therefore interesting in themselves. As we happen to live in (approximately) flat three-dimensional space, thus, for the time being, we postpone the study of infinitesimal affine bodies in curved manifolds (Lobachevski space and spherical space) till later, although they seem to be applicable beyond the purely academic field. Our results may be physically applicable in mechanics of media with microstructure. We mean micromorphic media which are continua of infinitesimal affinely-rigid bodies. Namely, surfaces of such bodies will behave as two-dimensional continua with the effective microstructure induced by the usual three-dimensional microstructure. There are also other possibilities like continua with the layered molecular structure or surface defects. Obviously, the classical curved space results might be applicable in geophysical problems or in ecological applications. Realize e.g. catastrophes like those of tankers and their consequences like the resulting motion of two-dimensional polution "spots" on the oceanic surface [2], or the sliding motion of continental plates. However, such applications, based on classical mechanics are outside our interest in this paper. No doubt, the surface motion of exotic molecules like fullerens also may be analysed according to our quantum models of two-dimensional affine bodies, adapted to curved manifolds; obviously the constant curvature spaces provide the simplest and analytically treatable example. The quantized curved space motion will be the subject of forthcoming papers.

The special stress is laid on models with "large" symmetry groups, i.e. with the doubly-isotropic potentials, depending only on deformation invariants. Kinetic energy models (metrics on the configuration space) are assumed to be:

(i) affinely invariant in the physical and material space,
(ii) spatially isotropic and affinely invariant in the material space,
(iii) affinely invariant in space and materially isotropic.

As usual when Hamiltonians have "large" symmetry groups, such models are completely or to a large extent analytically treatable. Some more detailed arguments, both geometrical and physical, for studying the models (i), (ii) and (iii) quoted above were given in [7, 8].

Roughly speaking, the classical configuration space Q of our model is identified with the group $\mathrm{GL}^+(2,\mathbb{R})$. Obviously, in n-dimensional physical space it would be the group $\mathrm{GL}^+(n,\mathbb{R})$. This identification is a kind of technicality. It is more correct to define the configuration space as a manifold of linear isomorphisms from some material space U onto the physical space V. But when calculating anything, it is more convenient to identify (the choice of bases) U and V with $\mathbb{R}^2$ ($\mathbb{R}^n$ in general) and then just to identify the configuration space with $\mathrm{GL}(2,\mathbb{R})$ ($\mathrm{GL}(n,\mathbb{R})$ in general). If one does not pay any attention to geometric details, there are some dangers of mistakes, but the computational simplification is obvious. So primarily one "identifies" the configuration space with $\mathrm{GL}^+(2,\mathbb{R})$, but later on certain constraints will be imposed so as to make deformation invariants positive, as one always does in elasticity. Some peculiarities of the "Flatland" [1] case $n = 2$ enable one to obtain some analytical rigorous solutions both on the classical and quantum level. This is impossible in higher dimensions because the orthogonal group $\mathrm{SO}(n,\mathbb{R})$ is then simple; this results in the very malicious nonseparable mixing of deformation invariants. Both classical and quantum motion on $\mathrm{GL}^+(n,\mathbb{R})$ may be split into isochoric part $\mathrm{SL}(n,\mathbb{R})$ and one-dimensional dilatational part $\mathbb{R}^+$ (multiplicative group of positive real numbers). It turns out that affinely-invariant geodetic models on $\mathrm{SL}(n,\mathbb{R})$ admit an open subset of bounded trajectories (we mean a subset of the general solution) on the classical level and the discrete spectrum on the quantum level. Of course there is a "dissociation threshold" above which one has to do with unbounded classical trajectories and quantum continuous spectrum [7, 8] (like in the attractive Kepler problem). This is particularly easily seen just in the "Flatland" case $n = 2$. The existence of classical and quantum bounded situations in the geodetic (potential-free) motion on the noncompact manifold $\mathrm{SL}(n, \mathbb{R})$ might seem surprising. However, it is not so because in affinely-invariant models the "metric tensor" underlying the kinetic energy form is curved, essentially Riemannian, thus the superficial analogy with Euclidean space, where geodetics are unbounded, is completely misleading. The geodetic highly-symmetric models possess often (and so is in the situations we study below) rigorous solutions in terms of well-known special functions. And it is interesting in itself that something like oscillatory vibration regime of deformation invariants may be modelled without using the potential energy. Of course this is not the case with the dilatational part of motion. Therefore, to obtain a physically satisfactory vibrational motion (bounded trajectories, discrete quantum spectrum) we must introduce some phenomenological potential depending on one deformation invariant, just the extension ratio (or volume, i.e. surface area in the two-dimensional case). This combination of geodetic $\mathrm{SL}(2, \mathbb{R})$-invariant vibrational regime with some appropriately chosen physically realistic (and analytically treatable) model mechanism of stabilizing dilatations is just the subject of this paper. Strictly speaking, we discuss here the Schrödinger quantization procedure for such a problem. The separation of variables is performed and then the corresponding one-dimensional Schrödinger equations are solved using the Sommerfeld polynomial method [4].

