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Defect Identification in Electrical Circuits via the Virtual Distortion Method. Part 1: Steady-state Case

MAREK KOKOT*,** AND JAN HOLNICKI-SZULC

Department of Intelligent Technologies, Institute of Fundamental Technological Research
Swietokrzyska 21, 00-049 Warsaw, Poland

ABSTRACT: Virtual Distortion Method, a numerical technique originally developed and applied to various optimization problems in structural mechanics, is adapted to DC/AC circuit analysis. Electro-mechanical analogies with discrete models of plain truss structures are utilized to introduce and implement the main concepts of the method. Simulation of conductance modifications in circuit elements by the equivalent set of virtual sources is a foundation of numerically effective algorithms enabling fast re-analysis and sensitivity analysis. Inverse problem of defect identification is discussed and solution based on distortion approach is provided.

Key Words: optimization, embedded intelligence, structural health monitoring.

INTRODUCTION

VIRTUAL DISTORTION METHOD (VDM) (Holnicki-Szulc and Gierliński, 1995; Kołakowski et al., 2007) is a technique of fast linear reanalysis, originally formulated with respect to discrete mechanical systems. The method introduces the concept of *virtual distortions* as certain additional input functions which imitate the influence of local structural modifications on global system response. Interpreted as the states of initial deformations in finite elements, distortions enable to simulate modifications of various structural parameters (mass, stiffness, damping) or system nonlinearities (plasticity). Relations between modified parameters, distortions and structural responses are formulated as local systems of linear equations which can be effectively computed. Moreover, gradients of response functions with respect to modification parameters can be derived, which enable to formulate and solve various optimization and inverse problems. The VDM is most effective when applied to models consisting of simple finite elements, i.e. truss and frame structures. The method has been formulated both in statics and dynamics, in discrete-time (Kołakowski et al., 2004) and frequency domain (Świercz et al., 2008). The scope of applications in structural mechanics includes progressive collapse analysis, topological optimization, sensitivity analysis, damage, and load identification. Recently, an effort was made to adapt the method to analysis of other engineering systems. In the paper by Holnicki-Szulc et al. (2005), an application of the VDM to detection of leakages in water networks was demonstrated.

This article is intended to be the first part of a two-part work describing an adaptation of the VDM to electrical circuit analysis. The scope is hereby limited to the steady-state cases (DC and AC circuits), with the main focus on introduction and interpretation of VDM concepts based on analogies between circuits and truss structures. Second part will cover transient analysis in discrete time domain. The main objective of the work is to provide a solution for the inverse problem of defect identification in linear RLC circuits of complex topology. The work has been motivated by the development of Structural Health Monitoring system in which sensing layer includes an embedded network of electrical sensors. The problem is to identify damage-induced changes of resistance or capacitance within the network when outputs of sensors cannot be measured directly. It is expected that the VDM, formulated in dynamic case, with it's ability to define local–global interrelations, could provide a feasible solution.

TRUSS-CIRCUIT ANALOGIES

In the original formulation, the VDM operates on the discrete, finite element model of mechanical system. An important case in the view of further considerations is the model of a plane truss structure, where only one type of finite element (truss member) and a single deformation mode (axial strain in elastic range) is considered. For such a model, a clear and consistent system of analogies with all-resistive circuit supplied by independent

*Author to whom correspondence should be addressed.
E-mail: mkokot@ippt.gov.pl
**Additional Address: CONTEC ST, www.smart-tech.pl

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sources can be formulated. Let consider one-element truss structure and single-resistor electrical circuit shown in Figure 1. Truss member of initial length L and stiffness k is presented in two configurations: subjected to external force P and fixed initial deformation ε^0. As a result of the load, the member undergoes deformation and action of internal forces, described accordingly by the strain ε and axial force N. In the first load scenario $N=P$ while $\varepsilon=\varepsilon^0$ in the second. Constitutive relation described by Hooke's law, relation between strain and nodal displacements x (in the local coordinate system) and boundary condition related to the blocked degree of freedom are gathered in Equation (1).

$$N = k\ \varepsilon; \qquad \varepsilon = \frac{x_2 - x_1}{L}; \qquad x_1 = 0 \tag{1}$$

On the other hand, an ideal resistor of conductance G (the reciprocal of resistance) is presented in configuration with a current source J or a voltage source E. The result of the supply is the state of voltage U and current flow I in the element. In the first case of supply $I=J$ while $U=E$ in the second. Constitutive relation described by Ohm's law, relation between voltage and electric potentials v across element terminals and boundary condition related to the grounded node are gathered in the Equation (2).

$$I = G\ U; \qquad U = v_2 - v_1; \qquad v_1 = 0 \tag{2}$$

Comparing the above relations, a system of analogous quantities can be established, where strains correspond to voltages, forces to currents and nodal displacements to electric potentials. The analogy complies also with load/supply conditions (external force $\leftrightarrow$ current source, initial deformation $\leftrightarrow$ voltage source) and global rules governing system behavior (equilibrium of forces in nodes and continuity of deformations along closed paths $\leftrightarrow$ Kirchhoff's laws).

