ON SOME THERMOELASTIC BRESSE SYSTEMS FREE OF THE SECOND SPECTRUM

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Outline

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1. Introduction
2. Existence and uniqueness of solutions
3. Exponential stability
   - Functionals and related lemmas
   - Main theorem and short proof
4. Polynomial Stability (case $\kappa_3\rho_1 - \kappa_1\rho_3 \neq 0$)
5. Numerical approximation
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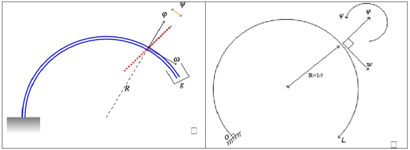
Bresse system I

☞ Bresse system is given by the following equations (Bresse 1959)

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_1\omega_{tt} = N_x - lQ + F_3, \end{cases} \tag{1}$$

- φ: transverse displacement, ψ: shear angle displacement, ω: longitudinal displacement, F_i: external forces
- N: axial force, Q: shear force, M: bending moment



Bresse system II

with $\rho_1 = \rho A$, $\rho_2 = \rho I$ and $l = R^{-1}$, where

- ρ: density, A: cross-sectional area, I: second moment of area of the cross-sectional area
- R: radius of curvature of the beam

☞ Constitutive relations:

$$Q = \kappa_1 (\varphi_x + \psi + l\omega), \quad M = \kappa_2 \psi_x, \quad N = \kappa_3 (\omega_x - l\varphi), \tag{2}$$

where $\kappa_1 = kAG$, $\kappa_2 = EI$ and $\kappa_3 = EA$, such that

- E: Young modulus of elasticity
- G: modulus of rigidity
- k: transverse shear factor

Bresse system III

- In the absence of heat effect, (1) and (2) yield the evolution Bresse system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa_1\left(\varphi_x + \psi + l\omega\right)_x - l\kappa_3\left(\omega_x - l\varphi\right) = F_1, \\ \rho_2\psi_{tt} - \kappa_2\psi_{xx} + \kappa_1\left(\varphi_x + \psi + l\omega\right) = F_2, \\ \rho_1\omega_{tt} - \kappa_3\left(\omega_x - l\varphi\right)_x + l\kappa_1\left(\varphi_x + \psi + l\omega\right) = F_3. \end{cases} \tag{3}$$

- This system has been investigated by many researchers, using various dampings, for example: Affilal, Guesmia and Soufyane ∗ Alabau-Boussouira, Muñoz Rivera and Almeida Júnior ∗ Alves, Fatori, M. Jorge Silva and Monteiro Filippo Dell.Oro ∗ Keddi, Tijani and Messaoudi ∗ Said-Houari and Soufyane ∗ Messaoudi and Hassan ∗ Guesmia ∗ Al-Arwadi, Messaoudi and Hassan ∗ Wehbe and Youssef and many others.

Summary of results

- In the presence of a linear frictional damping, viscoelastic or Fourier heat acting on the shear equation, the exponential stability is only obtained if the equal wave speeds hold:

$$\frac{\rho_1}{\kappa_1} = \frac{\rho_2}{\kappa_2} \text{ and } \kappa_1 = \kappa_3.$$

- If a linear frictional damping, viscoelastic or Fourier heat acting on the transverse displacement, the exponential stability can be obtained under appropriate relations between the parameters.
- If a linear frictional damping, viscoelastic or Fourier heat acting on the longitudinal displacement equation, the exponential stability cannot be obtained regardless to the parameters.

Timoshenko system I

☞ If the curvature l is zero, (3) reduces to the well-known Timoshenko system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa_1\left(\varphi_x + \psi\right)_x = F_1, \\ \rho_2\psi_{tt} - \kappa_2\psi_{xx} + \kappa_1\left(\varphi_x + \psi\right) = F_2 \end{cases} \tag{4}$$

and an independent wave equation

$$\rho_1\omega_{tt} - \kappa_3\omega_{xx} = F_3.$$

Timoshenko system II

- The same property was observed in the Timoshenko system (4); namely in the presence of only damping, the exponential stability holds if and only if

$$\frac{\rho_1}{\kappa_1} = \frac{\rho_2}{\kappa_2}.$$

- For example: Akil, Chitour, Ghader and Wehbe ∗ Apalara, Messaoudi and Keddi ∗ Tatar ∗ Jùnior, Santos and Rivera ∗ Messaoudi and Mustafa ∗ Messaoudi and Said-Houari ∗ Lasiecka et al and many others.

Truncated Timoshenko system I

- From the Physics point of view, it was shown that Timoshenko system (4) is characterized by two natural frequencies that yield to a paradox known as the second spectrum.
- The rotatory inertia and shear deformation included in the Timoshenko beam model lead to a hyperbolic equation with a finite wave propagation velocity starting from the first spectrum of natural frequencies.
- Nevertheless, the interaction of shear deformation and rotatory inertia creates a second spectrum, which is a new frequency range from which waves propagate at a lower frequency with infinite velocity.