2. Classical description

Let us begin with the classical description. When some reference configuration and Cartesian coordinates are fixed, the configuration space of the two-dimensional affinely-rigid body may be identified with $\mathrm{GL}^+(2, \mathbb{R})$ (when translational degrees of freedom are neglected). This means that the Lagrange (material, reference) coordinates a^A and the Euler (spatial, current) variables x^i are interrelated as follows: $x^i(t, a) = \varphi^i{}_A(t)a^A$, where $\varphi^i{}_A$ are generalized coordinates of internal motion (rotations and deformations).

The most adequate description of internal degrees of freedom is that based on the two-polar decomposition of matrices,

$$\varphi = LDR^{-1}, \tag{1}$$

where L, $R \in \mathrm{SO}(2, \mathbb{R})$, D is diagonal and positive. The term two-polar decomposition is not very fortunate. We use it only because in some of our earlier papers [6, 8] it was used. This decomposition is connected with the algebraic Gram–Schmid orthogonalization. It is also know in literature as the "singular value decomposition".

This decomposition is unique up to the permutation of diagonal elements of D accompanied by the simultaneous multiplication of L and R on the right by the appropriate special orthogonal matrix. This implies that the potential energy of doubly-isotropic models depends only on D and is invariant with respect to the permutation of its nonvanishing matrix elements [5]. The natural parametrization of the problem is as follows:

$$L = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \qquad R = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix},$$

$$D = \begin{bmatrix} Q^1 & 0 \\ 0 & Q^2 \end{bmatrix} = \begin{bmatrix} \exp q^1 & 0 \\ 0 & \exp q^2 \end{bmatrix}.$$

The angular velocities of L- and R-rotators are given, respectively, by

$$\chi = \frac{dL}{dt}L^{-1} = L^{-1}\frac{dL}{dt} = \hat{\chi} = \frac{d\alpha}{dt}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\vartheta = \frac{dR}{dt}R^{-1} = R^{-1}\frac{dR}{dt} = \hat{\vartheta} = \frac{d\beta}{dt}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and their conjugate momenta are respectively as follows:

$$\rho = \hat{\rho} = p_\alpha\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad -\tau = -\hat{\tau} = p_\beta\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where p_α, p_β are canonical momenta conjugate to α, β, respectively. The group $\mathrm{SO}(2,\mathbb{R})$ is commutative, therefore $\rho = \hat{\rho} = S$, $\tau = \hat{\tau} = -V$ [8]. Thus, the quantities $\hat{\rho}$, $\hat{\tau}$ are constants of motion. It is not the case for $n > 2$, where only spin S and vorticity V (generators of spatial and material rotations, respectively) are constants of motion (for invariant geodetic models and, more generally, for doubly-isotropic models). But it is exactly the use of $\hat{\rho}$, $\hat{\tau}$ that simplifies the problem and leads to a partial separation of variables. So, the problem may be effectively reduced to the dynamics of two deformation invariants both on the classical and quantum level.