In the following analysis, capacitors and inductors (coils) will be also taken into consideration, although no mechanical equivalents will be defined. In the general case, constitutive relations for these elements are described by integral or differential equations, but in the steady-state analysis they can be reduced to algebraic equations (Table 1). In the steady-state DC analysis, ideal capacitor can be treated as an element of zero conductance (break – no current flow) while ideal coil as an element of infinite conductance (short-circuit – no voltage drop). In the AC analysis, current and voltage, expressed as complex amplitudes, can be related by the admittance Y. According to Euler's formula, sinusoidal signal of amplitude A, angular frequency ω and phase φ can be expressed as follows:

$$x(t) = A\ \sin(\omega t + \varphi) = \mathrm{Im}[A\, e^{j\varphi}\, e^{j\omega t}] \tag{3}$$

The quantity $\underline{A} = A\, e^{j\phi}$ (j – imaginary unit) is defined as a complex amplitude. It is a complex number whose modulus and argument are associated with amplitude and phase of the signal. In the steady-state of response, the time factor $e^{j\omega t}$ is the same for all considered signals, hence all relation can be written with respect to complex amplitudes. Admittances specific to the elements are:

$$Y_r = G; \qquad Y_c = j\omega C; \qquad Y_l = \frac{1}{j\omega L} \tag{4}$$

In order to illustrate some of the introduced notions, references to simple exemplary circuit shown in Figure 2 will be made. The circuit consists of 5 resistors ($R_1 \div R_5$), 2 capacitors (C_1, C_2) and a coil L, and is supplied by the voltage source E and the current source J. Six nodes have been distinguished, with the node number 1 being grounded.

To describe the topology of the circuit, the notions of directed graph and incidence matrix $\mathbf{M}$ will be applied. If elements of the circuit are represented by the edges of fixed orientation, linking pair of adjacent nodes, then an entry of the matrix $M_{ij}=1$ denotes that jth edge enters ith node, $M_{ij}=-1$ denotes that the edge leaves the node and $M_{ij}=0$ states that the given edge

Table 1. Constitutive relations for different electrical elements.

	Resistor	Capacitor	Coil
General case	$i(t)=Gu(t)$	$i(t)=C\frac{du(t)}{dt}$	$i(t)=\frac{1}{L}\int u(t)dt$
DC case	$I=G\ U$	$I=0$	$U=0$
AC case	$\underline{I}=G\ \underline{U}$	$\underline{I}=j\omega C\underline{U}$	$\underline{I}=\frac{1}{j\omega L}\ \underline{U}$

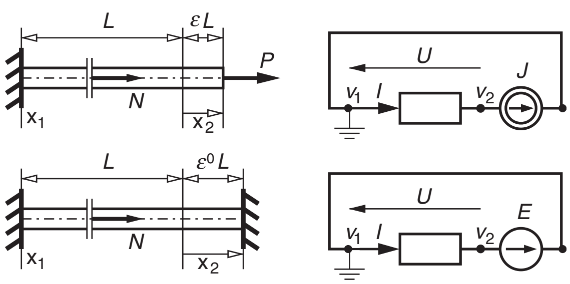


Figure 1. *Truss – circuit analogy.*

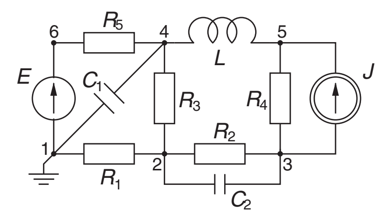


Figure 2. *Exemplary circuit.*

and node are disjointed. The assumed orientation of edges (pointing the node of higher index) establishes reference for directions of current flow and polarity of voltage. Figure 3 presents a subgraph obtained for resistors from the exemplary circuit and the corresponding incidence matrix. Separate matrices for every kind of source and element will be further used, with subscript denoting the set of components the matrix represents ($\mathbf{M}_r$ for resistors, $\mathbf{M}_c$ for capacitors etc.).

Incidence matrices enable to formulate relations between quantities referring to nodes and elements. Let arrange all quantities in column vectors denoted by bold small letters: $\mathbf{u}$ and $\mathbf{i}$ are accordingly vectors of voltages and currents in elements, $\mathbf{v}$ – electric potentials in nodes, $\mathbf{j}$ – intensities of current sources and $\mathbf{e}$ – electromotive forces. The underlined vectors will denote complex amplitudes (AC case). Making use of the incidence matrices, voltages across elements can be related with electric potentials:

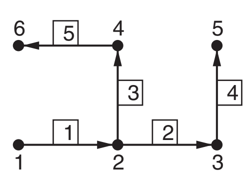
$$\mathbf{u} = \mathbf{M}^{\mathrm{T}}\mathbf{v} \tag{5}$$

while the product: $[\mathbf{M}\ \mathbf{i}]$ enables to sum up currents in nodes. The constitutive relations can be formulated in the following way:

$$\text{DC: } \mathbf{i} = -\mathbf{G}\,\mathbf{u}; \quad \text{AC: } \underline{\mathbf{i}} = -\mathbf{Y}\,\underline{\mathbf{u}}, \tag{6}$$

Matrices of parameters $\mathbf{G}$ (for resistors) and $\mathbf{Y}$ (specific to every passive element) store values of conductances or admittances on the main diagonals. The negative signs results from the assumed conventional flow notation: arrow of voltage points toward higher potential while current (i.e., movement of the positive charge) flows toward lower potential.

