Struct Multidisc Optim (2008) 36:217–234
DOI 10.1007/s00158-007-0158-7

REVIEW ARTICLE

# The virtual distortion method—a versatile reanalysis tool for structures and systems

**Przemysław Kołakowski · Marcin Wikło · Jan Holnicki-Szulc**

Received: 15 December 2006 / Revised: 15 May 2007 / Accepted: 28 May 2007 / Published online: 14 August 2007


**Abstract** For 20 years of development, the virtual distortion method (VDM) has proved to be a versatile reanalysis tool in various applications, including structures and truss-like systems. This article presents a summary of principal achievements, demonstrating the capabilities of the VDM both in statics and dynamics, in linear and nonlinear analysis. The major advantage of VDM is its exactness and no need for matrix inversion in the reanalysis algorithm. The influence matrix—numerical core of the VDM—contains the whole mechanical knowledge about a structure, by looking at all global responses due to local disturbances. The strength of the method is demonstrated for truss structures.

**Keywords** Exact structural reanalysis · Sherman–Morrison–Woodbury formulas · Nonlinear statics and dynamics · System analysis

## 1 Introduction

The virtual distortion method (VDM) has been extensively developed since mid-1980s in the Institute of Fundamental Technological Research (see the authors' affiliation). The term *distortions* was adopted from the book entitled "Theory of elasticity" written by Nowacki (1970), who used them to model material dislocations and to describe thermoelastic interactions. Then, Holnicki-Szulc and Gierlinski (1989) initiated the VDM as such, introducing the notion of *virtual distortions* and proposing the idea of *influence matrix*, which is the essence of the method.

The concept of VDM is similar to the previously existing approach, mainly *initial strains*, which was first theoretically considered by Kroner (1958). Subsequently, Argyris (1965) and Maier (1970) used initial strains to model the phenomenon of plasticity in structures. The introduction of an initial strain to the structure causes disturbance to the total equilibrium condition. As there is no relation between the imposed strain and the global response of the structure, the redistribution of stresses in the initial strains approach takes place in iterations. This is essentially the difference between initial strains and VDM, where local–global relations between elements of the structure are gathered in the influence matrix and further utilized in computations. The influence matrix stores information about the whole mechanical knowledge of the structure (topology, materials, and boundary conditions). Thanks to this, the redistribution of stresses due to introduction of a virtual distortion (equivalent to initial strain) is performed in simply one step, without iterations.

The VDM belongs to fast reanalysis methods, which basically means that a primary response of the structure (obtained via a finite element method (FEM) analysis) is further modified by introducing fields of virtual distortions in a fast and efficient way. It was proved by Akgun et al. (2001) that there is an equivalence between the VDM and the general Sherman–Morrison–Woodbury (SMW) formulas derived in 1949–1950, telling how to compute efficiently an inverse of a matrix

P. Kołakowski (✉) · M. Wikło · J. Holnicki-Szulc
Smart-Tech Centre,
Institute of Fundamental Technological Research,
Świętokrzyska 21, Warsaw 00-049, Poland
e-mail: pkolak@ippt.gov.pl
URL: http://smart.ippt.gov.pl

subject to a variation. A brief overview of other fast reanalysis methods can be found in the following section.

The VDM was first applied by Holnicki-Szulc to induce prestress in elastic structures. His works include analysis, design (e.g., remodeling), and control applications (Holnicki-Szulc 1991; Holnicki-Szulc and Gierlinski 1995). The same formal framework has been recently used in other applications related to smart structure technologies, e.g., adaptive structures or inverse problems of identification. Static and dynamic mechanical applications of VDM are demonstrated in this paper. The VDM can effectively work in the plastic regime, provided that the nonlinearity is approximated by piecewise linear sections. The greatest advantage of VDM is its versatility. It is interesting to note that the framework of VDM is general enough to solve problems from other technical fields too, e.g., hydraulic or electrical engineering, thanks to some analogies.

The purpose of the paper is to demonstrate the capabilities of VDM by providing an overview of developments of the method done so far in statics and dynamics. Sections 5.3 and 5.4 briefly describe new, unpublished ideas of handling plasticity and mass modifications in dynamics, respectively. However, they were incorporated into this article for providing a complete view of VDM at the current stage of development.

One-dimensional models (trusses and beams) are the most effective in VDM, as the number of distortions to be imposed in a finite element is small (just one for trusses and three for beams). Plate or shell elements require more distortion states and consequently the composition of the influence matrix becomes more complex and time-consuming. Thanks to the analogies between trusses and nonstructural systems, i.e., water or electrical networks, the VDM has recently been extended to model these systems. Truss structures are quite popular in civil engineering, so the method presented in the paper is readily applicable for optimal design and health monitoring of real structures. For all these reasons, the strength of VDM is further demonstrated for the truss model.

## 2 Overview of reanalysis methods

A few articles (Abu Kassim and Topping 1987; Barthelemy and Haftka 1993) reviewing the static methods of structural reanalysis have appeared in the literature in the last 20 years. The most recent one has been published by Akgun et al. (2001), who describe and compare three methods of structural reanalysis—the combined approximations (CA), theorems of structural variation (TSV), and VDM. It is shown that all the methods stemming from structural analysis are equivalent to the Sherman and Morrison (1949) and Woodbury (1950) formulas originating from purely mathematical considerations for linear modifications of matrices. Akgun et al. admit that the capability of handling physically nonlinear problems by VDM was the incentive for them to extend the SMW formulas to nonlinear range as well. Unlike in VDM, the nonlinear reanalysis by SMW formulas requires an iteration procedure (e.g., Newton-like methods).

Fox and Miura (1971) and Noor and Lowder (1974) presented the idea of the reduced basis approach (also called the Ritz vector approach in model reduction or eigenproblem) in structural reanalysis. The point is that the displacement vector of the modified structure is approximated with a linear combination of only a few (significantly less than the number of the degrees of freedom) linearly independent vectors (similar to influence vectors in VDM) of a previously analyzed structure. Kirsch and Liu (1995) continued to develop the reduced basis idea in his CA method. The basis vectors in the CA method are calculated from a recurrent formula using an inverse of a decomposed stiffness matrix. The number of basis vectors in reanalysis is arbitrarily selected, however, rarely exceeding ten even for large problems. Satisfactory accuracy of response of the system reanalyzed by the CA method is usually assured with only a few basis vectors. If the basis vectors come close to being linearly dependent, then the solution becomes nearly exact. The approach was primarily developed for linear static analysis. An extension of CA to geometrically nonlinear problems (Kirsch 2003) is worth noting.

TSV (Majid and Elliott 1973) are in fact very similar to VDM and were initiated at the same time. Instead of applying unit strains for building the influence matrix, unit loadings are used. Like VDM, the method provides exact results. The first theorem expresses element forces and nodal displacements in a modified structure in terms of forces for the original structure and forces due to unit loadings. The second theorem concerns analogous expression for displacements. The TSV method has been extended to 2D (Topping and Kassim 1987) and 3D (Saka 1998) finite elements. Elastoplastic analysis can be performed by TSV (Majid and Celik 1985), too. No development of the TSV method in dynamics has yet been done, as far as the authors know.

Deng and Ghosn (2001) develop pseudoforce method (PM) to perform reanalysis. The concept of pseudoforces, analogous to virtual distortions in VDM and pseudoloads in TSV, is used to model structural modifications. Basing on the SMW formula, which requires the inverse of an initial stiffness matrix, an algorithm is proposed for solving both linear and nonlinear reanalysis problems. It is noted that at some point of nonlinear incremental analysis, factorization of the stiffness matrix may be necessary. Otherwise, the PM solution will prove costlier than a standard solver. Linear reanalysis of optimal placement of bracing for a 2D frame and an elastoplastic analysis of a bridge deck are presented.

Bae and Grandhi (2004) use successive matrix inversion (SMI) method for reanalysis of structural systems. For initialization, the inverse of an initial stiffness matrix $K$ is required. Subsequently, the applied structural modification $\Delta K$ is decomposed into submodifications $\Delta K_j$ ($j$=1, DOF (degree of freedom)), each one with only the $j$th nonzero column for the DOF $\times$ DOF system. This allows for taking advantage of the Neumann (binomial) series expansion at the element level to obtain a recursive formula for finding the inverse of the modified stiffness matrix $K + \Delta K$ instead of inverting it directly. The SMI method is applied to a truss, frame, and plate in linear statics. Approximated (not exact) solutions are obtained.

An approach proposing improvement of accuracy to the Neumann series expansion was proposed by Hurtado (2002). To this end, Shanks transformation (ST) is used to handle large modifications effectively. A significant improvement compared to the Pade approximation, described in Chen et al. (2000), is demonstrated. Comparison with CA shows that the presented method is equally accurate, however, exhibits faster convergence with the increase of expansion terms in the Neumann series. Linear examples of trusses are presented.

The term reanalysis in the nonlinear range may be understood in two ways. The first way is the standard modification to a structural parameter like in linear problems. The second way is different—it is rather an improvement (reduction of operations) of the Newton–Raphson procedure, which performs iterations to follow a nonlinear path. Examples of the different understanding of reanalysis are applications of the ST method and the Leu and Tsou (2000) method.

Most of the existing reanalysis methods in dynamics concentrate on resolving the modal problem, in which only modifications to eigenvalues and eigenmodes are considered. This problem is solved quasistatically in the frequency domain (no dependence on time is investigated). A review of some eigenvalue reanalysis methods can be found in Chen et al. (2000).

Recent methods dealing with reanalysis of the eigenproblem are generally named in the literature as structural dynamic modification (SDM). For solving the SDM problem, Ravi et al. (1998) propose single-step perturbation method as an alternative to a previously developed multiplestep perturbation. The single step approach seems to outrank the multistep one, both in terms of accuracy for large modifications and computational effort. Yap and Zimmermann (2002) prove that their iterative SDM method provides better estimates of both natural frequencies and mode shapes than sensitivity-based methods. It can also provide a reasonable trade-off between accuracy and computational effort. McDonnell–Douglass test space structure was used to demonstrate the validity of their approach. Chen (2006) proposes an efficient iterative SDM for large modifications of modal parameters, basing only on limited knowledge of the original mode shapes (neither the original stiffness nor mass matrix is required). His noniterative high-order approximation approach also gives good estimations of the modified modal parameters. Reduced eigenvalue reanalysis presented by Grissom et al. (2005) is used to predict the behaviour of a structure with multiple absorbers on the basis of the response of the structure without absorbers. The method is confronted with impedance-based approaches. The obtained results agree with the ones produced by the FE code NASTRAN and measured in experiment. Recently, Kirsch et al. (2006) extended the CA method to nonlinear dynamic problems. Similar to SDM, the CA approach in structural dynamics is also limited to recalculation of an eigenproblem. The procedure involves shifts of the basis vectors and Gram–Schmidt orthogonalizations. The effectiveness very much depends on a proper choice of the basis vectors. The approach has been validated against the FE code ADINA. Huang et al. (2000) propose a reanalysis method based on Rayleigh–Ritz analysis, which handles extension of the basis vectors. This enables performance of an eigenproblem reanalysis in case of topological changes, i.e., addition of members and joints to the structure. Accuracy highly depends upon the number of eigenmodes analyzed for the original structure. All the above-mentioned SDM methods neglect the damping matrix in the analysis. With the perturbation approach, proposed by Cronin (1990) and Tang and Wang (1996), it is possible to analyze modifications to the damping characteristics of a structure as well. The assumption is that the original structure exhibits classical (proportional) damping, which means that it has the same modes as the corresponding undamped structure. Thus, the perturbation reanalysis can be performed in the configuration space by using the known real modes.