It is convenient to introduce the new variables:

$$q = \frac{q^1 + q^2}{2}, \qquad x = q^2 - q^1,$$

their conjugate canonical momenta are, respectively,

$$p = p_1 + p_2, \qquad p_x = \frac{p_2 - p_1}{2}.$$

This splits Q into pure dilatational and isochoric parts, i.e., $\mathrm{GL}^+(2,\mathbb{R}) = \mathbb{R}^+\mathrm{SL}(2,\mathbb{R})$.

REMARK. Some comments are necessary here. Namely, the primary thing is the decomposition $\mathrm{GL}^+(2,\mathbb{R}) = \mathbb{R}^+\mathrm{SL}(2,\mathbb{R})$ ("centre times simple part"). But we have used above the coordinates q^1, q^2, thus representing the diagonal elements Q^1, Q^2 in exponential form:

$$Q^1 = \exp q^1, \qquad Q^2 = \exp q^2.$$

This is not quite correct due to the peculiarity of the even dimension $n = 2$. Namely, unlike in the above representation, Q^1 and Q^2 may be simultaneously negative and this does not violate the condition that the determinant of the diagonal part is positive. It would be more correct to use two charts so that

$$Q^1 = \pm \exp q^1, \qquad Q^2 = \pm \exp q^2$$

(coincidence of signs assumed). Another possibility would be to use the "plus signs" only and simultaneously to replace $\mathbb{R}^+$ by $\mathbb{R} \setminus \{0\}$. But in elasticity for certain reasons one assumes the deformation invariants to be positive, and then the use of the (q, x) representation actually denotes imposing some constraints onto $\mathrm{GL}^+(2, \mathbb{R})$.

Thus, the expressions for the classical affine-affine, metric-affine and affine-metric kinetic energies in Hamiltonian representation are as follows [8]:

$$T_{\mathrm{int}}^{\mathrm{aff-aff}} = \frac{p^2}{4(A+2B)} + \frac{p_x^2}{A} + \frac{(p_\beta - p_\alpha)^2}{16A\sinh^2\frac{x}{2}} - \frac{(p_\beta + p_\alpha)^2}{16A\cosh^2\frac{x}{2}},$$

$$T_{\mathrm{int}}^{\mathrm{met-aff}} = \frac{p^2}{4(I+A+2B)} + \frac{p_x^2}{I+A} + \frac{(p_\beta - p_\alpha)^2}{16(I+A)\sinh^2\frac{x}{2}} - \frac{(p_\beta + p_\alpha)^2}{16(I+A)\cosh^2\frac{x}{2}} + \frac{Ip_\alpha^2}{I^2 - A^2},$$

$$T_{\mathrm{int}}^{\mathrm{aff-met}} = \frac{p^2}{4(I+A+2B)} + \frac{p_x^2}{I+A} + \frac{(p_\beta - p_\alpha)^2}{16(I+A)\sinh^2\frac{x}{2}} - \frac{(p_\beta + p_\alpha)^2}{16(I+A)\cosh^2\frac{x}{2}} + \frac{Ip_\beta^2}{I^2 - A^2},$$

where I, A, B are inertial constants. Those alternative choices are a bit enlightened by the remarks below.