An approach based on the modified nodal analysis (MNA) will be used to formulate global system of circuit equations. The method is founded on Kirchhoff's current law, with electric potentials in nodes assigned as the unknown variables. The principle of current



$$\mathbf{M_r} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 3. *Oriented graph and the corresponding incidence matrix.*

equilibrium can be formulated in the following way (both DC and AC case):

$$\mathbf{M}_r\,\mathbf{i}_r + \mathbf{M}_c\,\mathbf{i}_c + \mathbf{M}_l\,\mathbf{i}_l + \mathbf{M}_e\,\mathbf{i}_e + \mathbf{M}_j\,\mathbf{j} = [\mathbf{0}]. \tag{7}$$

Making use of Equations (5) and (6), currents can be expressed in terms of nodal potentials. In the DC case, the following relation is obtained:

$$\mathbf{M}_r\mathbf{G}\mathbf{M}_r^{\mathrm{T}}\,\mathbf{v} - \mathbf{M}_l\,\mathbf{i}_l - \mathbf{M}_e\,\mathbf{i}_e = \mathbf{M}_j\,\mathbf{j} \tag{8}$$

While currents in coils and voltage sources cannot be directly related with potentials, they are treated as unknown variables and additional nodal constraints are imposed, namely $V_j - V_i = 0$ for coils and $V_j - V_i = E$ for voltage sources. Ultimately, the following system of equations can be derived:

$$\begin{bmatrix} \tilde{\mathbf{G}} & -\mathbf{M}_l & -\mathbf{M}_e \\ \mathbf{M}_l^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_e^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{i}_l \\ \mathbf{i}_e \end{bmatrix} = \begin{bmatrix} \mathbf{M}_j\,\mathbf{j} \\ \mathbf{0} \\ \mathbf{e} \end{bmatrix} \tag{9}$$

where $\tilde{\mathbf{G}} = \mathbf{M}_r\mathbf{G}\mathbf{M}_r^{\mathrm{T}}$. In the AC case, the system has the following form:

$$\begin{bmatrix} \tilde{\mathbf{Y}} & -\mathbf{M}_e \\ \mathbf{M}_e^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{v}} \\ \underline{\mathbf{i}}_e \end{bmatrix} = \begin{bmatrix} \mathbf{M}_j\,\underline{\mathbf{j}} \\ \underline{\mathbf{e}} \end{bmatrix} \tag{10}$$

where the global matrix of admittance is:

$$\tilde{\mathbf{Y}} = \mathbf{M}_r\mathbf{Y}_r\mathbf{M}_r^{\mathrm{T}} + \mathbf{M}_c\mathbf{Y}_c\mathbf{M}_c^{\mathrm{T}} + \mathbf{M}_l\mathbf{Y}_l\mathbf{M}_l^{\mathrm{T}}$$

The last step of the procedure is to impose constraints related with the grounded nodes. At least one node needs to be grounded (value of potential set to zero) in order to obtain non-singularity of the main matrix (which results from the fact that the number of independent Kirchhoff's current laws is less by one then the number of nodes while Equation (7) has been formulated with respect to all nodes). To implement the condition $V_i = 0$, all entries in the ith row and the ith column of the main matrix, as well as the ith entry in the excitation vector should be set to zero, except the diagonal entry of the matrix which should be set to one.

After imposing the constrains, Equations (9) or (10) can be solved using standard routines and algorithms for systems of linear equations (following the rules of linear algebra in the complex number domain in the AC case). The obtained variables (nodal potentials, currents in voltage sources) enable to calculate all other circuit responses (making use of Equation (5) and constitutive relations). Not coincidentally, the assumed system of analogies between trusses and circuits results in distinct similarities between the MNA and the Finite Element Method (equivalent unknown variables, aggregation of the main matrices, handling boundary conditions). It can be also noted that the MNA-based approach enables to integrate many other electrical components (like dependent sources, ideal transformers or operational amplifiers), but that issue will not be taken into account in the further considerations.

VDM FORMULATION

Basic notions and definitions of the VDM will be now introduced with reference to linear model of a plane truss structure (statically undetermined), subjected to static load (nodal forces), where small deformations (elastic range) are assumed. A very simplified description of the method is provided, more information can be found in the cited references. Global state of nodal displacements, internal forces, and strains in truss members, generated by the external load, is called the *linear response* of a structure. If stiffness properties (like Young's modulus or cross-sectional area) of some elements have been modified, then the same boundary and load conditions would generate the *modified response* of a structure. VDM postulates that the influence of stiffness modifications can be simulated by the field of initial strains (*virtual distortions*) imposed on the original structure. These virtual deformations of elements generate a certain state of strain/forces called the *residual response*. It is assumed that superposition of linear and residual response is globally equivalent to the modified response of the structure. The second important notion of the VDM is the *influence matrix*. The column of the matrix stores the structural responses on unit distortion imposed on the selected element (assuming lack of external load and active boundary conditions). Residual response can be computed as a linear combination of distortions, where coefficients of the combination are components of the influence matrix.

Based on the truss–circuit analogy from the previous section, the concept of *electrical distortion* can be introduced as a virtual source which imitate the modification of conductance in the element of a circuit. Following the analogy, an equivalent to the initial strain ε^0 would be a voltage source e^0 inserted in series with the element (Figure 4). However, since the initial strain cannot be directly included into the FEM-based model of the system, distortion may be also considered as a pair of virtual nodal forces p^0 which realize the initial strain ($p^0 = k\varepsilon^0$).