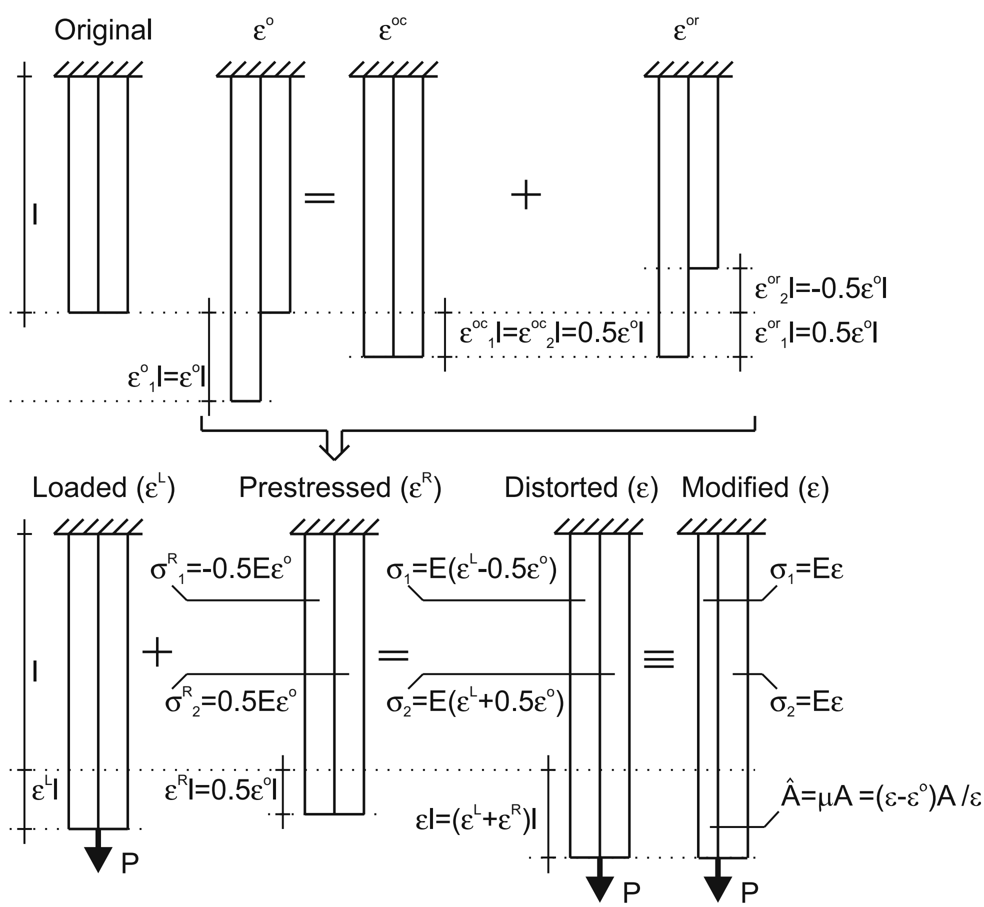
The only reanalysis method, known to the authors, producing a dynamic response in the time domain is the one based on the dynamic modification method proposed by Muscolino (1996). Cacciola et al. (2005) continue to develop this method proving its numerical efficiency and accuracy. For performing dynamic analysis, the equations of motion for a classically damped structure are uncoupled by the modal coordinate transformation, which also reduces the modal space (similarly to the Ritz vector approach). As a result, diagonal instead of full matrices enter the equations of motion. The second step is reformulation of the reduced problem in the state space. A tridiagonal transition matrix has to be defined. This allows for employing a relatively simple solution procedure for the state variables involving operations (including inversions) on tridiagonal matrices. Finding the solution back in the original modal space is straightforward. For performing a dynamic reanalysis with this method, an analogous procedure is used in which the increment of modification has to be specified explicitly. All other matrices appearing in the reanalysis are related to the original structure. It is claimed that nonproportional damping can be handled by the method by treating it as a system modification. Accurate results of a response in the time domain with only five modes in the reduced basis are presented for a truss structure. Both deterministic and stochastic loads are considered. The approach fails only for drastic modifications of the original system, for which the order of the reduced modal space is not significantly lower than the original one and the numerical gain vanishes.

## 3 Virtual distortion method—main idea

In the whole paper, the lowercase subindices refer to elements in the local coordinate system. The uppercase subindices refer to nodes in the global coordinate system. Einstein's summation rule is used. Underlined indices are exempt from summation.

A simple two-bar truss has been chosen for demonstration of the main idea. Figure 1 schematically depicts an *original structure* consisting of two parallel truss elements, suppressed at the upper common node and exhibiting identical deformation at the lower common node. Assume that the left-hand element of the original structure has been subject to a modification (e.g., due to a change of its cross-sectional area).

**Fig. 1** VDM scheme



Let us call the corresponding *initial strain* of the left-hand element (in isolation, i.e., out of structure) a *virtual distortion* $\varepsilon_1^0$. This initially deformed member has to comply with the continuity constraints of the structure. Thus, placing the element back into the structure provokes a self-equilibrated state of residual stresses $\sigma_i^{\mathrm{R}}$ and a compatible state of strains $\varepsilon_i^{\mathrm{R}}$ (see the *prestressed structure* in Fig. 1).

Then, let us apply external force-type load P to the analyzed structure. It generates the deformation denoted by $\varepsilon_i^{\mathrm{L}}$ in the *loaded*, linearly elastic structure. Superposing these two states of the prestressed and the loaded structure, we get, as a result, a *distorted structure* (with combination of linearly elastic responses to initial strains and external load). It is postulated now that (as marked in Fig. 1) the distorted structure be identical in terms of final strains $\varepsilon_i$ and internal forces $A_{\underline{i}}\sigma_{\underline{i}}$ with a *modified structure* (with modified cross-sectional area in the left-hand element from $A_1$ to $\hat{A}_1$).

Virtual distortions can be used to simulate not only modifications of material distribution but also material nonlinearities, i.e., plastic effects.

An arbitrary state of distortions can be uniquely decomposed $\varepsilon_i^0 = \varepsilon_i^{0c} + \varepsilon_i^{0r}$ (cf. Holnicki-Szulc and Gierlinski 1995, see Fig. 1). The component $\varepsilon^{0c}$ is responsible for the compatible, stress-free deformation of the structure (e.g., caused by homogenous heating of both elements of the truss) while the component $\varepsilon^{0r}$ causes the self-equilibrated, strain-free stress state in the structure (e.g., caused by heating of the left element with simultaneous cooling of the right one). The components $\varepsilon_i^{0r}$ are presented in Fig. 1 before satisfying the continuity constraints.

## 4 Static virtual distortion method

### 4.1 Influence matrix in statics

The main feature distinguishing VDM from the initial strains approach is the influence matrix $D_{ij}$. It describes strains in the truss member $i$ caused by the unit virtual distortion $\varepsilon_i^0 = 1$ (unit initial strain) applied to the member $j$. The unit virtual distortion is practically imposed as a pair of *self-equilibrated compensative forces* of reverse signs (equivalent to a unit strain as in Fig. 2) applied to the nodes of the strained element. The influence matrix $D_{ij}$ collects $m$ influence vectors, where $m$ denotes the number of truss elements. To build an influence vector, a solution of a standard linear elastic problem by the FEM has to be found:

$$K_{MN}u_N = f_M \tag{1}$$

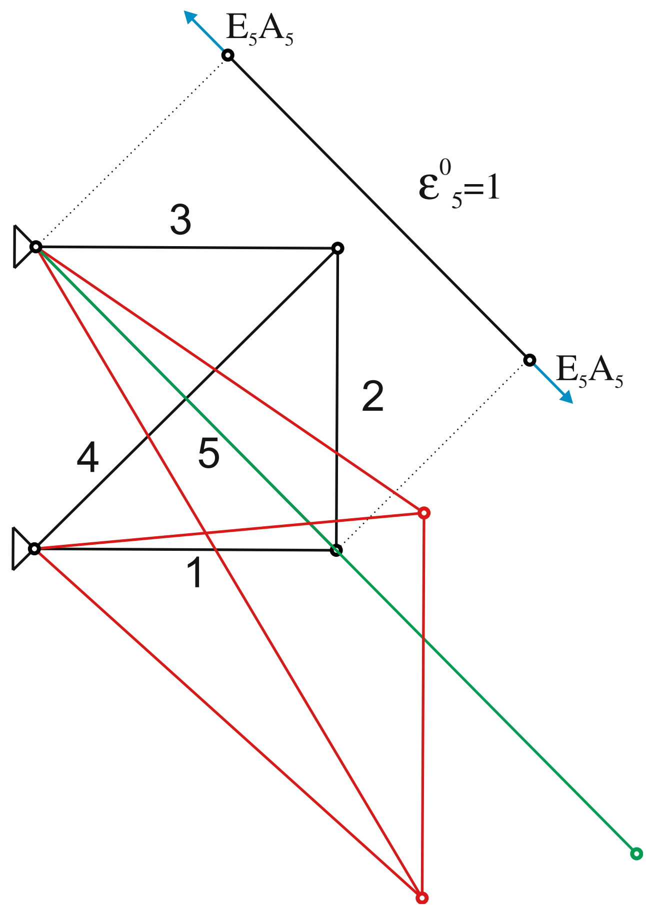


**Fig. 2** Influence of the unit distortion applied in chosen location

with K being the stiffness matrix. Usually, the obtained displacements serve to calculate a corresponding response in strains:

$$\varepsilon_i = G_{iN}u_N \tag{2}$$

with $G_{iN}$ being the geometric matrix, which transforms global degrees of freedom to local strains. The response in strains is a standard for building an influence vector. However, storage of any other required response is also useful, i.e., displacements, stresses, or forces.

The external force vector f in (1) corresponds to two compensative forces (axial forces in case of truss structures) applied to a structural member, equivalent with application of a unit strain to the unconstrained member (see the diagonal element in Fig. 2 after applying the pair of forces). The response of the structure to the imposition of the unit virtual distortion $\varepsilon_5^0 = 1$ is depicted by the deformed configuration in Fig. 2.

Thus, to build the influence matrix $D_{ij}$, $m$ solutions of a linear elastic problem have to be found. The set (1) has to be solved with $m$ different right-hand sides corresponding to $m$ pairs of compensative forces applied successively in each structural member. This way, the influence matrix stores information about the entire structure properties, including topology, material characteristics, and boundary conditions in calculation of structural response.

Note that the static influence matrix for statically determinate structures becomes identity matrix (zero redundancy means no interrelations between members) and the VDM loses its major tool.