One can wonder if there exists any essential difference between the above models of kinetic energy, $T_{\mathrm{int}}^{\mathrm{aff-aff}}$, $T_{\mathrm{int}}^{\mathrm{met-aff}}$, $T_{\mathrm{int}}^{\mathrm{aff-met}}$. From the purely analytical point of view they are in a sense isomorphic. There exists however an important physical distinction between them. The model $T_{\mathrm{int}}^{\mathrm{aff-aff}}$ does not "see" the metric tensor both in the physical and material space. It is invariant under the group $\mathrm{GL}(2, \mathbb{R})$ acting in the physical and material space. Unlike this, $T_{\mathrm{int}}^{\mathrm{met-aff}}$ depends explicitly on the spatial metric tensor but is independent on the material space metric tensor. Therefore, its invariance group is not any longer $\mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R})$ but $\mathrm{SO}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R})$. The hyperspin (affine spin), i.e. Hamiltonian of the right-acting linear group is a conserved quantity, but the spatial hyperspin, i.e. generator of the left-acting $\mathrm{GL}(2, \mathbb{R})$ is not such. However, its skew-symmetric part, i.e. spatial angular momentum, is a constant of motion in virtue of the invariance under the group of left-acting (spatial) rotation $\mathrm{SO}(2, \mathbb{R})$. Such a model may be interpreted as a discretization (in two dimensions) of the Arnold model of fluid, $\mathrm{SDiff}(2, \mathbb{R})$ being restricted to $\mathrm{SL}(2, \mathbb{R})$. And the situation is exactly reciprocal in the case of model

$T_{\rm int}^{\rm aff-met}$. Now the spatial hyperspin will be conserved, but only the skew-symmetric part of the material affine momentum, i.e. the vorticity (Dyson's term [3]) will be a constant of motion. The situation in this respect is more readable in the n-dimensional space $\mathbb{R}^n$, because it is not obscured then by peculiarities ("pathology") of dimension $n = 2$. But it is always true that the models $T_{\rm int}^{\rm aff-aff}$, $T_{\rm int}^{\rm met-aff}$, $T_{\rm int}^{\rm aff-met}$ have identical solutions on the level of variables (angular momentum, deformation invariants and their conjugate momenta, vorticity), i.e. p_α, q, x, p, p_x, p_β. Nevertheless they are different when all degrees of freedom including also (α, β) are taken into account. It is also important that the models $T_{\rm int}^{\rm met-aff}$ and $T_{\rm int}^{\rm aff-met}$ have the maximal symmetry compatible with the one-side affine invariance, when translational degrees of freedom are taken into account. Indeed, nondegenerate metric tensors on the total affine group $\mathrm{GAf}(n, \mathbb{R}) \simeq \mathrm{GL}(n, \mathbb{R}) \times_s \mathbb{R}^n$ do not exist. This follows from the rather malicious nonsemisimplicity of the affine group. Just this fact fixed our attention on the metric–affine and affine–metric model. This kind of symmetry is compatible with including translational degrees of freedom.

We can notice that on the level of new variables all these geodetic models have identical dynamics. The difference appears only on the level of angular variables α and β. And, just as for the general n, the same is true if we introduce to Hamiltonians some doubly-isotropic potentials $V(q, x) = V_{\rm dil}(q) + V_{\rm sh}(x)$,

$$H = T + V(q, x),$$

where T is any of the kinetic energy terms described above. In particular, this is true for dilatation-stabilizing potentials $V_{\rm dil}(q)$, i.e. in a sense, for geodetic invariant models on $\mathrm{SL}(2, \mathbb{R})$ (incompressible bodies).

3. Schrödinger quantization

The mathematical framework of Schrödinger quantization is based on the Hilbert space $L^2(Q, \lambda)$. The Haar measure λ on $\mathrm{GL}(2, \mathbb{R})$ is given as [8]:

$$d\lambda(\alpha; q, x; \beta) = |\sinh x| d\alpha d\beta dq dx.$$

The Fourier expansion of wave functions with respect to α, β is given by

$$\Psi(\alpha; q, x; \beta) = \sum_{m,n} f^{mn}(q, x) e^{im\alpha} e^{in\beta},$$

where $m, n \in \mathbb{Z}$.