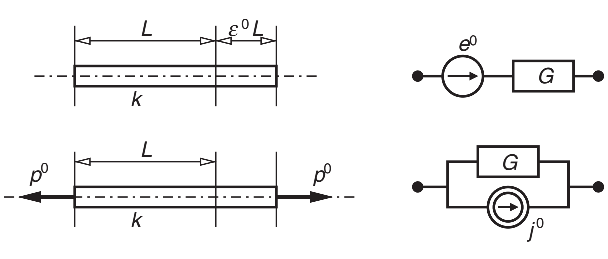


Figure 4. *Distortions in trusses and circuits.*

In this case, the electrical equivalent would be a current source j^0 inserted in parallel with the resistor.

Voltage and current distortions are interchangeable ($j^0 = Ge^0$), hence in the further considerations only the current distortion j^0 will be used. In the MNA-based formulation of circuit equations, current distortions are much easier to integrate. Taking into account systems of Equations (9) or (10), aggregation of voltage distortions would require to extend system dimension by imposing the additional nodal constraints, while aggregation of current distortions requires only to update the right-hand side vector of excitations. Since current distortions are inserted in parallel with resistors, an additional input vector can be formulated as:

$$\mathbf{z} = \mathbf{M}_{\mathrm{r}}\,\mathbf{j^0} \tag{11}$$

Taking into account the superposition principle, the response of a circuit with imposed distortions can be considered as a sum of *linear response* (generated by real sources) and *residual response* (induced by distortion). For example, system of Equation (9) with imposed distortions (and assumed absence of coils and voltage sources for the sake of simplicity), would have the following form:

$$\tilde{\mathbf{G}}\,\mathbf{v} = \mathbf{M}_{\mathrm{j}}\,\mathbf{j} + \mathbf{z} \tag{12}$$

Vector of nodal potentials could be then calculated as:

$$\mathbf{v} = \tilde{\mathbf{G}}^{-1}\left(\mathbf{M}_{\mathrm{j}}\,\mathbf{j} + \mathbf{M}_{\mathrm{r}}\,\mathbf{j^0}\right) = \mathbf{v}^{\mathrm{L}} + \mathbf{D}^{\mathrm{v}}\,\mathbf{j^0} \tag{13}$$

The linear part $\mathbf{v}^{\mathrm{L}}$ is the response of a circuit in the original configuration while the residual part can be considered as a linear combination of the distortions. Following the VDM nomenclature, the matrix $\mathbf{D}^{\mathrm{v}}$ will be called the (potential) influence matrix. A column of the matrix can be interpreted as a vector of nodal potentials generated by the unit distortion imposed on the selected element (solution to the system of circuit equations with input vector equal to the column of incidence matrix $\mathbf{M}_{\mathrm{r}}$). In the AC case, unit distortion means harmonic current source of unit amplitude, zero phase, and frequency compliant with real sources.

The concepts of influence matrix and residual response can be applied to arbitrary circuit responses:

$$\mathbf{f} = \mathbf{f}^{\mathrm{L}} + \mathbf{D}^{\mathrm{f}}\,\mathbf{j^0}; \quad \mathbf{f}^{\mathrm{L}} = \mathbf{T}\,\mathbf{v}^{\mathrm{L}}; \quad \mathbf{D}^{\mathrm{f}} = \mathbf{T}\,\mathbf{D}^{\mathrm{v}} \tag{14}$$

where vector $\mathbf{f}$ stores circuit responses of arbitrary kind (currents, voltages, potentials) and $\mathbf{T}$ is a certain matrix of linear transformation. For example, voltages across resistors can be calculated as:

$$\mathbf{u} = \mathbf{u}^{\mathrm{L}} + \mathbf{D}^{\mathrm{u}}\,\mathbf{j}^{\mathbf{0}}; \quad \mathbf{u}^{\mathrm{L}} = \mathbf{M}_{\mathrm{r}}^{\mathrm{T}}\,\mathbf{v}^{\mathrm{L}}; \quad \mathbf{D}^{\mathrm{u}} = \mathbf{M}_{\mathrm{r}}^{\mathrm{T}}\,\mathbf{D}^{\mathrm{v}} \tag{15}$$

The matrices $\mathbf{D}^{\mathrm{u}}$ (voltage influence matrix) and $\mathbf{D}^{\mathrm{f}}$ (general influence matrix) are important notions in regard to VDM algorithms. First of them will be used to calculate the distortions corresponding to the assumed modifications of conductance while the second to update the circuit responses.

Having the concepts of linear responses and influence matrices defined, the relation between the distortions and the conductance modifications can be now derived. According to the concept, an element of the modified value of conductance $\tilde{G}$ can be substituted by the element of the original value G and the distortion j^0 inserted in parallel. It is assumed that the global state of circuit responses is the same in both cases. Relations between the current I and the voltage U across element terminals can be written as follows:

$$\text{Modified element: } I = -\tilde{G}\,U \tag{16a}$$

$$\text{Element with distortion: } I - j^0 = -G\,U \tag{16b}$$

Assuming equality of currents and voltages, the following condition for the value of the distortion can be obtained:

$$j^0 = (G - \tilde{G})\,U = (1 - \mu)\,G\,U, \tag{17}$$

where the parameter of conductance modification μ is introduced as:

$$\mu = \frac{\tilde{G}}{G} \tag{18}$$

Relation (17) between the distortion j^0 and the modification parameter μ is not linear as the voltage response U refers to the modified configuration of the circuit (it depends on all other introduced distortions). Distortions are mutually related hence they need to be considered simultaneously. Equation (17) written in the matrix form, with the voltage response expressed in terms of distortions, is as follows:

$$\mathbf{j^0} = \left(\mathbb{I} - [\boldsymbol{\mu}]\right)\mathbf{G}\left(\mathbf{u}^{\mathrm{L}} + \mathbf{D}^{\mathrm{u}}\,\mathbf{j^0}\right) \tag{19}$$

where $\mathbb{I}$ denotes the identity matrix and $[\boldsymbol{\mu}]$ – the diagonal matrix with the parameters μ on the main diagonal. Organizing variables with respect to distortions, Equation (19) can be transformed into the following system of linear equations:

$$\mathbf{A}^0\,\mathbf{j^0} = \mathbf{b} \tag{20}$$

where:

$$\mathbf{A}^0 = \mathbb{I} - \left(\mathbb{I} - [\boldsymbol{\mu}]\right)\mathbf{G}\,\mathbf{D}^{\mathrm{u}} \tag{21a}$$

$$\mathbf{b} = \left(\mathbb{I} - [\boldsymbol{\mu}]\right)\mathbf{G}\,\mathbf{u}^{\mathrm{L}} \tag{21b}$$

The system of Equation (20) enables to calculate vector of distortions $\mathbf{j^0}$ corresponding to the given vector of modifications $\boldsymbol{\mu}$. The system can be formulated locally, i.e., it may comprise only the modified elements (for every $\mu_k = 1$ meaning no modification of conductance $\to A^0_{kj} = \mathbb{I}_{kj}$, $b_k = 0$ and immediately $j^0_k = 0$). In the AC case, the system has the same form, but linear responses, influence matrix and the resulting distortions are expressed as complex amplitudes.

The basic algorithm of the VDM, which enables to calculate system responses for the given set of modifications, is a two step procedure. In the first step distortions are calculated from Equation (20), while in the second circuit responses are updated from Equation (14). To illustrate, let us consider the exemplary circuit from Figure 2, with the following values of parameters assumed: $R_1 \div R_5 = 5\,\mathrm{k\Omega}$, $C_1 = C_2 = 1\,\mu\mathrm{F}$, $L = 2\,\mathrm{mH}$. In the DC case, sources are of constant values $E = 5\,\mathrm{V}$ and $J = 0.002\,\mathrm{A}$. In the AC case, sources are of frequency 1kHz and complex amplitudes $\underline{E} = 5j\,\mathrm{V}$ and $\underline{J} = 0.002\,\mathrm{A}$. Suppose that changes of conductance are introduced into resistors R_1 and R_3, described by modifications parameters $\mu_1 = 0.5$ and $\mu_3 = 5$ (i.e., the modified values of resistances are $\tilde{R}_1 = 10\,\mathrm{k\Omega}$ and $\tilde{R}_3 = 1\,\mathrm{k\Omega}$).

To calculate the distortions corresponding to the assumed modifications, the linear voltage responses and the voltage influence matrix need to be precomputed from the model of the original circuit:

$$\mathbf{u}^{\mathrm{L}} = \begin{bmatrix} U_1^{\mathrm{L}} \\ U_3^{\mathrm{L}} \end{bmatrix} = \begin{bmatrix} 0.625 \\ 3.75 \end{bmatrix} \qquad \mathbf{D}^{\mathrm{u}} = \begin{bmatrix} 3125 & -1250 \\ -1250 & 2500 \end{bmatrix}$$

$$\underline{\mathbf{u}}^{\mathrm{L}} = \begin{bmatrix} -3.218 - 0.135j \\ 3.385 + 0.038j \end{bmatrix}$$

$$\underline{\mathbf{D}}^{\mathrm{u}} = \begin{bmatrix} 1672 - 88j & -1666 - 18j \\ -1666 - 18j & 1668 - 35j \end{bmatrix}$$

Columns of influence matrices correspond to voltage responses on unit distortions imposed in turn on resistors R_1 and R_3. Substituting modification parameters and the computed quantities into the system of Equation (20), the following values of distortions are obtained:

$$\mathbf{j^0} = \begin{bmatrix} 0.0002571 \\ -0.0009143 \end{bmatrix};$$

$$\underline{\mathbf{j}}^{\mathbf{0}} = \begin{bmatrix} -0.000139 - 0.000006j \\ -0.001239 - 0.000032j \end{bmatrix}$$

Let us suppose that a current in the coil and voltages across both capacitors are of interest in a certain problem. In the DC case, the linear responses and the general influence matrix are:

$$\mathbf{f}^{\mathrm{L}} = \begin{bmatrix} I_L \\ U_{C_1} \\ U_{C_2} \end{bmatrix} = \begin{bmatrix} -0.000625 \\ 4.375 \\ -3.125 \end{bmatrix} \quad \mathbf{D}^{\mathrm{f}} = \begin{bmatrix} -0.125 & 0.25 \\ 1875 & 1250 \\ -625 & 1250 \end{bmatrix}$$

Making use of Equation (14), the updated values of responses are:

$$\mathbf{f} = \mathbf{f}^{\mathrm{L}} + \mathbf{D}^{\mathrm{f}}\,\mathbf{j}^{\mathbf{0}} = \begin{bmatrix} -0.000886 \\ 3.714 \\ -4.429 \end{bmatrix}$$

It can be noted that the VDM may find application as an efficient method of circuit re-analysis, especially for large system, where only a few elements need to be modified and a few responses need to be re-calculated. The problem reduces to solving two locally formulated systems of linear equations (calculation of distortions and the update of responses). The additional numerical cost related with the calculation of linear responses and influence matrices is paid only at the initial stage of calculations. However, the more important application of the concept of virtual distortions is the possibility to derive accurate formulas for the gradient of circuit response and to solve the inverse problem of identification. These issues are covered in the next sections.

