Saturation of Estimates for the Maximum Enstrophy Growth in a Hydrodynamic System as an Optimal Control Problem

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Thanks to Ch. Doering (University of Michigan)
& D. Pelinovsky (McMaster)

Funded by Early Researcher Award (ERA)

November 2, 2011
Agenda

Background
  Regularity Problem for Navier–Stokes Equation
  Enstrophy Estimates

Saturation of Estimates as Optimization Problem
  Instantaneous Estimates
  Finite-Time Estimates
  Burgers Problem

Results
  Optimal Solutions for Wavenumber $m = 1$
  Envelopes & Power Laws
  Solutions for Other Initial Guesses $m = 2, 3, \ldots$
Navier–Stokes equation \((\Omega = [0, L]^d, d = 2, 3)\)

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p - \nu \Delta \mathbf{v} &= 0, \quad \text{in } \Omega \times (0, T] \\
\nabla \cdot \mathbf{v} &= 0, \quad \text{in } \Omega \times (0, T] \\
\text{Initial Condition} & \quad \text{on } \Gamma \times (0, T] \\
\text{Boundary Condition (periodic)} & \quad \text{in } \Omega \text{ at } t = 0
\end{align*}
\]

**2D Case**
- Existence Theory Complete — smooth and unique solutions exist for arbitrary times and arbitrarily large data

**3D Case**
- Weak solutions (possibly nonsmooth) exist for arbitrary times
- Classical (smooth) solutions (possibly nonsmooth) exist for *finite* times only
- Possibility of “blow–up” (finite–time singularity formation)
- One of the Clay Institute “Millennium Problems” ($1M!$)
  
  [http://www.claymath.org/millennium/Navier-Stokes_Equations](http://www.claymath.org/millennium/Navier-Stokes_Equations)
What is known? — Available Estimates

- **A Key Quantity — Enstrophy**
  \[
  \mathcal{E}(t) \equiv \int_{\Omega} |\nabla \times \mathbf{v}|^2 d\Omega \quad (= \|\nabla \mathbf{v}\|_2^2)
  \]

- **Smoothness of Solutions ⇔ Bounded Enstrophy**
  (Foias & Temam, 1989)
  \[
  \max_{t \in [0, T]} \mathcal{E}(t) < \infty
  \]

- **Can estimate \(\frac{d\mathcal{E}(t)}{dt}\)** using the momentum equation, Sobolev’s embeddings, Young and Cauchy–Schwartz inequalities, ...
  - **Remark:** incompressibility not used in these estimates ....
2D Case:

\[ \frac{d\mathcal{E}(t)}{dt} \leq \frac{C^2}{\nu} \mathcal{E}(t)^2 \]

- Gronwall’s lemma and energy equation yield \( \forall t \mathcal{E}(t) < \infty \)
- smooth solutions exist for all times

3D Case:

\[ \frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3} \mathcal{E}(t)^3 \]

- corresponding estimate not available ....
- upper bound on \( \mathcal{E}(t) \) blows up in finite time

\[ \mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4\frac{C\mathcal{E}(0)^2}{\nu^3} t}} \]

- singularity in finite time cannot be ruled out!
Problem of Lu & Doering (2008), I

- Can we actually find solutions which “saturate” a given estimate?
- Estimate $\frac{dE(t)}{dt} \leq cE(t)^3$ at a fixed instant of time $t$

\[
\max_{\mathbf{v} \in H^1(\Omega), \nabla \cdot \mathbf{v} = 0} \frac{dE(t)}{dt}
\]

subject to $E(t) = E_0$

where

- $\frac{dE(t)}{dt} = -\nu \|\Delta \mathbf{v}\|^2_2 + \int_\Omega \mathbf{v} \cdot \nabla \mathbf{v} \cdot \Delta \mathbf{v} d\Omega$
- $E_0$ is a parameter
- Solution using a gradient–based descent method
Problem of Lu & Doering (2008), II

\[ \left[ \frac{d\mathcal{E}(t)}{dt} \right]_{max} = 8.97 \times 10^{-4} \mathcal{E}_0^{2.997} \]
How about solutions which saturate $\frac{dE(t)}{dt} \leq cE(t)^3$ over a finite time window $[0, T]$?

