On symmetric tensors of second order and fourth order in mechanics

with focus on Cauchy stress and Hooke tensors

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Educational Materials

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Foreword

The present work contains materials developed in the form of a monograph and presented during a series of lectures I gave at the Institute of Fundamental Technological Problems of the Polish Academy of Sciences, in the spring semester of 2023.

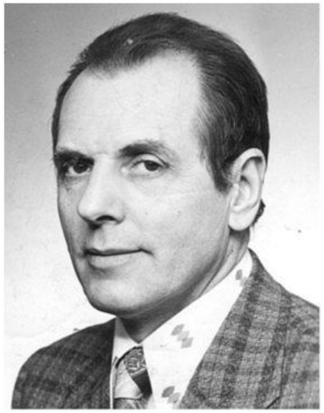
Thales of Miletus is considered the most outstanding of the seven Sages who, in the late seventh and by the middle of the sixth century B.C., in Greece, instigated the transition from the previously dominant emotional and fantastic, mythological and religious thinking about the real world to its rational and logical explanation through reflection and abstraction – by appealing to reason, and then generalization of real situations – by building models of reality. However, it was Aristotle in the 3rd century BC who laid the foundations for modern scientific methodology in his work Metaphysics. Aristotle considered as a prime principle the inquiry into the “essence of things” and not their accidentals through the application of a certain procedure (methodology), proposing the proceedings that schematically can be described by a cyclic sequence: existence → senses → reflection → abstraction → existence. At the same time, Aristotle concluded that without formulating the questions posed to nature in the form of planned observations of reality, to resolve the problems that arise when investigating the nature of the Universe it is very easy to fall into false notions. For it is not enough (passive) observation of phenomena, reflection and abstraction with disregard or extreme distrust of the universe as perceived through and as a result of the action of human senses. He believed that it was necessary to properly plan the observations of phenomena (thought and/or real experiments) in such a way that their results would make it possible to unambiguously resolve the issues being studied, and thus avoid false conclusions. A certain “golden mean” is needed, moderation in every aspect of the process of scientific investigation.

Foreword

Both ancient Greek thinkers and all subsequent generations of natural scientists and philosophers up to the present day have observed, experienced and continue to notice that the observed by them regularities – the laws governing phenomena in the Universe – remain constant in time and space. The problem has always been to find a tool enabling description, documentation and exploitation of the observed laws of nature in the manner as simple, clear, concise and convenient as possible. Flexible and versatile colloquial language allowed and allows conveying the "essence of things", but does not allow achieving the precision required if only in engineering applications. It was necessary to develop logic, mathematics enabling the invention and development of a tool convenient for the description, documentation and practical use of the laws of the Universe, which took mankind almost two and a half thousand years. This tool turned out to be tensor calculus, which was developed only at the turn of the 20th century, and which makes it possible to write down physical laws in an lucid (abstract) invariant form, independent of the coordinate system (absolute tensor notation) and in a practical form that depends on the adopted coordinate system convenient, e.g., in engineering calculations (index notation).

The present monograph discusses the achievements of tensor calculus as applied to mechanics. Elements of its historical development are provided, its current state of advancement is given, as well as information rarely found in standard academic textbooks. Finally, several open research problems are formulated aimed at further development of tensor calculus as applied to mechanics.

I dedicate this work to Professor Jan Rychlewski, philosopher of science, Master of Polish School of Mechanics, my teacher and tutor in tensorial calculus.



Jan Rychlewski
(1934-2011)

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Elements of philosophy and practice of science.

1.1 Elements of philosophy and practice of science.
1.2 Aims and targets of the present study

Elements of philosophy and practice of Science.

Our present examination of elements of tensorial calculus belongs to the fundamental problems of science in this sense that modeling tool is developed useful and convenient, and precise for formulation of scientific problems, description of physical phenomena, documentation and proliferation of scientific results.

Plenty of misunderstandings exists regarding fundamental tasks and targets of scientific activities and the place and role of science in human life. Due to that before we move on to discuss the main theme, i.e., tensors, let us outline and prepare the place of the drama and its scenery, i.e., discuss what are activities, problems, conditions and limitations involved with Science.

Science consists in activities leading to cognition and understanding of universe. What is the motivation for running scientific activities, what is the target of science and what are its results?

Motivation	CURIOSITY of the universe
Target	COGNITION and UNDERSTANDING of the universe
Results	KNOWLEDGE about the universe

So, Science means acquiring knowledge about the *Universe*.
(from Latin *scientia*, meaning cognition, knowledge of things).

Elements of philosophy and practice of Science.

Science *is an effort to discover and to know* - and thus broaden the human understanding on *how the physical world works*. Science no less important task is *documentation* and *public dissemination* of knowledge about universe (and not keeping it secret or confidential).

Science, *scientific activities* are based on a number of *assumptions*, some of them of fundamental importance are as follows,

– the universe is *at all cognizable* with senses available to humans,

– there exist *general principles governing the universe* that are *invariable in space and time*,

– the universe always *provides true data*, "doesn't cheat", when giving answers to asked her questions, e.g., in the form of *experimental tests results*.

The main method of *scientific activities* is based (out of necessity) on *Abduction* (abductive reasoning), *Not Deduction* (deductive reasoning).

Critical for abductive reasoning is operation in conditions of *irremovable uncertainty*, *limited cognitiveness* (limited availability of information/ knowledge).

Elements of philosophy and practice of Science.

When examining any physical phenomenon there are *always present* consciously known and/or unknown factors that,

a) do not affect the course of the phenomenon, and/or
b) are constant and due to that have a constant influence,
(indistinguishable from the influence of other controlled factors).

There are many reasons for such situation, e.g., limitation of our senses.

Methodology used in applied sciences, e.g., *engineering*, in majority of cases is based on *deduction* and *not abduction* (already existing knowledge is used).

Note On the modes of reasoning
Deduction: allows you to *derive b* as a *consequence* of *a* : in other words, deduction is the process of deriving conclusions from what is already known.
Induction: allows you to derive *a* as a *precondition* of *b* : in other words, induction is the process of deriving causes from known effects in the *casual connection* "*a* implies *b*".
Abduction: allows you to pose a hypothesis (general rule) *a* as an explanation for *b (specific case/-es)*: in other word, abduction is the process of *formulating the best guess hypothesis* in the conditions of incomplete, limited information/knowledge.

Elements of philosophy and practice of Science.

The result of the abductive methodology is the *best result* (*assessment*) based on the knowledge and experience to which we have access.
All the time we assume so called good will in conduct of scientist.

A very good example of *abductive methodology result* taken from our daily life is *medical diagnosis*. The medical examination uses abductive methodology but has different target and rationale than scientific activity.

Employment (*out of necessity*) of abductive methodology in scientific research has a *very strong consequence*. Namely, *all scientific knowledge* about the universe, the main result of scientific activities can be reasonably considered to *represents models* of reality and *not reality itself*. Among it all scientific theories. A very astute opinion on scientific research expressed professor Jan Rychlewski,

"… The goal of Natural Sciences (Science)*, … , is not to search for absolute truths, but to construct models describing reality in a way that is completely internally consistent, as clear and elegant as possible and sufficiently accurate. ..."*

after Jan Rychlewski, Dimensions and Similarity, PWN,1991; Appendix C: Newton's Mechanics.

Elements of philosophy and practice of Science.

So, how actually scientists run research?
Scientific research process consists in collecting,

Data -> Information -> Knowledge.

Usually, it is a long-term and cyclic process.

Using controlled methods (*research*), *scientists* use observable *physical evidence of natural phenomena* to collect *data*, next analyze and order it to create *information*, and next structure this last to build *predictive models* to gain *knowledge* that logically explains the processes ongoing in universe.

Practice of scientific methods is based on *properly designed real and/or thought experiments*, results of which are to enable revealing the actual causes and course of physical phenomena and their effects.

It is worth to gain a broader perspective and to give some thought on what is the role of science in humanity.

In *ancient Greece* where science was born there were *a lot of philosophers*, in *modern times* there are *many scientists* but philosophers are very hard to find.

Elements of philosophy and practice of Science.

Ancient great Greek thinkers practiced Natural Philosophy under which they conducted activities aimed at penetrating the *secrets of nature*, not only to get to know them *but to become wise people*.

Science (source of knowledge) was treated as an intermediate (partial) step in this process. (lat. Philosophia < gr. φιλοσοφία < gr. φιλέω (love, cherish, adore) and gr. σοφία (wisdom); in verbatim translation "love of wisdom".). It is worth to explicitly show the difference between aims and targets of scientists and philosophers,

Scientist: Data -> Information -> Knowledge.
Philosopher: Data -> Information -> Knowledge -> Wisdom.

It is also worth to indicate difference between knowledge and wisdom,

Knowledge: is the entirety of reliable information and understanding about real world along with the ability to use these assets.

Wisdom: is the ability and willingness to use knowledge to do G*ood*.
discernment about right and wrong (Socrates);
the ability to use reason accurately and do what is best (R. Descartes);

The source for recognition what is "Good" is *Religion* as the source of *Ethics.*

Contrasting and/or interchanging *Science* and *Religion* is a misunderstanding. These are two different categories complementary to each other and not opposing/excluding each other.

Elements of philosophy and practice of Science.

Knowledge (together with education) is often equated with *wisdom*. This is a *misjudgment*. A person can be *well educated*, possess *deep knowledge* and still *can be not wise*.

Contemporary scientists extremely often terminate their activity on acquiring/gaining knowledge. It seems that it would be sensible to return to the original Greek thought and treat *scientific activities* as measures to reach the goal of *acquiring wisdom* (personal and collective) and *not acquiring knowledge only.*

Polish philosopher Feliks Koneczny proposed as he called it *Quincunx* (pentnomial of being) as the tool (*measure*) enabling reliable comparison of *different civilizations* (systems of collective life).

The *Quincunx* contains five factors, *Good* (*Ethics* finding its source in religion), *Truth* (*Knowledge* finding its source in Science), *Health*, *Well-being* and *Beauty*. Good an Truth are considered *spiritual elements*, Health and Well-being are *considered physical* elements and Beauty is considered connecting link between spiritual and physical factors of collective (and individual) life.

In Koneczny's reflections one can find a clue that science, scientific activities, is not exercised for itself or for gaining knowledge (truth) about universe only but that it is a necessary measure and intermediate step towards acquiring wisdom (personal and collective), i.e., using knowledge for attaining good.

Koneczny F., O wielości cywilizacji (in Polish), Kraków, 1935.
There exist English translation of this work by Toynbee A., On plurality of civilizations, London 1962.

Aims and targets of the present study

In the present study we will be interested in development of elements of tensorial calculus, as it will be shown a natural language of science.

We shall try to address such questions as,

- what factors caused such widespread use and popularity of *tensorial calculus* (*tensors*)?
- when and how the notion of a tensor came to existence?
- what tensors actually are, and/or how they can be understood/ interpreted?
- what are the specific features, eigenproperties, of tensors and what is the best manner to deal with them?

An effort to address listed above and some other issues will be done taking *Cauchy stress tensor* (*second order symmetric tensor*) as a generic example.

While the problems are discussed here with the use of *Cauchy stress tensor*, all the obtained here results *mutatis mutandis* transfer to *all second order symmetric tensors*, having possibly miscellaneous interpretations in wide variety of pure science, engineering and/or other fields of application.

Lecture 2

Perception and description of physical phenomena.

2.1 Tensorial calculus, native language of natural sciences. Algebraic-Geometric dualism of tensors
2.2 Cauchy's concept of pressures non-perpendicular to the surfaces.
2.3 Cauchy's stress as macroscopic constitutive model describing force interactions in materials.
2.4 Perception of physical phenomena. Limitations or capabilities of human senses?
2.5 What tensors are, how they can be understood, some historical information.

Tensorial calculus, native language of natural sciences.

When contemplating on the place and role of *tensorial calculus* and *tensors* in human life. One can readily come to the conclusion that it makes a *native language* (lingua franca) for *describing* and *documenting* in convenient and quantitatively exact manner knowledge about all *natural and engineering sciences,* e.g., physics or mechanics.

Care must be taken *not to entangle* the *phenomenon description* with the *phenomenon itself*. This is a methodological error.

Juliusz Słowacki „Beniowski", Pieśń 5

Chodzi mi o to, aby język giętki	*My point is that the gift of the gab*
Powiedział wszystko,	*Said everything,*
co pomyśli głowa:	*what the head will think:*
A czasem był jak piorun jasny, prędki,	And sometimes be like thunder bright, swift
A czasem smutny jako pieśń stepowa,	And sometimes sad as a steppe song
A czasem jako skarga nimfy miętki,	And sometimes as a nymph's complaint tender
A czasem piękny jak aniołów mowa...	And sometimes beautiful as Angels' speech..
Aby przeleciał wszystka ducha skrzydłem	To flash over all the soul on wing.
Strofa być winna taktem, nie wędzidłem.	Stanza should be tact, not the snaffle.

Algebraic-Geometric Dualism of tensors.

Pragmatically, science deals with various phenomena *space-time relations* in their *cause and effect* aspect. Such relations tell *what*, *where* and *when* is happening.

For centuries, also today, "*where*" has been expressed in the language of *geometry*. However, it turned out that there are great advantages when the quantitative methods and language of *algebra* are used for such a description.

Tensor calculus (tensors) is, in a way, the crowning achievement of the process of *algebraization of geometry*.

However, it turns out that the reverse process of *geometrization of tensors*, i.e., perceiving tensors as geometric objects proves to be very useful and convenient in building *models of universe*.

Geometric-algebraic dualism of tensors proves to be very convenient and useful in first formulating and documenting general laws of universe and next their use, for example, in quantitative numerical computations.

Cauchy's concept of pressures non-perpendicular to the surfaces.

Augustin Cauchy's landmark lecture before the Paris Académie in 1822 and publication of the abstracts from the lecture in 1823 deliver sound reason to judge that Cauchy's tetrahedron argument paved the way to the concept of tensor (Cauchy stress) and its subsequent overwhelming introduction and presence in science and engineering.

Cauchy invented, came to the idea of a *tensor* upon realizing that classical Euler's *notion of pressure* can and should be extended to embrace "pressures" *not perpendicular to the planes on which they act*, and demonstrated this with his tetrahedron argument.

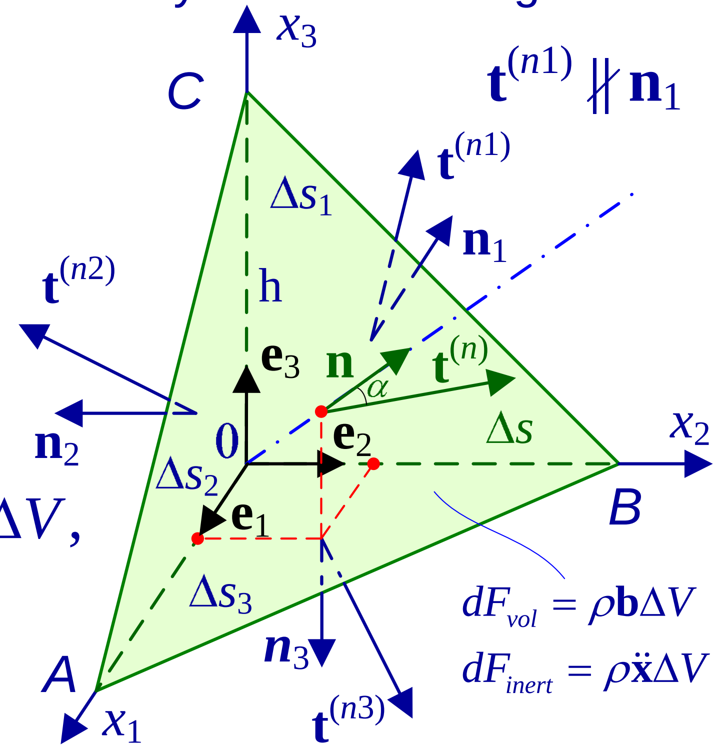
$$\int_{S=\partial V} \boldsymbol{t}\,ds + \int_V \rho \boldsymbol{b}\,dV = \int_V \rho \ddot{\mathbf{x}}\,dV, \;\; \Delta V = \tfrac{1}{3} h \Delta s,$$

$$\boldsymbol{t}^{(n)}\Delta s + \boldsymbol{t}^{(n1)}\Delta s_1 + \boldsymbol{t}^{(n2)}\Delta s_2 + \boldsymbol{t}^{(n3)}\Delta s_3 = \rho(\ddot{\mathbf{x}} - \boldsymbol{b})\Delta V,$$

$$\Delta s_i = n_i\,\Delta s, \quad \boldsymbol{n} = n_i \boldsymbol{e}_i, \;\; \boldsymbol{n}_i = -\boldsymbol{e}_i \rightarrow \boldsymbol{t}^{(n_i)} = -\boldsymbol{t}^{(e_i)};$$

$\boldsymbol{t}(\boldsymbol{n}_i) = -\boldsymbol{t}(-\boldsymbol{n}_i)$ – Cauchy's lemma.

Cauchy tetrahedron argument



Cauchy A., Recherches sur l'equilibre et le mouvement interieur des corps solides ou uides, elastiques ou non-elastiques (in French), Bull Soc Filomat, Paris 913, 1823.
See, also e.g., Azadi E, Cauchy tetrahedron argument and the proofs of the existence of stress tensor, a comprehensive review, challenges, and improvements, 2017 pp. 1-34, arXiv:1706.08518v3

Cauchy's concept of pressures non-perpendicular to the surfaces.

Upon decreasing the volume of tetrahedron ($\Delta V \to 0$) the body and inertia forces become negligible in comparison to surface forces. Then, taking advantage of geometrical relations and *Cauchy lemma* simple transformations lead to the following balance of traction forces,

$$\boldsymbol{t}^{(n)} = \boldsymbol{t}^{(e1)} + \boldsymbol{t}^{(e2)} + \boldsymbol{t}^{(e3)}; \quad \boldsymbol{t}^{(e1)} = \sigma_{11}\boldsymbol{e}_1 + \sigma_{12}\boldsymbol{e}_2 + \sigma_{13}\boldsymbol{e}_3,$$

$$\boldsymbol{t}^{(e2)} = \sigma_{21}\boldsymbol{e}_1 + \sigma_{22}\boldsymbol{e}_2 + \sigma_{23}\boldsymbol{e}_3, \; \boldsymbol{t}^{(e3)} = \sigma_{31}\boldsymbol{e}_1 + \sigma_{32}\boldsymbol{e}_2 + \sigma_{33}\boldsymbol{e}_3,$$

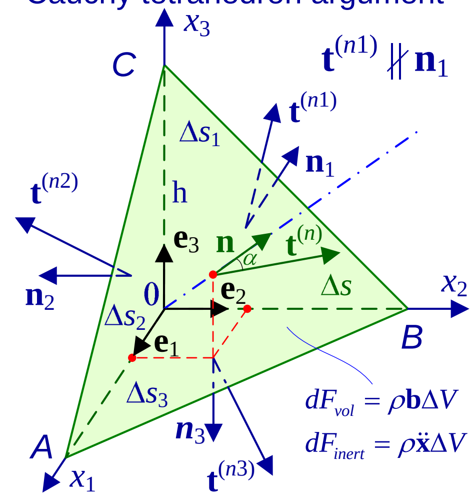
and finally upon rearrangement of terms the notion of a stress tensor can be reached

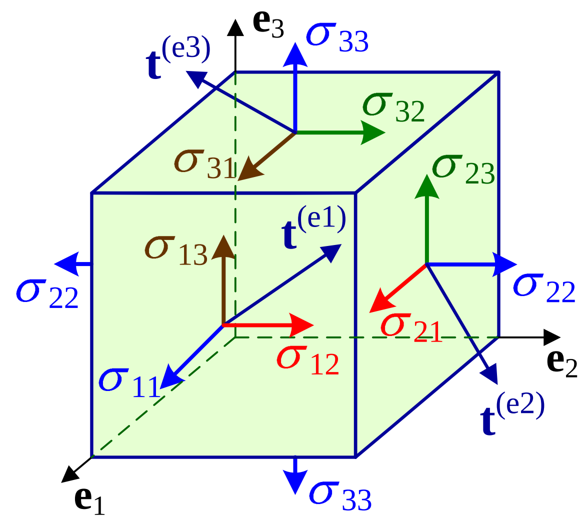
$$\boldsymbol{t}^{(n)} = \begin{bmatrix} t_1^{(n)} \\ t_2^{(n)} \\ t_3^{(n)} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}^T \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}, \quad \boldsymbol{t}^{(n)} = \boldsymbol{\sigma}^T \boldsymbol{n}, \quad \boldsymbol{n} = n_i \boldsymbol{e}_i$$

where $\boldsymbol{n}$ can be any conceivable surface direction.

Balance of moments leads to the condition $\sigma_{ij} = \sigma_{ji}$.

Cauchy tetrahedron argument





Stress components shown in positive directions

See also, Chapter 4 in Truesdell C., Essays in the History of Mechanics. Springer-Verlag New York Inc, 1968

Cauchy's stress as macroscopic constitutive model describing force interactions in materials.

It is worth reflection what actually Cauchy has done.

From *philosophical point of view* the Cauchy's contribution can be treated as a step towards development of *constitutive theory of forces*.

The original motivation behind Cauchy's work was development of *theory of elasticity*, what he achieved by a very original move of generalizing the Euler's notion of *pressure* acting always perpendicularly on the surface element to introduce the idea of *force traction,* which can be *oblique* to the *surface* on which it acts.

The Cauchy's tetrahedron argument can be understood as a certain *continuum model of force* describing in an averaged manner the microscopic interactions between molecules (through surfaces) – a transition from *molecular interactions* towards *continuous medium interactions*. The model actually laid the foundations of *continuum mechanics*. Cauchy stress can be rationally interpreted as a macroscopic measure of internal forces interaction between particles in microscale of observation.