SENSITIVITY ANALYSIS

An important issue in many optimization problems is to determine how system responses are influenced by the minor modifications of structural parameters. In a general sense, to perform sensitivity analysis, the gradient of the response with respect to the selected variables need to be calculated. In this section, the VDM notions will be used to find the gradient of an arbitrary circuit response f_i, with respect to the parameter of conductance modification μ_j, for any state of circuit modifications ($\boldsymbol{\mu} \neq \mathbf{1}$).

The classical formulation of the problem, based on the definition of the derivative, has the following form:

$$\frac{\partial f_i(\boldsymbol{\mu})}{\partial \mu_j} = \lim_{\Delta\mu \to 0} \frac{f_i(\boldsymbol{\mu};\ \mu_j + \Delta\mu) - f_i(\boldsymbol{\mu})}{\Delta\mu} \tag{22}$$

The above formula provides only numerical approximation of gradient value, of accuracy strongly dependent on proper selection of the finite increment $\Delta\mu$. The VDM, making use of relations between system responses, distortions and modification parameters, enables to derive accurate analytical formulas.

It has been demonstrated in the previous section that arbitrary circuit response can be expressed as a linear function of distortions:

$$\mathbf{f} = \mathbf{f}^{\mathrm{L}} + \mathbf{D}^{\mathrm{f}}\,\mathbf{j}^{\mathbf{0}} \tag{23}$$

The linear responses and the components of the influence matrix are independent of the introduced modifications, hence the Jacobi matrix of the first derivatives of responses with respect to the modification parameters $\boldsymbol{\mu}$ can be calculated from the following relation:

$$\frac{\partial\,\mathbf{f}}{\partial \boldsymbol{\mu}} = \mathbf{D}^{\mathrm{f}}\,\frac{\partial\,\mathbf{j}^{\mathbf{0}}}{\partial \boldsymbol{\mu}} \tag{24}$$

The unknown quantity $\partial \mathbf{j}^{\mathbf{0}}/\partial \boldsymbol{\mu}$ on the right-hand side of the equation is defined as a *gradient of distortions*. Similarly as the distortions, the components of the gradient are mutually interrelated and need to be considered globally. To derive the formula for the gradient of distortions, both sides of Equation (19) need to be differentiated:

$$\frac{\partial\,\mathbf{j}^{\mathbf{0}}}{\partial \boldsymbol{\mu}} = \frac{\partial}{\partial \boldsymbol{\mu}}\left[\left(\mathbb{I} - [\boldsymbol{\mu}]\right)\mathbf{G}\left(\mathbf{u}^{\mathrm{L}} + \mathbf{D}^{\mathrm{u}}\,\mathbf{j}^{\mathbf{0}}\right)\right] \tag{25}$$

Calculating the derivative of the right-hand side of the equation and organizing variables with respect to components of the gradient, the following system of linear equations can be formulated:

$$\mathbf{A}^{0}\,\frac{\partial\,\mathbf{j}^{\mathbf{0}}}{\partial \boldsymbol{\mu}} = \mathbf{B} \tag{26}$$

where:

$$\mathbf{A}^{0} = \mathbb{I} - \left(\mathbb{I} - [\boldsymbol{\mu}]\right)\mathbf{G}\,\mathbf{D}^{\mathrm{u}} \tag{27a}$$

$$\mathbf{B} = \mathrm{diag}\left\{-\mathbf{G}\left(\mathbf{u}^{\mathrm{L}} + \mathbf{D}^{\mathrm{u}}\,\mathbf{j}^{\mathbf{0}}\right)\right\} \tag{27b}$$

Matrix $\mathbf{A}^{0}$ is the same matrix as in the system of equations used to calculate distortions (Equation (21a)). $\mathbf{B}$ is a diagonal matrix dependent on the modeled voltage response. In the case of original circuit configuration ($[\boldsymbol{\mu}] = \mathbb{I}$), components of gradient of distortions can be calculated from the simplified relation:

$$\frac{\partial j_i^{0}}{\partial \mu_k} = \begin{cases} -G_i\,U_i^{L} & \text{for} \quad i = k \\ 0 & \text{for} \quad i \neq k \end{cases} \tag{28}$$

In order to calculate the gradient of circuit response, distortions (if necessary) have to be calculated from Equation (20), then the gradient of distortions from Equation (26) and in the end, the gradient of the response can be obtained from Equation (24). All the above relations have been derived for the DC case, but are also valid for the AC case (provided that responses and distortions are expressed as complex amplitudes and complex derivatives are considered).