$$\max_{v \in H^1(\Omega), \nabla \cdot v = 0} \left[ \max_{t \in [0, T]} E(t) \right]$$

subject to $E(t) = E_0$

where

$\triangleright$

$$E(t) = \int_0^t \frac{dE(\tau)}{d\tau} \, d\tau + E_0$$

$\triangleright$ $E_0$ is a parameter

$\triangleright$ $\max_{t \in [0, T]} E(t)$ nondifferentiable w.r.t initial condition $\implies$ non–smooth optimization problem

$\triangleright$ In principle doable, but will try something simpler first ...
Burgers equation ($\Omega = [0, 1]$, $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}$)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } \Omega$$

$$u(x) = \phi(x) \quad \text{at } t = 0$$

Periodic B.C.

Enstrophy: $\mathcal{E}(t) = \frac{1}{2} \int_0^1 |u_x(x, t)|^2 \, dx$

Solutions smooth for all times

Questions of sharpness of enstrophy estimates still relevant

$$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2} \left( \frac{1}{\pi^2 \nu} \right)^{1/3} \mathcal{E}(t)^{5/3}$$

Best available finite-time estimate

$$\max_{t \in [0, T]} \mathcal{E}(t) \leq \left[ \mathcal{E}_0^{1/3} + \left( \frac{L}{4} \right)^2 \left( \frac{1}{\pi^2 \nu} \right)^{4/3} \mathcal{E}_0 \right]^{3} \rightarrow \mathcal{C}_2 \mathcal{E}_0^3$$
“Small” Problem of Lu & Doering (2008), I

Estimate \( \frac{d\mathcal{E}(t)}{dt} \leq c\mathcal{E}(t)^{5/3} \) at a \textit{fixed} instant of time \( t \)

\[
\max_{u \in H^1(\Omega)} \frac{d\mathcal{E}(t)}{dt}
\]

subject to \( \mathcal{E}(t) = \mathcal{E}_0 \)

where

\[
\frac{d\mathcal{E}(t)}{dt} = -\nu \left\| \frac{\partial^2 u}{\partial x^2} \right\|_2^2 + \frac{1}{2} \int_0^1 \left( \frac{\partial u}{\partial x} \right)^3 d\Omega
\]

\( \mathcal{E}_0 \) is a parameter

Solution (maximizing field) found analytically! (in terms of elliptic integrals and Jacobi elliptic functions)
"Small" Problem of Lu & Doering (2008), II

\[
\left[ \frac{d\mathcal{E}(t)}{dt} \right]_{\text{max}} = 0.2476 \frac{\mathcal{E}_0^{5/3}}{\nu^{1/3}}
\]

Instantaneous estimate is sharp

\[
\max_{t \in [0, T]} \mathcal{E}(t) \leq C\mathcal{E}_0^{1.048}
\]

Finite–time estimate far from saturated
Finite–Time Optimization Problem (I)

▶ Statement

\[ \max_{u \in H^1(\Omega)} \mathcal{E}(T) \]

subject to \( \mathcal{E}(t) = \mathcal{E}_0 \)

\( T, \mathcal{E}_0 \) — parameters

▶ Optimality Condition

\[ \forall \phi' \in H^1 \quad \mathcal{J}'(\phi; \phi') = -\int_0^1 \frac{\partial^2 u}{\partial x^2} \bigg|_{t=T} u' \bigg|_{t=T} dx - \lambda \int_0^1 \frac{\partial^2 \phi}{\partial x^2} \bigg|_{t=0} u' \bigg|_{t=0} dx \]
Finite–Time Optimization Problem (II)

- **Gradient Descent**

\[ \phi^{(n+1)} = \phi^{(n)} - \tau^{(n)} \nabla J(\phi^{(n)}), \quad n = 1, \ldots, \]
\[ \phi^{(0)} = \phi_0, \]

where \( \nabla J \) determined from adjoint system via \( H^1 \) Sobolev preconditioning

\[ -\frac{\partial u^*}{\partial t} - u \frac{\partial u^*}{\partial x} - \nu \frac{\partial^2 u^*}{\partial x^2} = 0 \quad \text{in } \Omega \]
\[ u^*(x) = -\frac{\partial^2 u}{\partial x^2}(x) \quad \text{at } t = T \]

Periodic B.C.