Truncated Timoshenko system II

- This is a nonphysical condition for vibrating beams, of course, and the second spectrum plays a significant role in the literature on vibrating beam models.
- Over the years, a great deal of research has been done on this topic. To deal with this paradox and eliminate the second spectrum anomaly, other models have been proposed.
- In 2009, Elishakoff [1] proposed the following truncated form

$$\begin{cases} \rho_1\varphi_{tt} - \kappa_1\left(\varphi_x + \psi\right)_x = F_1, \\ -\boldsymbol{\rho_2\varphi_{xtt}} - \kappa_2\psi_{xx} + \kappa_1\left(\varphi_x + \psi\right) = F_2. \end{cases} \tag{5}$$

Truncated Timoshenko system III

☞ Under several damping terms, various stability results of system (5) have been established irrespective of the system parameters. For example:

➠ Almeida Júnior, Ramos and Freitas [2] looked into

$$\begin{cases} \rho_1\varphi_{tt} - \kappa_1(\varphi_x + \psi)_x + \mu\varphi_t = 0, \\ -\kappa_2\psi_{xx} + \kappa_1(\varphi_x + \psi) = 0. \end{cases} \tag{6}$$

- Discussed briefly the well-posedness.
- Proved an exponential decay without imposing any relationship between the coefficients.

Truncated Timoshenko system IV

- Apalara et al. [3] considered the following thermoelastic truncated Timoshenko system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) + \gamma\theta_x = 0, \\ \rho_3\theta_t - \beta\theta_{xx} + \gamma\psi_{xt} = 0, \end{cases} \tag{7}$$

  - Discussed briefly the well-posedness.
  - Proved an exponential decay without imposing any relationship between the coefficients.

Truncated Timoshenko system V

➠ Keddi, Messaoudi and Alahyane [4] studied the system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) + \delta\theta_x = 0, \\ c\theta_t + q_x + \delta\psi_{xt} = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases} \tag{8}$$

- Established the well-posedness, using some nonclassical techniques.
- Proved an exponential decay without imposing any relationship between the coefficients.
- Provided some numerical illustrations.

Truncated Timoshenko system VI

- Messaoudi, Keddi and Alahyane studied the system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \gamma\left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_x = 0, \\ \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_t - \kappa\left(\tau_\theta\theta_{xt} + \theta_x\right)_x + \gamma\theta^0\psi_{xt} = 0, \end{cases} \quad (9)$$

  - Established the well-posedness, using some nonclassical techniques.
  - Proved an exponential decay without imposing any relationship between the coefficients.
  - Provided some numerical illustrations.
- See other works by Ahmima et al., Ben Moussa et al., Ramos et al., Messaoudi and Keddi, Zougheib and El Arwadi, ...

Truncated Bresse system I

- Taking into account the truncated shear beam model (6) and considering the Bresse system as an extension of the Timoshenko model, we consider the shear Bresse system:

$$\begin{cases} \rho_1\varphi_{tt} - \kappa_1\left(\varphi_x + \psi + l\omega\right)_x - l\kappa_3\left(\omega_x - l\varphi\right) + \beta\theta_x = 0, \\ -\kappa_2\psi_{xx} + \kappa_1\left(\varphi_x + \psi + l\omega\right) = 0, \\ \rho_3\omega_{tt} - \kappa_3\left(\omega_x - l\varphi\right)_x + l\kappa_1\left(\varphi_x + \psi + l\omega\right) = 0, \\ c\theta_t - \kappa\theta_{xx} + \beta\varphi_{xt} = 0, \end{cases} \tag{10}$$

in $(0, L) \times (0, \infty)$, where $\rho_1, \rho_3, \kappa_1, \kappa_3, b, c, l, \beta, \kappa$ are positive constants.

Truncated Bresse system II

- We take the following boundary conditions

$$\begin{matrix} \varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0, \\ \omega_x(0,t) = \omega_x(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0, \end{matrix} \quad \text{for } t \in [0,\infty) \tag{11}$$

and the initial conditions

$$\begin{matrix} \varphi(x,0) = \varphi_0(x), & \omega(x,0) = \omega_0(x), \\ \varphi_t(x,0) = \varphi_1(x), & \omega_t(x,0) = \omega_1(x), \\ & \theta(x,0) = \theta_0(x), \end{matrix} \quad \text{for } x \in (0,L). \tag{12}$$

Outline

Auxiliary system I

Before discussing the well-posedness of our problem, we introduce, by eliminating ψ, the following auxiliary system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa_1\mathfrak{B}\left(\varphi_x + l\omega\right)_x - l\kappa_3\left(\omega_x - l\varphi\right) + \beta\theta_x = 0, \\ \rho_3\omega_{tt} - \kappa_3\left(\omega_x - l\varphi\right)_x + l\kappa_1\mathfrak{B}\left(\varphi_x + l\omega\right) = 0, \\ c\theta_t - \kappa\theta_{xx} + \beta\varphi_{xt} = 0, \end{cases} \tag{13}$$

with $\mathfrak{B}$ is defined as follows:

Auxiliary system II

$$\mathfrak{B} = I - \kappa_1 \mathscr{S}^{-1} = -\kappa_2 \left(\mathscr{S}^{-1} \circ \partial_{xx}\right),$$

where

$$\mathscr{S} = -\kappa_2 \partial_{xx} + \kappa_1 I.$$

Semigroup Setting I

Let us consider the following spaces

$$\begin{aligned} L_*^2(0,L) &= \left\{ u \in L^2(0,L) : \int_0^L u(x)\,dx = 0 \right\}, \\ H_*^1(0,L) &= H^1(0,L) \cap L_*^2(0,L), \\ H_*^2(0,L) &= \left\{ u \in H^2(0,L) : u_x \in H_0^1(0,L) \right\} \end{aligned}$$

and the Hilbert space

$$\mathscr{H} = H_0^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times L_*^2(0,L) \times L_*^2(0,L)$$

Semigroup Setting II

equipped with the inner product

$$\begin{aligned}\langle U, U^*\rangle_{\mathscr{H}} &= \rho_1 \langle \phi, \phi^*\rangle + \kappa_2\kappa_1 \left\langle \mathscr{S}^{-\frac{1}{2}} (\varphi_x + l\omega)_x , \mathscr{S}^{-\frac{1}{2}} (\varphi_x^* + l\omega^*)_x \right\rangle \\ &+\rho_3 \langle w, w^*\rangle + \kappa_3 \langle (\omega_x - l\varphi), (\omega_x^* - l\varphi^*)\rangle + c\,\langle \theta, \theta^*\rangle ,\end{aligned}$$

for $U = (\varphi, \phi, \omega, w, \theta)^T, U^* = (\varphi^*, \phi^*, \omega^*, w^*, \theta^*)^T \in \mathscr{H}$.
Therefore, system (13) can be written in the operator form

$$\begin{cases} U_t = \mathscr{A} U, \quad \forall t > 0, \\ U(0) = U_0 = (\varphi_0, \varphi_1, \omega_0, \omega_1, \theta_0)^T , \end{cases} \tag{14}$$

Semigroup Setting III

where the operator $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ is defined by

$$\mathscr{A}U = \begin{pmatrix} \phi \\ -\rho_1^{-1}\left[-\kappa_1\mathfrak{B}(\varphi_x + l\omega)_x - l\kappa_3(\omega_x - l\varphi) + \beta\theta_x\right] \\ w \\ -\rho_3^{-1}\left[-\kappa_3(\omega_x - l\varphi)_x + l\kappa_1\mathfrak{B}(\varphi_x + l\omega)\right] \\ -c^{-1}\left[-\kappa\theta_{xx} + \beta\phi_x\right] \end{pmatrix}.$$

and

$$\begin{aligned} D(\mathscr{A}) &= \left[H^2(0,L) \cap H_0^1(0,L)\right] \times H_0^1(0,L) \times \left[H_*^2(0,L) \cap H_*^1(0,L)\right] \\ &\quad \times H_*^1(0,L) \times \left[H_*^2(0,L) \cap H_*^1(0,L)\right]. \end{aligned}$$

Auxiliary well-posedness result I

Thus, we have the following existence and uniqueness result of the auxiliary system (13) .

Theorem 1

For any initial data $U_0 \in \mathscr{H}$, the problem (14) has a unique weak solution $U \in C\left([0,+\infty);\mathscr{H}\right)$. Moreover, if $U_0 \in D\left(\mathscr{A}\right)$, then $U \in C\left([0,+\infty);D\left(\mathscr{A}\right)\right) \cap C^1\left([0,+\infty);\mathscr{H}\right)$.

Lemma 2

The operator $\mathscr{A}$ is dissipative.

Auxiliary well-posedness result II

Proof.

For any $U \in D(\mathscr{A})$, using the inner product, the definition of the operator $\mathfrak{B}$, the symmetry property of the operator $\mathscr{S}^{-\frac{1}{2}}$, we get

$$\langle \mathscr{A}U, U\rangle_{\mathscr{H}} = -\kappa \int_0^L \theta_x^2 dx \leq 0,$$

□

Lemma 3

The operator $\mathscr{A}$ is maximal.

Auxiliary well-posedness result III

Proof.

The application of the variational approach, the Lax-Milgram theorem and the regularity theory for the linear elliptic equations guarantees that the operator $I - \mathscr{A}$ is surjective. □

Therefore, the result of Theorem 1 follows by Hille-Yosida theorem.

Main well-posedness result I

Now, we return to the original problem. We introduce the Hilbert space

$$\mathbb{H} = H_0^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times H_*^1(0,L) \times L_*^2(0,L) \times L_*^2(0,L)$$

and

$$\begin{aligned} \mathbb{D} &= \left[H^2(0,L) \cap H_0^1(0,L)\right] \times H_0^1(0,L) \times \left[H_*^2(0,L) \cap H_*^1(0,L)\right] \\ &\quad \times \left[H_*^2(0,L) \cap H_*^1(0,L)\right] \times H_*^1(0,L) \times \left[H_*^2(0,L) \cap H_*^1(0,L)\right]. \end{aligned}$$

Then, the main well-posedness result in this section is given by the following theorem

Main well-posedness result II

Theorem 4

For any initial data $(\varphi_0, \varphi_1, \omega_0, \omega_1, \theta_0)^T \in \mathscr{H}$, *the problem* (10)-(12) *has an unique weak solution*

$$(\varphi, \varphi_t, \psi, \omega, \omega_t, \theta) \in C\left([0, +\infty)\,; \mathbb{H}\right).$$

Moreover, if $(\varphi_0, \varphi_1, \omega_0, \omega_1, \theta_0)^T \in D\left(\mathscr{A}\right)$, *then*

$$(\varphi, \varphi_t, \psi, \omega, \omega_t, \theta) \in C\left([0, +\infty)\,; \mathbb{D}\right) \cap C^1\left([0, +\infty)\,; \mathbb{H}\right).$$

Main well-posedness result III

Proof.