INVERSE ANALYSIS AND DEFECT IDENTIFICATION

The concept of virtual distortions is well suited to formulate and solve inverse problems related with modifications of structural parameters. If the main task is to identify the modifications of parameters which realize the desired system performance (system optimization) or which invoke changes in response conforming with the behavior of a real system (model calibration, defect identification), then the problem can be formulated as a search for the equivalent distribution of distortions and solved from the transformed system of linear equations or by means of a gradient-based optimization. This section is devoted to the problem of defect identification, where the term *defect* should be understood as a modification of conductance (in a range from break to short-circuit) in resistive element of a circuit. It is assumed that input data include full numerical model of the circuit in the original configuration and a certain set of responses obtained for the modified configuration. These *reference responses*, presumably measured in the real system, may be of any kind: nodal potentials, voltages or currents in arbitrary elements. It is also assumed that locations of defects can be initially limited to selected circuit elements, although all elements may be considered as well.

Vector of the reference responses $\mathbf{f}^{\mathrm{ref}}$ (acquired from the modified circuit) corresponds to a certain unknown state of modification parameters $\boldsymbol{\mu}$, which can be simulated by virtual distortions $\mathbf{j}^{\mathbf{0}}$:

$$\mathbf{f}^{\mathrm{ref}} = \mathbf{f}(\boldsymbol{\mu}) = \mathbf{f}(\mathbf{j}^{\mathbf{0}}) \tag{29}$$

The response of a circuit with imposed distortions can be decomposed into linear and residual part:

$$\mathbf{f}(\mathbf{j}^{\mathbf{0}}) = \mathbf{f}^{\mathrm{L}} + \mathbf{D}^{\mathrm{f}}\mathbf{j}^{\mathbf{0}} \tag{30}$$

The linear responses and the influence matrix are obtained from the model of the original circuit. By simple substitution and transformation, the vector of distortions corresponding to the unknown modifications, can be calculated as:

$$\mathbf{j}^{\mathbf{0}} = (\mathbf{D}^{\mathrm{f}})^{-1}(\mathbf{f}^{\mathrm{ref}} - \mathbf{f}^{\mathrm{L}}) \tag{31}$$

Naturally, this operation is allowed and produce a unique solution only if the influence matrix is square and non-singular. Hence, the procedure of inverse analysis based on Equation (31) demands that the selected reference responses have to fulfill the following conditions:

1. The number of the reference responses (corresponding to the number of rows in the influence matrix) is equal to the number of assumed possible distortion locations (corresponding to the number of columns).
2. The reference responses depend on the modifications (they are not determined exclusively by the supply or boundary conditions).
3. The reference responses are mutually independent of each other (one cannot be determined from the others using constitutive relations or Kirchhoff's laws).

To explain the second and third condition, let us turn back again to the exemplary circuit from Figure 2. A full graph of the circuit comprising all its elements is presented in Figure 5(a). The third condition requires that the reference responses cannot include voltages in all elements comprising a single loop, nor currents in all elements connected to a single node. Another demand is that the voltage and current in the same element cannot be assigned as reference responses if the element is excluded from the set of possible defect locations. If these conditions are not satisfied then one or more rows of the incidence matrix can be expressed as a linear combination of the others (using Kirchhoff's or Ohm's laws) and the matrix is singular. The resulting general rule is that the number of independent voltage responses is equal to the number of independent nodes while the number of independent current responses is equal to the number of elementary loops. In the full graph of the exemplary circuit, six nodes and five loops can be distinguished. However, the second condition postulates that nodes and loops for which potentials and currents can be determined from the supply or boundary conditions, need to be excluded (Figure 5(b)). These are nodes number one ($V_1 = 0$) and six ($V_6 = V_1 + E$) and a loop including the current source J. Additional dependencies can be pre-determined in the DC case (Figure 5(c)): the coil introduces a short-circuit ($V_5 = V_4$) while the capacitors introduce breaks (loop currents equal zero). Altogether, in the DC case there are only three independent nodes and two elementary loops, while in the AC

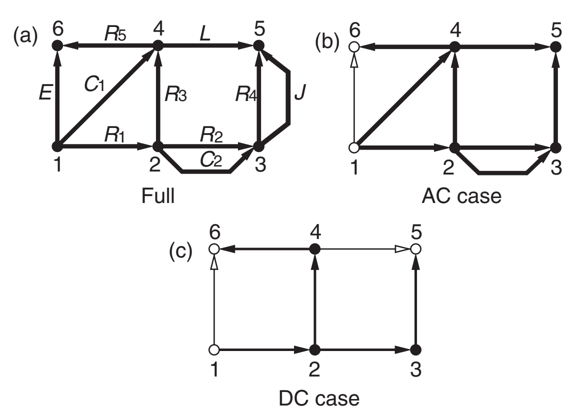


Figure 5. *Graphs of exemplary circuit.*

case there are four nodes and four loops. If possible defect locations are assumed in all resistive elements of the exemplary circuit (five unknowns), then five mutually independent reference responses need to be assigned. In the DC case the only option is to choose voltages on three different resistors (or potentials in independent nodes) and two currents in every loop. In the AC case there are eight possible responses to choose from (four voltages and four currents, provided that the third condition is not violated).

It was assumed in the above analysis that the numerical model of the circuit and the reference responses are accurate and error-free. In practice, parameters of circuit elements are always specified within a certain tolerance and reference responses are disturbed by the measurement errors. Moreover, if the constraints need to be imposed on the unknowns, then Equation (31) cannot be used directly to find the solution. However, the VDM enables to solve the problem iteratively, through the gradient-based optimization.