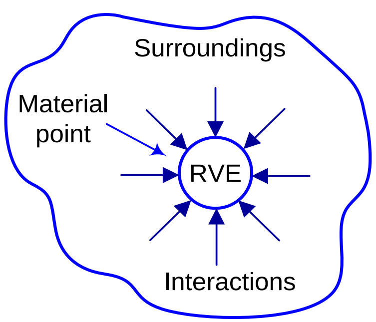
In his original presentation and publications Cauchy *was not talking about stress tensor* but about *pressures*. One of the reasons for this surely must have been that the tensor notion did not exist yet at that time.

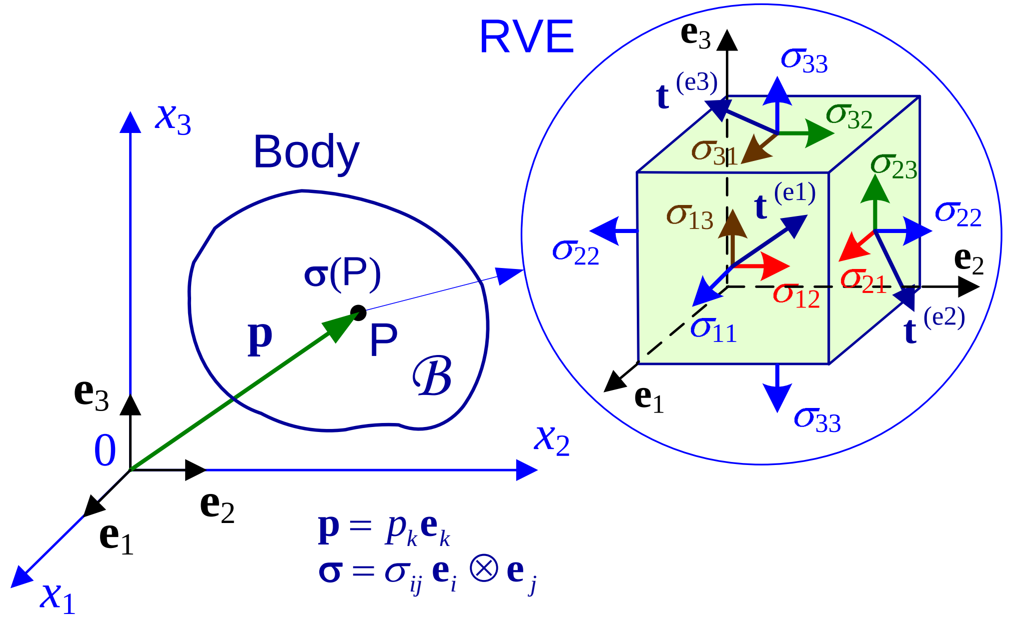
Cauchy's stress as macroscopic constitutive model of force interactions. The concept of representative volume element

Pragmatic justification for development and introduction of quite complex and abstract apparatus of *tensorial calculus* is creation of a tool for "*mathematization*" of real physical space and real physical phenomena.

For example, for the purpose that a sentence: "Stress state $\boldsymbol{\sigma}$ exists at material location P." had *precise quantitative mathematical and physical sense*. The other motivation is that *physical laws* expressed *in absolute tensorial notation have the same (invariant) form* regardless of the coordinate system. This is an immense advantage and achievement of tensorial calculus.

One of the most *ingenious idea of science* is ability to skillfully distinguish *isolated systems* and their division into *material point* - Representative Volume Element – (RVE), and its *surroundings*, which mutually interact.





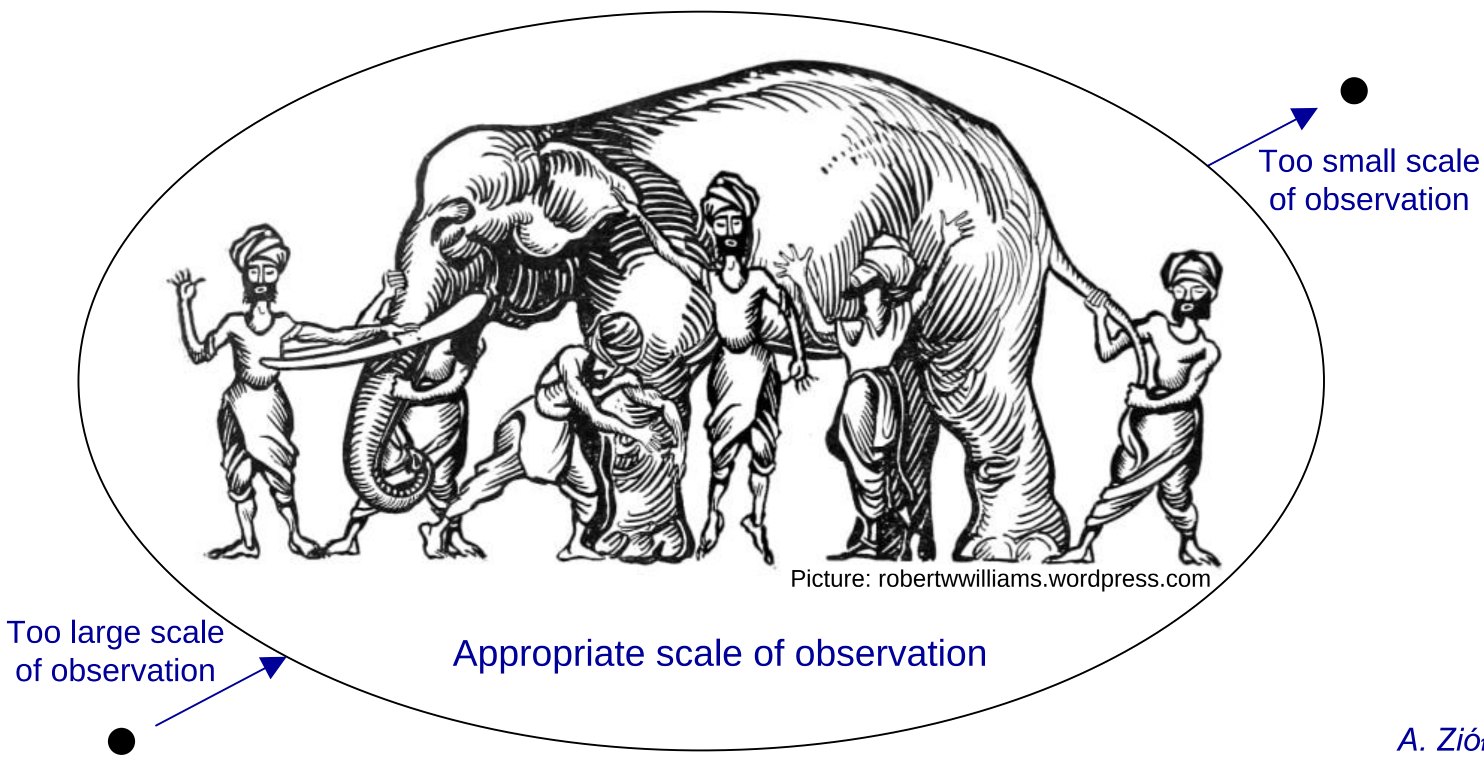
Perception of physical phenomena.
Limitations or capabilities of human senses?

So, what the tensors actually are, and how we can understand them from more pragmatic point of view?

Anecdotally, we can say that we are in a position of six blind man trying to find out what an elephant is ?

We must adopt a right scale of observation and right viewpoint to get information that we want.

Choosing scale and focus to find out what something is?



Perception of physical phenomena. Limitations or capabilities of human senses?

Human senses have interesting limitations or maybe capabilities? that we (objectively) *do see objects differently* depending on *how we look*.

Interesting limitations or maybe capabilities
of human senses

Picture: Fliegende Blatter, a German humor magazine
published in Munich, October 23, 1892, p.147

Depending on the focus humans *objectively do see the same object differently*.

Another interesting property of human senses is that *at specific instant* we can have *only one specific interpretation* of how we do see some object.

What tensors are?, how they can be understood? Some historical information.

Tensors gained its today omnipresence in science and technology because they proved to be excellent *modeling objects* of reality.

The same type of tensors enable description of, among the other,

- *state of real objects* (e.g., temperature, velocity, deformation, stress, strain, energy),
- properties *of real objects* (e.g., thermal expansion, piezoelectric effects, elastic stiffness or compliance),
- loading *of real objects* (e.g., force or displacement load).

Tensors of various orders proved to be very convenient and reliable modeling tools in description and/or prediction of broad range of various real phenomena. This delivers motivation to understand as best as possible what tensors actually are and what are their *eigenproperties* – represented by their various *invariants*. Fine comprehension of tensor objects themselves can deliver better insight and facilitate deep understanding of specific real physical situations modeled with their aid.

What tensors are?, how they can be understood? Various defintions.

There exist at least several definitions of a tensor notion,

- *algebraic definition*, in which tensor is treated as *algebraic object* an element of advanced algebraic structure called tensorial space ($\mathcal{T}_q$),

- *operational definition*, in which tensor is understood as a *linear operator* transforming one tensorial object into another tensorial object linearly,

- *geometric definition*, in which tensor is envisioned as a *geometrical object* with specific *"shape"* and *orientation* with respect to a certain fixed reference/ coordinate frame.

Depending on *specific targets* of research and/or analysis one or the other interpretation of tensors can be more useful and/or convenient.

For example, for purposes of modeling of *real physical space* and *real physical phenomena* it is convenient to treat tensors as *geometrical objects* but in order to obtain some precise *quantitative results* using tensorial calculus, it is appropriate to treat tensors as *algebraic objects.* This very useful feature of tensors can be called *algebraic-geometric dualism* of tensors.

Algebraic definition of a tensor.

Formal algebraic definition of tensors that

"Tensors are elements of tensorial linear spaces $\boldsymbol{T}_q$ ",

is actually very similar to the entry devoted to headword *horse, i.e.,*

"Horse, how it is everybody sees."

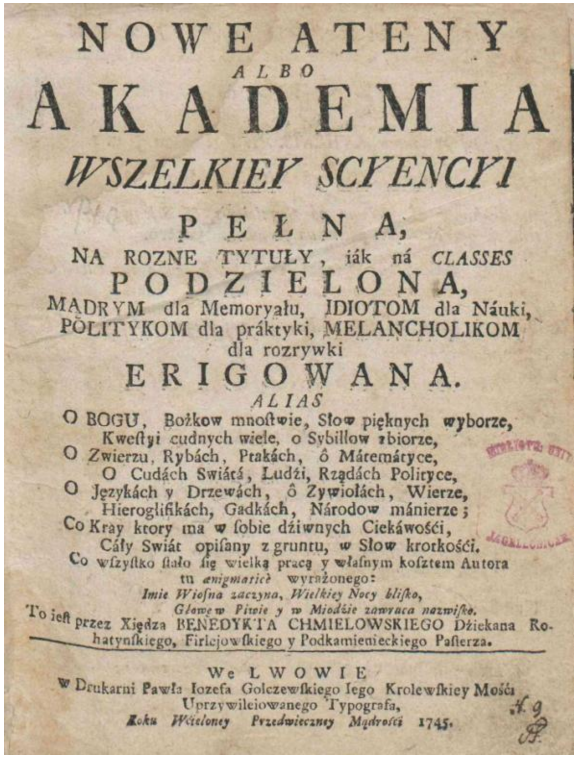
presented in the historically first Polish Encyclopedia entitled "New Athens" authored by Benedykt Chmielowski's.

Both recalled above definitions are *practically meaningless* unless broad background information is already available to the readers of them.

Here and attempt is undertaken to bring such background information regarding tensors so that one could say at the end of this work,

"Tensor, how it is everybody sees" ,

and this with profound understanding of the matter.


NOWE ATENY
ALBO
AKADEMIA
WSZELKIEY SCYENCYI
PEŁNA,
NA ROZNE TYTUŁY, iák ná CLASSES
PODZIELONA,
MĄDRYM dla Memoryału, IDIOTOM dla Náuki,
POLITYKOM dla práktyki, MELANCHOLIKOM
dla rozrywki
ERIGOWANA.
ALIAS
O BOGU, Bożkow mnoſtwie, Słow pięknych wyborze,
Kweſtyi cudnych wiele, o Sybillow zbiorze,
O Zwierzu, Rybách, Ptakách, ô Mátemátyce,
O Cudách Swiátá, Ludźi, Rządách Polityce,
O Językách y Drzewách, ô Zywiołách, Wierze,
Hieroglifikách, Gadkách, Národow mánierze;
Co Kray ktory ma w ſobie dźiwnych Ciekáwośći,
Cáły Swiát opiſany z gruntu, w Słow krotkośći.
Co wſzyſtko ſtało ſię wielką pracą y włáſnym koſztem Autora
tu ænigmaticè wyrażonego:
Imie Wioſna zaczyna, Wielkiey Nocy bliſko,
Głowę w Piwie y w Miodźie zawraca nazwiſko.
To ieſt przez Xiędza BENEDYKTA CHMIELOWSKIEGO Dźiekana Rohatynſkiego, Firlejowſkiego y Podkamienieckiego Paſterza.

We LWOWIE
w Drukarni Pawła Iozefa Golczewſkiego Iego Krolewſkiey Mośći
Uprzywileiowanego Typografa,
Roku Wćielonej Przedwieczney Mądrośći 1745.

While not easily accessible and quite hermetic at first sight, actually the algebraic tools of tensor calculus are indispensable in obtaining any *quantitative results*.

Algebraic definition of tensors.
Gregorio Ricci-Curbastro's absolute differential calculus.

The mathematical grounds of *tensor calculus/analysis* with all the fundamental underlying formal mathematical apparatus was originally developed by Gregorio Ricci-Curbastro in the years 1888-1892.
More information on life and work of Ricchi-Curbastro can be found in Angelo Tonolo commemoration paper "Commemorazione di Gregorio Ricci-Curbastro nel primo centenario della nascita" were also a list of key works of Ricchi-Curbastro devoted to development of absolute calculus can be found.
The main four of them are as follows:

G. RICCI-CURBASTRO

1. *Delle derivazioni covarianti e controvarianti e del loro uso nella Analisi applicata*, «Studi editi dalla Università di Padova a commemorare l'ottavo centenario della origine della Università di Bologna», Vol. III, [1888].
2. *Sopra certi sistemi di funzioni*, «Rend. Acc. Lincei», Vol. V, [1889].
3. *Di un punto della teoria delle forme quadratiche ternarie*, «Rend. Acc. Lincei», Vol. V, [1889].
4. *Résumé de quelques travaux sur les systèmes variables de fonctions*, « Bull. Sc. Math.», T. XVI, [1892].

Symbol of Ricci-Curbastro family vineyard.
(Hedgehogs under the Oak tree [quercus in Latin])

Algebraic definition of tensors.
Gregorio Ricci-Curbastro's absolute differential calculus.

The motivation of Gregorio Ricci-Curbastro standing behind development of formal mathematical apparatus of tensorial calculus was completely different from this of Augustin Cauchy, namely it was investigation on *invariance of quadratic forms* and Ricci-Curbastro called the technique *absolute differential calculus*.

Ricci-Curbastro can be regarded as father of *tensor notion* as it is the feature of *invariance with respect to change of coordinate systems*, which is the essence and profound sense of tensorial objects.

Label of wine produced to this day by Ricci-Curbastro family. *In vino veritas*.

The term *tensor* in its contemporary meaning was coined by Woldemar Voigt in his work from 1898.

Tonolo A., Commemorazione di Gregorio Ricci-Curbastro nel primo centenario della nascita, Rendiconti del Seminario Matematico della Università di Padova (in Italian), tome 23 (1954), p. 1-24.
Voigt W., Die fundamentalen physikalischen Eigenschaften der Kristalle in elementarer Darstellung (in German), Verlag Von Veit & Comp. 1898, Leipzig.

Lecture 3

Algebraic, operational and geometric definitions of Tensors.

3.1 Algebraic definition of tensors.
3.2 Algebraic structures of tensorial calculus.
3.3 Translation rules between absolute notation and indicial notation of tensors.
3.4 Operational definition of a tensor.
3.5 Geometrical definition of a tensor.

Comprehensive, accessible and mathematically precise presentations of foundations and elements of tensorial calculus in application to mechanics can be found, e.g., in Ray Ogden , A. J. M. Spencer , Janina Ostrowska-Maciejewska textbooks.

Ogden R.W., Non-linear elastic deformations. Dover Publications, Inc. 1997.
Ostrowska-Maciejewska J., Podstawy i Zastosowania Rachunku Tensorowego, IPPT PAN, Reports, Warsaw, 2007.
Spencer A.J.M., Continuum Mechanics, Dover Publications Inc.(first Edition 1980), 2004.
https://mathshistory.st-andrews.ac.uk/Miller/mathword/ Earliest Uses of Some Words of Mathematics

Algebraic definition of a tensor.

Definition 3.1. An *Euclidean tensor* of order q and dimension n is and *algebraic structure*, an *element of linear tensorial space* $\mathcal{T}_q$, which is generated by the q-fold tensorial product of n dimensional *Euclidean vector spaces* E_n. When the same basis is accepted in all spaces E_n then any tensor $\mathbf{T}$ belonging to $\mathcal{T}_q$ can be presented in the form,

$$\mathbf{T} = \underbrace{T_{ij\ldots m}}_{components} \underbrace{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_m}_{basis\ (q\text{-}fold)} \quad i, j, \ldots, m = 1, .., n, \quad \mathbf{T} \in \mathcal{T}_q, \mathbf{e}_i \in E_n$$

where a set of q versors $\mathbf{e}_i \in E_n$ is an orthonormal basis of Euclidean vector space E_n, a set of n^q simple tensors $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_m$ (q-fold tensorial product of versors $\mathbf{e}_i$) is a basis of space $\mathcal{T}_q \equiv E_n^{(1)} \otimes \ldots \otimes E_n^{(q)}$, and numbers $T_{ij\ldots m}$ are called *components* of tensor $\mathbf{T}$ – its *representation* in basis $\{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_m\}$.

By *algebraic structure*, it is understood a set composed of finite number of possibly infinite sets of *elements* and *mappings* of Cartesian products of these sets into these products. The mappings are called *operations*.

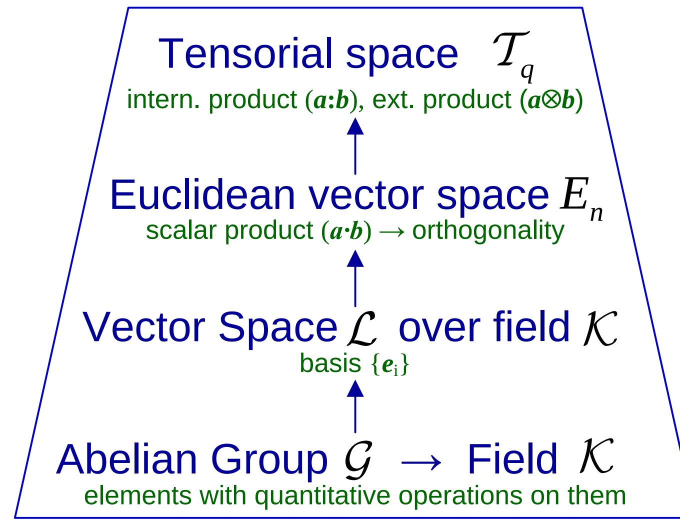
Tensorial linear space is higher level *(complex) algebraic structure* composed of lower level (simpler) algebraic structures.

Algebraic definition of a tensor.

Note The most important information to acquire from *algebraic definition* of tensors (elements of a linear space) is that *tensor makes an inseparable integrity of a basis and components* – its *representation* in the basis.

When the basis is fixed, *isomorphism* exists between *tensor* and its *representation (components)* in this basis, i.e., the *tensor* can be identified with its *components*.

$$\mathbf{T} = \underbrace{T_{ij\ldots m}}_{\textit{Components}} \underbrace{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_m}_{\textit{basis}}$$



The components of a tensor transform in *linear manner* with change of the *coordinate system*.

For example,

$$\mathbf{e}_i' = \mathbf{R}\mathbf{e}_i,\ \boldsymbol{\sigma} = \sigma_{ij}\,\mathbf{e}_i \otimes \mathbf{e}_j = R_{ki}\sigma_{ij}R_{jl}\,\mathbf{e}_k' \otimes \mathbf{e}_l' = \sigma_{kl}'\,\mathbf{e}_k' \otimes \mathbf{e}_l' \rightarrow \sigma_{kl}' = R_{ki}\sigma_{ij}R_{jl}$$

where $\mathbf{R}$ is so called rotation tensor $\mathcal{R} \equiv \{\,\mathbf{R} \in \mathcal{T}_2;\quad \mathbf{R}\mathbf{R}^T = \mathbf{1},\ \det(\mathbf{R}) = +1\,\}$.

Note Care must be exercised because in general the *basis of a tensor* can be changed *in non-linear manner.* This issue will be discussed later in Lecture 9 with the concept of *isometric tensorial bases*.

Algebraic structures of tensorial calculus.

The following *algebraic structures* (ideas/concepts) of growing complexity, from the most simple to the most complex, can be identified to make structural building blocks enabling construction of algebraic definition of *tensorial linear space* (*tensors*) and quantitative (numerical) operations on them (*tensor calculus*).