Let $(\varphi, \varphi_t, \omega, \omega_t, \theta)$ be the solution of the problem 14. Considering the results of Theorem 1 and defining ψ as a solution to the problem

$$\begin{cases} \mathscr{S}\psi = -\kappa_1\left(\varphi_x + l\omega\right), \\ \psi_x\left(0\right) = \psi_x\left(L\right) = 0, \end{cases}$$

the desired result is achieved. □

Outline

Energy functional I

Lemma 5

The energy functional of system (10) − (12), *given by*

$$E(t) = \frac{1}{2}\int_0^L \Big[\rho_1\varphi_t^2 + \kappa_1(\varphi_x + \psi + l\omega)^2 + \kappa_2\psi_x^2 + \kappa_3(\omega_x - l\varphi)^2 + \rho_3\omega_t^2 + c\theta^2\Big]\,dx$$

satisfies, along the solution,

$$E'(t) = -\kappa\int_0^L \theta_x^2 dx. \qquad (15)$$

Energy functional II

Proof.

Multiplying the four equations of system (10) by $\varphi_t, \psi_t, \omega_t$ and θ, respectively, integrating over $(0, L)$, using integration by parts and the boundary conditions (11), then adding the results at the end, we obtain (15). □

Technical lemmas I

Lemma 6

Let $(\varphi, \psi, \omega, \theta)$ be the solution of problem $(10)-(12)$. Then, the functional

$$\mathscr{F}_1(t) = c\rho_1 \int_0^L \left(\int_0^x \theta(y)\, dy \right) \varphi_t dx$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$\begin{aligned} \mathscr{F}_1'(t) \quad \leq \quad & -\frac{\beta\rho_1}{2} \int_0^L \varphi_t^2 dx + \varepsilon_1 \int_0^L (\varphi_x + \psi + l\omega)^2 dx \\ & + l^2\varepsilon_1 \int_0^L (\omega_x - l\varphi)^2 dx + m\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^L \theta_x^2 dx, \end{aligned} \tag{16}$$

for some constant $m > 0$, independent of ε_1 and l.

Technical lemmas II

Lemma 7

Let $(\varphi, \psi, \omega, \theta)$ *be the solution of problem* $(10)-(12)$*. Then, the functional*

$$\mathscr{F}_2(t) = \rho_1 \int_0^L \varphi_t \left(\int_0^x (\varphi_x + l\omega)(y)\, dy \right) dx$$

satisfies the estimate

$$\begin{aligned}\mathscr{F}_2'(t) \leq & -\frac{\kappa_1}{2}\int_0^L (\varphi_x + \psi + l\omega)^2\, dx - \frac{\kappa_2}{2}\int_0^L \psi_x^2 dx + l^2 \int_0^L \omega_t^2 dx \\ & + l^2 m \int_0^L (\omega_x - l\varphi)^2\, dx + m\int_0^L \varphi_t^2 dx + m \int_0^L \theta_x^2 dx, \quad (17)\end{aligned}$$

for some constant $m > 0$*, independent of* l*.*

Technical lemmas III

Lemma 8

Let $(\varphi, \psi, \omega, \theta)$ *be the solution of problem* $(10) - (12)$. *Then, the functional*

$$\mathscr{F}_3(t) = -\rho_1\kappa_3 \int_0^L \varphi_t(\omega_x - l\varphi)\,dx - \kappa_1\rho_3 \int_0^L \omega_t(\varphi_x + \psi + l\omega)\,dx$$

satisfies, for any $\varepsilon_3 > 0$, *the estimate*

$$\begin{aligned} \mathscr{F}_3'(t) \leq\ & (\kappa_3\rho_1 - \kappa_1\rho_3) \int_0^L \varphi_{xt}\omega_t dx - l\left(\kappa_3^2 - \varepsilon_3\right) \int_0^L (\omega_x - l\varphi)^2\,dx \\ & + l\varepsilon_3 \int_0^L \omega_t^2 dx + lm\left(1 + \frac{1}{l^2\varepsilon_3}\right) \int_0^L \varphi_t^2 dx \\ & + l\kappa_1^2 \int_0^L (\varphi_x + \psi + l\omega)^2\,dx + \frac{1}{l\varepsilon_3} m \int_0^L \theta_x^2 dx, \end{aligned} \qquad (18)$$

for some constant $m > 0$, *independent of* ε_3 *and* l.