The reduced Hamiltonians corresponding to our dynamical affine models are as follows:

$$\begin{aligned} \hat{H}^{mn}_{\rm aff-aff} f^{mn} = &-\frac{\hbar^2}{A}\hat{D}_x f^{mn} - \frac{\hbar^2}{4(A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2} \\ &+ \frac{\hbar^2 (n-m)^2}{16A\sinh^2\frac{1}{2}x} f^{mn} - \frac{\hbar^2 (n+m)^2}{16A\cosh^2\frac{1}{2}x} f^{mn} + V_{\rm dil}(q) f^{mn}, \end{aligned}$$

$$\hat{H}^{mn}_{\text{met-aff}} f^{mn} = -\frac{\hbar^2}{I+A}\hat{D}_x f^{mn} - \frac{\hbar^2}{4(I+A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2} + \frac{\hbar^2 (n-m)^2}{16(I+A)\sinh^2\frac{1}{2}x} f^{mn}$$
$$- \frac{\hbar^2 (n+m)^2}{16(I+A)\cosh^2\frac{1}{2}x} f^{mn} + \frac{I\hbar^2 m^2}{I^2 - A^2} f^{mn} + V_{\text{dil}}(q) f^{mn},$$

$$\hat{H}^{mn}_{\text{aff-met}} f^{mn} = -\frac{\hbar^2}{I+A}\hat{D}_x f^{mn} - \frac{\hbar^2}{4(I+A+2B)}\frac{\partial^2 f^{mn}}{\partial q^2} + \frac{\hbar^2 (n-m)^2}{16(I+A)\sinh^2\frac{1}{2}x} f^{mn}$$
$$- \frac{\hbar^2 (n+m)^2}{16(I+A)\cosh^2\frac{1}{2}x} f^{mn} + \frac{I\hbar^2 n^2}{I^2 - A^2} f^{mn} + V_{\text{dil}}(q) f^{mn},$$

where
$$\hat{D}_x f^{mn} = \frac{1}{|\sinh x|}\frac{\partial}{\partial x}\left(|\sinh x|\frac{\partial f^{mn}}{\partial x}\right).$$

In all these expressions the complete separation between dilatational and incompressible motion is very effectively described in analytical terms just due to the use of new variables q, x.

Solutions of the corresponding stationary Schrödinger equations may be sought in the form
$$f^{mn}(q, x) = \phi^{mn}(q)\chi^{mn}(x);$$
the problem with the potential $V_{\text{dil}}(q)$ reduces then to one-dimensional Schrödinger equations for ϕ^{mn} and χ^{mn}:

(i) affine–affine models:
$$\frac{d^2\phi^{mn}}{dq^2} + \frac{4(A+2B)}{\hbar^2}\left(E_q - V_{\text{dil}}(q)\right)\phi^{mn} = 0, \tag{2}$$
$$\frac{d}{dx}\left(|\sinh x|\frac{d\chi^{mn}}{dx}\right) - \left(\frac{(n-m)^2}{16\sinh^2\frac{1}{2}x} - \frac{(n+m)^2}{16\cosh^2\frac{1}{2}x} - \frac{A}{\hbar^2}E_x\right)|\sinh x|\chi^{mn} = 0; \tag{3}$$

(ii) metric–affine models:
$$\frac{d^2\phi^{mn}}{dq^2} + \frac{4(I+A+2B)}{\hbar^2}\left(E_q - V_{\text{dil}}(q)\right)\phi^{mn} = 0, \tag{4}$$
$$\frac{d}{dx}\left(|\sinh x|\frac{d\chi^{mn}}{dx}\right) - \left(\frac{(n-m)^2}{16\sinh^2\frac{1}{2}x} - \frac{(n+m)^2}{16\cosh^2\frac{1}{2}x} + \frac{Im^2}{I-A} - \frac{I+A}{\hbar^2}E_x\right)|\sinh x|\chi^{mn} = 0; \tag{5}$$

(iii) affine–metric models:
$$\frac{d^2\phi^{mn}}{dq^2} + \frac{4(I+A+2B)}{\hbar^2}\left(E_q - V_{\text{dil}}(q)\right)\phi^{mn} = 0, \tag{6}$$
$$\frac{d}{dx}\left(|\sinh x|\frac{d\chi^{mn}}{dx}\right) - \left(\frac{(n-m)^2}{16\sinh^2\frac{1}{2}x} - \frac{(n+m)^2}{16\cosh^2\frac{1}{2}x} + \frac{In^2}{I-A} - \frac{I+A}{\hbar^2}E_x\right)|\sinh x|\chi^{mn} = 0, \tag{7}$$

where E_q and E_x are fixed values of the energy.