Elements Operations, "Tools"

growing complexity ↑

			"Tools"
→ Tensorial linear space $\mathcal{T}_q$	Ricci Curbastro (1888-92),	$\mathcal{T}_q \equiv (E_n; \otimes, :)$	absolute notation → indpendence from coordinate systems
→ Euclidean point (affine) space $\mathcal{E}_n$	Euler (1748), Weyl (1918),	$\mathcal{E}_n \equiv (\{P\}; \varphi_O), \varphi_O(O,X) \to \overrightarrow{OX} = \boldsymbol{x}$	homogernity (no distinguished point)
→ Euclidean vector (linear) space E_n	Gibbs (1881), Heaviside (1893),	$E_n \equiv (\mathcal{L}; \cdot)$	scalar and vector product $\mathbf{a}\cdot\mathbf{b} \in R$, $\mathbf{a}\times\mathbf{b}$
→ Vector space $\mathcal{L}$ over field $\mathcal{K}$	Peano (1888),	$\mathcal{L} \equiv ((\{L\}; \oplus), \odot, (\{K\}; +, *))$	basis $\{\mathbf{e}_i\}$
→ Field $\mathcal{K}$	Weber (1893),	$\mathcal{K} \equiv (\{K\}; +, *)$	
→ Abelian Group $\mathcal{G}_A$	Jordan (1870),	$g \lozenge h = h \lozenge g$ (*operation* "$\lozenge$" *is comutative*)	
→ Group $\mathcal{G}$	Galois (1830),	$\mathcal{G} \equiv (\{G\}, \lozenge)$	
→ Cartesian coordinate system	Descartes (1637),	$(x_1, \dots, x_n)$	

In establishing the thinkers who were the first to create and develop a certain concepts and/or ideas it was very helpful information contained in the web page *Earliest Uses of Some Words of Mathematics*

Group $\mathcal{G}$

An algebraic structure $\mathcal{G}\equiv(\{G\}, \lozenge)$ is called a Group $\mathcal{G}$ when $\{G\}$ is a non-empty set of *elements*, and $\lozenge$ is *operation* (mapping) assigning an element from $\{G\}$ to any pair of elements from $\{G\}$

$$\lozenge : \ (g,h)\in\{G\}\times\{G\}\Rightarrow g\lozenge h\in\{G\}$$

Operation must satisfy the following axioms,

$$\bigwedge_{g_1,g_2,g_3\in G} g_1\lozenge(g_2\lozenge g_3)=(g_1\lozenge g_2)\lozenge g_3$$ it is associative

$$\bigvee_{e\in G}\ \bigwedge_{g\in G} e\lozenge g=g\lozenge e=g$$ exists neutral element of the group

$$\bigwedge_{g\in G}\ \bigvee_{h\in G} g\lozenge h=h\lozenge g=e$$ exists inverse element of the group

Abelian Group (commutative group) is a group, which operation is commutative,

$$\bigwedge_{g,h\in G} g\lozenge h=h\lozenge g$$

For example, a set of all rotations of real space around fixed axis is a commutative group.

Field $\mathcal{K}$

An algebraic structure $\mathcal{K}$≡($\{K\}$, +, $*$) is called a *Field* $\mathcal{K}$ when $\{K\}$ is a non-empty set of elements, and +, $*$ are operations (mappings) assigning an element from $\{K\}$ to any pair of elements from $\{K\}$

$$+:\ (\alpha,\beta)\in\{K\}\times\{K\}\rightarrow\alpha+\beta\in\{K\},$$

$$*:\ (\alpha,\beta)\in\{K\}\times\{K\}\rightarrow\alpha*\beta\in\{K\}.$$

The following axioms must be true,

$(\{K\},+)$ set $\{K\}$ with operation "+" is Abelian group,

$(\{K\}-0,\cdot)$ set $\{K\}$ without neutral element of Abelian group ($\{K\}$,+), i.e., "0" is also Abelian group,

operation $*$ is distributive with respect to operation +,

$$\bigwedge_{\alpha,\beta,\gamma\in\{\mathcal{K}\}} \alpha*(\beta+\gamma)=\alpha*\beta+\alpha*\gamma\ .$$

For example, set of all real numbers with summation and multiplication operations is a field ($\{R\}$,+,$*$).

Vector (Linear) space $\mathcal{L}$

Algebraic structure $\mathcal{L} \equiv (\ (\{L\},\oplus),\ \odot,\ (\{K\},+,*)\ )$ is called linear space $\mathcal{L}$ over field $\mathcal{K}$ when the following axioms are true:

$(\{L\},\oplus)$ is Abelian group

$(\{K\},+,*)$ is field

Operation $\odot$ of multiplication of elements from $\{L\}$ by elements from set $\{K\}$

$$\odot:\ (\alpha, A) \in \{K\} \times \{L\} \Rightarrow \alpha \odot A = \alpha A \in \{L\}$$

has the following properties

$\bigwedge_{\alpha\in\mathcal{K}} \bigwedge_{A,B\in\mathcal{L}} \alpha(A+B) = \alpha A + \alpha B$ + is distributive with respect to $\odot$

$\bigwedge_{\alpha,\beta\in\mathcal{K}} \bigwedge_{A\in\mathcal{L}} (\alpha+\beta)A = \alpha A + \beta A$ $\odot$ is distributive with respect to +

$\bigwedge_{\alpha,\beta\in\mathcal{K}} \bigwedge_{A\in\mathcal{L}} \alpha(\beta A) = (\alpha\beta)A$ $\odot$ is associative

$\bigvee_{1\in\mathcal{K}} \bigwedge_{A\in\mathcal{K}} 1A = A$ neutral element of field $\mathcal{K}$ exists and is unique.

For example, a set of all ordered pairs of points – vectors, in 2D plane with their summation and multiplication by real numbers is a linear space.

Algebraic structures of tensorial calculus. Vector (linear) space $\mathcal{L}$.

Basis and *dimension* of *vector (linear) space* $\mathcal{L}$

Each set of elements $A_1, A_2, \ldots, A_n \in \mathcal{L}$ such that from equality

$$\alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_n A_n = 0$$

it results that

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

is called *linearly independent set of order n.*

Linear space $\mathcal{L}$ is called *n* – dimensional and denoted by $\mathcal{L}_n$, when it *does exists* linearly independent set of elements of order *n* in it, and *it does not exist* linearly independent set of elements of order greater than *n*.

Basis of space $\mathcal{L}_n$, it is called every *linearly independent set* of elements $A_1, A_2, \ldots, A_n \in \mathcal{L}_n$.

When the set $A_1, A_2, \ldots, A_n \in \mathcal{L}_n$ is a basis in $\mathcal{L}_n$ then each element A of space $\mathcal{L}_n$ can be expressed in the form

$$\bigwedge_{A \in \mathcal{L}_n} \bigvee_{\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{K}} A = \alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_n A_n$$

The set of coefficients α_i is unique for a specific fixed basis. Thus any set $A_1, A_2, \ldots, A_n$ being basis allows for generation of all elements of space $\mathcal{L}_n$

Algebraic structures of tensorial calculus. Euclidean vector space E_{n}.

Euclidean (linear) vector space E_n

Linear space $\mathcal{L}_n$ (*n*-dimensional) over field of real numbers R, equipped with *scalar product* operation defined on its elements, is called Euclidean vector space E_n.

Bilinear form is called *scalar product*

$$\cdot \ : \ (\boldsymbol{a},\boldsymbol{b}) \in \{L\} \times \{L\} \Rightarrow \boldsymbol{a} \cdot \boldsymbol{b} \in R$$

when it has the following properties

$$\bigwedge_{\boldsymbol{a},\boldsymbol{b} \in E_n} \quad \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

$$\bigwedge_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in E_n} \quad \boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}, \quad (\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}$$

$$\bigwedge_{\boldsymbol{a},\boldsymbol{b} \in E_n} \quad \bigwedge_{\alpha \in \mathcal{R}} \quad \alpha\, \boldsymbol{a} \cdot \boldsymbol{b} = \alpha\,(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot (\alpha \boldsymbol{b})$$

$$\bigwedge_{\boldsymbol{a} \in E_n} \quad \boldsymbol{a} \cdot \boldsymbol{a} \geq 0 \quad \mathit{and} \quad \boldsymbol{a} \cdot \boldsymbol{a} = 0 \quad \mathit{for} \quad \boldsymbol{a} = 0$$

Scalar product enables introduction of the concepts of *vector norm (modulus)* and the concept of *angle* between vectors.

Algebraic structures of tensorial calculus. Euclidean vector space E_n.

Norm and angle between vectors in Euclidean vector space E_n.

Norm (modulus) of vector $\boldsymbol{a}$, it is called a real number a satisfying the following conditions

$$a = |\boldsymbol{a}| \equiv (\boldsymbol{a}\cdot\boldsymbol{a})^{1/2}$$

$$|\boldsymbol{a}| \geq 0, \quad |\boldsymbol{a}| = 0 \quad \textit{for} \quad \boldsymbol{a} = 0, \qquad |\alpha\,\boldsymbol{a}| = |\alpha|\,|\boldsymbol{a}|$$

$$\bigwedge_{\boldsymbol{a},\boldsymbol{b}\in E_n} \quad |\boldsymbol{a}+\boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}| \quad - \text{ Schwartz inequality}$$

Vector $\boldsymbol{a}$, for which $|\boldsymbol{a}|=1$ is called a *versor*.

Angle φ between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be determined from the formula

$$\cos(\varphi) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}, \quad 0 \leq \varphi \leq \pi$$

Scalar product enables introduction of the notion of *orthonormal basis*.

Basis $\boldsymbol{e}_i$ is orthonormal when the following conditions are satisfied

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}, \quad |\boldsymbol{e}_i| = 1, \quad i, j = 1,\ldots,n$$

Vectors of orthonormal basis are mutually orthogonal and are versors.

Algebraic structures of tensorial calculus. Euclidean point (Affine) space $\mathcal{E}_n$.

Euclidean point (Affine) space n -dimensional is an *algebraic structure* $\mathcal{E}_n \equiv (\{P\}, E_n, \varphi)$, where $\{P\}$ is a set of elements (points), and φ is operation (mapping) uniquely (in one-to-one manner) assigning to each ordered pair of points $A, B \in \mathcal{E}_n$ a vector from vector space E_n

$$\varphi : \ (A, B) \in \{P\} \times \{P\} \Rightarrow \overrightarrow{AB} \in E_n$$

Mapping φ satisfies the following axioms,

$$\bigwedge_{A,B \in \mathcal{E}_n} \quad \overrightarrow{AB} = -\overrightarrow{BA},$$

$$\bigwedge_{A,B,C \in \mathcal{E}_n} \quad \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC},$$

$$\bigwedge_{O \in \mathcal{E}_n} \ \bigwedge_{X \in \mathcal{E}_n} \ \bigvee_{\mathbf{x} \in E_n} \quad \overrightarrow{OX} = \mathbf{x}$$

$\mathbf{x}$ *denotes vector radius* of point X with respect to selected, fixed point O.

Algebraic structures of tensorial calculus. Euclidean point (affine) space $\mathcal{E}_n$.

Upon completely arbitrarily distinguishing of a certain point O from space $\mathcal{E}_n$, it can be defined one-to-one mapping between points from *Euclidean point space* $\mathcal{E}_n$ and vectors from *Euclidean vector space* E_n, i.e., arithmetization of space $\mathcal{E}_n$ can be attained

$$\varphi_O : \varphi_O(O, X) = \overrightarrow{OX} = \mathbf{x}, \ \mathcal{E}_n \leftrightarrow E_n, \quad \varphi_O{}^{-1} : \varphi_O{}^{-1}(\mathbf{x}) = O \curlywedge \mathbf{x} = X$$

where symbol φ_o denotes operation of assigning points to vectors, and symbol $\curlywedge$ denotes opposite operation of assigning vectors to points.

Affine space is *homogeneous*, in this sense that all its elements, point, are fully equivalent and it does not exist in it whatever special element which could be distinguished by its special features, like e.g., "zero" element in linear (vector) space E_n.

Alternative more simple conceptually definition of *affine space* can be given as follows,

Affine space is an *algebraic structure* $\{\mathcal{E}_3, E_3, \curlywedge\}$ composed of a set of points $\mathcal{E}_3$, associated with this set vector space E_3, and operation $\curlywedge$ defining adding of vectors to points.

Algebraic structures of tensorial calculus. Euclidean point space $\mathcal{E}_n$. Commonly accepted, convenient, model of real physical space.

Three dimensional Euclidean Point Space $\mathcal{E}_3$ - *Affine Space*, is at present commonly accepted as convenient *mathematical model of real physical space*.

Euclidean point space $\mathcal{E}_n$ *is not a vector (linear) space*, because it does not posses required structure of linear space, e.g., addition of its elements is not defined as required in definition of vector (linear) space.

Mapping φ defining addition of vectors to points enables acquiring functionality of vector space E_n - taking advantage of elements of its structure.

Each pair (O, $\boldsymbol{e}_\mathrm{i}$), where $O \in \mathcal{E}_n$, $\boldsymbol{e}_\mathrm{i} \in E_n$ is called *coordinate system* of point space $\mathcal{E}_n$ associated with vector space E_n. Point O is a hooking point of the coordinate system and $\boldsymbol{e}_\mathrm{i}$ are basis versors of space E_n.

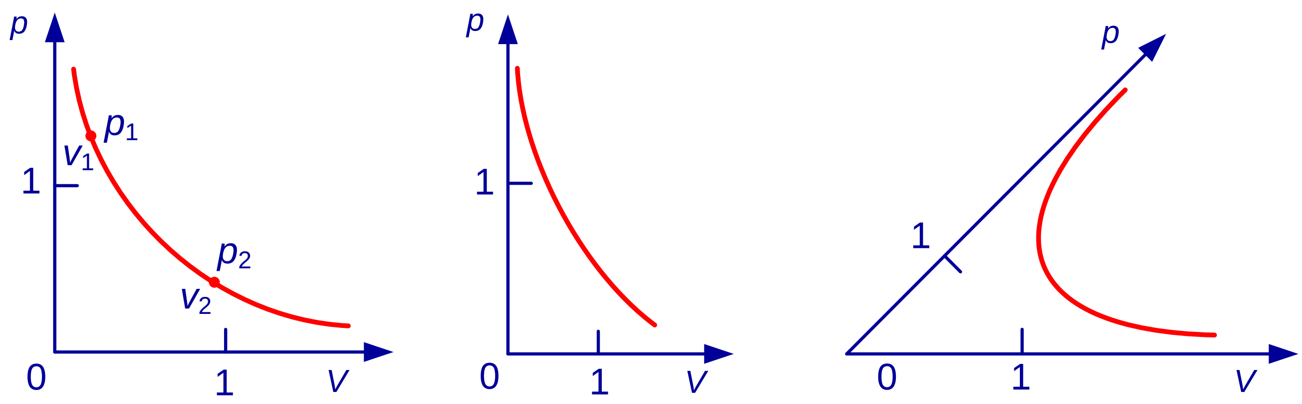
$$\bigwedge_{\boldsymbol{a} \in E_n} \quad \overset{1}{\bigvee_{\alpha_1, \alpha_2, \alpha_3 \in \mathcal{R}}} \quad \boldsymbol{a} = \alpha_1 \boldsymbol{e}_1 + \alpha_2 \boldsymbol{e}_2 + \alpha_3 \boldsymbol{e}_3 = \alpha_k \boldsymbol{e}_k$$

In continuum mechanics we are mainly working with *Euclidean (vector) space* E_3 when modeling real physical phenomena. This in view of existence of *bijection* between spaces E_3 and $\mathcal{E}_3$ - upon fixing reference point O.

Algebraic structures of tensorial calculus. Dimensional Affine Space.

Note *Care must be exercised* because there exists another concept/definition of affine space, in which *affinity results from assigning different dimensions* to the individual coordinate axes of the space.

Geometry in which properties of figures are examined *invariant* with respect to *units of measure* adopted on individual axes of *coordinate system* is called *affine geometry*. *Space* subject to *affine geometry* is called *affine vector space* (after Edmund Karaśkiewicz, p.355)



Operation of change of *physical units* of measure, and/or change of *length scale of line segments* and/or transition into *non-orthogonal coordinate systems* does not change the (depicted) *physical relation*.

Length of vector in such defined *(dimensional) affine space* does not have physical sense.

Karaśkiewicz E., Zarys teorii wektorów i tensorów, II Wyd., PWN, Warszawa, 1974.
Rychlewski J., Wymiary i Podobieństwo, PWN, Warszawa, 1991.

Algebraic structures of tensorial calculus. Tensorial linear spaces $\mathcal{M}_{nm}$.

Tensorial Product of linear spaces

Linear space $\mathcal{M}_{nm} = \mathcal{L}_n \oplus \mathcal{N}_m$ arising from Cartesian product $\mathcal{L}_n \otimes \mathcal{N}_m$ is called Tensorial Product of linear spaces $\mathcal{L}_n$ and $\mathcal{N}_m$ over field $\mathcal{K}$. Dimension of space $\mathcal{M}_{nm}$ is m·n.

Product operation $\otimes$ of elements from set $\{L\}$ by elements from set $\{N\}$,

$$\otimes: \ (\boldsymbol{A},\boldsymbol{a}) \in \{L\} \times \{N\} \Rightarrow \boldsymbol{A} \otimes \boldsymbol{a} \in \mathcal{L}_n \otimes \mathcal{N}_m \quad (\mathcal{L}_n \times \mathcal{N}_m \to \mathcal{L}_n \otimes \mathcal{N}_m)$$

by conjecture has the following properties,

$$\bigwedge_{A \in \mathcal{L}_n} \bigwedge_{a,b \in \mathcal{N}_m} A \otimes (a+b) = A \otimes a + A \otimes b$$ + is distributive with respect to $\otimes$

$$\bigwedge_{A,B \in \mathcal{L}_n} \bigwedge_{a \in \mathcal{N}_m} (A+B) \otimes a = A \otimes a + B \otimes a$$ $\otimes$ is distributive with respect to +

$$\bigwedge_{A \in \mathcal{L}_n} \bigwedge_{b \in \mathcal{N}_m} \bigwedge_{\alpha \in \mathcal{K}} \alpha A \otimes b = \alpha (A \otimes b)$$ $\otimes$ is associative

Tensorial linear space $\mathcal{T}_q$ and Euclidean tensors $\boldsymbol{T}$

The q-tuple tensorial product of n-dimensional vector Euclidean spaces E_n *is called tensorial space* $\mathcal{T}_q$ *of Euclidean tensors with dimension n and order q,*

$$\mathcal{T}_q = \overset{(1)}{E}_n \otimes \overset{(2)}{E}_n \otimes \ldots \otimes \overset{(q)}{E}_n = \bigotimes_{i=1}^{q} \overset{(i)}{E}_n, \quad \mathbf{T}_q \in \mathcal{T}_q .$$

Algebraic structures of tensorial calculus. Tensorial linear space $\mathcal{T}_q$.

Elements of Euclidean tensorial spaces are called *Euclidean Tensors*.

In continuum mechanics the most widespread use gained Euclidean tensors with dimension 3. Examples of tensorial spaces with tensors of different order (rank) are:

$\mathcal{T}_q = R$ space of real numbers,

$\mathcal{T}_1 = E_3$ space of 3D vectors,

$\mathcal{T}_2 = E_3 \otimes E_3$ space 3D second order tensors.

Any *second order symmetric tensor* can be expressed in the form,

$$\mathbf{A} = A_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j \in \mathcal{T}_2^{\,s} \equiv E_3 \otimes E_3, \quad where \quad A_{ij} = A_{ji} \Leftrightarrow \mathbf{A} = \mathbf{A}^T.$$

Ostrowska-Maciejewska J., Podstawy i Zastosowania Rachunku Tensorowego, IPPT PAN, Reports, Warsaw, 2007.

Translation rules between absolute notation and indicial notation of tensors.

A very convenient and useful tool of tensorial calculus is *absolute notation (index free notation)* of tensors. When writing down, documenting, various generally valid physical laws and principles involving tensors, we need and require the utmost simplicity, clarity and transparency and in such situation we are neither interested nor is it important what is the specific *coordinate system*. This is possible upon use of absolute notation of tensors in which physical laws and principles have the same, invariant form, independent on coordinate systems. Naturally, in particular examined physical situation the form of physical laws and principles depends on adopted specific coordinate system. Hence, in order to use them in practical calculations/ computations the absolute notation must be transformed into the *indicial notation*, i.e., into the form using indicial representations of tensors proper for the adopted specific, fixed coordinate system - *tensorial basis*.

Translation rules between absolute notation and indicial notation of tensors.