Technical lemmas IV

Lemma 9

Let $(\varphi, \psi, \omega, \theta)$ *be the solution of problem* $(10) - (12)$. *Then, the functional*

$$\begin{aligned}\mathscr{F}_4(t) &= -\rho_3 \int_0^L \omega_t \omega dx - \rho_1 \int_0^L \varphi_t \varphi dx \\ &\quad - \frac{c\beta}{\kappa} \int_0^L \varphi \left(\int_0^x \theta(y)\, dy \right) dx - \frac{\beta^2}{2\kappa} \int_0^L \varphi^2 dx\end{aligned}$$

satisfies the estimate

$$\begin{aligned}\mathscr{F}_4'(t) &\leq -\rho_3 \int_0^L \omega_t^2 dx - \frac{\rho_1}{2} \int_0^L \varphi_t^2 dx + \kappa_1 \int_0^L (\varphi_x + \psi + l\omega)^2 dx \\ &\quad + \kappa_3 \int_0^L (\omega_x - l\varphi)^2 dx + \kappa_2 \int_0^L \psi_x^2 dx + m \int_0^L \theta_x^2 dx, \quad (19)\end{aligned}$$

for some constant $m > 0$, *independent of* l.

Main theorem (Exponential decay) I

This part is devoted to establishing our exponential stability result. Before that, we present some necessary results.

Proposition 10 (Gearhart-Herbst-Prüss-Huang Theorem)

Let $T(t) = e^{At}$ be a C_0-semigroup of contractions on a Hilbert space. Then $T(t)$ is exponentially stable if and only if

$$i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(A)$$

and

$$\overline{\lim}_{|\lambda| \to \infty} \left\| (i\lambda I - A)^{-1} \right\| < \infty$$

hold.

Main theorem (Exponential decay) II

The main result of this part is

Theorem 11

Let $(\varphi, \psi, \omega, \theta)$ be the solution of problem $(10) - (12)$. Then, the energy functional E is exponentially stable if and only if $\kappa_3\rho_1 - \kappa_1\rho_3 = 0$.

Proof

① Assume that $\kappa_3\rho_1 - \kappa_1\rho_3 = 0$ and define the following Lyapunov functional

$$\mathscr{L}(t) = NE(t) + n_1\mathscr{F}_1(t) + n_2\mathscr{F}_2(t) + 2l^{-1}\mathscr{F}_3(t) + \kappa_3\mathscr{F}_4(t)$$

where N and n_i are positive constants.

❊ We choose carefully the constants ε_i, n_i and N so that the functional $\mathscr{L}$ satisfies

- for two positive constants λ_1 and λ_2

$$\lambda_1 E(t) \leq \mathscr{L}(t) \leq \lambda_2 E(t) \;\; \forall t \geq 0. \tag{20}$$

- for some $\lambda_3 > 0$,

$$\mathscr{L}'(t) \leq -\lambda_3 E(t). \tag{21}$$

❊ we use (20) and (21) to easily find the first direction of proof.

Proof

① Assume that $\kappa_3\rho_1 - \kappa_1\rho_3 = 0$ and define the following Lyapunov functional

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Proof

① Assume that $\kappa_3\rho_1 - \kappa_1\rho_3 = 0$ and define the following Lyapunov functional

$$\mathscr{L}(t) = NE(t) + n_1\mathscr{F}_1(t) + n_2\mathscr{F}_2(t) + 2l^{-1}\mathscr{F}_3(t) + \kappa_3\mathscr{F}_4(t)$$

where N and n_i are positive constants.

❊ We choose carefully the constants ε_i, n_i and N so that the functional $\mathscr{L}$ satisfies

- for two positive constants λ_1 and λ_2

$$\lambda_1 E(t) \leq \mathscr{L}(t) \leq \lambda_2 E(t) \ \ \forall t \geq 0. \tag{20}$$

- for some $\lambda_3 > 0$,

$$\mathscr{L}'(t) \leq -\lambda_3 E(t). \tag{21}$$

❊ we use (20) and (21) to easily find the first direction of proof.

Proof

② By contrast, if $\kappa_3\rho_1 - \kappa_1\rho_3 \neq 0$, by using Proposition 10, we can then show the lack of exponential stability thanks to the following procedure: we show that there exists a sequence of real values λ_n such that

$$\left\|\left(i\lambda_n I - \mathbb{A}\right)^{-1}\right\|_{\mathscr{L}(\mathbb{H})} \to +\infty.$$