Let the vector of distance functions $\mathbf{d}$ contain differences between the responses simulated by a temporary state of distortions and the reference responses:

$$\mathbf{d} = \mathbf{f}(\mathbf{j^0}) - \mathbf{f}^{\text{ref}} \tag{32}$$

The objective function g is defined as the least square problem:

$$g = (\mathbf{d})^{\text{H}}\mathbf{d} \tag{33}$$

where $(\cdot)^{\text{H}}$ denotes the conjugate transpose. Function g is a real-valued function of real (DC case) or complex arguments (AC case). Optimization variables, in regard to which function g will be minimized, may be the distortions $\mathbf{j^0}$ or the modification parameters $\boldsymbol{\mu}$. In the latter case, constraints can be easily defined: they may specify physical conditions ($\mu_k \geq 0$), but also expected type of defects (like $\mu_k \in [\,0;\,1]$ in the case of breaks identification). Making use of the steepest-descent approach, in every iteration of the optimization procedure, the modification parameters are updated according to the following formula:

$$\boldsymbol{\mu}^{(p+1)} = \boldsymbol{\mu}^{(p)} - \lambda^{(p)}\,\nabla g \tag{34}$$

where p denotes the iteration step and $\lambda^{(p)}$ is a non-negative factor normalizing the step length. The gradient of the objective function ∇g with respect to the modification parameters $\boldsymbol{\mu}$ can be calculated from the following relation:

$$\nabla g = 2\left(\frac{\partial \mathbf{d}}{\partial \boldsymbol{\mu}}\right)^{\text{H}}\mathbf{d} \tag{35}$$

where $\partial\mathbf{d}/\partial\boldsymbol{\mu}$ is the Jacobi matrix of partial derivatives of the distance functions, equal to the gradient of response (Equation (24)):

$$\frac{\partial \mathbf{d}}{\partial \boldsymbol{\mu}} = \frac{\partial \mathbf{f}}{\partial \boldsymbol{\mu}} = \mathbf{D}^{\text{f}}\,\frac{\partial \mathbf{j^0}}{\partial \boldsymbol{\mu}} \tag{36}$$

However, the procedure of gradient-based optimization usually also demands that the number of the assumed defect locations is not greater than the number of independent reference responses. If not, the obtained vector of modifications $\boldsymbol{\mu}^{\text{opt}}$, although minimizes the objective function, may not be equal with the actual solution ($\boldsymbol{\mu}^{\text{opt}} \neq \boldsymbol{\mu}$). The corresponding vector of distortions consists then of the additional components called the impotent states of distortions:

$$\boldsymbol{\mu}^{\text{opt}} \Longleftrightarrow \mathbf{j^0} + \sum \mathbf{j}^{\mathbf{0}(\text{imp})} \tag{37}$$

$\mathbf{j^0}$ represents the distortions corresponding with the actual solution ($\mathbf{j^0} \Leftrightarrow \boldsymbol{\mu}$). The remaining components are called the impotent states because they do not generate voltage or current response, either globally or locally with regard to the reference responses. Impotent states of distortions can be explained on the base of principle of inserting ideal sources known from the circuit theory:

- Ideal current sources (current distortions) of equal intensity inserted in parallel with all elements comprising a loop will not affect the state of voltages in the circuit. This results from the fact that every node belonging to the loop is connected with two current sources of opposite direction (which mutually cancel in Kirchhoff's current law).
- Ideal voltage sources (voltage distortions) of equal value inserted in series with every element connected with the selected node will not affect the state of currents in the circuit. This results from the fact that every loop containing the given node includes two voltage sources of opposite polarity (which mutually cancel in Kirchhoff's voltage law).

Impotent states of distortions may be generated during the optimization if the set of the assumed defect locations includes elements comprising a loop or elements isolating a node. To avoid generation of these specific configurations of distortions, the set of the reference responses needs to include current in at least one of the elements comprising the loop and voltage in one of the elements isolating the node.

SUMMARY

It has been demonstrated that the concept of virtual distortions, to the well-defined system of analogies, can be easily adapted to the steady-state analysis of simple electrical circuits. Modifications of conductance can be simulated by the equivalent set of virtual current sources (distortions) which enable to formulate effective algorithms for re-analysis, derive a formula for the gradient of response and solve the inverse problem of defect identification. However, the procedure of defect identification for the steady-state cases is limited in the case of the global approach. If a large number of possible defect locations is assumed then the same number of independent reference responses need to be acquired in order to solve the problem (measurements of both voltages and currents might be required). Additionally in the AC case, since the reference responses are considered to be complex amplitudes, both amplitude and phase of the signal need to be measured.

A continuation of this work will be implementation of Impulse Virtual Distortion Method (IVDM) to the transient analysis in discrete time domain. Combined with the concept of *piezodiagnostic* (i.e., the monitored structure is equipped with piezo-electric sensors registering propagation of elastic wave), the IVDM enables to solve the dynamic problem of damage identification (solution through the gradient-based optimization). The IVDM not only enables to reduce the required number of sensors, but also to identify other structural modifications (for example reduction of mass). It is expected that analogous defect identification procedure may be formulated for the transient states in electrical circuits.

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