A vocabulary specified below delivers "recipes" for transforming any tensorial formula written down in *absolute notation* into the formula written down in (matrix) *indicial representation form* in a fixed Cartesian basis, i.e., *(Cartesian) index notation,*

$\boldsymbol{\alpha}\boldsymbol{\tau}$	$\leftrightarrow$	$\alpha_{ij}\tau_{jk}$
$\boldsymbol{\alpha}\cdot\boldsymbol{\beta}$	$\leftrightarrow$	$\alpha_{ij}\beta_{ij}$
$\mathbf{C}\cdot\boldsymbol{\omega}$	$\leftrightarrow$	$C_{ijkl}\omega_{kl}$
$\boldsymbol{\alpha}\cdot\mathbf{C}\cdot\boldsymbol{\beta}$	$\leftrightarrow$	$C_{ijkl}\alpha_{ij}\beta_{kl}$
$\mathbf{A}\cdot\mathbf{B}$	$\leftrightarrow$	$A_{ijkl}B_{ijkl}$
$\mathbf{C}\circ\mathbf{S}$	$\leftrightarrow$	$C_{ijkl}S_{klpq}$
$\mathbf{Q}*\boldsymbol{\omega}\equiv\mathbf{Q}\boldsymbol{\omega}\mathbf{Q}^T$	$\leftrightarrow$	$Q_{ij}Q_{kl}\omega_{jl}, \quad (\mathbf{Q}\mathbf{Q}^T=\mathbf{1})$
$\mathbf{Q}*\mathbf{C}$	$\leftrightarrow$	$Q_{ij}Q_{kl}Q_{pq}Q_{st}C_{jlqt}$
$\sigma\times\mathbf{T}$	$\leftrightarrow$	$T_{12\ldots p}\rightarrow T_{\sigma(1)\,\sigma(2)\ldots\,\sigma(p)}$

$\mathbf{n}, \boldsymbol{\omega}, \mathbf{C}$	$\leftrightarrow$	$n_i\ , \omega_{ij}\ , C_{ijkl}$
$\mathbf{1}$	$\leftrightarrow$	δ_{ij}
$\mathbf{n}\otimes\mathbf{m}, \mathbf{n}\otimes\boldsymbol{\omega}$	$\leftrightarrow$	$n_i m_j\ , n_i\omega_{jk}$
$\boldsymbol{\omega}\otimes\boldsymbol{\tau}$	$\leftrightarrow$	$\omega_{ij}\tau_{kl}$
$\boldsymbol{\omega}^2, \boldsymbol{\omega}^3$	$\leftrightarrow$	$\omega_{ij}\omega_{jk}\ , \omega_{ij}\omega_{jk}\omega_{kl}$
$\boldsymbol{\omega}\mathbf{n}, \mathbf{n}\boldsymbol{\omega}\mathbf{m}$	$\leftrightarrow$	$\omega_{ij}n_j\ , \omega_{ij}n_i m_j$
$\lvert\boldsymbol{\omega}\rvert=(\boldsymbol{\omega}\cdot\boldsymbol{\omega})^{1/2}$	$\leftrightarrow$	$(\omega_{ij}\omega_{ij})^{1/2}$
$\lvert\mathbf{C}\rvert$	$\leftrightarrow$	$(C_{ijkl}C_{ijkl})^{1/2}$

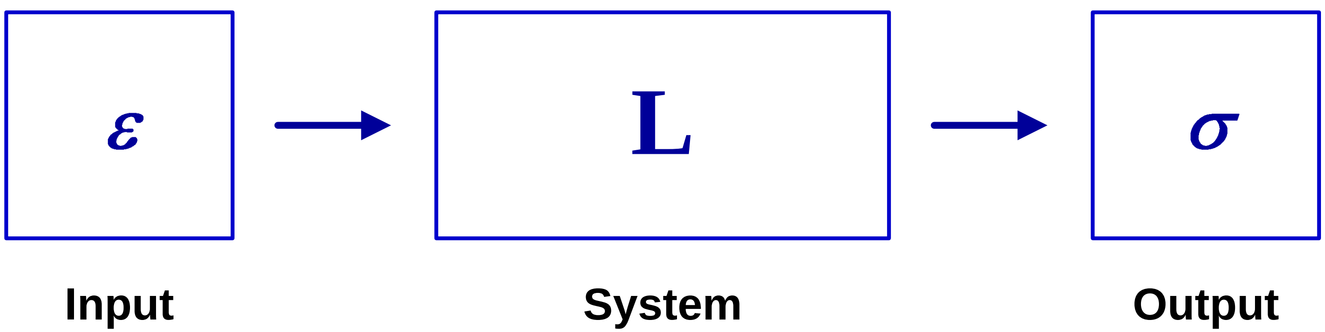
"   " – (no sign) denotes contraction over two indices, " $\cdot$ " – (dot) denotes full contraction, " $\circ$ " – (empty dot) denotes contraction over two indices (higher order tensors), " $*$ " – denotes orthogonal transformation, " $\sigma\times$ " – denotes permutation operation, " $\otimes$ " – denotes tensorial product operation.

Operational definition of a tensor. Constitutive models of materials are based on this interpretation.

Tensors can be treated as a *linear operators* transforming one tensorial object (space) into another tensorial object (space) linearly.

For example, fourth order tensor $\mathbf{L} \in \mathcal{T}_4$ upon its multiplication (contraction) with second order tensor $\boldsymbol{\varepsilon} \in \mathcal{T}_2$ transforms this tensor into some other second order tensor $\boldsymbol{\varepsilon} \to \mathbf{L}\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \in \mathcal{T}_2$.

$$\boldsymbol{\varepsilon} \Rightarrow \mathbf{L} \Rightarrow \boldsymbol{\sigma}; \quad \boldsymbol{\sigma} = \mathbf{L}\boldsymbol{\varepsilon}$$

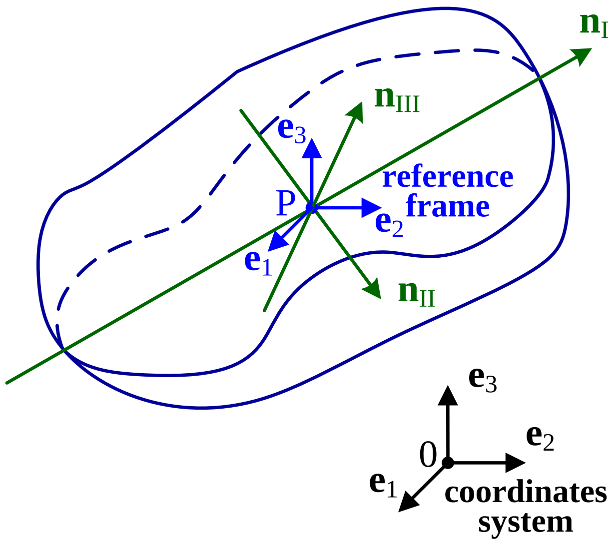


The *linear operators* were and are subject of broad and vivid research activities, documented in rich literature on the subject.
For example, actually such domains as *linear optimization*, *linear control theory* or *linear stability analysis* are varieties of linear operators analysis.
The most common approach to tensors treated as linear operations takes the form of *matrix calculus*.

Geometrical definition of a tensor.

Tensors can be interpreted as *geometrical objects* possessing specific *orientation* in physical space (*reference frame*) and specific features (*eigenproperties*) - number of which depends e.g., on order of the tensor.

Tensors information content and complexity grows with their increasing order and dimension of vector space generating respective tensor space.



Scalars (zero order tensors) does not bear information on their orientation in reference frame (physical space).

Vectors (first order tensors) carry information on their orientation in reference frame, and on *one* feature (eigenproperty) expressed by their moduli.

Second order symmetric tensors carry information on their orientation in reference frame and on up to *three* linearly independent eigenproperties.

Fourth order symmetric tensors (with Hooke's tensor symmetries - in fixed basis taking the form $\mathbf{H}=\mathbf{H}^{<1234>}=\mathbf{H}^{<2134>}=\mathbf{H}^{<4312>} \sim H_{ijkl}=H_{jikl}=H_{klij}$) can describe up to *eighteen* linearly independent eigenproperties.

Lecture 4

Symmetries of tensors and convenient notations of symmetric tensors.

A concept and definition of symmetry.

Symmetry concept plays *pivotal role* in tensorial calculus (*tensors*) and its applications.
A general, very capacious modern definition of symmetry can be formulated as follows (by the present author) :

Definition 4.1

Symmetry is the *invariance* (constancy, steadiness, stability) of a certain *feature* (geometric, physical, biological, information, etc.) of an *object* (an object can be a geometric system, a material object, a natural phenomenon, a physical law, a social relationship, a process in time, a physical field, etc.) after subjecting it to *action of transformations* from a certain *group* (transformations can be shifts, mirror images, rotations, changes of order, etc.) with respect to which symmetry is considered.

Symmetry can be perceived as a certain *universal philosophical category* (property) characterizing the *organization structure* of all systems existing in the universe.

See also, Weyl H., *Symmetry*, Princeton University Press, New Jersey, 1952.

Internal symmetry of tensors.

Before we proceed further Let us deliver more information on the concepts of internal and external symmetries of tensors. These properties are extensively used in modeling real phenomena with the aid of tensors.

Definition 4.2 *Permutation operation* $\sigma \times$ on a *tensor* $\mathbf{T}$ of p-th order is a linear mapping defined by the following formula,

$$\sigma \times \mathbf{T}: \ \mathbf{T} = T_{12\ldots p}\,\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \ldots \otimes \mathbf{e}_p \rightarrow \sigma \times \mathbf{T} = T_{12\ldots p}\,\mathbf{e}_{\sigma(1)} \otimes \mathbf{e}_{\sigma(2)} \otimes \ldots \otimes \mathbf{e}_{\sigma(p)},$$

$$\sigma \equiv <\sigma(1), \sigma(2), \ldots, \sigma(p)>, \quad \mathbf{T}, \sigma \times \mathbf{T} \in \mathcal{T}_p,$$

where $\sigma(1),\ldots,\sigma(p)$ is a preset permutation of the first p natural numbers $(1,\ldots,p)$ and $T_{12\ldots p}$ are components of tensor $\mathbf{T}$.

The permutation operation can be equivalently interpreted/treated as a permutation of the components of the tensor $\mathbf{T}$ written out in fixed basis,

$$\sigma \times \mathbf{T} \equiv <\sigma(1), \sigma(2), \ldots, \sigma(p)> \times \mathbf{T} = T_{\sigma(1)\,\sigma(2)\ldots\,\sigma(p)}\,\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \ldots \otimes \mathbf{e}_p \in \mathcal{T}_p.$$

It is convenient to introduce the following more *compact notation* for permutation operation,

$$\sigma \times \mathbf{T} \equiv <\sigma(1), \ldots, \sigma(p)> \times \mathbf{T} \equiv \mathbf{T}^{<(\sigma(1)\,\sigma(2)\ldots\,\sigma(p)>}.$$

When it is known that only two indices permute it is convenient to use still more short denotation e.g., $\mathbf{T}^{<4,2>} \equiv \mathbf{T}^{<1432>}$.

Definition of internal symmetry of tensors.

The set of *all permutations operations* acting in the space of tensors of a fixed order (e.g., $\mathcal{T}_p$) constitutes the group $\mathcal{P}^\sigma$. This allows to introduce the concept of *internal symmetry of tensors*. The group $\mathcal{P}^\sigma$ is discreet, its size is finite and equals $p!$ elements, for example, for tensors of the 4-th order there are $4!=24$ elements of this group.

Definition 4.3 An *internal symmetry group* of a tensor $\boldsymbol{T} \in \mathcal{T}_p$ is a subset of the permutation group $\mathcal{P}^\sigma$ which elements satisfy the condition

$$\mathcal{P}_{\mathbf{T}}^{\sigma} \equiv \{\sigma \in \mathcal{P}^{\sigma};\quad \sigma \times \mathbf{T} = \mathbf{T}\};\quad \mathcal{P}_{\mathbf{T}}^{\sigma} \subset \mathcal{P}^{\sigma}$$

Tensors $\boldsymbol{T}$ satisfying the above condition are called (*internally*) *symmetric tensors* with respect to permutations of indices.

A tensor $\boldsymbol{T}$ is (*internally*) symmetric over a pair of indices (α,β), if the following equality holds, $\mathbf{T} = \mathbf{T}^{<\alpha,\beta>} = \mathbf{T}^{<\beta,\alpha>} \;\leftrightarrow\; T_{...\alpha.....\beta..} = T_{...\beta.....\alpha..}$, i.e., when the tensor $\boldsymbol{T}$ representation elements upon swapping their indices (α,β) are the same in any fixed basis,. In the case of fourth-order tensors, the symmetry with respect to permutation operation $<1324>\times$ means that $\mathbf{T}=\mathbf{T}^{<1234>}=\mathbf{T}^{<1324>}$, i.e., $T_{ijkl}=T_{ikjl}$ in any fixed basis.

Definition 4.4 A tensor is *absolutely (totally) internally symmetric* when the group of its symmetries is the entire set of permutations $\mathcal{P}_{\mathbf{T}}^{\sigma} = \mathcal{P}^{\sigma}$.

Definition of external symmetry of tensors. Orthogonal tensors of 2-order.

Definition 4.5 A set of second order tensors ***Q*** with properties,

$$\mathcal{O} = \{\mathbf{Q} \in \mathcal{T}^2;\ \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{1},\quad \det \mathbf{Q} = \pm 1\}$$

is a group and is called the *group of orthogonal tensors*.

Definition 4.6 A subset of orthogonal tensors for which det (***Q***)=+1

$$\mathcal{R} = \{\mathbf{Q} \in \mathcal{T}^2;\ \mathbf{Q}\mathbf{Q}^T = \mathbf{1},\ \det(\mathbf{Q}) = +1\},\quad \mathcal{R} \subset \mathcal{O}$$

is a group and is called the *special orthogonal group* or *rotation group (SO_3).*

Operation of *orthogonal tensors* on *second order symmetric tensors* takes the following forms in *absolute* and *indicial* notations,

$$\mathbf{Q} * \boldsymbol{\sigma} \equiv \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T \sim \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix}$$

$$\mathbf{Q} * \boldsymbol{\sigma} \equiv \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T \quad \leftrightarrow \quad \sim Q_{ij}Q_{kl}\sigma_{jl},\quad (\mathbf{Q}\mathbf{Q}^T = \mathbf{1})$$

$$\mathbf{Q} * \mathbf{C} \quad \leftrightarrow \quad \sim Q_{ij}Q_{kl}Q_{pq}Q_{st}C_{jlqt}$$

Definition of external symmetry of tensors.

The *orthogonal tensors* treated as transformation operators of other tensors realize their *rotation* and/or *mirror reflection*, this last when $\det \mathbf{Q} = -1$. The set of *orthogonal tensors* is used to define *external symmetry* of tensors.

Definition 4.7 A *group of external symmetry* of tensor $\boldsymbol{T} \in \mathcal{T}_p$ (p denotes order of the tensor) we call a subset of all orthogonal tensors $\boldsymbol{Q}$, which satisfy the following condition

$$\mathcal{O}_T = \{\mathbf{Q} \in \mathcal{O};\ \ \mathbf{Q} * \mathbf{T} = \mathbf{T} \in \mathcal{T}^p\}, \quad \mathcal{O}_T \subseteq \mathcal{O}; \quad (\mathbf{Q} = \boldsymbol{Q} * \mathbf{T} \leftrightarrow T_{ij\ldots k} = Q_{ia} Q_{jb} \ldots Q_{kc} T_{ab\ldots c})$$

On the other hand tensors $\boldsymbol{T}$ that satisfy the above condition are called (externally) *symmetric* with respect to group $\mathcal{O}_T$.

***Definition* 4.8** Tensor is *isotropic* when the group of its external symmetry is the whole set of orthogonal tensors $\mathcal{O}_T = \mathcal{O}$.

***Definition* 4.9** Tensor is *hemitropic* (also called *proper-isotropic*) when the group of its external symmetry is the whole set of proper orthogonal tensors $\mathcal{O}_T = \mathcal{R}$.

Note The symmetry property is a characteristic of a *tensor* as an integrated entity of the *basis* and *representation* (components in the basis) and not the matrix of tensor components (its representation in a fixed basis) only.

Second and fourth order symmetric tensors notations and interpretations.

According to general representation theorem the basis of *second order Eulerian tensors* space $\mathcal{T}_2$ can be constructed from *nine* so called dyads $\{\mathbf{i}_i \otimes \mathbf{i}_j\}$, $(i, j=1,2,3)$, where $\mathbf{i}_\mathrm{i}$ are versors of basis of 3-dimensional linear vector space E_3 - generating tensor space $\mathcal{T}_2$.

For the versors $\mathbf{i}_\mathrm{i}$ usually there are adopted *orthonormal versors* $\mathbf{e}_\mathrm{i}$, $\mathbf{i}_i \otimes \mathbf{i}_j \rightarrow \mathbf{e}_i \otimes \mathbf{e}_j$. In such a case customarily *Eulerian tensors* are called *Cartesian tensors*. The second order tensors can be expressed with the aid of nine dyads $\mathbf{e}_i \otimes \mathbf{e}_j$ $(i, j = 1..3)$ as follows,

$$\boldsymbol{T} = \sum_{i,j=1,3} T_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j \in \mathcal{T}_2$$

In the case of so called *symmetric second order tensors* their components fulfill condition $\boldsymbol{T}=\boldsymbol{T}^T$ $(T_{ij}=T_{ji})$. This means that their 3x3 representation matrix has only 6 linearly independent components.

Symmetric second order tensors make *six-dimensional* subspace of general second order tensors space $(\mathcal{T}_2^{\,s} \subset \mathcal{T}_2)$.

It is important to carefully distinguish between different *notations* used for tensors, as information may be differently distributed between the tensor *basis* and its *components* depending on the notation.

Cauchy tensor in standard, Kelvin and Principal values notation.

The following notations are very commonly used in the case of second order symmetric tensors (Cauchy tensor is used here as working example)

$$\boldsymbol{\sigma} = \sum_{i,j=1,3} \sigma_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j, \qquad \boldsymbol{\sigma} = \sum_{K=1,6} \sigma_K^{Ke}\, \mathbf{a}_K, \qquad \boldsymbol{\sigma} = \sigma_I\, \mathbf{N}_I + \sigma_{II}\, \mathbf{N}_{II} + \sigma_{III}\, \mathbf{N}_{III},$$

$$\mathbf{N}_J \equiv \mathbf{n}_J \otimes \mathbf{n}_J, \quad \mathbf{n}_J = \mathbf{n}_J(\mathbf{e}_i),$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}, \quad \left[\sigma_1^{Ke}, \sigma_2^{Ke}, \sigma_3^{Ke}, \sigma_4^{Ke}, \sigma_5^{Ke}, \sigma_6^{Ke}\right], \qquad \left[\sigma_I, \sigma_{II}, \sigma_{III}\right].$$

where (σ_I, σ_{II}, σ_{III}) denote principal values and ($\mathbf{n}_I$, $\mathbf{n}_{II}$, $\mathbf{n}_{III}$) are principal directions (eigenvectors) of the tensor $\boldsymbol{\sigma}$. The principal axes $\mathbf{n}_J$ are rotated with respect to laboratory frame axes ($\mathbf{e}_1$,$\mathbf{e}_2$,$\mathbf{e}_3$) by three (Euler) angles (θ_1,θ_2,θ_3).

The first notation is a *standard (Cartesian) notation*. The second notation is called *Kelvin notation*. In explicit form it takes the form,

$$\boldsymbol{\sigma} = \sigma_1 \boldsymbol{a}_1 + \sigma_2 \boldsymbol{a}_2 + \sigma_3 \boldsymbol{a}_3 + \sqrt{2}\sigma_4 \boldsymbol{a}_4 + \sqrt{2}\sigma_5 \boldsymbol{a}_5 + \sqrt{2}\sigma_6 \boldsymbol{a}_6; \quad \sigma_1 = \sigma_{11},\ \sigma_2 = \sigma_{22},\ \sigma_3 = \sigma_{33},\ \sigma_4 = \sigma_{23},\ \sigma_5 = \sigma_{13},\ \sigma_6 = \sigma_{12},$$

$$ij \to K: \quad 11 \to 1, 22 \to 1, 33 \to 1, 23 \to 4, 13 \to 5, 12 \to 6, \qquad i, j = 1,2,3, \ K = 1,..,6.$$

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{\boldsymbol{a}_1\equiv}_{\mathbf{e}_i \otimes \mathbf{e}_j}, \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{\boldsymbol{a}_2\equiv}_{\mathbf{e}_i \otimes \mathbf{e}_j}, \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\boldsymbol{a}_3\equiv}_{\mathbf{e}_i \otimes \mathbf{e}_j}, \overbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}^{\boldsymbol{a}_4\equiv}_{\mathbf{e}_i \otimes \mathbf{e}_j}, \overbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}^{\boldsymbol{a}_5\equiv}_{\mathbf{e}_i \otimes \mathbf{e}_j}, \overbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{\boldsymbol{a}_6\equiv}_{\mathbf{e}_i \otimes \mathbf{e}_j}.$$

The third notation is called *principal values notation* because it uses principal values and principal directions.

Hooke's tensor in standard notation.

Let us take linear elasticity law – Hooke's constitutive law, for another illustrative example of absolute and indicial notation,

$$\boldsymbol{\sigma} = \mathbf{S}\cdot\boldsymbol{\varepsilon} \sim \sigma_{ij} = S_{ijkl}\varepsilon_{kl}, \quad i,j,k,l = 1,..,3; \quad \boldsymbol{\sigma},\boldsymbol{\varepsilon} \in \mathcal{T}_2^{\,s}{}_{(n=3)}, \; \mathbf{S} \in \mathcal{T}_4^{\,s}{}_{(n=3)}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & S_{1123} & S_{1132} & S_{1113} & S_{1131} & S_{1112} & S_{1121} \\ S_{2211} & S_{2222} & S_{2233} & S_{2223} & S_{2232} & S_{2213} & S_{2231} & S_{2212} & S_{2221} \\ S_{3311} & S_{3322} & S_{3333} & S_{3323} & S_{3332} & S_{3313} & S_{3331} & S_{3312} & S_{3321} \\ S_{2311} & S_{2322} & S_{2333} & S_{2323} & S_{2332} & S_{2313} & S_{2331} & S_{2312} & S_{2321} \\ S_{3211} & S_{3222} & S_{3233} & S_{3223} & S_{3232} & S_{3213} & S_{3231} & S_{3212} & S_{3221} \\ S_{1311} & S_{1322} & S_{1333} & S_{1323} & S_{1332} & S_{1313} & S_{1331} & S_{1312} & S_{1321} \\ S_{3111} & S_{3122} & S_{3133} & S_{3123} & S_{3132} & S_{3113} & S_{3131} & S_{3112} & S_{3121} \\ S_{1211} & S_{1222} & S_{1233} & S_{1223} & S_{1232} & S_{1213} & S_{1231} & S_{1212} & S_{1221} \\ S_{2111} & S_{2122} & S_{2133} & S_{2123} & S_{2132} & S_{2113} & S_{2131} & S_{2112} & S_{2121} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{13} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{bmatrix}$$

The tensors which possess the following internal symmetries are called Hooke's tensors,

$$\mathbf{H}^{<1234>} = \mathbf{H}^{<2134>} = \mathbf{H}^{<3412>} \quad \sim \quad H_{ijkl} = H_{jikl} = H_{klij}\ .$$

The elastic stiffness and compliance tensors **S**, **C** appearing in the linear law of elasticity - Hooke's law, are both Hooke's tensors.