In our case, there exists a discrete spectrum (bounded situations) for χ-functions, i.e. for the isochoric $SL(2,\mathbb{R})$-problem, even in the purely geodetic case without any potential $V_{sh}(x)$. And this is true in spite of the noncompactness of the $SL(2,\mathbb{R})$-configuration space. Everything depends on the relationship between the quantum numbers n, m. If $|n+m|>|n-m|$, the spectrum is discrete. In the opposite case, if $|n+m|<|n-m|$, it is continuous.

It is natural to expect that for dilatation-stabilizing potentials $V_{dil}(q)$ the resulting Schrödinger equations should be rigorously solvable in terms of some standard special functions. The most convenient way of solving them is to use the Sommerfeld polynomial method [4].

4. Some models

In this method the solutions are expressed by the usual or confluent Riemann P-functions. They are deeply related to the hypergeometric functions (respectively, usual F or confluent F_1, F_2). If the usual convergence demands are imposed, then the hypergeometric functions become polynomials and our solutions are expressed by elementary functions. At the same time the energy levels and separation constants are expressed by the eigenvalues of the corresponding operators. There exists some special class of potentials to which the Sommerfeld polynomial method is applicable. The restriction to solutions expressible in terms of Riemann P-functions is reasonable, because this class of functions is well investigated and many special functions used in physics may be expressed by them. There is also an intimate relationship between these functions and representations of Lie groups [9].

Eqs. (2), (4) and (6) may be solved only when the explicit form of potential $V_{dil}(q)$ is specified. It is clear that simple solutions in terms of known special functions may be expected only when the potential has some particular geometric interpretation.

4.1. Harmonic oscillator

Here we consider the model of the harmonic oscillator potential as a "stabilizer" of dilatational motion, i.e.

$$V_{dil}(q)=\frac{\gamma}{2}q^2, \qquad \gamma>0. \tag{8}$$

Applying the Sommerfeld polynomial method we obtain the energy levels $E=E_q+E_x$ as follows:

(i) affine–affine models:

$$E=\hbar\Omega\left(l+\frac{1}{2}\right)-\frac{\hbar^2}{4A}\left[\left(2k+1+\frac{|n+m|}{2}+\frac{|n-m|}{2}\right)^2-1\right], \tag{9}$$

where l, $k=0,1,\ldots$ and

$$\Omega=\sqrt{\frac{\gamma}{2(A+2B)}}.$$

After some calculations we obtain the functions ϕ^{mn} and χ^{mn} in the form

$$\phi^{mn} = e^{-\kappa\frac{q^2}{2}}(\sqrt{\kappa}q)^l F_2\left(-\frac{1}{2}l, -\frac{1}{2}(l-1); -\frac{1}{\kappa q^2}\right), \tag{10}$$

$$\chi^{mn} = \left(\cosh^2\frac{x}{2}\right)^{\mu}\left(\sinh^2\frac{x}{2}\right)^{\nu} F\left(-k, 1+k+2\mu+2\nu; 1+2\mu; \cosh^2\frac{x}{2}\right), \tag{11}$$

where

$$\kappa = \sqrt{\frac{2\gamma(A+2B)}{\hbar^2}}, \qquad \mu = \frac{|n+m|}{4}, \qquad \nu = \frac{|n-m|}{4},$$