- **Step size** \( \tau^{(n)} \) found via *arc minimization*
Two parameters: $T, E_0 \quad (\nu = 10^{-3})$

Optimal initial conditions corresponding to initial guess with wavenumber $m = 1$ (local maximizers)

Fixed $E_0 = 10^3$, different $T$

Fixed $T = 0.0316$, different $E_0$
Background
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Solutions for Other Initial Guesses \(m = 2, 3, \ldots\)

\[
\text{argmax}_{t \in [0, T]} \mathcal{E}(t) \sim C \mathcal{E}_0^{-0.5}
\]

\[
\max_{t \in [0, T]} \mathcal{E}(t) \sim C \mathcal{E}_0^{1.5}
\]
Sol’ns found with initial guesses $\phi^{(m)}(x) = \sin(2\pi mx), \ m = 1, 2, \ldots$

$m = 1, \mathcal{E}_0 = 10^3$

$m = 2, \mathcal{E}_0 = 10^3$

Change of variables leaving Burgers equation invariant ($L \in \mathbb{Z}^+$):

$x = L\xi, \ (x \in [0, 1], \ \xi \in [0, 1/L]), \ \tau = t/L^2$

$\nu(\tau, \xi) = Lu(x(\xi), t(\tau)),$

$\mathcal{E}_\nu(\tau) = L^4\mathcal{E}_u \left( \frac{t}{L^2} \right)$
Background
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Results

- **Solutions for** $m = 1$ and $m = 2$, after rescaling

  ![Graphs showing solutions for $m = 1$ and $m = 2$.]

- **Using initial guess**:
  
  $\phi^{(0)}(x) = \sin(2\pi mx)$, $m = 1$, or $m = 2$

  $\phi^{(0)}(x) = \epsilon \sin(2\pi mx) + (1 - \epsilon) \sin(2\pi nx)$, $m \neq n$, $\epsilon > 0$

- **All local maximizers with** $m = 2, 3, \ldots$ are rescaled copies of the $m = 1$ maximizer
Location of Singularities in $\mathbb{C}$ from the Fourier spectrum

$$|\hat{u}_k| \sim C |k|^{-\alpha} e^{iz^*} \quad \text{as} \quad k \to \infty$$

Analyticity strip for a meromorphic function
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\[ \Im\{z^*(t)\} \]

\[ \mathcal{E}(t) \]

- **RED** — instantaneously optimal (Lu & Doering, 2008)
- **BOLD BLUE** — finite–time optimal ($T = 0.1$)
- **DASHED BLUE** — finite–time optimal ($T = 1$)
Summary & Conclusions

- Some evidence that optimizers found are in fact *global*
- Exponents in \( \max_{t \in [0, T]} E(t) = CE_0^\alpha \) as \( E_0 \to \infty \)

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<th>( \alpha )</th>
<th>theoretical estimate</th>
<th>optimal (instantaneous)</th>
<th>optimal (finite–time)</th>
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<tr>
<td>3</td>
<td>[Lu &amp; Doering, 2008]</td>
<td>([\text{present study}])</td>
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- More rapid enstrophy build–up in finite–time optimizers than in instantaneous optimizers
- Theoretical estimate *not sharp* \( \implies \) finite–time optimizers offer insights re: refinements required (work in progress)

- Finite–time maximizers saturate Poincaré’s inequality (largest kinetic energy for a given enstrophy)
- Future work: Navier–Stokes 2D, 3D...