Hooke's tensor in Kelvin notation.

Internal symmetry of second order tensors $\boldsymbol{\omega}=\boldsymbol{\omega}^{\mathrm{T}}$, $\omega_{\mathrm{ij}}=\omega_{\mathrm{ji}}$ and internal symmetry of Hooke's tensors make it possible, in full accordance with formal rules of tensor calculus, to interpret such symmetric second order and fourth order tensors defined in three dimensional space as vectors and second order tensors in six dimensional space, and the opposite,

$$\boldsymbol{\omega}\in\mathcal{T}_{1\,(n=3)}\otimes\mathcal{T}_{1\,(n=3)} \leftrightarrow \boldsymbol{\omega}\in\mathcal{T}_{1\,(n=6)}, \qquad \omega_{ij}=\omega_{ji} \qquad \leftrightarrow \omega_K, \quad i,j,k,l=1,2,3,\ K,L=1,..,6,$$

$$\mathbf{H}\in\overset{4}{\otimes}\mathcal{T}^{\,s}_{1\ (n=3)} \qquad \leftrightarrow \mathbf{H}\in\mathcal{T}^{\,s}_{2\ (n=6)},\ H_{ijkl}=H_{jikl}=H_{klij}\leftrightarrow H_{KL}=H_{LK}.$$

This reinterpretation is called *Kelvin notation*. Taking advantage of Kelvin notation Hooke's law can be written out in much more compact form as follows,

$$\boldsymbol{\sigma}=\mathbf{S}\cdot\boldsymbol{\varepsilon},\ \sim\sigma_{ij}=S_{ijkl}\varepsilon_{kl}, \qquad i,j,k,l=1,..,3;\quad \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathcal{T}^{\,s}_{2\,(n=3)},\ \mathbf{S}\in\mathcal{T}^{\,s}_{4\,(n=3)}\Leftrightarrow$$

$$\sigma^{Ke}_K\mathbf{a}_K=S^{Ke}_{KL}\mathbf{a}_K\otimes\mathbf{a}_L\cdot\varepsilon^{Ke}_L\mathbf{a}_L,\ \sim\sigma^{Ke}_K=S^{Ke}_{KL}\varepsilon^{Ke}_K,\ K,L=1,...,6;\ \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathcal{T}_{1\,(n=6)},\mathbf{S}\in\mathcal{T}^{\,s}_{2\ (n=6)},$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sqrt{2}\sigma_{23}\\ \sqrt{2}\sigma_{13}\\ \sqrt{2}\sigma_{12}\end{bmatrix}^{Ke}=\begin{bmatrix}S_{1111}&S_{1122}&S_{1133}&\sqrt{2}S_{1123}&\sqrt{2}S_{1113}&\sqrt{2}S_{1112}\\ S_{2211}&S_{2222}&S_{2233}&\sqrt{2}S_{2223}&\sqrt{2}S_{2213}&\sqrt{2}S_{2212}\\ S_{3311}&S_{3322}&S_{3333}&\sqrt{2}S_{3323}&\sqrt{2}S_{3313}&\sqrt{2}S_{3312}\\ \sqrt{2}S_{2311}&\sqrt{2}S_{2322}&\sqrt{2}S_{2333}&2S_{2323}&2S_{2313}&2S_{2312}\\ \sqrt{2}S_{1311}&\sqrt{2}S_{1322}&\sqrt{2}S_{1333}&2S_{1323}&2S_{1313}&2S_{1312}\\ \sqrt{2}S_{1211}&\sqrt{2}S_{1222}&\sqrt{2}S_{1233}&2S_{1223}&2S_{1213}&2S_{1212}\end{bmatrix}^{Ke}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \sqrt{2}\varepsilon_{23}\\ \sqrt{2}\varepsilon_{13}\\ \sqrt{2}\varepsilon_{12}\end{bmatrix}^{Ke},$$

$$\mathbf{a}_1\equiv\mathbf{e}_1\otimes\mathbf{e}_1,\ \mathbf{a}_2\equiv\mathbf{e}_2\otimes\mathbf{e}_2,\ \mathbf{a}_3\equiv\mathbf{e}_3\otimes\mathbf{e}_3,\ \mathbf{a}_4\equiv\tfrac{1}{\sqrt{2}}[\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_2],$$

$$\mathbf{a}_5\equiv\tfrac{1}{\sqrt{2}}[\mathbf{e}_1\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_1],\ \mathbf{a}_6\equiv\tfrac{1}{\sqrt{2}}[\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1];\quad \mathbf{a}_K\cdot\mathbf{a}_L=\delta_{KL},\ \mathbf{a}_K\in\mathcal{T}_{1\,(n=6)}.$$

Hooke's tensor on Voigt notation.

In *computational mechanics* so called *Voigt (matrix) notation* is very frequently employed to express Hooke's law as follows,

$$\boldsymbol{\sigma} = \mathbf{S}\cdot\boldsymbol{\varepsilon}, \sim \sigma_{ij} = S_{ijkl}\,\varepsilon_{kl} \leftrightarrow \sigma_K = S_{KL}\,\gamma_L \;\; (S_{KL} = S_{LK}) \leftrightarrow \sigma_K = S^{Vo}_{KL}\,\varepsilon_L \;\; (S^{Vo}_{KL} \neq S^{Vo}_{LK}),$$

$$\gamma_1 \equiv \varepsilon_{11},\; \gamma_2 \equiv \varepsilon_{22},\; \gamma_3 \equiv \varepsilon_{33},\; \gamma_4 \equiv 2\varepsilon_{23},\; \gamma_5 \equiv 2\varepsilon_{13},\; \gamma_6 \equiv 2\varepsilon_{12},$$

$$j \to K:\;\; 11 \to 1,\, 22 \to 1,\, 33 \to 1,\, 23 \to 4,\, 13 \to 5,\, 12 \to 6, \qquad i, j = 1,2,3,\;\; K = 1,..,6.$$

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & S_{1123} & S_{1113} & S_{1112}\\ S_{2211} & S_{2222} & S_{2233} & S_{2223} & S_{2213} & S_{2212}\\ S_{3311} & S_{3322} & S_{3333} & S_{3323} & S_{3313} & S_{3312}\\ S_{2311} & S_{2322} & S_{2333} & S_{2323} & S_{2313} & S_{2312}\\ S_{1311} & S_{1322} & S_{1333} & S_{1323} & S_{1313} & S_{1312}\\ S_{1211} & S_{1222} & S_{1233} & S_{1223} & S_{1213} & S_{1212} \end{bmatrix} \begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\\ \gamma_4\\ \gamma_5\\ \gamma_6\end{bmatrix} \leftrightarrow \begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 2S_{14} & 2S_{15} & 2S_{16}\\ S_{21} & S_{22} & S_{23} & 2S_{24} & 2S_{25} & 2S_{26}\\ S_{31} & S_{32} & S_{33} & 2S_{34} & 2S_{35} & 2S_{36}\\ S_{41} & S_{42} & S_{43} & 2S_{44} & 2S_{45} & 2S_{46}\\ S_{51} & S_{52} & S_{53} & 2S_{54} & 2S_{55} & 2S_{56}\\ S_{61} & S_{62} & S_{63} & 2S_{64} & 2S_{65} & 2S_{66} \end{bmatrix}^{Vo} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix}$$

Note Voigt (matrix) notation is computationally correct *but it is not consistent with formal rules of tensorial calculus*.

Hooke's tensor on Voigt notation.

This can be straightforwardly found out upon noticing that norms of Voigt's vectors and matrices representing stress, strain and Hooke's tensors are not equal to the norms of these tensors,

$$\boldsymbol{\sigma} \sim \sigma_{ij} \;\rightarrow \|\boldsymbol{\sigma}\|^2 = \|\sigma_{ij}\|^2 = \sigma_{11}{}^2 + \sigma_{22}{}^2 + \sigma_{33}{}^2 + 2(\sigma_{23}{}^2 + \sigma_{13}{}^2 + \sigma_{12}{}^2),$$

$$\boldsymbol{\sigma} \sim \sigma_K \;\rightarrow\; \|\sigma_K\|^2 = \sigma_{11}{}^2 + \sigma_{22}{}^2 + \sigma_{33}{}^2 + \sigma_{23}{}^2 + \sigma_{13}{}^2 + \sigma_{12}{}^2,$$

$$\boldsymbol{\varepsilon} \sim \varepsilon_K \;\rightarrow\; \|\varepsilon_K\|^2 = \varepsilon_{11}{}^2 + \varepsilon_{22}{}^2 + \varepsilon_{33}{}^2 + \varepsilon_{23}{}^2 + \varepsilon_{13}{}^2 + \varepsilon_{12}{}^2,$$

$$\boldsymbol{\varepsilon} \sim \gamma_K \;\rightarrow\; \|\gamma_K\|^2 = \varepsilon_{11}{}^2 + \varepsilon_{22}{}^2 + \varepsilon_{33}{}^2 + 4\varepsilon_{23}{}^2 + 4\varepsilon_{13}{}^2 + 4\varepsilon_{12}{}^2,$$

$$\|\sigma_K\|^2 \neq \|\boldsymbol{\sigma}\|^2, \quad \|\varepsilon_K\|^2 \neq \|\varepsilon_{ij}\|^2 = \|\boldsymbol{\varepsilon}\|^2, \quad \|\gamma_K\|^2 \neq \|\boldsymbol{\varepsilon}\|^2, \quad \|S^{Vo}_{KL}\|^2 \neq \|S_{ijkl}\|^2 = \|\mathbf{S}\|^2 .$$

The Voigt's vectors and matrices representing stress, strain and Hooke's tensors are not tensorial representations of these respective tensors in a certain tensorial basis. It is interesting and convenient that elastic energy is correctly expressed by product of Voigt's stress and strain vectors,

$$E = \tfrac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} = \tfrac{1}{2}\sigma_K \gamma_K .$$

Unit tensors of 0,1, 2 and 4-th order.

Very important roles in modeling of physical phenomena with the aid of tensorial calculus play *unit tensors* of various orders, where by unit tensors there are understood tensors composed of Kronecker deltas, see e.g., Ogden. Unit tensors of zero, first, second and fourth order are defined as follows,

$$\mathbf{1} = \mathbf{1}^{(2)} \; (\sim \delta_{ij}),$$

$$\mathbf{1}\otimes\mathbf{1} = \mathfrak{id}\times(\mathbf{1}\otimes\mathbf{1}) \;\; (\sim \delta_{ij}\delta_{kl}), \quad (\mathbf{1}\otimes\mathbf{1})^{<3,2>} \;\; (\sim \delta_{ik}\delta_{jl}), \quad (\mathbf{1}\otimes\mathbf{1})^{<4,2>} \;\; (\sim \delta_{il}\delta_{kj}),$$

$$\boldsymbol{I}^{(4s)} \equiv \mathfrak{c}\times(\mathbf{1}\otimes\mathbf{1}) = \tfrac{1}{2}[(\mathbf{1}\otimes\mathbf{1})^{<3,2>} + (\mathbf{1}\otimes\mathbf{1})^{<4,2>}] \;\; (\sim \delta_{KL} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})), \;\; K,L = 1,\ldots,6,$$

$$\boldsymbol{I}^{(4skew)} \equiv \tfrac{1}{2}[(\mathbf{1}\otimes\mathbf{1})^{<3,2>} - (\mathbf{1}\otimes\mathbf{1})^{<4,2>}] \;\; (\sim \tfrac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kj}), \qquad i,j,k,l = 1,2,3.$$

Appearing in the above formulas symbols $\mathfrak{id}$, $\mathfrak{c}$ denote *permutation operators* of identity and symmetrization over indices (2,3) and (2,4),

$$\mathfrak{id} \equiv <1,2,3,4>, \quad \mathfrak{c} \equiv \tfrac{1}{2}[<1324> + <1432>], \quad \mathfrak{s} \equiv \tfrac{1}{3}[\mathfrak{id} + 2\mathfrak{c}], \quad \mathfrak{t} \equiv \tfrac{2}{3}[\mathfrak{id} - \mathfrak{c}]$$

the $\mathfrak{s}$, $\mathfrak{t}$ denote permutation operators of total symmetrization (symmetry over all indices) and orthogonal complement to total symmetrization, respectively.

Ogden R.W., On isotropic tensors and elastic moduli, Proceedings of Cambridge Philosophical Society 75, 1974, p. 427.

Unit tensors of 0,1, 2 and 4-th order.

The unit tensors have the following representations in standard bases $\mathbf{e}_\mathrm{i}$, $\mathbf{e}_i \otimes \mathbf{e}_j$ and in Kelvin bases $\mathbf{a}_K$, $\mathbf{a}_K \otimes \mathbf{a}_{l.}$,

$$\text{scalar, versor, 2-order unit tensor,} \qquad \text{4-order unit tensors}$$

$$1 \sim [1],\ \mathbf{e}_1 \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{1} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{1} \otimes \mathbf{1} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ \boldsymbol{I}^{(4s)} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{e}_i \qquad \mathbf{e}_i \otimes \mathbf{e}_j \qquad \mathbf{a}_K \otimes \mathbf{a}_L \qquad \mathbf{a}_K \otimes \mathbf{a}_L$$

Unit tensors of the fourth order are used for construction of several extre-mely useful *projectors*, i.e., tensors which have the following property,

$$\mathbf{P} \circ \mathbf{P} = \mathbf{P} \ \sim P_{MN} P_{NL} = P_{ML}, \quad (M, N, L = 1, \ldots, 6), \quad \mathbf{P} \in \mathcal{T}_2{}^{s}{}_{(n=6)}.$$

Unit tensors of 0,1, 2 and 4-th order. Tensorial projectors.

Namely, *identity projector*, *isotropic projector* and *deviatoric projector*,

$$\mathbf{I}^{(4s)} = \tfrac{1}{2}[(\mathbf{1}\otimes\mathbf{1})^{<3,2>} + (\mathbf{1}\otimes\mathbf{1})^{<4,2>}] \;-\; \text{identity projector} \quad (\mathbf{I}^{(4s)}\boldsymbol{\sigma} = \boldsymbol{\sigma}),$$

$$\mathbf{J} = \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1} \;\; (\sim \tfrac{1}{3}\delta_{ij}\delta_{kl}) \;-\; \text{isotropic projector} \quad (\mathbf{J}\boldsymbol{\sigma} = \tfrac{1}{3}tr(\boldsymbol{\sigma})\mathbf{1} = \sigma_m\mathbf{1}),$$

$$\mathbf{K} = \mathbf{I}^{(4s)} - \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1} \qquad -\; \text{deviatoric projector} \quad (\mathbf{K}\boldsymbol{\sigma} = \boldsymbol{\sigma} - \sigma_m\mathbf{1} = \mathbf{s}),$$

$$\mathbf{J}\circ\mathbf{J} = \mathbf{J}, \quad \mathbf{K}\circ\mathbf{K} = \mathbf{K}, \quad \mathbf{J}\circ\mathbf{K} = 0, \quad \mathbf{I}^{(4s)} \equiv \mathbf{J} + \mathbf{K}, \quad \mathbf{I}^{(4s)}, \mathbf{J}, \mathbf{K} \in \mathcal{T}_{2\;(n=6)}^{\;s} \; (\mathcal{T}_{4\;(n=3)}^{\;s})$$

The projectors **J**, **K** have the following representations in Kelvin basis $\mathbf{a}_K \otimes \mathbf{a}_L$,

$$\begin{bmatrix}\sigma_m\\ \sigma_m\\ \sigma_m\\ 0\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}\tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})\\ \tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})\\ \tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})\\ 0\\ 0\\ 0\end{bmatrix} = \overbrace{\frac{1}{3}\begin{bmatrix}1&1&1&0&0&0\\1&1&1&0&0&0\\1&1&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}}^{\mathbf{J}} \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sqrt{2}\sigma_{23}\\ \sqrt{2}\sigma_{13}\\ \sqrt{2}\sigma_{12}\end{bmatrix}, \quad \begin{bmatrix}s_{11}\\ s_{22}\\ s_{33}\\ \sqrt{2}s_{23}\\ \sqrt{2}s_{13}\\ \sqrt{2}s_{12}\end{bmatrix} = \begin{bmatrix}\tfrac{1}{3}(2\sigma_{11}-\sigma_{22}-\sigma_{33})\\ \tfrac{1}{3}(-\sigma_{11}+2\sigma_{22}-\sigma_{33})\\ \tfrac{1}{3}(-\sigma_{11}-\sigma_{22}+2\sigma_{33})\\ \sqrt{2}\sigma_{23}\\ \sqrt{2}\sigma_{13}\\ \sqrt{2}\sigma_{12}\end{bmatrix} = \overbrace{\begin{bmatrix}\tfrac{2}{3}&-\tfrac{1}{3}&-\tfrac{1}{3}&0&0&0\\-\tfrac{1}{3}&\tfrac{2}{3}&-\tfrac{1}{3}&0&0&0\\-\tfrac{1}{3}&-\tfrac{1}{3}&\tfrac{2}{3}&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}}^{\mathbf{K}} \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sqrt{2}\sigma_{23}\\ \sqrt{2}\sigma_{13}\\ \sqrt{2}\sigma_{12}\end{bmatrix}.$$

Lecture 5

Invariants and decompositions of 2-nd order symmetric tensors.

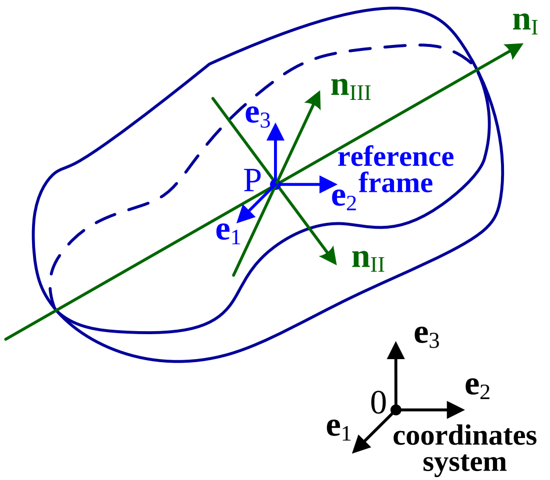
5.1 coordinate system versus reference frame.
5.2 Various bases of second order symmetric tensors.
5.3 Invariants (eigenproperties) of second order symmetric tensors.
5.4 Various decompositions of second order symmetric tensors.

coordinate system versus Reference frame.

A pair composed of a point O belonging to *Euclidean point space* ($O \in \mathcal{E}_3$) and a set of basis versors $\{\mathbf{e}_i\}$ of associated with it *Euclidean vector space* ($\mathbf{e}_i \in E_3$) is called *coordinate system* (*coordinate frame*). Frequently, in shortcut, the coordinate system is denoted by the set of basis vectors only.

In continuum mechanics besides *coordinate system* it is used concept of *reference frame.* "Physically" both sets are composed of a certain anchoring point and a set of basis vectors, e.g., $\{O, \mathbf{e}_i\}$.

The *difference* between *coordinate system* and *reference frame* is in their *functionality*. The coordinate system makes a reference, usually global, for determination of vector (tensor) location and components, while the reference frame makes a reference for examination of e.g., motions (kinematics) at a certain material point *P* of a body.



Depending on the *need* and *convenience* in examination of specific problem the *same* pair $\{O, \mathbf{e}_i\}$ can be adopted for coordinate frame and reference frame or *different* pairs can be adopted.

Various bases of second order symmetric tensors.

1. Standard (Cartesian) basis

$$\mathbf{e}_i \otimes \mathbf{e}_j, \quad i, j = 1,3$$

2. Kelvin basis

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{e}_i \otimes \mathbf{e}_j}^{\boldsymbol{a}_1 \equiv}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{e}_i \otimes \mathbf{e}_j}^{\boldsymbol{a}_2 \equiv}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{e}_i \otimes \mathbf{e}_j}^{\boldsymbol{a}_3 \equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\mathbf{e}_i \otimes \mathbf{e}_j}^{\boldsymbol{a}_4 \equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{e}_i \otimes \mathbf{e}_j}^{\boldsymbol{a}_5 \equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{e}_i \otimes \mathbf{e}_j}^{\boldsymbol{a}_6 \equiv}$$

3. Principal directions basis

$$\mathbf{N}_J \equiv \mathbf{n}_J \otimes \mathbf{n}_J, \quad \mathbf{n}_J = \mathbf{n}_J(\mathbf{e}_i), \quad J = I, II, II$$

4. Hooke's tensor eigenstresses basis

$$\boldsymbol{\omega}_K \in \mathcal{T}_{1(n=6)}\,(\mathcal{T}^{\,s}_{2\,(n=3)}); \;\; \boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_K = \delta_{KL}, \;\; \mathbf{H} \in \mathcal{T}^{\,s}_{2\;(n=6)}(\mathcal{T}^{\,s}_{4\,(n=3)}), \;\; K, L = 1,..,6,$$

$$\mathbf{H} \cdot \boldsymbol{\omega} = \lambda\, \boldsymbol{\omega} \;\to\; \det(\mathbf{H} - \lambda \mathbf{I}^{(4s)}) = 0 \;\to\; \lambda_K, \boldsymbol{\omega}_K, \quad \mathbf{I}^{(4s)} \in \mathcal{T}^{\,s}_{2\;(n=6)} \sim diag[1,1,1,1,1,1]$$

Ziółkowski A. G. Parametrization of Cauchy Stress Tensor Treated as Autonomous Object Using Isotropy Angle and Skewness Angle, *Engineering Transactions*, **70**, 3, pp. 239–286, 2022, doi: 10.24423/EngTrans.2210.20220809. Full text available freely at the link: https://et.ippt.gov.pl/index.php/et/article/view/2210

Various bases of second order symmetric tensors.