For truss structures, the strain influence matrix $D_{ij}$ is quadratic, nonsymmetric and singular. Making use of Betti's mutual work principle, one can easily prove that the matrix $A_{\underline{i}} l_{\underline{i}} D_{\underline{i}j}$ becomes symmetric, where $A_i$ and $l_i$ denote initial cross-sectional area and element length, respectively. The rank of the $m \times m$, symmetric matrix is $\text{rank}[A_{\underline{i}} l_{\underline{i}} D_{\underline{i}j}] = m - k$, where $m$ and $k$ denote the number of all elements and the structural redundancy, respectively. It means that there are *m-k* linearly independent components $\varepsilon_i^{0c}$ causing stressless compatible strains $\varepsilon_i^{\mathrm{R}}$ and $k$ linearly independent components $\varepsilon_r^{0r}$ causing strainless self-equilibrated stresses $\sigma_i^{\mathrm{R}}$. All non-vanishing eigenvalues of the matrix $A_{\underline{i}} l_{\underline{i}} D_{\underline{i}j}$ are positive, thus, it is nonnegative definite.

Analogously, one can prove that the influence matrix storing responses in stresses $A_{\underline{i}} l_{\underline{i}} D_{\underline{i}j}^{\sigma} = A_{\underline{i}} l_{\underline{i}} E_{\underline{i}} (D_{\underline{i}j} - \delta_{\underline{i}j})$ is symmetric, nonpositive definite of the rank $[A_{\underline{i}} l_{\underline{i}} D_{\underline{i}j}] = k$. $E_i$ denotes Young's modulus and $\delta_{ij}$ is the Kronecker's delta. This matrix (originally called Z) was introduced by Maier (cf. Maier 1970) and applied to initial strains approach allowing for elastoplastic analysis of stress redistribution through a quadratic programming procedure. In the VDM approach, however, the distortions $\beta_j^0$ modeling plastic permanent deformations can be simply determined by solving a set of linear equations while satisfying the condition that yield stress be reached in all overloaded members (cf. Holnicki-Szulc and Gierlinski 1995).

### 4.2 Stiffness remodeling in statics

Let us confine our considerations to truss structures in the elastic range first. Let us consider introducing a field of virtual distortions $\varepsilon_j^0$ into a truss structure. This action will induce *residual strains* and *stresses* in the structure, expressed as follows (cf. Holnicki-Szulc 1991; Holnicki-Szulc and Gierlinski 1995):

$$\varepsilon_i^{\mathrm{R}} = D_{ij} \varepsilon_j^0 \tag{3}$$

$$\sigma_i^{\mathrm{R}} = E_{\underline{i}} \left( D_{\underline{i}j} - \delta_{\underline{i}j} \right) \varepsilon_j^0 \tag{4}$$

Assume that application of external load to the structure provokes elastic *linear response* $\varepsilon_i^{\mathrm{L}}$ and $\sigma_i^{\mathrm{L}}$, which will be superposed with the residual response $\varepsilon_i^{\mathrm{R}}$ and $\sigma_i^{\mathrm{R}}$. Thus, in view of (3) and (4), we get:

$$\varepsilon_i = \varepsilon_i^{\mathrm{L}} + \varepsilon_i^{\mathrm{R}} = \varepsilon_i^{\mathrm{L}} + D_{ij} \varepsilon_j^0 \tag{5}$$

$$\sigma_i = \sigma_i^{\mathrm{L}} + \sigma_i^{\mathrm{R}} = E_{\underline{i}} \varepsilon_{\underline{i}}^{\mathrm{L}} + E_{\underline{i}} \left( D_{\underline{i}j} - \delta_{\underline{i}j} \right) \varepsilon_j^0 = E_{\underline{i}} \left( \varepsilon_{\underline{i}} - \varepsilon_{\underline{i}}^0 \right) \tag{6}$$

Relation between element forces $p_i$ and stresses $\sigma_i$ is known via the cross-sectional areas $A_i$:

$$p_i = A_{\underline{i}} \sigma_{\underline{i}} \tag{7}$$

Let us now take into account structural stiffness modifications exemplified by changes of Young's modulus. This means considering a modified value $\hat{E}_i$. In view of (6) and (7) we can express element forces in the modified structure and original structure with introduced virtual distortion field (i.e., distorted structure), as follows:

$$\hat{p}_i = \hat{E}_{\underline{i}} A_{\underline{i}} \hat{\varepsilon}_{\underline{i}} \tag{8}$$

$$p_i = E_{\underline{i}} A_{\underline{i}} \left( \varepsilon_{\underline{i}} - \varepsilon_{\underline{i}}^o \right) \tag{9}$$

The main postulate of the VDM in static remodeling requires that local strains (including plastic strains) and forces in the modified and distorted structure are equal:

$$\hat{\varepsilon}_i = \varepsilon_i \tag{10}$$

$$\hat{p}_i = p_i \tag{11}$$

This postulate leads to the following relation:

$$\hat{E}_{\underline{i}} A_{\underline{i}} \varepsilon_{\underline{i}} = E_{\underline{i}} A_{\underline{i}} \left( \varepsilon_{\underline{i}} - \varepsilon_{\underline{i}}^0 \right) \tag{12}$$

Equation (12) provides the coefficient of the stiffness change for each truss element $i$ as the ratio of the modified Young's modulus to the original one:

$$\mu_i^{\mathrm{E}} \stackrel{\text{def}}{=} \frac{\hat{E}_{\underline{i}}}{E_{\underline{i}}} = \frac{\varepsilon_{\underline{i}} - \varepsilon_{\underline{i}}^0}{\varepsilon_{\underline{i}}} = \frac{\hat{A}_{\underline{i}}}{A_{\underline{i}}} \stackrel{\text{def}}{=} \mu_i^{\mathrm{A}} \tag{13}$$

Note that the coefficient $\mu_i^{\mathrm{E}}$ may be equivalently expressed as the ratio of the original to modified cross-sectional area of a truss element $\mu_i^{\mathrm{A}}$. If $\mu_i^{\mathrm{E}} = 1$, we deal with an intact structure. Variation of the coefficient in the range $0 \leq \mu_i^{\mathrm{E}} \leq 1$ means reduction of stiffness and in the range $\mu_i^{\mathrm{E}} \geq 1$ increase of stiffness. Substituting (5) into (13), we get a set of equations for $\varepsilon_i^0$, which must be solved for an arbitrary number of modified elements (usually small compared to all elements in the structure), described by a coefficient $\mu_i^{\mathrm{E}}$ different than 1:

$$\left( D_{\underline{i}j} - \frac{\delta_{\underline{i}j}}{1 - \mu_{\underline{i}}^{\mathrm{E}}} \right) \varepsilon_j^0 = -\varepsilon_i^{\mathrm{L}} \tag{14}$$

### 4.3 Plasticity in statics

Virtual distortion field introduced in the structure may be twofold. We shall distinguish between purely virtual distortions $\varepsilon_i^0$ (having no physical meaning) used for modeling structural geometry modifications (e.g., changes of cross-sectional area) and plastic-like distortions $\beta_i^0$ used for simulating physical nonlinearities in the structure. The plastic-like distortions are identified with plastic strains:

$$\beta_i^0 \equiv \varepsilon_i^{\mathrm{pl}} \tag{15}$$

and have no virtual character. Thus, the plastic behavior of members is effectively included in the strain and stress formulas in the following way [cf. (5) and (6)]:

$$\varepsilon_i = \varepsilon_i^{\mathrm{L}} + D_{ik}\beta_k^0 \tag{16}$$

$$\sigma_i = E_{\underline{i}}\varepsilon_{\underline{i}}^{\mathrm{L}} + E_{\underline{i}}\left(D_{\underline{i}k} - \delta_{\underline{i}k}\right)\beta_k^0 = E_{\underline{i}}\left(\varepsilon_{\underline{i}} - \beta_{\underline{i}}^0\right) \tag{17}$$

The VDM can be used to model nonlinear constitutive relation provided that it is piecewise linear (see Fig. 3).

Consequently, let us assume the behaviour of material, after reaching the yield limit $\sigma_i^\star$, as a linear section with inclination $\gamma_{\underline{i}}E_{\underline{i}}$ to the horizontal axis less than the original Young's modulus $E_i$:

$$\sigma_i - \sigma_i^\star = \gamma_{\underline{i}}E_{\underline{i}}\left(\varepsilon_{\underline{i}} - \varepsilon_{\underline{i}}^\star\right) \tag{18}$$

The coefficient $\gamma_i$ determines isotropic hardening of material. If $\gamma_i = 0$, perfectly plastic behaviour occurs.

Substituting (16) and (17) to (18), a local set of equations is assembled to be solved for plastic distortions $\beta_i^0$:

$$\left(D_{\underline{i}k} - \frac{\delta_{\underline{i}k}}{1-\gamma_{\underline{i}}}\right)\beta_k^0 = -\varepsilon_i^L + \varepsilon_i^\star \tag{19}$$

### 4.4 Example no. 1 in statics

The five-element, $1 \times 1$ m truss structure, shown in Fig. 4, has been chosen for demonstration of the VDM capabilities in statics. All elements have the same Young's modulus E=210 GPa and cross-sectional area A=1.0e-05 m$^2$. The structure is subjected to a static vertical force of F=2.5 kN in node no. 2. Buckling is not taken into account.

The strain influence matrix D of the structure and an equivalent of the stress influence matrix D-I (see static influence matrix) take the following values:

$$D = \begin{bmatrix} 0.8845 & -0.1155 & -0.1155 & 0.2310 & 0.2310 \\ -0.1155 & 0.8845 & -0.1155 & 0.2310 & 0.2310 \\ -0.1155 & -0.1155 & 0.8845 & 0.2310 & 0.2310 \\ 0.1634 & 0.1634 & 0.1634 & 0.6733 & -0.3267 \\ 0.1634 & 0.1634 & 0.1634 & -0.3267 & 0.6733 \end{bmatrix}$$

$$D - I = \begin{bmatrix} -0.1155 & -0.1155 & -0.1155 & 0.2310 & 0.2310 \\ -0.1155 & -0.1155 & -0.1155 & 0.2310 & 0.2310 \\ -0.1155 & -0.1155 & -0.1155 & 0.2310 & 0.2310 \\ 0.1634 & 0.1634 & 0.1634 & -0.3267 & -0.3267 \\ 0.1634 & 0.1634 & 0.1634 & -0.3267 & -0.3267 \end{bmatrix}$$

Note that the degree of redundancy of the structure is 1, which is also the rank of the stress influence matrix