Indeed, there exists a sequence of vectors $F_n = (0,0,0,0,f_n,0)^T \in \mathbb{H}$, where $f_n = \rho_3^{-1}\cos\left(\frac{n\pi x}{L}\right)$, and a sequence of numbers $\lambda_n \in \mathbb{R}$, where $\lambda_n^2\rho_3 = \kappa_3\left(\frac{n\pi}{L}\right)^2 + l^2\kappa_1$, with $\|F_n\|_{\mathbb{H}} < \infty$ such that

$$\left\|\left(i\lambda_n I - \mathbb{A}\right)^{-1} F_n\right\|_{\mathbb{H}} = \|U_n\|_{\mathbb{H}} \to +\infty,$$

where $U_n \in D(\mathbb{A})$ be the solution of

$$\left(i\lambda_n I - \mathbb{A}\right) U_n = F_n.$$

Proof

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where $U_n \in D(\mathbb{A})$ be the solution of

$$(i\lambda_n I - \mathbb{A})\, U_n = F_n.$$

Outline

Main theorem (Polynomial decay) I

We assume $\kappa_3\rho_1 - \kappa_1\rho_3 \neq 0$ and define the second-order energy functional by

$$\begin{aligned}\mathscr{E}(t) &= \frac{1}{2}\int_0^L \Big[\rho_1\varphi_{tt}^2 + \kappa_1(\varphi_x + \psi + l\omega)_t^2 + \kappa_2\psi_{xt}^2 \\ &\quad +\rho_3\omega_{tt}^2 + \kappa_3(\omega_x - l\varphi)_t^2 + c\theta_t^2\Big]\,dx.\end{aligned}$$

As in the proof of Lemma 5, it follows that $\mathscr{E}$ satisfies

$$\mathscr{E}'(t) = -\kappa\int_0^L \theta_{xt}^2 dx. \tag{22}$$

Main theorem (Polynomial decay) II

Here, with the same notations and Lemmas of the previous section, we define the Lyapunov functional $\tilde{\mathscr{L}}(t)$ as follows

$$\begin{aligned}\tilde{\mathscr{L}}(t) &= N(E(t)+\mathscr{E}(t)) + n_1\mathscr{F}_1(t) + n_2\mathscr{F}_2(t) \\ &\quad +2l^{-1}\tilde{\mathscr{F}}_3(t) + \beta\kappa_3\mathscr{F}_4(t),\end{aligned}$$

where

$$\tilde{\mathscr{F}}_3(t) = \beta\mathscr{F}_3(t) + \kappa(\kappa_3\rho_1 - \kappa_1\rho_3)\int_0^L \theta_x(\omega_x - l\varphi)\,dx.$$

Main theorem (Polynomial decay) III

The functional $\tilde{\mathscr{F}}_3$ satisfies, for any $\eta > 0$, the estimate

$$\begin{aligned}\tilde{\mathscr{F}}_3'(t) \leq & -l\beta\kappa_3^2\int_0^L(\omega_x - l\varphi)^2\,dx + l\eta\int_0^L(\omega_x - l\varphi)^2\,dx \\ & + l\eta\int_0^L\omega_t^2dx + l\beta\kappa_1^2\int_0^L(\varphi_x+\psi+l\omega)^2\,dx \\ & + lm\left(1+\frac{1}{l^2\eta}\right)\int_0^L\varphi_t^2dx + lm\left(1+\frac{1}{l^2\eta}\right)\int_0^L\theta_x^2dx \\ & + \frac{m}{l\eta}\int_0^L\theta_{xt}^2dx, \qquad (23)\end{aligned}$$

for some constant $m > 0$, independent of η and l.
The main result of this part is given by

Main theorem (Polynomial decay) IV

Theorem 12

Let $(\varphi, \psi, \omega, \theta)$ be the strong solution of problem $(10) - (12)$ and assume that $\kappa_3\rho_1 - \kappa_1\rho_3 \neq 0$. Then, the energy functional E satisfies

$$E(t) \leq \frac{\sigma}{t}, \quad \forall t > 0, \tag{24}$$

where σ is a positive constant.

Proof

- We choose carefully the constants N, n_i, ε_1 and η so that the functional $\tilde{\mathscr{L}}(t)$ satisfies
  - $$\tilde{\mathscr{L}}(t) \geq 0, \ \ \forall t \geq 0.$$
  - for some $\sigma_0 > 0$,
    $$\tilde{\mathscr{L}}'(t) \leq -\sigma_0 E(t), \ \ \forall t > 0.$$
- Recalling that E is decreasing, we have
  $$tE(t) \leq \frac{\tilde{\mathscr{L}}(0) - \tilde{\mathscr{L}}(t)}{\sigma_0} \leq \frac{\tilde{\mathscr{L}}(0)}{\sigma_0}, \ \ \forall t > 0,$$
  which implies the estimate of polynomial stability (24).

Proof

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  - $$\tilde{\mathscr{L}}(t) \geq 0, \ \ \forall t \geq 0.$$
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- We choose carefully the constants N, n_i, ε_1 and η so that the functional $\tilde{\mathscr{L}}(t)$ satisfies
  -
  $$\tilde{\mathscr{L}}(t) \geq 0, \quad \forall t \geq 0.$$
  - for some $\sigma_0 > 0$,
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Proof

- We choose carefully the constants N, n_i, ε_1 and η so that the functional $\tilde{\mathscr{L}}(t)$ satisfies
  - $$\tilde{\mathscr{L}}(t) \geq 0, \quad \forall t \geq 0.$$
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$$tE(t) \leq \frac{\tilde{\mathscr{L}}(0) - \tilde{\mathscr{L}}(t)}{\sigma_0} \leq \frac{\tilde{\mathscr{L}}(0)}{\sigma_0}, \quad \forall t > 0,$$

which implies the estimate of polynomial stability (24).