For example, for isotropic elastic materials it is

$$\{\mathbf{S}^{iso}\cdot\mathbf{h}=\lambda\,\mathbf{h}\sim S^{iso}_{\alpha\beta}\,h_\beta=\lambda\,h_\alpha\}\quad\Leftrightarrow\quad \det(\mathbf{S}^{iso}-\lambda\mathbf{I}^{(4s)})=0\ \rightarrow\ \lambda_K,\mathbf{h}_K,$$

$$\overbrace{\frac{1}{\sqrt{3}}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{\mathbf{e}_i\otimes\mathbf{e}_j}}^{\lambda_1=3K\leftrightarrow\mathbf{h}_1\equiv},\ \overbrace{\frac{1}{\sqrt{6}}\underbrace{\begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}}_{\mathbf{e}_i\otimes\mathbf{e}_j}}^{\lambda_2=2\mu\leftrightarrow\mathbf{h}_2\equiv},\ \overbrace{\frac{1}{\sqrt{2}}\underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}}_{\mathbf{e}_i\otimes\mathbf{e}_j}}^{\lambda_3=2\mu\leftrightarrow\mathbf{h}_3\equiv},\ \overbrace{\frac{1}{\sqrt{2}}\underbrace{\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{\mathbf{e}_i\otimes\mathbf{e}_j}}^{\lambda_4=2\mu\leftrightarrow\mathbf{h}_4\equiv},\ \overbrace{\frac{1}{\sqrt{2}}\underbrace{\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}}_{\mathbf{e}_i\otimes\mathbf{e}_j}}^{\lambda_5=2\mu\leftrightarrow\mathbf{h}_5\equiv},\ \overbrace{\frac{1}{\sqrt{2}}\underbrace{\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{\mathbf{e}_i\otimes\mathbf{e}_j}}^{\lambda_6=2\mu\leftrightarrow\mathbf{h}_6\equiv}.$$

The first eigenstate $\mathbf{h}_1$ is a spherical tensor and corresponding to it Kelvin moduli has the value $\lambda_1=3K=3\lambda+2\mu$. The remaining eigenstates $\mathbf{h}_\mathrm{i}$ are deviators and corresponding to them Kelvin moduli have the same value $\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5=2\mu$. The eigenstates $\mathbf{h}_\mathrm{i}$, $i=1..6$ make an orthonormal basis for second order symmetric deviators. So all the deviators must be eigenstates of isotropic Hooke's stiffness tensor $\mathbf{S}^{iso}$.

Note Problem for eigenstresses in full notation (9x9) is equivalent to problem for eigenstresses in Kelvin notation (6x6) but not in Voigt notation (6x6),

$$\{\mathbf{S}\cdot\boldsymbol{\omega}=\lambda\,\boldsymbol{\omega}\sim S_{ijkl}\omega_{kl}=\lambda\,\omega_{ij}\}\quad\Leftrightarrow\quad\{\mathbf{S}^{Ke}\cdot\boldsymbol{\omega}^{Ke}=\lambda\,\boldsymbol{\omega}^{Ke}\sim S^{Ke}_{\alpha\beta}\,\omega^{Ke}_\beta=\lambda\,\omega^{Ke}_\alpha\}.$$

Ziółkowski A. G. Parametrization of Cauchy Stress Tensor Treated as Autonomous Object Using Isotropy Angle and Skewness Angle, *Engineering Transactions*, **70**, 3, pp. 239–286, 2022, doi: 10.24423/EngTrans.2210.20220809. Full text available freely at the link: https://et.ippt.gov.pl/index.php/et/article/view/2210

Various bases of second order symmetric tensors.

It was already mentioned that *basis of tensor space* can be freely selected. However, depending on the target of the analysis a certain bases are more convenient than the other, e.g., suitable selection of the basis makes analytical and/or numerical computations more simple and/or effective.

Let us list some convenient bases of second order symmetric tensors space useful in the further discussion:

i) *Laboratory frame basis*. Such a basis is selected for example in view of convenient expression of imposed boundary conditions, e.g., loadings and/or constraints.

ii) *Symmetries oriented basis*. Such basis is selected to be collinear with axes of some kind of material symmetry or geometrical shape/layout of examined engineering structure/device. For example, it is chosen to be collinear with natural axes of symmetry of (anisotropic) material.

iii) *Principal axes basis*. Such a basis is selected when it is subject matter justified or convenient to work with principal values of second order symmetric tensor only.

iv) *Eigenstates (Eigenstresses) basis*. Such a basis might be convenient in formulation of strength of materials criteria.

Invariants (Eigenproperties) of second order symmetric tensors.

The *second order symmetric tensor* is fully characterized (defined) by *six components/parameters* (linearly independent) being its representation in a certain *fixed coordinate system* (*basis*). The components of a tensor change in *linear manner* with rotation of coordinate system.

From six components of second order symmetric tensor, there:

- can be constructed *infinite number of sets*, each consisting of *three invariants of the tensor* (linearly independent). Such invariants *do not change* when basis (coordinate system) of the tensor is changed.

- can be extracted a set of complementary parameters, three *Euler angles*, characterizing *orientation of the tensor object*, treated as *geometric entity*, with respect to axes $\mathbf{e}_i$ of the specific coordinate system (usually collinear with a certain convenient laboratory/reference frame) and generating basis of tensor space $\mathbf{e}_i \otimes \mathbf{e}_j$. *Euler angles do change (are not invariants)* with change of coordinate system (tensor space basis).

Note. The notion of *invariants of a tensor*, the *tensor itself being invariant* with respect to a change of coordinate system sounds like "butterfish butter" but actually it rather delivers a hint that the idea of a tensor is quite complex.

Various useful sets of invariants (eigenproperties).

Three linearly independent tensor invariants constructed from second order symmetric tensor components, invariant under change of coordinate system (basis), can be treated as *characteristic features (eigenproperties)* of the specific tensor. They deliver convenient specification (description) of the tensor when it is treated as a *geometrical object.*

Actually, *infinite number of triad sets of second order symmetric tensor invariants* can be constructed $\{I_1^\sigma, I_2^\sigma, I_3^\sigma\}$.

Construction (selection) of specific set of invariants and their usefulness depends on the area of study and/or specific examined problem.

For example this might be interpreted as ("Length", Width", "Height") or ("Hue", "Brightness", "Saturation").

Various useful sets of invariants (eigenproperties).

Tensorial modeling description *Reality*

Growing information content with growing tensor order:

Scalar description (1 datum), e.g., $a \sim [a_1]$

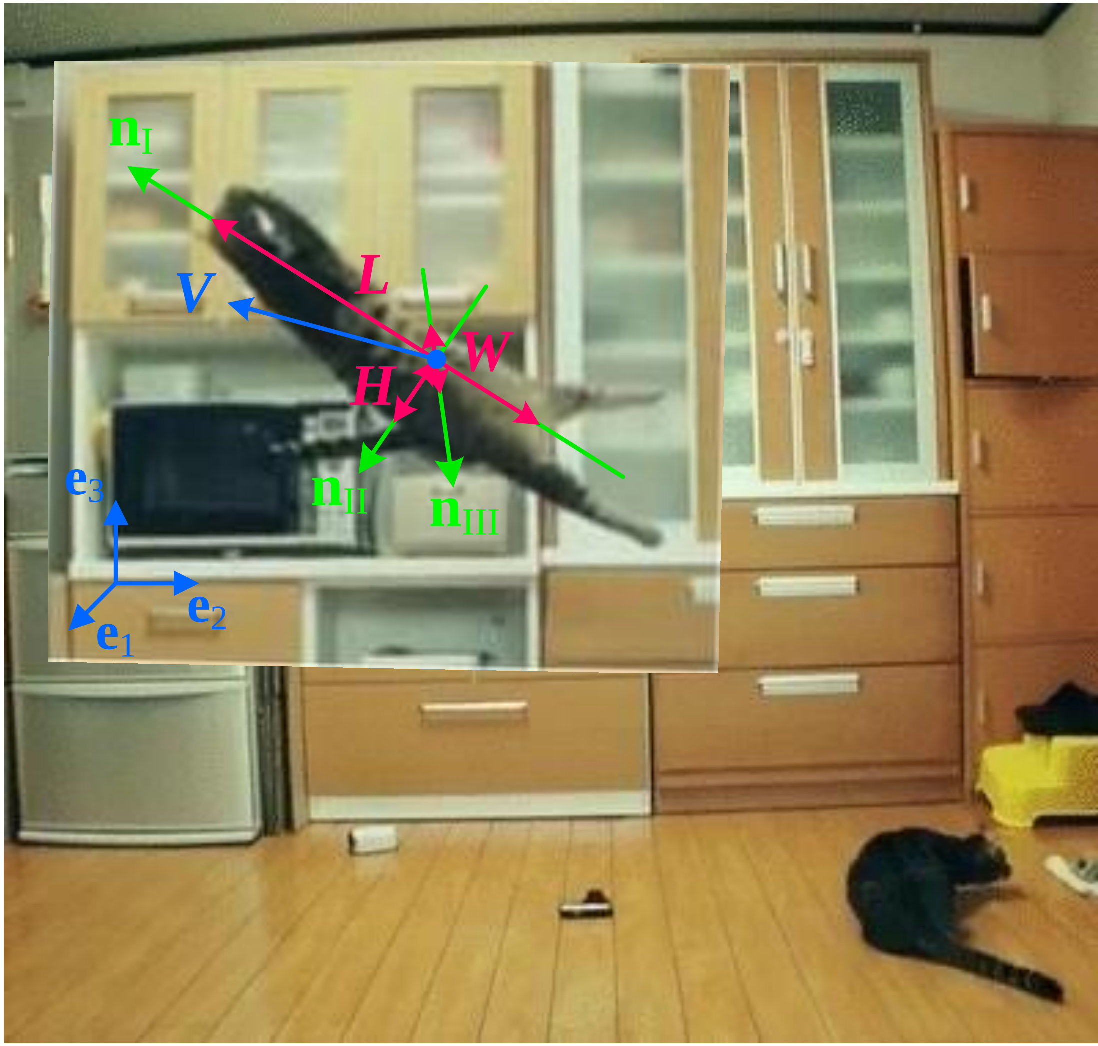
Vector description (3 data), e.g., $\mathbf{v} \sim \mathrm{v}_1\mathbf{e}_1+\mathrm{v}_2\mathbf{e}_2+\mathrm{v}_3\mathbf{e}_3$

2nd order symmetric tensor description (6 data), e.g.,

$$\mathbf{A} = \begin{bmatrix} L & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & W \end{bmatrix}$$

three invariants Length,Height,Width

three characteristic directions $\mathbf{n}_i \otimes \mathbf{n}_j$



Various useful sets of tensor invariants (eigenproperties)

Various invariants of second order symmetric tensors find useful physical interpretations and applications in different scientific and/or technological research areas. For example,

- trace of *stress tensor* has very important physical interpretation of pressure

$$⅓ tr(\boldsymbol{\sigma}) = ⅓ \sigma_{ii} = -p,$$

- trace of *small strains tensor* only approximately describes volumetric changes of the material

$$⅓ tr(\boldsymbol{\varepsilon}) = ⅓ \varepsilon_{ii} \approx dV/dV_0 \text{ ,}$$

- determinant of *deformation gradient tensor* delivers exact measure of volumetric changes

$$det(\mathbf{F}) = dV/dV_0.$$

Different *sets of tensor invariants* can be treated as *various parameterizations* of tensor *eigenproperties*.

Second order symmetric tensors, set of basic (main) invariants.

The common set of second order symmetric tensor invariants $\{I_{b1}, I_{b2}, I_{b3}\}$ most frequently encountered in mathematical studies are so called *basic invariants,* in some publications also called *main invariants*,

$$I_{b1} \equiv tr(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot \mathbf{1}, \qquad I_{b2} \equiv tr(\boldsymbol{\sigma}^2) = \|\boldsymbol{\sigma}\|^2 = \boldsymbol{\sigma}^2 \cdot \mathbf{1}, \qquad I_{b3} \equiv tr(\boldsymbol{\sigma}^3) = \boldsymbol{\sigma}^3 \cdot \mathbf{1}$$

where $||\boldsymbol{\sigma}||=(\sigma_{ij}\sigma_{ij})^{1/2}$ denotes *norm of a tensor*, $\mathbf{1}$ (δ_{ij}) denotes *unit tensor* in second order tensors space. The dot symbol denotes full (double) contraction of second order tensors $\boldsymbol{a}\cdot\boldsymbol{b}$ $(a_{ij}b_{ij})$.

The popularity of basic invariants comes from *computational effectiveness* of their determination, which requires only multiplication of tensor matrix representation, for which very effective numerical algorithms exist.

In continuum mechanics alternative set of linearly independent invariants so called *principal values* $\{\sigma_{I}, \sigma_{II}, \sigma_{III}\}$ of second order symmetric tensor gained popularity and is in widespread use. The reason for that is their physical interpretation, e.g., in the case of stress tensor they very well characterize the *effort state* of a medium under specific mechanical loading.

Various decompositions of second order symmetric tensors.

A number of various *decompositions of tensors* exist.

For example:

- decomposition into symmetric and antisymmetric part $\boldsymbol{A} = \boldsymbol{A}^{sym} + \boldsymbol{A}^{asym}$
- decomposition into spherical and deviatoric part $\boldsymbol{A} = \boldsymbol{A}^{sph} + \boldsymbol{A}^{dev}$
- isotropic decomposition $\boldsymbol{A} = \boldsymbol{A}^{iso} + \boldsymbol{A}^{aniso}$
- spectral decomposition $\boldsymbol{A} = \Sigma_{i=1,3}\, \lambda_i\, \mathbf{n}_i \otimes \mathbf{n}_i$
- polar decomposition $\boldsymbol{A} = \boldsymbol{RU} = \boldsymbol{VR}$

They are found to be useful for various purposes.

Spectral decomposition of second order symmetric tensor.

The *principal values* and *principal directions* of stress tensor are determined from so called *characteristic equation*,

$$\boldsymbol{\sigma}\,\mathbf{n} = \sigma\,\mathbf{n} \rightarrow (\boldsymbol{\sigma} - \sigma\,\mathbf{1})\mathbf{n} = 0 \rightarrow \quad \det(\boldsymbol{\sigma} - \sigma\,\mathbf{1}) = 0 \rightarrow$$

$$\sigma^3 - I_1\sigma^2 + I_2\,\sigma - I_3 = 0 \rightarrow$$

$$\sigma_I, \sigma_{II}, \sigma_{III}, \;\; \boldsymbol{\sigma} = \sigma_I\,\mathbf{n}_I \otimes \mathbf{n}_I + \sigma_{II}\,\mathbf{n}_{II} \otimes \mathbf{n}_{II} + \sigma_{III}\,\mathbf{n}_{III} \otimes \mathbf{n}_{III},$$

$$\boldsymbol{\sigma}\,\mathbf{n}_J = \sigma_J\,\mathbf{n}_J \;(!J) \rightarrow \boldsymbol{\sigma}^n\mathbf{n}_J = \sigma_J{}^n\mathbf{n}_J \;(!J), \;\; J = I, II, III$$

where $\mathbf{n}_{\mathrm{I}}$, $\mathbf{n}_{\mathrm{II}}$, $\mathbf{n}_{\mathrm{III}}$ are *principal directions* (eigenvectors) of a tensor $\boldsymbol{\sigma}$.

Note It is adopted naming convention here that $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$.

The following denotation was introduced for principal invariants

$$I_1 \equiv tr(\boldsymbol{\sigma}) = \sigma_{ii} = 3\sigma_m,\; I_2 \equiv \tfrac{1}{2}[(tr\,\boldsymbol{\sigma})^2 - tr(\boldsymbol{\sigma}^2)] = \tfrac{1}{2}[\sigma_{ii}{}^2 - \sigma_{ij}\sigma_{ij}],$$

$$I_3 \equiv \det(\boldsymbol{\sigma}) = \tfrac{1}{6}\,\varepsilon_{ijk}\varepsilon_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr}$$

symbol $\det(\cdot)$ denotes determinant operation, σ_m is average value of principal values, and ε_{ijk} is permutation symbol.

The set of three coefficients $\{I_1, I_2, I_3\}$ appearing in characteristic equation for determination of principal values of second order symmetric tensor are called *principal invariants*.

Decomposition into spherical (isotropic) and deviatoric (anisotropic) parts.

Any second order symmetric tensor can be decomposed into *direct sum* of *spherical part* and *deviatoric part*

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{sph} + \boldsymbol{s}, \quad \boldsymbol{\sigma}^{sph} \equiv \sigma_m \mathbf{1} \ \ (\sigma_m = \tfrac{1}{3}\sigma_{ii}), \quad \boldsymbol{s} \equiv \boldsymbol{\sigma} - \sigma_m \mathbf{1} \ \ (s_{ij} \equiv \sigma_{ij} - \sigma_m),$$

$$tr(\boldsymbol{s}) = 0, \quad \boldsymbol{\sigma}^{sph} \cdot \boldsymbol{s} = 0 \Leftrightarrow \boldsymbol{\sigma}^{sph} \perp \boldsymbol{s}, \quad \| \boldsymbol{\sigma}^{sph} \| \equiv ((\boldsymbol{\sigma}^{sph})^2 \cdot \mathbf{1})^{1/2} = \tfrac{1}{\sqrt{3}} | I_1 | = \sqrt{3} | \sigma_m |$$

where $\boldsymbol{\sigma}^{sph}$ denotes spherical part, $\boldsymbol{s}$ is deviator of the tensor. The dot symbol denotes full (double) contraction of second order tensors $\boldsymbol{a}_a \cdot \boldsymbol{b}_b$ $(\boldsymbol{a}_{ij}\boldsymbol{b}_{ji})$.

Thus, deviatoric decomposition leads to division of the space of second order symmetric tensors into two separate, complementary (orthogonal) subspaces $\mathcal{T}_2^s = \mathcal{P} \oplus \mathcal{D}$.

The *direct sum* decomposition means that sum of any two spherical tensors ($a\mathbf{1}+b\mathbf{1}$ always gives spherical tensor $(a+b)\mathbf{1}$, and that sum of any two deviatoric tensors $\boldsymbol{s}_a+\boldsymbol{s}_b$ always gives deviatoric tensor $\boldsymbol{s}_{ab}$.

Decomposition into spherical (isotropic) and deviatoric (anisotropic) parts.

Note Decomposition of *second order symmetric tensor* into

- *spherical part* $\sigma_m\mathbf{1}$ and *deviatoric part* $\boldsymbol{s}$,

is equivalent with its decomposition into

- *isotropic part* and *anisotropic part.*

The *spherical part* of the tensor $\sigma_m\mathbf{1}$ is *isotropic* in conventional sense, that is, it does not change under application of any proper orthogonal (rotation) tensor $\mathbf{Q}\in\mathcal{R}$, where $\mathcal{R}$ is a *group of all proper orthogonal tensors* ($\mathbf{Q}(\sigma_m\mathbf{1})\mathbf{Q}^{\mathrm{T}}=\sigma_m\mathbf{1}$).
Deviatoric part is *anisotropic*.

Various decompositions of second order symmetric tensors. Stress tensor deviator characteristic equation.

In analogy to characteristic equation for principal values of full stress tensor $\boldsymbol{\sigma}$, there can be formulated characteristic equation for eigenvalues of tensor deviator $\boldsymbol{s}$, whose coefficients make up a set of *principal invariants of tensor deviator* $\{J_1, J_2, J_3\}$ defined as follows

$$s^3 - J_1 s^2 - J_2 s - J_3 = 0 \rightarrow s_I, s_{II}, s_{III} \rightarrow \boldsymbol{s}^3 - J_2 \boldsymbol{s} - J_3 \mathbf{1} = 0,$$

$$J_1 \equiv tr(\boldsymbol{s}) = 0, \quad J_2 \equiv \tfrac{1}{2} tr(\boldsymbol{s}^2), \quad J_3 \equiv \det(\boldsymbol{s}) = \tfrac{1}{3} tr(\boldsymbol{s}^3),$$

$$J_1 = 0, \; J_2 = \tfrac{1}{2} \| \boldsymbol{s} \|^2 \geq 0 \; (\tfrac{1}{2} s_{ij} s_{ji}), \; J_3 = \det(\boldsymbol{s}) \; (\tfrac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} s_{ip} s_{jq} s_{kr})$$

The opposite sign in definition of second invariant of deviator ($J_2 \geq 0$) in comparison to definition of second invariant of full tensor (I_2) assures that it is always non negative. The J_2 invariant gained widespread use due to its physical interpretation of *shear stress intensity* measure.

Note In definition of second invariant of deviator the sign is changed to opposite in comparison to second invariant of the "full" tensor,

$$(s^3 - \cancel{J_1 s^2} - J_2 s - J_3 = 0 \quad \leftrightarrow \quad \sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0)$$

Various decompositions of second order symmetric tensors. Stress tensor deviator characteristic equation.