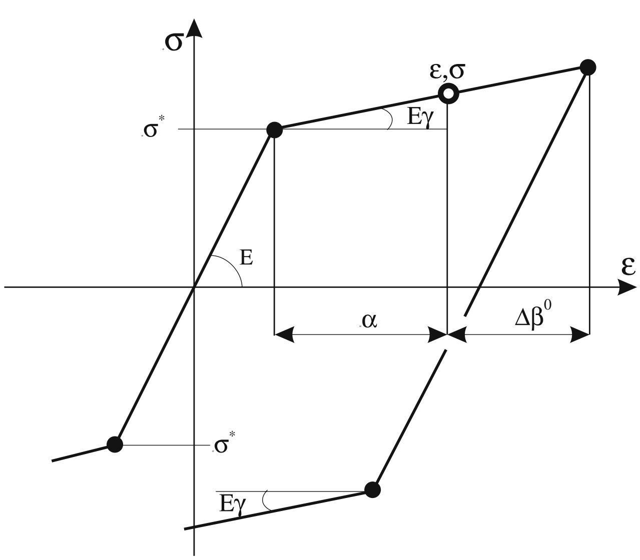



**Fig. 3** Piecewise nonlinear constitutive law

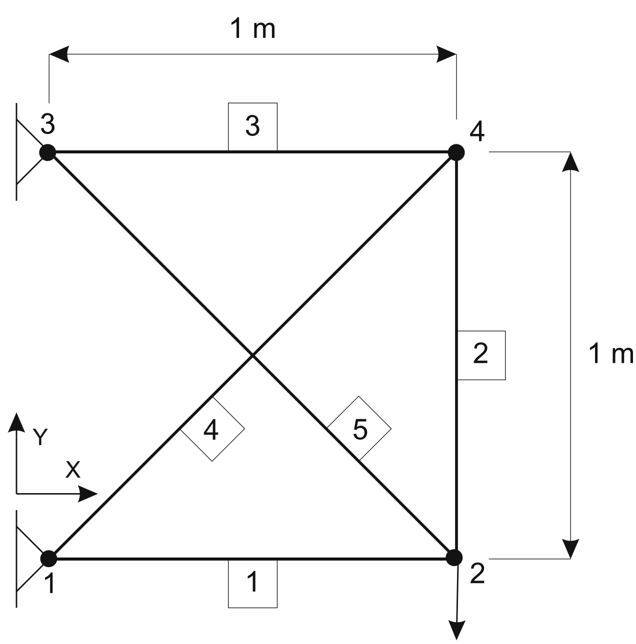



**Fig. 4** Five-element truss structure for testing VDM algorithms

**Table 1** Results of elimination of three members from the original truss structure

| | $\varepsilon^{L}$ | $\varepsilon^{R}$ | $\varepsilon$ | $\varepsilon^{0}$ |
|---|---|---|---|---|
| 1 | –0.664E–03 | –0.526E–03 | –0.119E–02 | 0.000E+00 |
| 2 | 0.526E–03 | 0.397E–03 | 0.923E–03 | 0.923E–03 |
| 3 | 0.526E–03 | 0.397E–03 | 0.923E–03 | 0.923E–03 |
| 4 | –0.744E–03 | –0.610E–03 | –0.135E–02 | –0.135E–02 |
| 5 | 0.938E–03 | 0.744E–03 | 0.168E–02 | 0.000E+00 |
| | $\sigma^{L}$ | $\sigma^{R}$ | $\sigma$ | $\mu$ |
| 1 | –0.139E+09 | –0.110E+09 | –0.250E+09 | 1.000 |
| 2 | 0.110E+09 | –0.110E+09 | –0.642E–07 | 0.000 |
| 3 | 0.110E+09 | –0.110E+09 | 0.104E–07 | 0.000 |
| 4 | –0.156E+09 | 0.156E+09 | –0.234E–07 | 0.000 |
| 5 | 0.197E+09 | 0.156E+09 | 0.353E+09 | 1.000 |

**Table 2** Results of elastoplastic analysis of the truss at the stage preceding collapse ($\alpha = 1.66$)

| | $\varepsilon^{L}$ | $\varepsilon^{R}$ | $\varepsilon$ | $\varepsilon^{0}$ |
|---|---|---|---|---|
| 1 | –0.110E–02 | 0.112E–03 | –0.990E–03 | 0.000E+00 |
| 2 | 0.873E–03 | 0.112E–03 | 0.985E–03 | 0.000E+00 |
| 3 | 0.873E–03 | 0.112E–03 | 0.985E–03 | 0.000E+00 |
| 4 | –0.124E–02 | –0.158E–03 | –0.139E–02 | 0.000E+00 |
| 5 | 0.156E–02 | 0.325E–03 | 0.188E–02 | 0.483E–03 |
| | $\sigma^{L}$ | $\sigma^{R}$ | $\sigma$ | $\mu$ |
| 1 | –0.231E+09 | 0.234E+08 | –0.208E+09 | 1.000 |
| 2 | 0.183E+09 | 0.234E+08 | 0.207E+09 | 1.000 |
| 3 | 0.183E+09 | 0.234E+08 | 0.207E+09 | 1.000 |
| 4 | –0.259E+09 | –0.332E+08 | –0.293E+09 | 1.000 |
| 5 | 0.327E+09 | –0.332E+08 | 0.294E+09 | 1.000 |

D-I (there is only one state of prestress available for the truss—all columns of the matrix D-I are linearly dependent). The rank of the strain influence matrix D is equal to 4 (there may be four states of strains accompanying the single prestress state).

Let us first demonstrate how we can quickly remodel the topology of the structure, simulating elimination of elements nos. 2, 3, and 4 by virtual distortions. To this end, the condition $\mu = 0$ is imposed in the mentioned members. This leads to a set of equations ($3 \times 3$) to be solved for $\varepsilon^{0}$ in one step [cf. (14)]. The results of the analysis are presented in Table 1.

The remaining members nos. 1 and 5 form a statically determinate structure, which can be further optimized to become isostatic (i.e., of zero redundancy and uniformly strained). To achieve this goal in this example, the stress in element no. 5 should be reduced (increasing the cross-section by $\sqrt{2}$) to match the stress in element no. 1.

Next, let us perform an elastoplastic analysis of the structure, assuming the yield limit $\sigma^{\star}$=294 MPa, and perfectly plastic ($\gamma = 0$) postcritical behaviour. The nominal load F is gradually increased by the factor $\alpha > 1$. Only one member can enter the plastic zone without violating the integrity of the structure. It is the diagonal element no. 5. Any other plastic hinge (element) will provoke kinematic mechanism. The results of the final stage (just before collapse at $\alpha = 1.66$ when the other diagonal no. 4 is very close to the yield limit $\sigma^{\star}$) are presented in Table 2.

The use of VDM for optimal static design of more complex structures, including beams and in-plane loaded plates, is amply exemplified in Holnicki-Szulc and Gierlinski (1995).

## 4.5 Example no. 2 in statics

The problem of optimal remodeling of truss structures in statics, using VDM, was previously presented in Kolakowski and Holnicki-Szulc (1997). From that article, a medium-size truss example was chosen to provide an insight into topological optimization capabilities of VDM. The *ground structure* approach, utilizing a regular $5 \times 5$ grid of nodes and considering 300 possible connections between them, was adopted. To reduce the computational effort associated with the literal ground structure, many members (i.e., overlapping and between supports) were disregarded to start up with only 136 connections. The horizontal-to-vertical aspect ratio of the grid is 8:5 ($24 \times 15$ m). Uniform cross-section A=2.55 cm$^2$ is assumed for all initial members.

The problem is posed in a classical manner as finding the minimum volume of the ground structure subjected

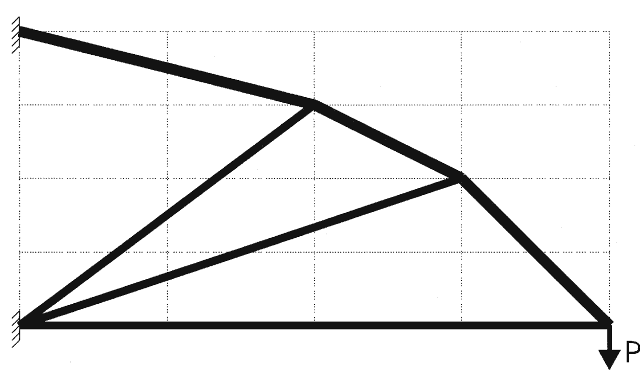



**Fig. 5** First optimal topology for the 136-element ground structure

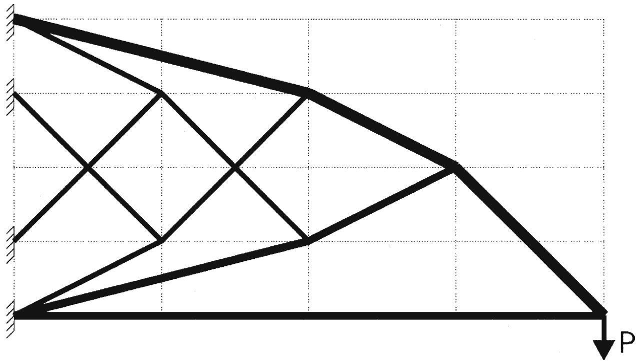



**Fig. 6** Second optimal topology for the 136-element ground structure

to one static force P, provided that limit stresses are not exceeded in any member. An equivalent formulation of the problem is to find maximum stiffness at constant volume. It is clear that most of the initial members should be eliminated in the process of structural remodeling and the remaining ones should be resized. For one load case, presented here, the resultant optimal truss is an isostatic (i.e., statically determinate, uniformly stressed) structure. Using VDM-based gradient optimization, two solutions, complying with the requirement, were found. The reason for having two solutions is that the threshold for elimination of members in the remodeling process—the coefficient of stiffness change [cf. (13)]—was arbitrarily adjusted in the range $0.05 \leq \mu \leq 0.10$. The first topology, shown in Fig. 5, consists of only six members and the corresponding final volume is 80.61 dm$^3$. The second topology, depicted in Fig. 6, consists of 12 members and the corresponding final volume is 80.26 dm$^3$. Formally, the performance of the algorithm was equally good in both cases. However, from the application point of view, the first topology is more attractive.

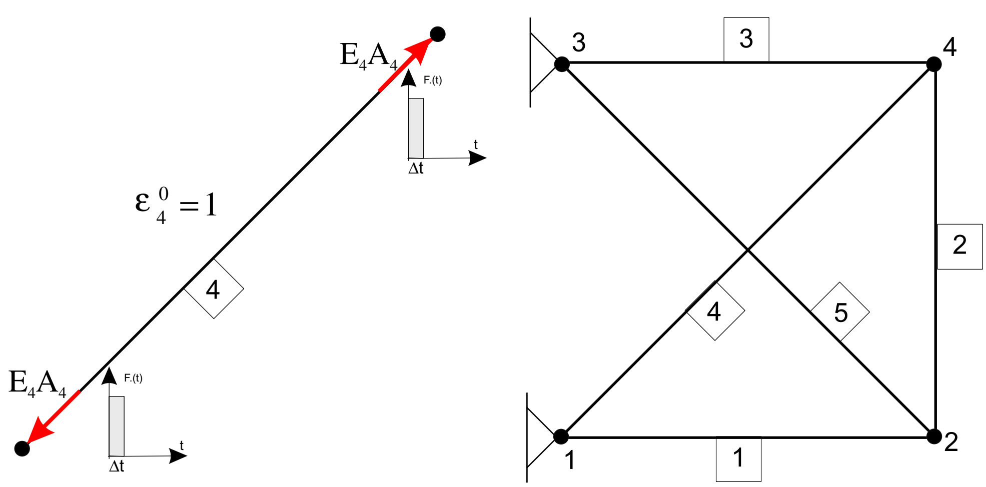
## 5 Dynamic virtual distortion method

### 5.1 Influence matrices in dynamics

For dynamic problems, the influence matrix has to be given one more dimension—time. The imposition of unit virtual distortion takes place in the first instant of an analyzed period of time. This corresponds to the first time step in numerical algorithms, where certain time discretization is assumed. Similar to statics, the unit virtual distortion $\varepsilon^0 = 1$ in the first time step is applied to an element as a pair of self-equilibrated compensative forces causing a unit strain of the element, when taken out of the structure (see Fig. 7). Such action has the character of an impulse excitation, which is consecutively imposed in all elements of the structure to compose the whole influence matrix. In practice, the response of the structure to the *impulse virtual distortion* in element is calculated using the Newmark integration algorithm over a chosen period of time. We are interested in two influence matrices, one storing the structural response in displacements (further denoted by $B^\varepsilon$) and the other—in strains (further denoted by $D^\varepsilon$).