Outline

Numerical scheme I

- Taking $L = 1$, we define the uniform partition of $(0, 1)$ by $0 = x_0 < x_1 < \cdots < x_{N_h} = 1$, denote the length of the interval (x_j, x_{j+1}) by $\Delta x = \frac{1}{N_h}$ and define

$$P_h^1 = \left\{u \in H_0^1(0, L), u|_{I_i} \text{ is a linear polynomial}\right\}.$$

- For the time discretization, we denote by $\Delta t = \frac{T}{N_t}$ the step time, where T is the total time and N_t is a positive integer.

Numerical scheme II

The finite-element approximate system for (10) is to find φ_h^n, ψ_h^n, ω_h^n, θ_h^n, such that, for all ξ_h, η_h, ζ_h, $\chi_h \in P_h^1$,

$$\left\{\begin{array}{l}
\frac{\rho_1}{\Delta t}(\varphi_{ht}^n - \varphi_{ht}^{n-1}, \xi_h) + \kappa_1(\varphi_{hx}^n + \psi_h^n + l\omega_h^n, \xi_{hx}) \\
-l\kappa_3(\omega_{hx}^n - l\varphi_h^n, \xi_h) + \beta(\theta_{hx}^n, \xi_h) = 0, \\
\kappa_2(\psi_{hx}^n, \eta_{hx}) + \kappa_1(\varphi_{hx}^n + \psi_h^n + l\omega_h^n, \eta_h) = 0, \\
\frac{\rho_3}{\Delta t}(\omega_{ht}^n - \omega_{ht}^{n-1}, \zeta_h) + \kappa_3(\omega_{hx}^n - l\varphi_h^n, \zeta_{hx}) + l\kappa_1(\varphi_{hx}^n + \psi_h^n + l\omega_h^n, \zeta_h) = 0, \\
\frac{c}{\Delta t}(\theta_h^n - \theta_h^{n-1}, \chi_h) + \kappa(\theta_x^n, \chi_{hx}) + \frac{\beta}{\Delta t}(\varphi_{hx}^n - \varphi_{hx}^{n-1}, \chi_h) = 0.
\end{array}\right.$$

with $\varphi_{ht}^n = (\varphi_h^n - \varphi_h^{n-1})/\Delta t$ and $\omega_{ht}^n = (\omega_h^n - \omega_h^{n-1})/\Delta t$.

Numerical scheme III

Then, the discrete energy is given by

$$\begin{aligned} E_h^n &= \frac{1}{2}\left(\rho_1\|\varphi_{ht}^n\|_2^2 + \kappa_1\|\varphi_{hx}^n + \psi_h^n + l\omega_h^n\|_2^2 + \kappa_2\|\psi_{hx}^n\|_2^2\right) \\ &\quad + \frac{1}{2}\left(\kappa_3\|\omega_{hx}^n - l\varphi_h^n\|_2^2 + \rho_3\|\omega_{ht}^n\|_2^2 + c\|\theta_h^n\|_2^2\right). \end{aligned}$$

Numerical experiments I

By using the following initial conditions:

$$\varphi_0(x) = 2\sin(\pi x), \quad \psi_0(x) = \cos(\pi x), \quad \omega_0(x) = \cos(\pi x),$$
$$\theta_0(x) = \cos(\pi x), \quad \varphi_1(x) = \sin(\tfrac{\pi}{2}x), \quad \omega_1(x) = \sin(\tfrac{\pi}{2}x),$$

Two numerical tests are done for different entries as follow:

- Test 1:

$$\rho_1 = \rho_3 = \kappa = \kappa_1 = \kappa_2 = c = \beta = 1, \ \kappa_3 = 2, \text{ and } l = 1$$

Numerical experiments II

- Test 2:

$$\begin{aligned} \rho_1 &= \rho_3 = 1,\ \kappa = 0.5,\ \kappa_1 = 1,\ \kappa_2 = 1.5, \\ c &= 2.3,\ \beta = 0.8,\ \kappa_3 = 2,\ \text{and } l = 0.5 \end{aligned}$$