The characteristic equation for principal values of deviator can be solved upon substitution of $s=(2/3)^{1/2}(2J_2)^{1/2}\cos(\theta)$ to obtain explicit formulas for stress deviator principal values $\{s_I, s_{II}, s_{III}\}$,

$$s_I = \tfrac{2}{3}\sigma_{ef}\cos(\theta_L), \quad s_{II} = \tfrac{2}{3}\sigma_{ef}\cos(\theta_L - 120^0), \quad s_{III} = \tfrac{2}{3}\sigma_{ef}\cos(\theta_L + 120^0).$$

The full stress tensor and its principal values can be expressed as follows,

$$\sigma_I = \sigma_m + s_I, \quad \sigma_{II} = \sigma_m + s_{II}, \quad \sigma_{III} = \sigma_m + s_{III},$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\sigma_m, \sigma_{ef}, \theta_L) = \sigma_m \mathbf{1} + s_I \, \mathbf{n}_I \otimes \mathbf{n}_I + s_{II} \, \mathbf{n}_{II} \otimes \mathbf{n}_{II} + s_{III} \, \mathbf{n}_{III} \otimes \mathbf{n}_{III},$$

$$\|\boldsymbol{\sigma}\|^2 = \|\sigma_m \mathbf{1}\|^2 + \|\mathbf{s}\|^2 = 3\sigma_m^{\ 2} + 2J_2, \quad r \equiv \sqrt{2J_2}, \quad \sigma_{ef} \equiv \sqrt{3J_2} = \sqrt{\tfrac{3}{2} s_{ij} s_{ij}}$$

where σ_{ef} denotes so called *effective stress*,

$$\cos(3\theta_L) \equiv \bar{J}_3, \ \theta_L \in \langle 0, \pi/3 \rangle, \quad \bar{J}_3 \equiv \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{\ 3/2}} = \frac{3\sqrt{6} J_3}{r^{3/2}} \in \langle -1, 1 \rangle$$

θ_L is called *Lode angle*, and $\bar{J}_3$ is called *normalized third invariant of deviator.* The following identities prove to be useful,

$$\cos^3(\theta) - \tfrac{3}{4}\cdot\cos(\theta) - \tfrac{1}{4}\cos(3\theta) = 0, \quad \cos(\theta)\cos(\theta + 120^0)\cos(\theta - 120^0) = \tfrac{1}{4}\cos(3\theta)$$

Relations between invariants of full tensor and invariants of its deviator.

The invariants I_α, J_α, $\alpha=1,2,3$ can be expressed in terms of general components of stress tensor and in terms of its principal values as follows

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}, \quad I_1 = \sigma_I + \sigma_{II} + \sigma_{III}, \quad J_1 = s_I + s_{II} + s_{III} = 0,$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}\sigma_{21} - \sigma_{23}\sigma_{32} - \sigma_{13}\sigma_{31} = \sigma_I\sigma_{II} + \sigma_I\sigma_{III} + \sigma_{II}\sigma_{III},$$

$$J_2 = \tfrac{1}{6}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2] + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 = \tfrac{1}{2} s_{ij} s_{ij}$$

$$J_2 = -(s_I s_{II} + s_{II} s_{III} + s_I s_{III}) = \tfrac{1}{2}(s_I{}^2 + s_{II}{}^2 + s_{III}{}^2) \geq 0, \qquad I_3 = \sigma_I\, \sigma_{II}\, \sigma_{III},$$

$$I_3 = \tfrac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr} = \sigma_{11}\sigma_{22}\sigma_{33} + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{32}\sigma_{21}\sigma_{13} - \sigma_{12}\sigma_{33}\sigma_{21} - \sigma_{13}\sigma_{22}\sigma_{31} - \sigma_{23}\sigma_{11}\sigma_{32}$$

$$J_3 = \tfrac{1}{3} s_{ij} s_{jk} s_{ki} = \tfrac{1}{3}(s_I{}^3 + s_{II}{}^3 + s_{III}{}^3) = s_I s_{II} s_{III} = (\sigma_I - \sigma_m)(\sigma_{II} - \sigma_m)(\sigma_{III} - \sigma_m).$$

The following relations are valid for basic (main), principal and deviator principal invariants $I_{b\alpha}$, I_α, J_α,

$$J_2(\mathbf{s}) = \tfrac{1}{2} tr(\boldsymbol{\sigma}^2) - \tfrac{1}{6}(tr\,\boldsymbol{\sigma})^2 = 3\sigma_m{}^2 - I_2(\boldsymbol{\sigma}) = \tfrac{1}{3} I_1{}^2(\boldsymbol{\sigma}) - I_2(\boldsymbol{\sigma}),$$

$$I_3(\boldsymbol{\sigma}) = \det(\boldsymbol{\sigma}) = \tfrac{1}{3} tr(\boldsymbol{\sigma}^3) - \tfrac{1}{2} tr(\boldsymbol{\sigma}^2) tr(\boldsymbol{\sigma}) + \tfrac{1}{6}(tr\,\boldsymbol{\sigma})^3 = J_3 - J_2 \cdot \sigma_m + \sigma_m{}^3,$$

$$J_3(\mathbf{s}) = \tfrac{1}{3} tr(\boldsymbol{\sigma}^3) - \tfrac{1}{3} tr(\boldsymbol{\sigma}^2) tr(\boldsymbol{\sigma}) + \tfrac{2}{27}(tr\,\boldsymbol{\sigma})^3 = I_3(\boldsymbol{\sigma}) - \tfrac{1}{3} I_2(\boldsymbol{\sigma}) I_1(\boldsymbol{\sigma}) + \tfrac{2}{27} I_1{}^3(\boldsymbol{\sigma}).$$

$$(\boldsymbol{\sigma}^3 - I_1\boldsymbol{\sigma}^2 + I_2\boldsymbol{\sigma} - I_3\mathbf{1} = 0) \;\Rightarrow\; (\mathbf{s}^3 = J_3\mathbf{1} + J_2\,\mathbf{s}) \;\Rightarrow\; (tr(\mathbf{s}^3) = 3J_3).$$

Lecture 6

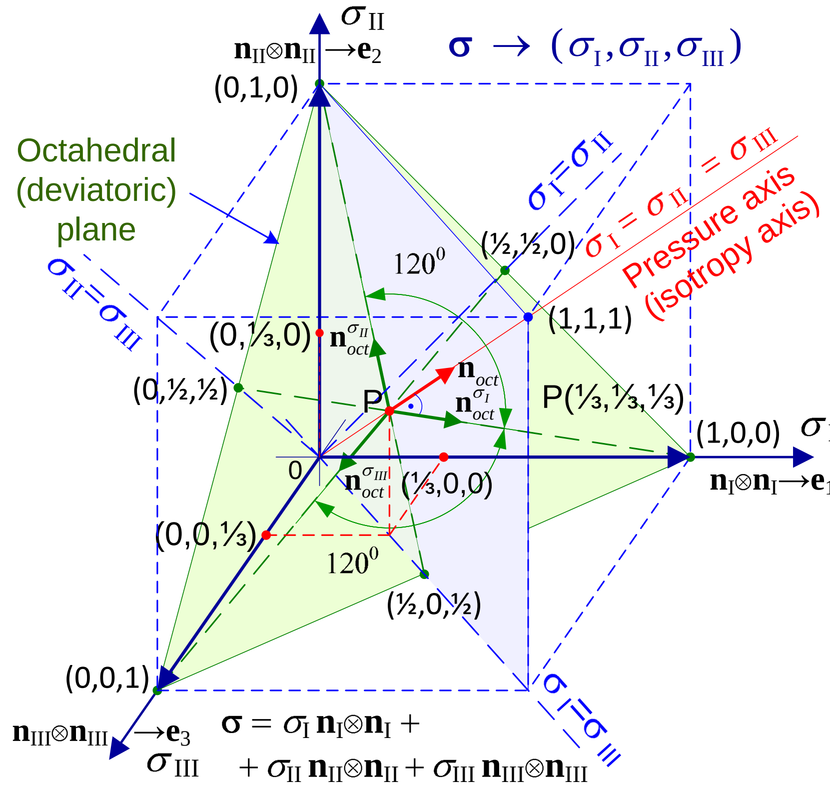
Haigh-Westergaard space.

6.1 Haigh-Westergaard space of stress tensor principal values.
6.2 Critical surfaces for isotropic materials in Haigh-Westergaard space.
6.3 Non-isomorphic and isomorphic coordinates in Haigh-Westergaard space.
6.4 Vectorial decomposition of octahedral traction vector.

Second order symmetric tensors, Haigh-Westergaard Cauchy stress principal values space.

Haigh and independently *Westergaard* in 1920 tried to establish some *criteria of material strength*, i.e., evaluate when the material will start to yield plastically when submitted to multiaxial loading. They intuitively conjectured that for isotropic elastic materials Euler angles of stress tensor, describing orientation of principal axes in laboratory frame, should not influence material strength, and can be neglected. Basing on this conjecture, they proposed to introduce three dimensional *Cauchy stress principal values space* (σ_I, σ_{II}, σ_{III}) with orthogonal coordinate system (axes) composed of *principal directions of stress tensor* ($\mathbf{n}_{\mathrm{I}}$, $\mathbf{n}_{\mathrm{II}}$, $\mathbf{n}_{\mathrm{III}}$). The principal values of stress tensor are Cartesian coordinates of points in this space.

Haigh-Westergaard space of Cauchy stress principal values



Haigh B. P., The strain-energy function and the elastic limit, Engineering (London), Jan. 30, 1920, pp. 158-160.
Westergaard H. M., On the resistance of ductile materials to combined stresses in two or three directions perpendicular to one another, J.F.I., May 1920, pp. 627-640.
see page 14, Maugin G., Thermomechanics of plasticity and Fracture, Cambridge University Press, 1992.

Second order symmetric tensors, Haigh-Westergaard (H-W) space of Cauchy stress principal values.

The *Cauchy stress principal values space* was coined the name *Haigh-Westergaard space*, according to Gerard Maugin.

The H-W space does not possess standard structure of vector (linear) space because summing of two vectors from this space is physically meaningless. Single point in H-W space represents an infinite number of non-coaxial stress tensors having the same principal values $(\sigma_I, \sigma_{II}, \sigma_{III})$ but different principal directions $(\mathbf{n}_{\mathrm{I}}, \mathbf{n}_{\mathrm{II}}, \mathbf{n}_{\mathrm{III}})$.

The H-W space can be interpreted as space of *numerical markers* of *Cauchy stress tensor orbits*. Precise mathematical definition of the concept of tensor orbit is as follows,

Orbit of a tensor is a set $\{\mathbf{T}^Q\}$ of all tensors that can be obtained by rotation of preset tensor $\mathbf{T}$ with any orthogonal tensor $\mathbf{Q} \in \mathcal{R}$,

$$\mathbf{T}^Q = \mathbf{Q} * \mathbf{T} \quad - \quad \text{any rotated tensor } \mathbf{T}, \quad (\mathbf{Q}\mathbf{Q}^T = \mathbf{I},\ \det \mathbf{Q} > 0),$$

$$\{\mathbf{T}^Q\} = \{all\ \mathbf{T}^Q\} \quad - \quad \text{orbit of a tensor } \mathbf{T}.$$

Second order symmetric tensors, critical surfaces for isotropic materials in Haigh-Westergaard space.

The concept of Haigh-Westergaard space proved to be extremely convenient in modeling behavior of isotropic materials.

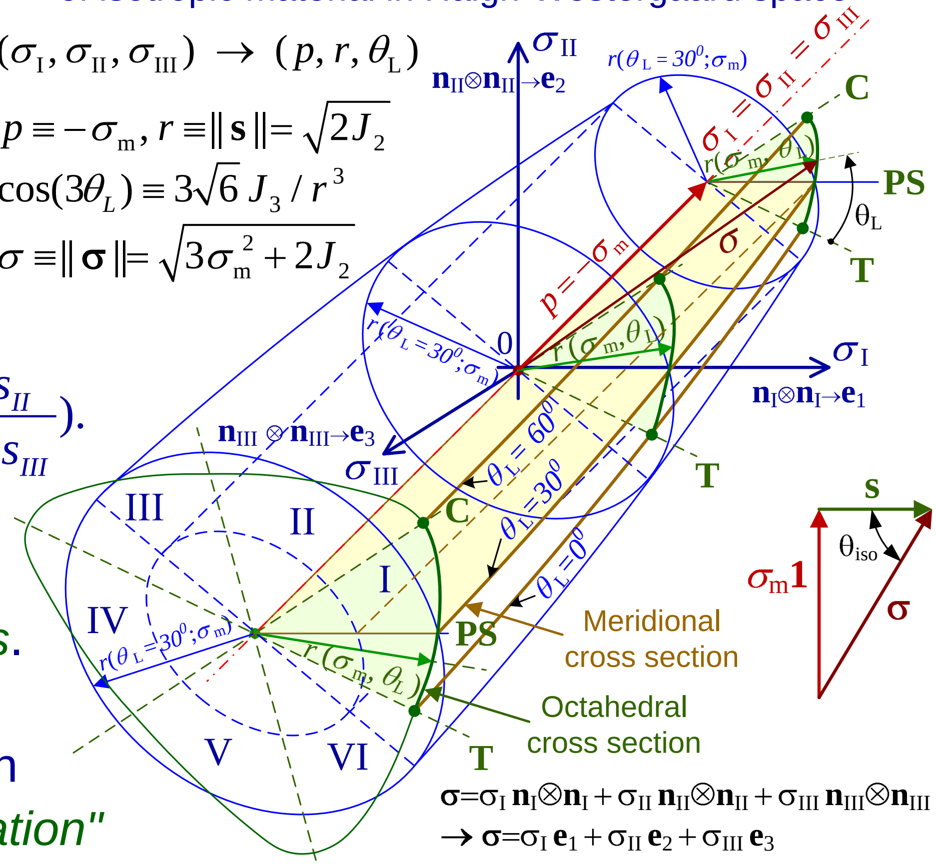
Ilustrative example of general critical surface of isotropic material in Haigh-Westergaard space

$$(\sigma_{\mathrm{I}},\sigma_{\mathrm{II}},\sigma_{\mathrm{III}}) \rightarrow (p,r,\theta_{\mathrm{L}})$$

$$p \equiv -\sigma_{\mathrm{m}},\ r \equiv \|\mathbf{s}\| = \sqrt{2J_2}$$

$$\cos(3\theta_L) \equiv 3\sqrt{6}\, J_3 / r^3$$

$$\sigma \equiv \|\boldsymbol{\sigma}\| = \sqrt{3\sigma_{\mathrm{m}}^2 + 2J_2}$$



$$\boldsymbol{\sigma} = \sigma_{\mathrm{I}}\, \mathbf{n}_{\mathrm{I}}\otimes\mathbf{n}_{\mathrm{I}} + \sigma_{\mathrm{II}}\, \mathbf{n}_{\mathrm{II}}\otimes\mathbf{n}_{\mathrm{II}} + \sigma_{\mathrm{III}}\, \mathbf{n}_{\mathrm{III}}\otimes\mathbf{n}_{\mathrm{III}}$$

$$\rightarrow \boldsymbol{\sigma} = \sigma_{\mathrm{I}}\, \mathbf{e}_1 + \sigma_{\mathrm{II}}\, \mathbf{e}_2 + \sigma_{\mathrm{III}}\, \mathbf{e}_3$$

In illustrative Figure besides standard for H-W space principal stresses (σ_{I}, σ_{II}, σ_{III}) coordinates also alternative cylindrical coordinate system (p, r, θ_{L}) is shown,

$$p = -\sigma_m = -\tfrac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III}),$$

$$r = \|\mathbf{s}\| = \sqrt{s_I^{\,2} + s_{II}^{\,2} + s_{III}^{\,2}},\ \ \theta_L = 30^0 + tg^{-1}\left(\frac{\sqrt{3}s_{II}}{s_I - s_{III}}\right).$$

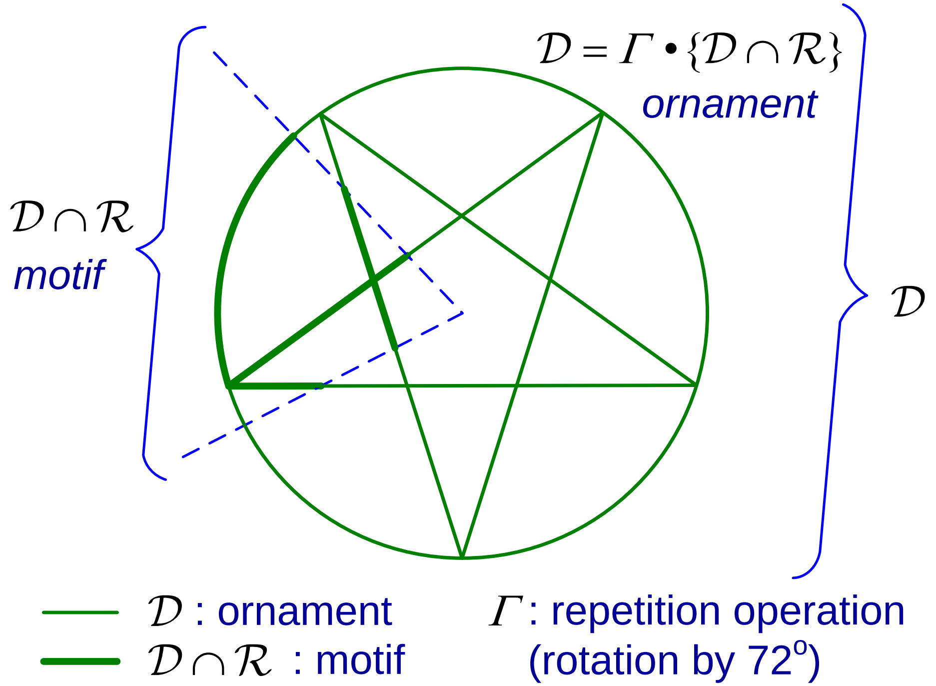
Sometimes in the literature it is raised a problem of ambiguity resulting from different *ordering of stress principal values*. This problem actually disappears when (p, r, θ_L) coordinates are used. Then, it can be immediately identified six-fold *"permutation" symmetry* exhibited by any and all *isotropic critical surfaces*, proving that ordering of principal stresses does not introduce any ambiguity, i.e., it is irrelevant. Alternatively, symmetry of isotropic critical surface in H-W space can be regarded as three-fold *mirror symmetry*.

Second order symmetric tensors, Principle of Ornament.

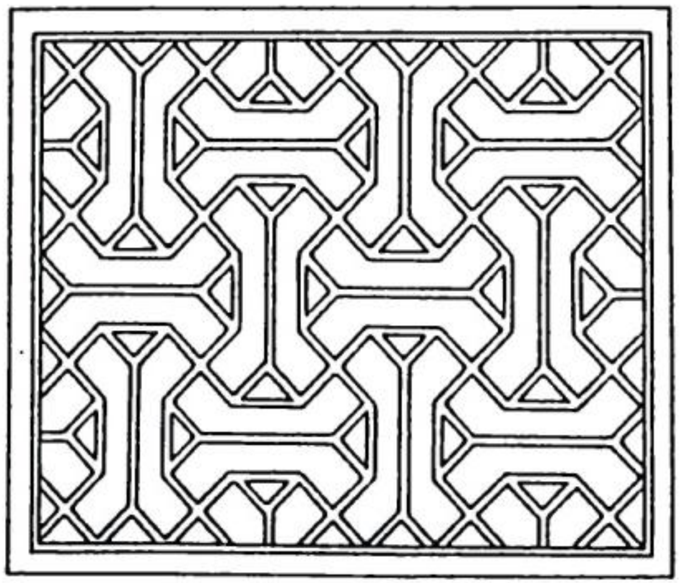
Isotropic critical surfaces in Haigh-Westergaard space make an excellent illustration of the symmetry *Principle of Ornament*, see Rychlewski, 1991.

Principle of Ornament, states that an *ornament* is generated using specific *motif* and specific *repetition operation*.

Principle of Ornament



An example of more complicated ornament of the type quite often encountered in architectural design, recalled after Hermann Weyl's book



Rychlewski J. *Symmetry of Causes and Effects*, PWN, 1991.
Weyl H., Symmetry, Princeton University Press, New Jersey, 1952

Second order symmetric tensors, different coordinate systems in Haigh-Westergaard space.

In Haigh-Westergaard space in place of *stress principal values* ($\sigma_I, \sigma_{II}, \sigma_{III}$) *coordinate system* any set of *three linearly independent stress tensor invariants* ($I^*_I, I^*_{II}, I^*_{III}$) may be adopted as a *system of coordinates*.

A very popular set of this kind are (cylindrical) coordinates − *pressure*, *effective stress*, *Lode angle* (p=-σ_m, σ_{ef}, θ_L),

$$p \equiv -\tfrac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III}),\ \sigma_{ef} \equiv \sqrt{\tfrac{3}{2}(s_I{}^2 + s_{II}{}^2 + s_{III}{}^2)},\ \theta_L \equiv 30^0 + tg^{-1}(\sqrt{3}s_{II} / (s_I - s_{III});$$

$$\sigma_I = \sigma_m + \tfrac{2}{3}\sigma_{ef}\cos(\theta_L),\ \ \sigma_{II} = \sigma_m + \tfrac{2}{3}\sigma_{ef}\cos(\theta_L - 120^0),\ \ \sigma_{III} = \sigma_m + \tfrac{2}{3}\sigma_{ef}\cos(\theta_L - 240^0)$$

They are used to present *plastic flow yield*, *damage*, *failure* or *phase transition start* critical surfaces for different materials in 2D cross sections,

- *octahedral* ($p=const$, σ_{ef}, θ_L) and/or
- *meridional* (p, σ_{ef}, θ_L=const).

In the continuum mechanics and material sciences literature often the coordinates (p, $r=(s_I{}^2+s_{II}{}^2+s_{III}{}^2)^{1/2}$, θ_L) or ($3p$, r, θ_L) are used in H-W space.

Note. The problem with all listed above sets of H-W space coordinates is that they *distort actual 3D shape* of the critical surfaces *in 2D meridional projections*. They are *non-isomorphic* coordinates.

Second order symmetric tensors, isomorphic coordinates in H-W space.

According to the present author literature survey Janusz Murzewski was the first researcher who in 1958 consciously introduced *isomorphic coordinates* in *Haigh-Westergaard space*, i.e., those preserving correct shapes (distances and angles) of critical surfaces in meridional cross sections.