The above-described influence matrices in dynamics are full analogies to the influence matrix in statics (expressed in strains), able to model stiffness changes and material nonlinearities. The second important parameter in dynamics is inertia and the ability of modeling mass changes. To this end, another kind of influence matrices (further denoted by $B^p$ and $D^p$) must be introduced. This time the virtual distortions are successively imposed in degrees of freedom of the structure (see Fig. 8). The principal difference is that the nodal virtual distortion has the form of an unequilibrated unit impulse force, contrary to unit virtual distortion applied

**Fig. 7** Impulse virtual distortion in an element, showing the process of matrix $B^\varepsilon$, $D^\varepsilon$ composition



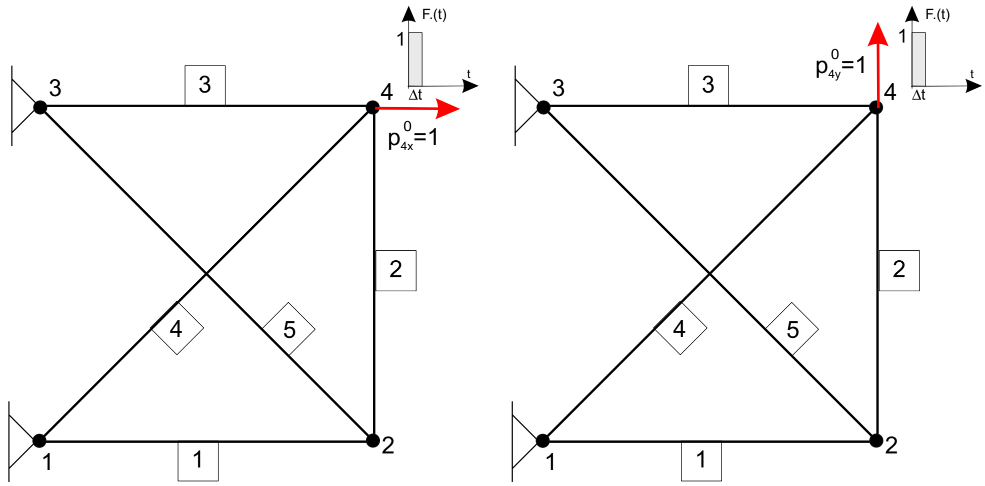



**Fig. 8** Impulse virtual distortion at a node, showing the process of matrices $B^p$ and $D^p$ composition

as a pair of self-equilibrated forces. This kind of *impulse force distortion* at node aims to capture the influence of inertia on structural response. Again, the response of the structure to the impulse force distortion at nodes is obtained by the Newmark algorithm.

It will be demonstrated in subsequent sections that by establishing the influence matrices, $B^\varepsilon$ , $D^\varepsilon$ , $B^p$, and $D^p$, the remodeling of stiffness and mass in the structure becomes feasible. Nonlinear constitutive relation can be also accounted for. However, linear geometric relations (small strains) are assumed.

### 5.2 Stiffness remodeling in dynamics

In signal processing performed in many fields of engineering, the output response of a system is expressed as an integral of the product of the input excitation and transfer function (i.e., system's response to an impulse function like Dirac's delta) over some period of time.

For a simple harmonic oscillator of mass $m$ and natural frequency $\omega$, the convolution of the two functions determines the displacement u(t) due to a series of impulses $f(\tau)d\tau$ over the time period $\langle 0, t\rangle$, as:

$$u(t) = \frac{1}{m\omega}\int_0^t f(\tau)\sin\omega(t-\tau)d\tau \tag{20}$$

Equation (20) is called the Duhamel's integral. Its range of validity is limited by the assumption of system's linearity, i.e., exhibiting small strains by a structural system.

Similar to the Duhamel's integral, the VDM residual response in displacement, modeling some modifications in the structure, can be written as a discrete convolution of the influence matrix (in displacements) $B^\varepsilon_{Mj}$ and virtual distortions $\varepsilon^0_j$. Superposing the residual with linear response (no modifications for an elastic structure), we get [cf. (5)]:

$$u_M(t) = u^{\mathrm{L}}_M(t) + \sum_{\tau=0}^{t} B^\varepsilon_{Mj}(t-\tau)\varepsilon^0_j(\tau) \tag{21}$$

The summation (not integral) over time in (21) indicates that the considered period of time $\langle 0, t\rangle$ was discretized to use the FEM. For performing time integration, the authors chose the Newmark algorithm. Except for the initial conditions at $t = 0$, all other quantities in the Newmark algorithm are calculated starting from the first time step onward. Therefore, to be consistent with the numerical integration, (21) should be formally modified to yield:

$$u_M(t) = u^{\mathrm{L}}_M(t) + \sum_{\tau=1}^{t} B^\varepsilon_{Mj}(t+1-\tau)\varepsilon^0_j(\tau) \tag{22}$$

where the summation starts from $\tau = 1$ [cf. (21)].

By using the geometrical relations between displacements and strains [cf. (2)], the latter can be expressed with the following formula:

$$\varepsilon_i(t) = G_{iM}u_M(t) \tag{23}$$

For the sake of simplifying subsequent strain and stress formulas, the following influence matrix (in strains) $D^\varepsilon_{ij}$ is specified:

$$D^\varepsilon_{ij} = G_{iM}B^\varepsilon_{Mj} \tag{24}$$

Using (24), the total strain, composed of the linear and residual one, can be conveniently written as:

$$\varepsilon_i(t) = \varepsilon^{\mathrm{L}}_i(t) + \sum_{\tau=1}^{t} D^\varepsilon_{ij}(t+1-\tau)\varepsilon^0_j(\tau) \tag{25}$$

The corresponding stresses take the following form:

$$\sigma_i(t) = E_{\underline{i}}\left(\varepsilon_{\underline{i}}(t) - \varepsilon^0_{\underline{i}}(t)\right) = \sigma^{\mathrm{L}}_i(t) + E_{\underline{i}}\left(\sum_{\tau=1}^{t-1} D^{\varepsilon}_{\underline{i}j}(t+1-\tau)\varepsilon^0_j(\tau) + \left(D^{\varepsilon}_{\underline{i}j}(1) - \delta_{\underline{i}j}\right)\varepsilon^0_j(t)\right) \tag{26}$$

Retrieving the valid static postulate [cf. (12)] of equivalence between the distorted and modified structure in terms of strains and internal forces, the following modification coefficient can be derived:

$$\mu^E_i \overset{\text{def}}{=} \frac{\hat{E}_{\underline{i}}}{E_{\underline{i}}} = \frac{\varepsilon_{\underline{i}}(t) - \varepsilon^0_{\underline{i}}(t)}{\varepsilon_{\underline{i}}(t)} = \frac{\hat{A}_i}{A_i} \overset{\text{def}}{=} \mu^A_i \tag{27}$$

Note that using the formula (27), structural stiffness can be modified here either as a change of Young's modulus or cross-sectional area, analogously to statics. Another observation is that the coefficient $\mu_i$, constant in time, is expressed in dynamics in terms of time-dependent components $\varepsilon_i(t)$ and $\varepsilon^0_i(t)$. After reshaping (27), the system of equations to be solved for virtual distortions $\varepsilon^0_i$ is obtained:

$$\left[\delta_{ij} - (1-\mu_{\underline{i}})D^{\varepsilon}_{\underline{i}j}(1)\right]\varepsilon^0_j(t) = (1-\mu^{\mathrm{E}}_{\underline{i}})\varepsilon^{\neq t}_{\underline{i}}(t) \tag{28}$$

where $\varepsilon^{\neq t}_i(t)$ denotes strains aggregated in all time steps preceding the current time instant t:

$$\varepsilon^{\neq t}_i(t) = \varepsilon^L_i(t) + \sum_{\tau=1}^{t-1} D^{\varepsilon}_{ij}(t+1-\tau)\varepsilon^0_j(\tau) \tag{29}$$

Note that the governing matrix on the left-hand side of (28) is time-independent, hence, remains constant throughout all time steps. Only the right-hand side vector varies. The set is local, i.e., limited to the elements (in local coordinates), for which stiffness is remodeled.

## 5.3 Plasticity in dynamics

Obtaining the VDM strain and stress formulas for the dynamic plastic range can be quickly done by replacing the virtual distortion $\varepsilon^0_j(t)$ in (25) and (26) with the plastic distortion $\beta^0_k(t)$ to produce:

$$\varepsilon_i(t) = \varepsilon^{\mathrm{L}}_i(t) + \sum_{\tau=1}^{t} D^{\varepsilon}_{ik}(t+1-\tau)\beta^0_k(\tau) \tag{30}$$

$$\sigma_i(t) = E_{\underline{i}}\left(\varepsilon_{\underline{i}}(t) - \beta^0_{\underline{i}}(t)\right) = \sigma^{\mathrm{L}}_i(t) + E_{\underline{i}}\left[\sum_{\tau=1}^{t-1} D^{\varepsilon}_{\underline{i}k}(t+1-\tau)\beta^0_k(\tau) + \left(D^{\varepsilon}_{\underline{i}k}(1) - \delta_{\underline{i}k}\right)\beta^0_k(t)\right] \tag{31}$$

Like in statics, a piecewise linear constitutive law (see Fig. 3) is adopted in dynamics, too. This time the relation is written in the incremental form, enabling to determine an increment of plastic distortion $\Delta\beta^0_k(t)$ in every time step:

$$\sigma_i(t) - \mathrm{sign}(\sigma^{\mathrm{TR}}_i)\sigma^{\star}_i = \gamma_{\underline{i}}E_{\underline{i}}\left(\mathrm{sign}(\sigma^{\mathrm{TR}}_{\underline{i}})\Psi_{\underline{i}} + \Delta\beta^0_{\underline{i}}(t)\right) \tag{32}$$

The vector $\sigma^{\mathrm{TR}}_i$ in (32) denotes trial stresses, necessary to determine elements entering the plastic zone, according to the formula:

$$\sigma^{\mathrm{TR}}_i = E_{\underline{i}}(\varepsilon_{\underline{i}}(t) - \beta^0_{\underline{i}}(t-1)) \tag{33}$$