(ii) metric–affine models:

$$E = \hbar\tilde{\Omega}\left(l+\frac{1}{2}\right) - \frac{\hbar^2}{4(I+A)}\left[\left(2k+1+\frac{|n+m|}{2}+\frac{|n-m|}{2}\right)^2 - \frac{4Im^2}{I-A} - 1\right], \tag{12}$$

where

$$\tilde{\Omega} = \sqrt{\frac{\gamma}{2(I+A+2B)}}.$$

The functions ϕ^{mn} and χ^{mn} are given respectively by (10) and (11), where

$$\kappa = \sqrt{\frac{2\gamma(I+A+2B)}{\hbar^2}}, \tag{13}$$

(iii) affine–metric models,

$$E = \hbar\tilde{\Omega}\left(l+\frac{1}{2}\right) - \frac{\hbar^2}{4(I+A)}\left[\left(2k+1+\frac{|n+m|}{2}+\frac{|n-m|}{2}\right)^2 - \frac{4In^2}{I-A} - 1\right]. \tag{14}$$

The functions ϕ^{mn} and χ^{mn} have the form (10) and (11) respectively, where κ is given by (13).

4.2. Oscillator combined with the inverse-square repulsion

We consider also another model for the dilatational potential combining the oscillatory attraction for large values of q and infinite repulsion from the singular collapsed state $q = 0$, i.e.

$$V_{\mathrm{dil}}(q) = \xi\left(\frac{1}{q^2} + q^2\right), \qquad \xi > 0. \tag{15}$$

It is seen that $q = 1$ corresponds to the stable equilibrium situation. Here the energy levels $E = E_q + E_x$ are as follows:

(i) affine–affine models:

$$E = \hbar\bar{\Omega}\left(l + \frac{1}{2} + \sqrt{\frac{\xi(A+2B)}{\hbar^2} + \frac{1}{16}}\right) - \frac{\hbar^2}{4A}\left[\left(2k + 1 + \frac{|n+m|}{2} + \frac{|n-m|}{2}\right)^2 - 1\right], \tag{16}$$

where

$$\bar{\Omega} = \sqrt{\frac{4\xi}{A+2B}}.$$

After some calculations we obtain the function ϕ^{mn} in the form

$$\phi^{mn} = e^{-\kappa\frac{q^2}{2}}(\sqrt{\kappa}q)^{a+\frac{1}{2}}F_1\left(-l; 1+a; \kappa q^2\right), \tag{17}$$

where

$$\kappa = \sqrt{\frac{4\xi(A+2B)}{\hbar^2}}, \qquad a = \sqrt{\frac{4\xi(A+2B)}{\hbar^2} + \frac{1}{4}},$$

and χ^{mn} is given by (11).

(ii) metric–affine models:

$$E = \hbar\check{\Omega}\left(l + \frac{1}{2} + \sqrt{\frac{\xi(I+A+2B)}{\hbar^2} + \frac{1}{16}}\right) - \frac{\hbar^2}{4(I+A)}\left[\left(2k + 1 + \frac{|n+m|}{2} + \frac{|n-m|}{2}\right)^2 - \frac{4Im^2}{I-A} - 1\right],$$

where

$$\check{\Omega} = \sqrt{\frac{4\xi}{I+A+2B}}.$$

The functions ϕ^{mn} and χ^{mn} are given respectively by (17) and (11), where

$$\kappa = \sqrt{\frac{4\xi(I+A+2B)}{\hbar^2}}. \tag{18}$$

(iii) affine–metric models

$$E = \hbar\check{\Omega}\left(l + \frac{1}{2} + \sqrt{\frac{\xi(I+A+2B)}{\hbar^2} + \frac{1}{16}}\right) - \frac{\hbar^2}{4(I+A)}\left[\left(2k + 1 + \frac{|n+m|}{2} + \frac{|n-m|}{2}\right)^2 - \frac{4In^2}{I-A} - 1\right].$$

The functions ϕ^{mn} and χ^{mn} have the form (17) and (11) respectively, where κ is given by (18).

5. Conclusions

The considered systems are completely nondegenerate. On the quantum level this fact is reflected by the existence of four quantum numbers labelling the energy levels. They cannot be combined into a single quantum number, i.e. there is no total quantum degeneracy, i.e. hyperintegrability, with respect to them. As yet it is not clear for us if some weaker degeneracy does occur. This is to be discussed later on.

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