Numerical experiments III

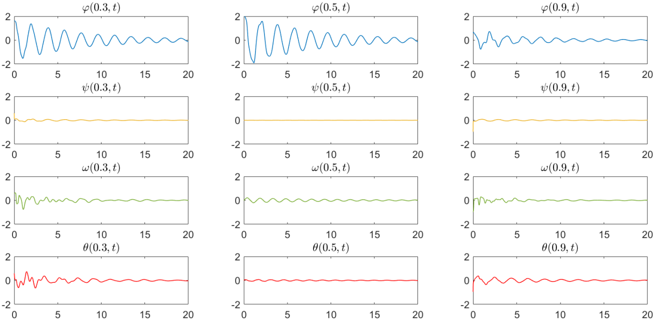

Figure 1: Damping cross section waves of Test 1

Numerical experiments IV

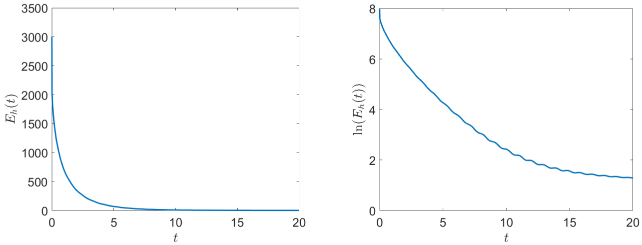

Figure 2: Energy decay in natural and log scales for Test 1

Numerical experiments V

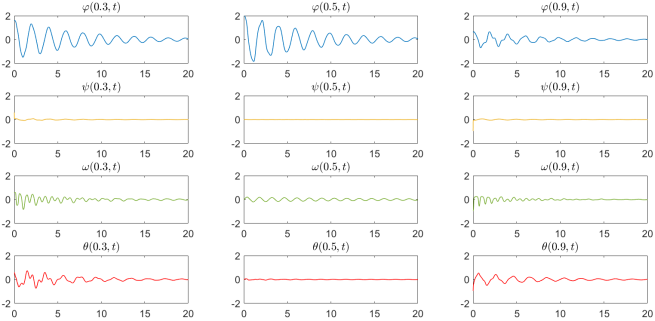

Figure 3: Damping cross section waves of Test 2

Numerical experiments VI

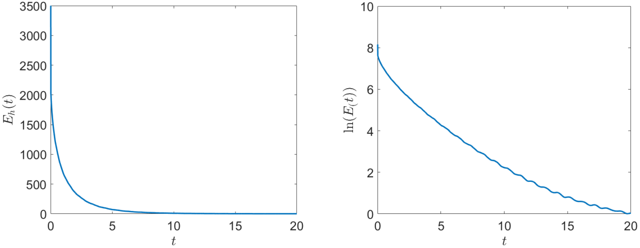

Figure 4: Energy decay in natural and log scales for Test 2

Numerical experiments VII

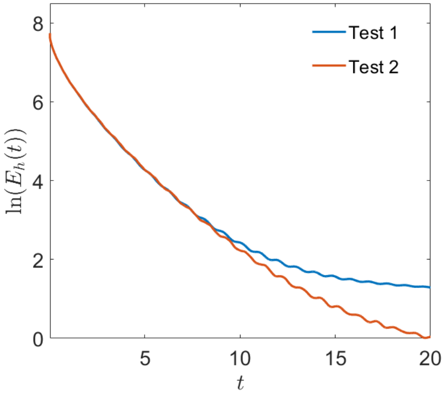

Figure 5: The evolution in time of $\ln(E_h)$

Numerical experiments VIII

As a conclusion, for both tests, we observed that the numerical solution converges to zero and the energy decay with different rates seems to be reached which is compatible with the theoretical results.

Outline

The shear Bresse system with Fourier's law I

- Two additional related works have been documented:
  - Keddi, Messaoudi and Alahyane [6] investigated a thermoelastic shear Bresse system, where the heat conduction is modeled by the Fourier's law acting on the shear force:

$$\begin{cases} \rho_1\varphi_{tt} - \kappa_1\left(\varphi_x + \psi + l\omega\right)_x - l\kappa_3\left(\omega_x - l\varphi\right) = 0, \\ -\kappa_2\psi_{xx} + \kappa_1\left(\varphi_x + \psi + l\omega\right) + \beta\theta_x = 0, \\ \rho_3\omega_{tt} - \kappa_3\left(\omega_x - l\varphi\right)_x + l\kappa_1\left(\varphi_x + \psi + l\omega\right) = 0, \\ c\theta_t - \kappa\theta_{xx} + \beta\psi_{xt} = 0. \end{cases} \tag{25}$$

    - Showed that the system is not exponentially stable.
    - Proved a polynomial decay of the solution in the case of the equal speeds of wave propagation of the two hyperbolic equations.

The shear Bresse system with Fourier's law II

- Keddi, Chabekh and Messaoudi (submitted) studied the effect of the thermal damping, where the coupling is via the axial displacement equation:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - lk_0\left(\omega_x - l\varphi\right) = 0, \\ -b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) = 0, \\ \rho_3\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\omega\right) + \beta\theta_x = 0, \\ c\theta_t - \kappa\theta_{xx} + \beta\omega_{xt} = 0. \end{cases} \tag{26}$$

  - Showed that the system is exponentially stable if and only if the speeds of wave propogation of the hyperbolic equations are equal.
  - Established a polynomial decay in the case of nonequal speeds.

The shear Bresse system with Fourier's law III

Conclusion.

The stability of the one-dimensional linear shear thermoelastic Bresse system, which consists of two hyperbolic equations and one elliptic equation coupled with a heat equation of Fourier type, is intrinsically linked to the coupling equation and the equality of the hyperbolic wave speeds.

The shear Bresse system with Fourier's law IV

Summary

	Equal Speed	Non-equal Speed
Dissipation via transverse displacement (First Equation)	Exponential decay	Polynomial decay
Dissipation via shear angle (Second Equation)	No exponential decay Only polynomial decay	Open question
Dissipation via longitudinal displacement (Third Equation)	Exponential decay	Polynomial decay

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Thank You