The vector $\Psi_i$ in (32) denotes an equivalent (total) plastic strain at isotropic hardening, expressed as:

$$\Psi_i = \sum_t \left|\Delta\beta^0_i(t)\right| \tag{34}$$

For performing stepwise plastic analysis, the increment of strains needs to be explicitly specified in the strain formula (30):

$$\varepsilon_i(t) = \varepsilon_i(t-1) + \Delta\varepsilon_i(t) \tag{35}$$

The strain increment in the current time step is given as:

$$\Delta\varepsilon_i(t) = \Delta\varepsilon^{\mathrm{L}}_i(t) + \sum_{\tau=1}^{t-1} D^{\varepsilon}_{ik}(t+1-\tau)\Delta\beta^0_k(\tau) + D^{\varepsilon}_{ik}(1)\Delta\beta^0_k(t) \tag{36}$$

The stresses (31), expressed in the incremental form, yield:

$$\sigma_i(t) = \sigma_i(t-1) + E_{\underline{i}}\Delta\varepsilon_{\underline{i}}(t) - E_{\underline{i}}\Delta\beta^0_{\underline{i}} \tag{37}$$

Substituting (36) to (37) and reshaping, using (33), we get a local set of equations (limited to plastic elements), which has to be solved for $\Delta\beta^0_k(t)$:

$$\left[E_{\underline{i}}(1+\gamma_{\underline{i}})\delta_{\underline{i}k} + E_{\underline{i}}D^{\varepsilon}_{\underline{i}k}(1)\right]\Delta\beta^0_k(t) = \sigma^{\mathrm{TR}}_i - \mathrm{sign}(\sigma^{\mathrm{TR}}_{\underline{i}})\left(\sigma^{\star}_{\underline{i}} + \gamma_{\underline{i}}E_{\underline{i}}\Psi_{\underline{i}}\right) \tag{38}$$

Equation (38) is known elsewhere as the return mapping algorithm for rate-independent plasticity.

### 5.4 Mass remodeling in dynamics

The inertia effects (mass remodeling) influencing structural behaviour are inherent to dynamic analysis. If we want to account for mass modifications it is necessary to introduce another quantity—an impulse force distortion $p^0$ (pseudoload). Unequilibrated distortions, each one in the form of a unit impulse force, are successively applied to global degrees of freedom, producing a corresponding influence matrix $B^p$ (in displacements) or $D^p$ (in strains)—see dynamic influence matrices. It is an important distinction from the matrices $B^\varepsilon$ and $D^\varepsilon$, in which a self-equilibrated pair of forces (equivalent to a unit strain) was applied. This time, however, it is necessary to collect the out-of-balance influences to reflect the changes of inertia.

Equations of motion for the modified structure subject to a change of mass and the structure modeled by impulse force distortions are given by:

$$\hat{M}_{MN}\ddot{u}_N(t) + K_{MN}u_N(t) = f_M(t) \tag{39}$$

$$M_{MN}\ddot{u}_N(t) + K_{MN}u_N(t) = f_M(t) + p^0_M(t) \tag{40}$$

Subtracting (40) from (39) we get:

$$\hat{M}_{MN}\ddot{u}_N(t) = M_{MN}\ddot{u}_N(t) - p^0_M(t) \tag{41}$$

Equation (41) constitutes the dynamic postulate of VDM [cf. static postulate (12)], saying that the inertia forces and accelerations in the modified and distorted structure are equal. Rearranging (41) leads to:

$$\Delta M_{MN}\ddot{u}_N(t) + p^0_M(t) = 0 \tag{42}$$

where

$$\begin{aligned}\Delta M_{MN} &= \hat{M}_{MN} - M_{MN} \\ &= \sum_i \left(\mu^\rho_{\underline{i}} - 1\right) A_{\underline{i}}\rho_{\underline{i}}l_{\underline{i}}\, \overset{i}{a}{}^T_{MK} M^{el}_{KL}\, \overset{i}{a}_{LN}\end{aligned} \tag{43}$$

defines a modification of the global mass matrix. The summation in (43) applies to all finite elements. The matrix $\overset{i}{a}$ denotes a local–global transformation and $M^{el}$ is the element consistent mass matrix. The coefficient $\mu^\rho$ defines the ratio of the modified density to the original one (or the modified cross-sectional area to the original one):

$$\mu^\rho_i \overset{\text{def}}{=} \frac{\hat{\rho}_i}{\rho_i} = \frac{\hat{A}_i}{A_i} \overset{\text{def}}{=} \mu^{\mathrm{A}}_i \tag{44}$$

Determination of the influence matrix $B^p$ and force distortions $p^0$ enables us to express nodal displacements for the mass remodeling problem in the following way:

**Table 3** Modifications to the five-element truss structure considered in dynamic reanalysis

| Element | $\mu^{\mathrm{E}}$ | $\mu^\rho$ | $\sigma^\star$ | $\gamma$ |
|---|---|---|---|---|
| 1 | 1.00 | 1.00 | 294 | 0.01 |
| 2 | 1.00 | 1.00 | 294 | 0.01 |
| 3 | 0.12 | 0.22 | 120 | 0.01 |
| 4 | 0.62 | 1.14 | 210 | 0.01 |
| 5 | 0.62 | 1.14 | 210 | 0.01 |

$$u_M(t) = u^{\mathrm{L}}_M(t) + \sum_{\tau=1}^{t} B^p_{MN}(t+1-\tau)p^0_N(\tau) \tag{45}$$

The corresponding nodal acceleration [second derivatives of (45) with respect to time] takes the form:

$$\ddot{u}_M(t) = \ddot{u}^{\mathrm{L}}_M(t) + \sum_{\tau=1}^{t} \ddot{B}^p_{MN}(t+1-\tau)p^0_N(\tau) \tag{46}$$

Substituting (46) to (42) and rearranging, the following set of equations is obtained:

$$\left[\delta_{MK} + \Delta M_{MN}\ddot{B}^p_{NK}(1)\right] p^0_K(t) = -\Delta M_{MN}\ddot{u}^{\neq t}_N(t) \tag{47}$$

where

$$\ddot{u}^{\neq t}_M(t) = \ddot{u}^{L}_M(t) + \sum_{\tau=1}^{t-1} \ddot{B}^p_{MN}(t+1-\tau)p^0_N(\tau) \tag{48}$$

collects the contribution from the preceding time steps. Note that again [cf. (28)], the governing matrix in (47)

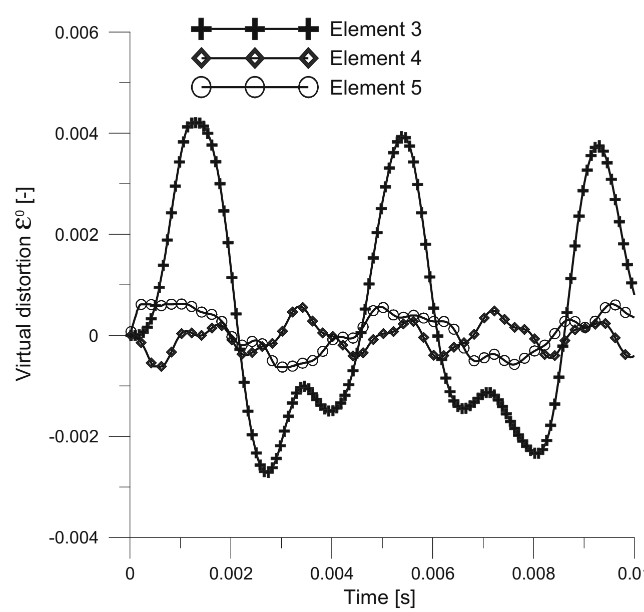



**Fig. 9** Virtual distortions modeling assumed stiffness modifications

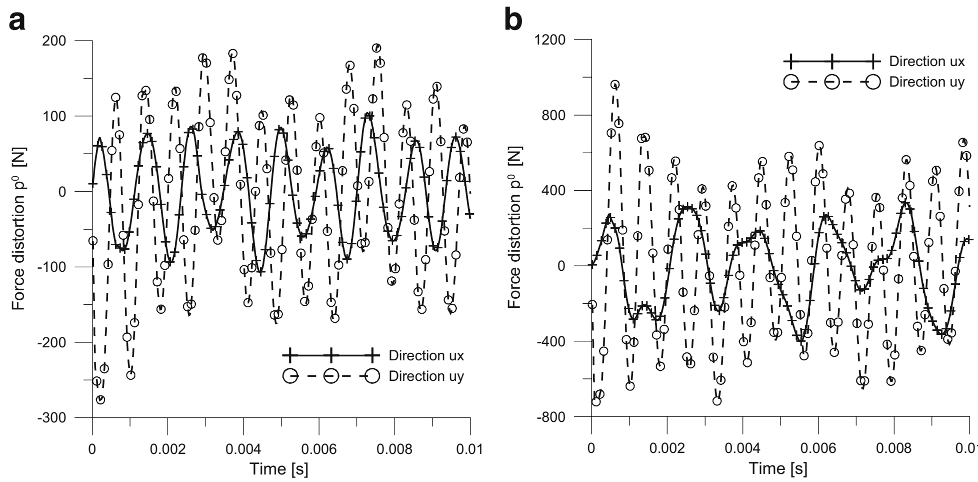



**Fig. 10** Force distortions modeling assumed mass modifications. (**a**) At node no. 2 and (**b**) at node no. 4

is constant in all time steps and only the right-hand side varies. The set is local, i.e., limited to the degrees of freedom (in global coordinates) corresponding to the remodeled mass in neighboring elements.

In combined reanalyses, where both stiffness and mass changes are considered and the nonlinear constitutive law is adopted (see the following example), it is necessary to define all relations on the element level. Therefore, there is a need to define the matrix $D^p$ (in strains), which is related to $B^p$ (in displacements) as follows [cf. (24)]:

$$D^p_{iN} = G_{iM} B^p_{MN} \tag{49}$$

The process of mass remodeling at structural nodes (e.g., modeling an impacting mass) proceeds analogously to the one presented above for elements.

## 5.5 Example in dynamics

The same five-element truss structure, already presented for statics (see Fig. 4), is considered in dynamic reanalysis. Instead of applying the static vertical force, the structure is excited with an initial vertical velocity of the value $v = 20\ m/s$ applied to node no. 2. The combined analysis was performed, i.e., stiffness and mass

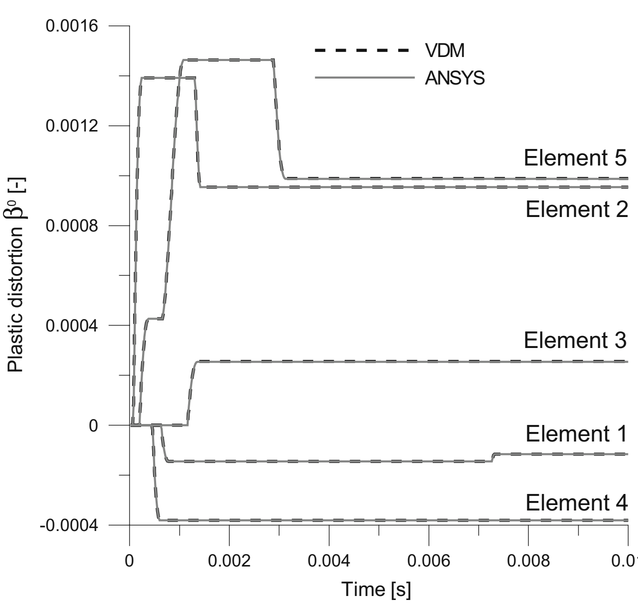



**Fig. 11** Plastic distortions modeling the assumed bilinear constitutive law

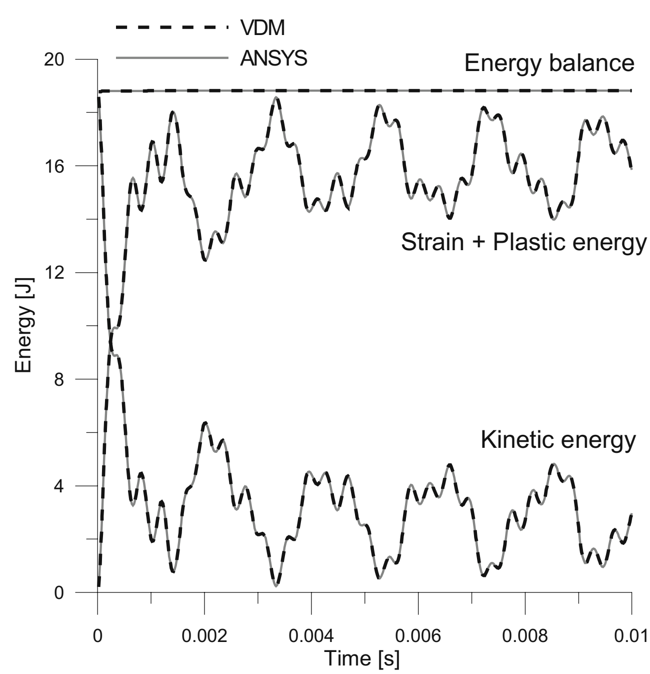



**Fig. 12** Energy balance of the five-element truss

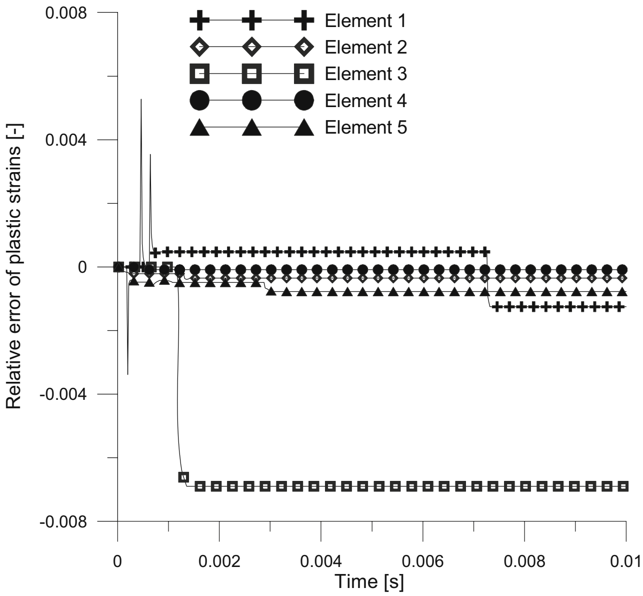



**Fig. 13** Relative error of plastic strains (VDM vs. ANSYS)

remodeling plus nonlinear constitutive law were included. The stiffness and mass modification coefficients as well as plastic parameters are listed in Table 3. The changes correspond to the replacement of material in element no. 3 from steel to magnesium and in elements nos. 4 and 5 from steel to copper.

The components of virtual distortion $\varepsilon^0$ modeling stiffness changes are depicted in Fig. 9. Force distortions $p^0$ corresponding to node nos. 2 and 4 are shown in Fig. 10a and b, respectively. Five components of plastic distortions $\beta^0$ are illustrated in Fig. 11. Energy balance is shown in Fig. 12. The plastic distortions and energy balance have been validated against the FE code ANSYS.

It can be observed in Figs. 11 and 12 that the VDM-generated curves follow the ones obtained from ANSYS very closely. For better visualization of differences between VDM and ANSYS, a relative error of plastic strains (with ANSYS results as reference) is depicted in Fig. 13. The error is 7 per mille for element no. 3 and less than 2 per mille for other elements.

It should be commented that the simultaneous modification of stiffness and mass is also possible by varying only the cross-sectional area. However, it is a special case because then only one coefficient $\mu$ can be used and the two fields of distortions $\varepsilon^0$, $p^0$ are closely correlated. If independent changes are of interest, it is convenient to use modifications to the Young's modulus E for stiffness and density $\rho$ for mass.

Other truss and beam examples were presented in papers (Jankowski and Wiklo 2006; Kolakowski et al. 2004, 2006; Swiercz et al. 2006), dealing with one of the most promising applications of VDM in dynamic analysis—the inverse problem of parameter identification.

## 6 Recapitulation

### 6.1 General remarks

The VDM has proved to be a versatile tool of structural and system reanalysis for 20 years of its development. In the authors' opinion, the principal advantages of the VDM, distinguishing it from other reanalysis methods, are:

- Exact, analytical formulation, capturing all system's features in the influence matrix;
- Handling nonlinear constitutive law; and
- Dynamic reanalysis in the time domain enabling modification of both stiffness and inertia parameters.

VDM is an analytical approach, producing exact (not approximated) results unlike the majority of structural reanalysis methods presently used. The exactness of the method is due to the influence matrix capturing all relations between a local disturbance and global response of a structure (or system). Any response of a structure subject to modifications is simply a linear combination of components of the influence matrix and virtual distortions (design variables) even for physically nonlinear problems. The exact formulation of VDM is especially important in calculating precise sensitivities, which are subsequently utilized in gradient-based optimization. This feature also enables effective handling of large modifications of parameters (see Sections 4.4 and 4.5). If the number of design variables to be modified is small compared to the degrees of freedom, the VDM solution is fast, as the set of equations to be solved is always local (referring only to the modified variables).

Inverse of the stiffness matrix is required in VDM only at the stage of building the influence matrix. For some applications, e.g., progressive collapse analysis, only selected influence vectors need to be built, for others, e.g., identification problems, it is convenient to have full influence matrix at start. In statics, once assembled for a structure with some redundancy, the influence matrix remains constant throughout the whole reanalysis. Nevertheless, building the full influence matrix, especially for dynamics, may involve some initial computational cost for large structures. However, creating the full influence matrix has the advantage of forming a computational basis for VDM. With this matrix and VDM-based analytical sensitivities, various types of analyses, sometimes addressing really complex problems (see Section 6.4), can be performed.

As shown in Akgun et al. (2001), reanalysis methods are variations of the SMW formulas. For VDM in the linear regime, the equivalence to SMW is restricted with the condition that only required influence vectors (not the whole influence matrix), corresponding to the modified locations, are constructed. For physical nonlinearities, the VDM algorithm progresses without iterations due to considering a piecewise linear (in particular bilinear) constitutive law. It seems that for the extension of SMW formulas to the nonlinear regime, proposed in Akgun et al. (2001), iterations would not be necessary either, if an analogous, piecewise linear relation were assumed instead of a general, nonlinear one.

Apart from the dynamic modification method (Muscolino 1996), the VDM appears to be exceptional in performing dynamic reanalysis in the time domain. Thanks to considering two fields of virtual distortions—one modeling stiffness modifications like in statics and the other modeling mass modifications—various changes of structural parameters can be effectively tracked in dynamics. The time-domain dynamic VDM turns out to be a powerful tool when analyzing combined problems of design and adaptation for structures subjected to impact loading (see Section 6.4).

Thus far the VDM is basically limited to skeletal structures. An extension of the method for continuum, similar to what has been done within the TSV approach (cf. Saka 1998; Topping and Kassim 1987), will be the subject of future research.

### 6.2 Applications of virtual distortion method to structures

The list of main application areas of VDM in structural mechanics includes:

- In statics
  - Stiffness remodeling (direct problem)—topology optimization and
  - Piecewise linear constitutive law (direct problem)—noniterative plasticity.
- In dynamics
  - Stiffness remodeling (inverse problem)—identification of stiffness degradation in structural health monitoring;
  - Mass remodeling (inverse problem)—identification of dynamic load history; and
  - Combined stiffness, mass remodeling, and piecewise linear plasticity (direct problem)—optimal design of adaptive structures for dynamic loads of known characteristics.

In static analysis, the VDM can be successfully used for remodeling of structures. Analytically derived sensitivities for trusses (Kolakowski and Holnicki-Szulc 1998) and frames (Putresza and Kolakowski 2001) are the basis for gradient-based topological optimization (Kolakowski and Holnicki-Szulc 1997). Another useful accomplishment in statics is the ability of analyzing a piecewise, nonlinear constitutive relation, practically employed in the progressive collapse analysis (Holnicki-Szulc and Gierlinski 1995). A preliminary quasistatic study of optimal design of adaptive structures was presented in Holnicki-Szulc et al. (1998). An overview of the VDM in statics can be found in Holnicki-Szulc and Bielecki (2000).

In dynamic analysis, the VDM, contrary to most reanalysis methods, is able to model the response of a modified structure in the time domain. Using stiffness degradation as a damage modeling parameter, an inverse dynamic analysis, examining response due to an impulse excitation, was proposed for damage identification (Kolakowski et al. 2004). This way, the VDM has been applied to a new field of structural health monitoring. Comparison of performance of the gradient-based VDM approach with a soft-computing method was made in Kolakowski et al. (2006). The VDM damage identification philosophy can also be transferred to the frequency domain by applying a harmonic excitation and looking only at amplitudes (Swiercz et al. 2006). A similar problem of load identification (location and magnitude) using VDM was successfully handled in Jankowski and Wiklo (2006). Using VDM, first attempts to evaluate the crashworthiness of structures (Holnicki and Knap 2004) and to design adaptive structures (Holnicki-Szulc et al. 2003), effectively dissipating the energy of impact load, were undertaken. However, the problems have to be analyzed with the assumption of nonlinear geometry (large strains) and still remain a research challenge.

The VDM has been the subject of study of other researchers, too. Makode et al. (1996) describe an extension of VDM to frame structures. In their subsequent paper (Makode et al. 1999) (with the name *Pseudo Distortion Method*), simultaneous modification of moment of inertia and cross-sectional area is included. Also, the elastoplastic analysis with multiple hinges in the structure, located either at one or two ends of the frame element, is presented. In the following companion paper, the VDM is used to account for secondary geometric effects caused by large axial forces influencing bending moments in frame structures.

Recently, an application of VDM to probabilistic analysis has been developed. Di Paola et al. (2004) and Di Paola (2004) use the VDM to model uncertainty of parameters in truss structures.

### 6.3 Applications of virtual distortion method to nonstructural systems

The VDM framework is general enough to encompass problems concerning systems other than structural as well. This is due to analogies between structural mechanics and network analysis discovered by Cross (1936) and to capturing all system's features in the influence matrix by VDM. Using the general system theory (Lind 1962) and oriented graph approach, it turns out that similar relations govern constitutive, continuity, and equilibrium conditions for truss structures, water networks, and electrical networks. Taking advantage of this fact, the VDM idea, originating from structural mechanics, was adapted to model both the types of nonstructural systems.

In Holnicki-Szulc et al. (2005), closed-loop water networks, assuming a steady-state flow, are modeled using VDM. For water heads measured in all nodes of the network, an algorithm for detection of leakage in the midpoint of a branch was proposed. VDM sensitivity enabled to employ a quadratic programming tool to solve the problem. As a result, the algorithm gives the location and intensity of leakage. Multiple leakage detection is feasible. Continuation of the research will include precise location of leakage along the branch, optimal location of measuring nodes, consideration of transient effects (unsteady flow).

In Kokot and Holnicki-Szulc (2005), closed-loop electrical networks are modeled using VDM. Static-like approach with constant current intensities is proposed. As an extension, quasistatic approach for harmonic current sources has been developed, where only amplitudes and phase shifts are analyzed in the complex numbers domain. Finally, dynamic-like, time-dependent approach, able to reflect transient behavior of electrical networks as a response to an impulse current, is described in Kokot and Holnicki-Szulc (2006). An "electrical finite element" has been elaborated, enabling to perform a FEM-like analysis. Defects in electrical networks are effectively modeled in all the mentioned approaches as loss of conductance in branches. The changes of conductance (defect coefficient) may be tracked in the continuous range from zero (break in the network) to one (intact conductance).

### 6.4 Future challenges

One of the benefits of using VDM is the possibility of analyzing complex problems. The advantages of the method mentioned at the beginning of Section 6 allow, for instance, to consider the following task: "Given a structure subjected to an impact load in several possible locations, find its optimal topological design and predict its best adaptation to the identified load." The first part of the problem belongs to topological optimization as a fully stressed design problem (multiload case), performed for dynamic loading in the time domain. The second part belongs to smart structures—assuming that the location of impact has been identified (by a sensor), the adaptation of the structure (meeting a defined criterion) through modification of nonlinear constitutive characteristics takes place. An example of such analysis for a truss structure subjected to static loading can be studied in Kolakowski and Holnicki-Szulc (1997). Dynamic case will be the topic of a future paper.

**Acknowledgements** The authors would like to acknowledge the financial support from the projects 3T11F00930 and PBZ-KBN-105/T10/2003 granted by the State Committee for Scientific Research in Poland. Some parts presented in the paper belong to the Ph.D. Thesis of the second author supervised by the third author. This article is intended to be incorporated into the book "Smart Technologies for Safety Engineering," edited by the third author, to be published by Wiley.

## References

Abu Kassim AM, Topping BHV (1987) Static reanalysis: a review. J Struct Eng ASCE 113(6):1029–1045

Akgun MA, Garcelon JH, Haftka RT (2001) Fast exact linear and non-linear structural reanalysis and the Sherman-Morrison-Woodbury formulas. Int J Numer Methods Eng 5:1587–1606

Argyris JH (1965) Elasto-plastic matrix displacement analysis of three-dimensional continua. J R Aeronaut Soc 69:633–636

Bae H-R, Grandhi RV (2004) Successive matrix inversion method for reanalysis of engineering structural systems. AIAA J 42(8):1529–1535

Barthelemy JFM, Haftka RT (1993) Approximation concepts for optimum design—a review. Struct Optim 5:129–144

Cacciola P, Impollonia N, Muscolino G (2005) A dynamic reanalysis technique for general structural modifications under deterministic and stochastic input. Comput Struct 83: 1076–1085

Chen H-P (2006) Efficient methods for determining modal parameters of dynamic structures with large modifications. J Sound Vib 298:462–470

Chen SH, Yang XW, Lian HD (2000) Comparison of several eigenvalue reanalysis methods for modified structures. Struct Multidisc Optim 20:253–259

Cronin DL (1990) Eigenvalue nad eigenvector determination for nonclassically damped dynamic systems. Comput Struct 36:133–138

Cross H (1936) Analysis of flow in networks of conduits or conductors. Engineering Experiment Station bulletin no. 286. University of Illinois, Chicago, IL, USA.

Di Paola M (2004) Probabilistic analysis of truss structures with uncertain parameters (virtual distortion method approach). Probab Eng Mech 19(4):321–329

Di Paola M, Pirrotta A, Zingales M (2004) Stochastic dynamics of linear elastic trusses in presence of structural uncertainties (virtual distortion approach). Probab Eng Mech 19(1–2): 41–51

Deng L, Ghosn M (2001) Pseudoforce method for nonlinear analysis and reanalysis of structural systems. J Struct Eng ASCE 127(5):570–578

Fox RL, Miura H (1971) An approximate analysis technique for design calculations. Technical note. AIAA J 9(1):177–179

Grissom MD, Belegundu AD, Koopmann GH (2005) A reduced eigenvalue method for broadband analysis of a structure with vibration absorbers possessing rotatory inertia. J Sound Vib 281:869–886

Holnicki-Szulc J (1991) Virtual distortion method. In: Lecture notes in engeneering, no.65. Springer, Heidelberg-Berlin

Holnicki-Szulc J, Bielecki T (2000) Virtual distortion method—review. Applications to fast redesign and sensitivity analysis. Int J Nonlinear Sci & Numer Simul 1(2):71–98

Holnicki-Szulc J, Gierlinski J (1989) Structural modifications simulated by virtual distortions. Int J Numer Methods Eng 28:645–666

Holnicki-Szulc J, Gierlinski JT (1995) Structural analysis, design and control by the virtual distortion method. Wiley, Chichester, UK

Holnicki J, Knap L (2004) Adaptive crashworthiness concept. Int J Impact Eng 30(6):639–663

Holnicki-Szulc J, Mackiewicz A, Kolakowski P (1998) Design of adaptive structures for improved load capacity. AIAA J 36(3):471–476

Holnicki-Szulc J, Pawlowski P, Wiklo M (2003) High-performance impact absorbing materials—the concept, design tools and applications. Smart Mater Struct 12(2003): 461–467

Holnicki-Szulc J, Kolakowski P, Nasher N (2005) Leakage detection in water networks. J Intell Mater Syst Struct 16(3): 207–219

Huang C, Chen S-H, Liu Z (2000) Structural modal reanalysis for topological modifications of finite element systems. Eng Struct 22:304–310

Hurtado JE (2002) Reanalysis of linear and nonlinear structures using iterated shanks transformation. Comput Methods Appl Mech Eng 191:4215–4229

Jankowski L, Wiklo M (2006) A posteriori impact identification. In: Proceedings of the 3rd European workshop on structural health monitoring, 5–7 July 2006. Granada, Spain, pp 675–682

Kirsch U (2003) A unified reanalysis approach for structural analysis, design and optimization. Struct Multidisc Optim 25:67–85

Kirsch U, Liu S (1995) Exact structural reanalysis by a first-order reduced basis approach. Struct Optim 10: 153–158

Kirsch U, Bogomolni M, Sheinman I (2006) Nonlinear dynamic reanalysis of structures by combined approximations. Comput Methods Appl Mech Eng 195:4420–4432

Kokot M, Holnicki-Szulc J (2005) Health monitoring of electric circuits. In: Proceedings of the 6th international conference on damage assessment of structures—DAMAS 2005, 4–6 July, 2005, Gdansk, Poland, pp 669–676

Kokot M, Holnicki-Szulc J (2006) SHM concept applied to dynamical defect identification in electrical circuits. In: Proceedings of the 3rd European workshop on structural health monitoring, 5–7 July 2006. Granada, Spain, pp 1111–1118

Kolakowski P, Holnicki-Szulc J (1997) Optimal remodelling of truss structures—simulation by virtual distortions. Comput Assist Mech Eng Sci 4(2):257–281

Kolakowski P, Holnicki-Szulc J (1998) Sensitivity analysis of truss structures—virtual distortion method approach. Int J Numer Methods Eng 43(6):1085–1108

Kolakowski P, Zielinski TG, Holnicki-Szulc J (2004) Damage identification by the dynamic virtual distortion method. J Intell Mater Syst Struct 15(6):479–493

Kolakowski P, Mujica LE, Vehi J (2006) Two approaches to structural damage identification: model updating vs. soft computing. J Intell Mater Syst Struct 17(1):63–79

Kroner E (1958) Kontinuumstheorie der Versetzungen und Eigenspannungen. In: Collatz L., Lösch F. (eds) Ergebnisse der Angewandte Mathematik, vol. 5. Springer, Berlin (in German)

Leu L-J, Tsou C-H (2000) Applications of a reduction method for reanalysis to nonlinear dynamic analysis of framed structures. Comput Mech 26:497–505

Lind NC (1962) Analysis of structures by system theory. J Struct Div ASCE 88:ST2 (Apr)

Maier G (1970) A matrix structural theory of piecewise-linear plasticity with interacting yield planes. Meccanica 7:51–66

Majid KI, Celik T (1985) The elastic-plastic analysis of frames by the theorems of structural variation. Int J Numer Methods Eng 21:671–681

Majid KI, Elliott DWC (1973) Forces and deflections in changing structures. Struct Eng 51:93–101

Makode PV, Ramirez MR, Corotis RB (1996) Reanalysis of rigid frame structures by the virtual distortion method. Struct Multidisc Optim 11(1–2):71–79

Makode PV, Corotis RB, Ramirez MR (1999) Non-linear analysis of frame structures by pseudodistortions. J Struct Eng ASCE 125(11):1309–1327

Muscolino G (1996) Dynamically modified linear structures: deterministic and stochastic response. J Eng Mech ASCE 122(11):1044–1051

Noor AK, Lowder HE (1974) Approximate techniques of structural reanalysis. Comput Struct 4:801–812

Nowacki W (1970) Teoria sprezystosci. PWN, Warsaw (in Polish)

Putresza JT, Kolakowski P (2001) Sensitivity analysis of frame structures—virtual distortion method approach. Int J Numer Methods Eng 50(6):1307–1329

Ravi SSA, Kundra TK, Nakra BC (1998) Reanalysis of damped structures using the single step perturbation method. J Sound Vib 211(3):355–363

Saka MP (1998) The theorems of structural variation for solid cubic finite elements. Comput Struct 68:89–100

Sherman J, Morrison WJ (1949) Adjustment of an inverse matrix corresponding to changes in the elements of a given column or a given row of the original matrix. Ann Math Stat 20:621

Swiercz A, Kolakowski P, Holnicki J (2006) Identification of damage using low-frequency harmonics in trusses and beams. In: Proceedings of the international conference on noise and vibration engineering, ISMA2006, CD-ROM edition, 18–20 September 2006. Leuven, Belgium, pp 2137–2144

Tang J, Wang W-L (1996) Structural reanalysis in configuration space by perturbation approach: a real mode method. Comput Struct 58(4):739–746

Topping BHV, Kassim AMA (1987) The use and efficiency of the theorems of structural variation for finite element analysis. Int J Numer Methods Eng 24:1900–1920

Woodbury M (1950) Inverting modified matrices. Memorandum report 42. Statistical Research Group, Princeton University, NJ, USA

Yap KC, Zimmermann DC (2002) A comparative study of structural dynamic modification and sensitivity method approximation. Mech Syst Signal Process 16(4):585–597