Murzewski's *isomorphic cylindrical coordinates* are as follows,

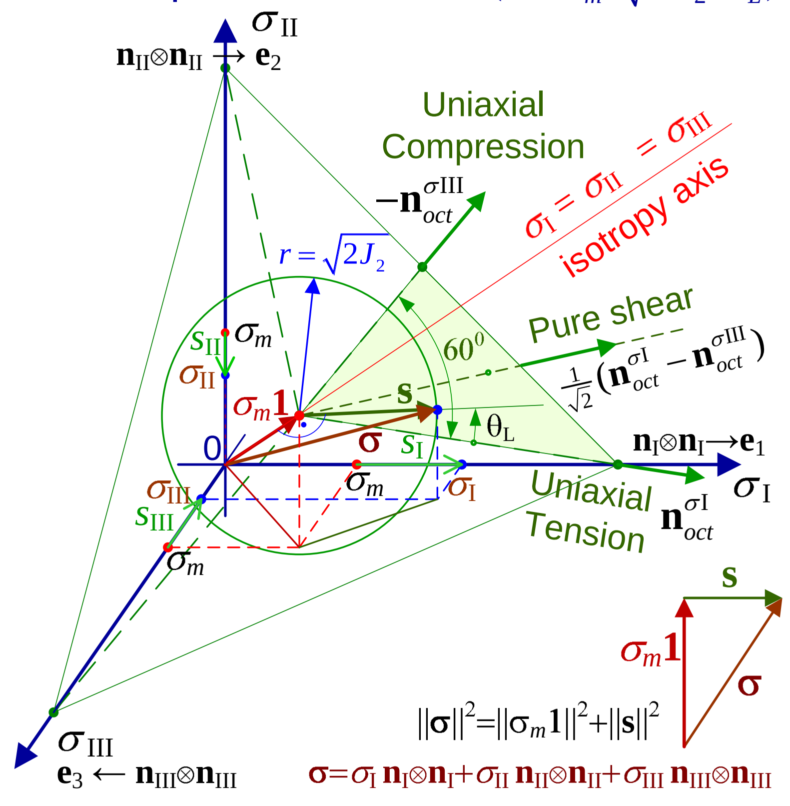
$$(z \equiv \sqrt{3}\,\sigma_m,\ \ r \equiv \|\mathbf{s}\| = \sqrt{2J_2},\ \ \theta_L\ ),$$

$$|\,z\,| \equiv \|\,\sigma^{sph}\,\| = (\boldsymbol{\sigma}\cdot\mathbf{1})^{1/2} = \sqrt{3}\,|\,\sigma_m\,| = \sqrt{3}\,|\,p\,|,$$

$$\theta_L \equiv \tfrac{1}{3}\cos^{-1}(3\sqrt{6}\,J_3\,/\,(2J_2)^{3/2}.$$

It can be noticed that behind the concept of *isomorphic coordinates* stands the (orthogonal) *decomposition* of second order symmetric tensors into (*spherical* + *deviatoric*) ↔ (*isotropic* + *anisotropic*) parts.

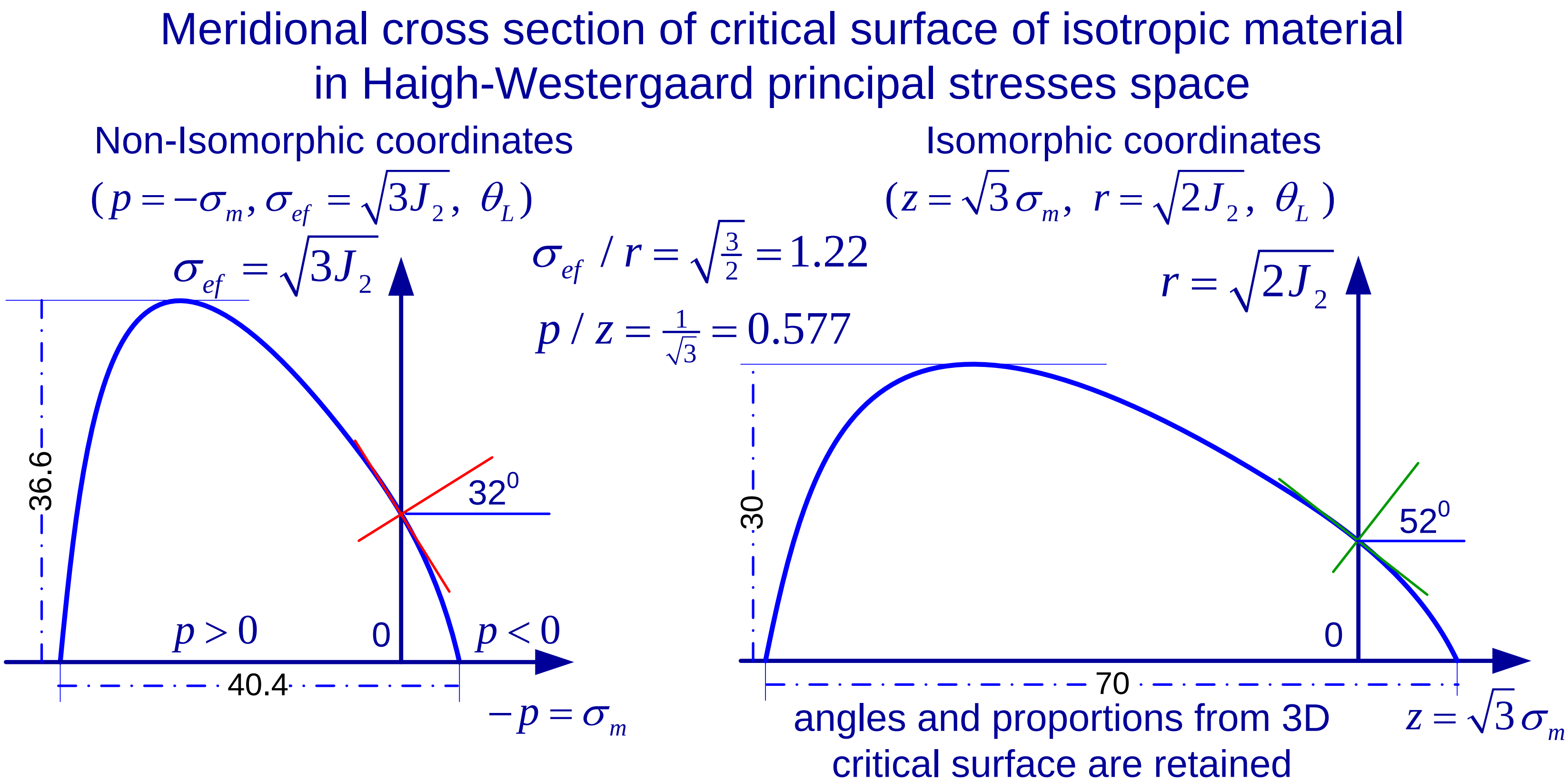
Haigh-Westergaard space
isomorphic coordinates $(\sqrt{3}\,\sigma_m, \sqrt{2J_2}, \theta_L)$



Murzewski J., A probabilistic theory of plastic and brittle behaviour of quasi-homogeneous materials, Archives of Mechanics, 1960 (Received 1st, December, 1958), 2, 12, pp. 203-227.

Second order symmetric tensors, non-isomorphic and isomorphic coordinates in Haigh-Westergaard space.

The size of distortion of 2D meridional projections of 3D critical surface upon using *non-isomorphic coordinates* $(p=-\sigma_m, \sigma_{ef}=\sqrt{3J_2},\ \theta_L)$ in comparison to using *isomorphic coordinates* $(z=\sqrt{3}\sigma_m,\ r=\sqrt{2J_2},\ \theta_L)$ in Haigh-Westergaard (H-W) space.



Second order symmetric tensors, isomorphic coordinates in H-W space.

Stress tensor and its principal values expressed in Murzewski's coordinates take the following form,

$$\sigma_I = \tfrac{1}{\sqrt{3}} z + \sqrt{\tfrac{2}{3}}\, r\cos(\theta_L), \;\; \sigma_{II} = \tfrac{1}{\sqrt{3}} z + \sqrt{\tfrac{2}{3}}\, r\cos(\theta_L - 120^0),$$

$$\sigma_{III} = \tfrac{1}{\sqrt{3}} z + \sqrt{\tfrac{2}{3}}\, r\cos(\theta_L - 240^0),$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(z, r, \theta_L) = z\,\mathbf{N}_{oct} + r\cos\theta_L\,\mathbf{N}_{oct}^{\sigma_I} + r\sin\theta_L\,\mathbf{N}_{oct}^{\sigma_I\perp}, \quad \boldsymbol{\sigma}_{sph} = z\,\mathbf{N}_{oct},$$

$$\mathbf{N}_{oct} = \tfrac{1}{\sqrt{3}}(\mathbf{N}_I + \mathbf{N}_{II} + \mathbf{N}_{III}) = \tfrac{1}{\sqrt{3}}\mathbf{1}, \qquad \|\mathbf{N}_{oct}\| = \|\mathbf{N}_{oct}^{\sigma_I}\| = \|\mathbf{N}_{oct}^{\sigma_I\perp}\| = 1,$$

$$\mathbf{N}_{oct}^{\sigma_I} = \tfrac{1}{\sqrt{6}}(2\mathbf{N}_I - \mathbf{N}_{II} - \mathbf{N}_{III}), \;\; \mathbf{N}_{oct}^{\sigma_I\perp} = \tfrac{1}{\sqrt{2}}(\mathbf{N}_{II} - \mathbf{N}_{III}); \quad \mathbf{N}_J \equiv \mathbf{n}_J \otimes \mathbf{n}_J.$$

An alternative to ($\sqrt{3}\sigma_m$, $r=\sqrt{(2J_2)}$, θ_L) isomorphic coordinates for meridional cross sections in Haigh-Westergaard space is pair of variables ($\sigma_{oct}=\sigma_m$, $\tau_{oct}=r/\sqrt{3}$). These last coordinates, with denotation ($\omega_1 \leftrightarrow \sigma_{oct}$, $\omega_2 \leftrightarrow \tau_{oct}$), were already used in 1929 by Burzyński in proposed by him extended plastic yield strength criterion for linearly elastic, isotropic solids.

Second order symmetric tensors, vectorial decomposition of octahedral traction vector.

Tensorial decomposition of tensor $\boldsymbol{\sigma}$ into *spherical (isotropic)* $\sigma_m \mathbf{1}$ and *deviatoric (anisotropic)* $\mathbf{s}$ parts,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{sph} + \mathbf{s} = \boldsymbol{\sigma}(z,r,\theta_L) = z\,\mathbf{N}_{oct} + r\cos\theta_L\,\mathbf{N}_{oct}^{\sigma_I} + r\sin\theta_L\,\mathbf{N}_{oct}^{\sigma_I\perp}, \quad \|\boldsymbol{\sigma}\| = [3\sigma_m{}^2 + r^2]^{1/2},$$

$$\boldsymbol{\sigma}^{sph} = \sigma_m \mathbf{1} = z\,\mathbf{N}_{oct}, \quad \mathbf{N}_{oct} = \tfrac{1}{\sqrt{3}}(\mathbf{N}_I + \mathbf{N}_{II} + \mathbf{N}_{III}) = \tfrac{1}{\sqrt{3}}\mathbf{1}, \quad \mathbf{N}_{oct}^{\sigma_I} = \tfrac{1}{\sqrt{6}}(2\mathbf{N}_I - \mathbf{N}_{II} - \mathbf{N}_{III}),$$

$$\mathbf{N}_{oct}^{\sigma_I\perp} = \tfrac{1}{\sqrt{2}}(\mathbf{N}_{II} - \mathbf{N}_{III}), \qquad \|\mathbf{N}_{oct}\| = \|\mathbf{N}_{oct}^{\sigma_I}\| = \|\mathbf{N}_{oct}^{\sigma_I\perp}\| = 1, \quad \mathbf{N}_J \equiv \mathbf{n}_J \otimes \mathbf{n}_J,$$

should be carefully distinguished from *vectorial decomposition* of *octahedral traction* (stress vector) $\boldsymbol{t}_{oct}$ into *octahedral normal traction* $\sigma_{oct}\mathbf{n}_{oct}$ and *octahedral shear traction* $\tau_{oct}\mathbf{n}_s$,

$$\boldsymbol{t}_{oct} = t_{oct}\,\mathbf{n}_t = \sigma_{oct}\,\mathbf{n}_{oct} + \tau_{oct}\,\mathbf{n}_s, \quad \mathbf{n}_t = \boldsymbol{t}_{oct}/\|\boldsymbol{t}_{oct}\|, \quad \mathbf{n}_s = \mathbf{s}/\|\mathbf{s}\|,$$

$$\boldsymbol{t}_{oct} \equiv \boldsymbol{\sigma}\,\mathbf{n}_{oct} = \tfrac{1}{\sqrt{3}}(\sigma_I\mathbf{n}_I + \sigma_{II}\mathbf{n}_{II} + \sigma_{III}\mathbf{n}_{III}), \quad \mathbf{n}_{oct} = \tfrac{1}{\sqrt{3}}(\mathbf{n}_I + \mathbf{n}_{II} + \mathbf{n}_{III}),$$

$$\boldsymbol{\sigma} = \sigma_I\mathbf{n}_I \otimes \mathbf{n}_I + \sigma_{II}\mathbf{n}_{II} \otimes \mathbf{n}_{II} + \sigma_{III}\mathbf{n}_{III} \otimes \mathbf{n}_{III}.$$

The moduli t_{oct}, σ_{oct} and τ_{oct} can be determined from the following formulas,

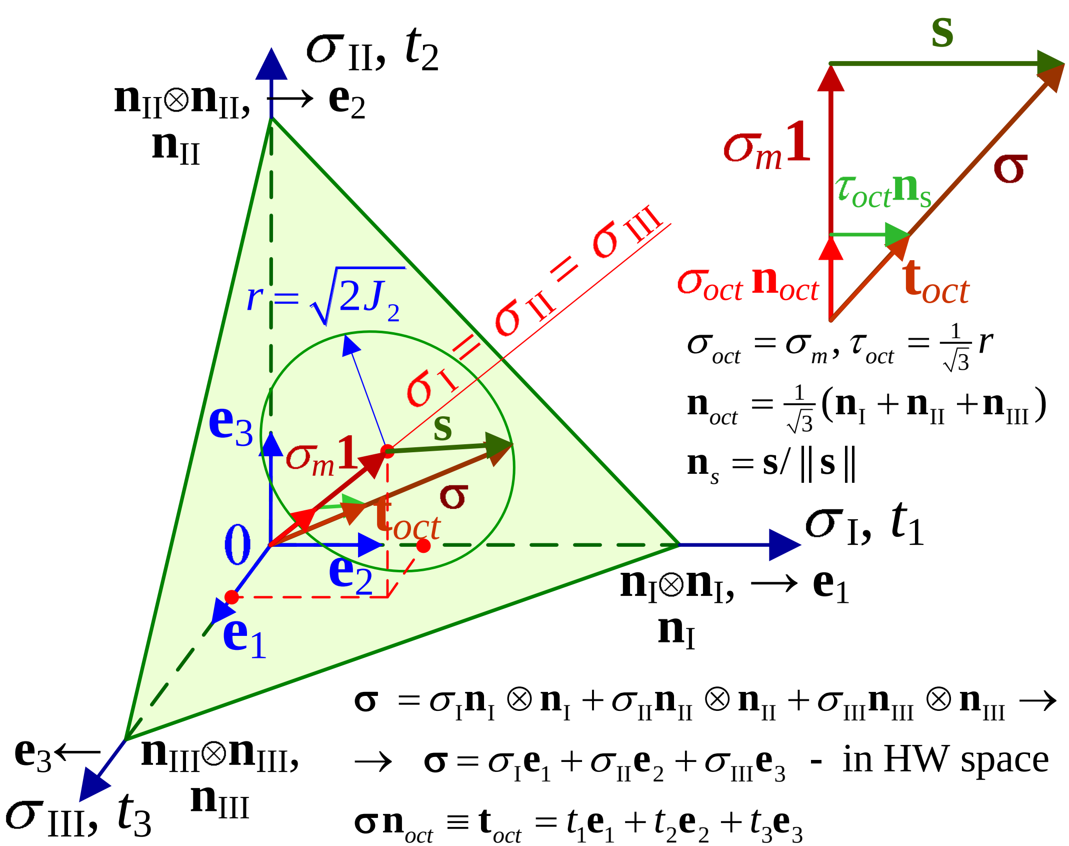
$$t_{oct} = \|\mathbf{t}_{oct}\| = [\sigma_{oct}{}^2 + \tau_{oct}{}^2]^{1/2} = \tfrac{1}{\sqrt{3}}\|\boldsymbol{\sigma}\|,$$

$$\sigma_{oct} = \boldsymbol{t}_{oct}\cdot\mathbf{n}_{oct} = \sigma_m, \quad \tau_{oct} = [\|\boldsymbol{t}_{oct}\|^2 - \sigma_{oct}{}^2]^{1/2} = \sqrt{2J_2/3} = r/\sqrt{3}.$$

Second order symmetric tensors, vectorial decomposition of octahedral traction vector.

Graphical illustration of *tensorial decomposition* of *stress tensor* $\boldsymbol{\sigma}$ into *spherical (isotropic)* part $\sigma_m\mathbf{1}$ and *deviatoric (anisotropic)* part $r\cdot\mathbf{s}/\|\mathbf{s}\|$, versus *vectorial decomposition* of *octahedral traction* $\boldsymbol{t}_{oct}$ into *octahedral normal traction* $(1/\sqrt{3})\sigma_{oct}\mathbf{1}$ and *octahedral shear traction* $\tau_{oct}\cdot\mathbf{s}/\|\mathbf{s}\|$.

Tensorial decomposition of stress tensor
and vectorial decomposition traction vector



$$\sigma_{oct} = \sigma_m,\ \tau_{oct} = \tfrac{1}{\sqrt{3}} r$$

$$\mathbf{n}_{oct} = \tfrac{1}{\sqrt{3}}(\mathbf{n}_{\mathrm{I}} + \mathbf{n}_{\mathrm{II}} + \mathbf{n}_{\mathrm{III}})$$

$$\mathbf{n}_s = \mathbf{s}/\|\mathbf{s}\|$$

$$\boldsymbol{\sigma} = \sigma_{\mathrm{I}}\mathbf{n}_{\mathrm{I}}\otimes\mathbf{n}_{\mathrm{I}} + \sigma_{\mathrm{II}}\mathbf{n}_{\mathrm{II}}\otimes\mathbf{n}_{\mathrm{II}} + \sigma_{\mathrm{III}}\mathbf{n}_{\mathrm{III}}\otimes\mathbf{n}_{\mathrm{III}} \rightarrow$$

$$\rightarrow \quad \boldsymbol{\sigma} = \sigma_{\mathrm{I}}\mathbf{e}_1 + \sigma_{\mathrm{II}}\mathbf{e}_2 + \sigma_{\mathrm{III}}\mathbf{e}_3 \ \text{ - in HW space}$$

$$\boldsymbol{\sigma}\mathbf{n}_{oct} \equiv \mathbf{t}_{oct} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3$$

Lecture 7 – synopsis

Pure shears as convenient comparison reference states.

7.1 Lode angle expressed in terms of principal values and principal invariants.

7.2 Lode angle lack of lucid physical interpretation.

7.3 Some properties of pure shears and their physical interpretations.

Second order symmetric tensors, Lode angle expressed in terms of principal values and principal invariants.

Originally Lode expressed *shear mode angle (today called Lode angle) in terms of stress tensor principal values*,

$$\mu_L \equiv \frac{2\sigma_{II} - \sigma_I - \sigma_{III}}{\sigma_I - \sigma_{III}} = \sqrt{3}\,tg(\theta_L - 30^0)$$

This formula is very *inconvenient* (inefficient) from numerical standpoint because *first there must be calculated* principal values of stress tensor (with the aid of principal invariants) and only later the value of Lode angle can be computed. Upon some reflection it is clear that *Lode angle can be calculated directly from stress tensor principal invariants* (without computing principal stresses). This gives large savings in computational effort.

The present author, upon historical survey of the literature, found that the first researcher who explicitly expressed Lode angle in terms of second and third stress deviator invariants was Valentin Novozhilov, in a paper from 1951. Actually, he published relevant formulas for *mode angle* ζ defined by him with sinus function, whereas Lode angle θ_L is defined with cosine function.

Given by him relation is as follows

$$\sin(\zeta) \equiv -\bar{J}_3, \quad \bar{J}_3 = 3\sqrt{6}\,J_3 / (2J_2)^{3/2} \in <-1,1>; \quad (\cos(\theta_L) \equiv \bar{J}_3)$$

Novozhilov V.V., О связи между напряжениями и деформациями в нелинейно упругой среде (in Russian) On relations between stresses and strains in non-linear elastic media, Prikl. Mat. Mekh. XV, 1951, pp. 183-194.

Second order symmetric tensors,
Lode angle lack of lucid physical interpretation.

In this place one can get impression that everything what could be done regarding parametrization of Cauchy stress tensor has already been done. However, more careful analysis proves that it is not the case.

The $\sigma_m = -p = \frac{1}{3} I_1$ and $r = \| \mathbf{s} \| = \sqrt{2J_2}$ coordinates *have clear physical interpretations* of pressure (with negative sign), and modulus (norm) of deviatoric (shear) part of stress tensor. However, *Lode angle* θ_{L} coordinate *does not have clear physical interpretation*.

From *mathematical standpoint* Lode angle describes angle between projection of specific stress tensor σ and projection of corresponding (having the same modulus) uniaxial tension tensor, on octahedral plane.

In a search for better *parameter* describing *mode of shear stress*, possessing lucid and meaningful *physical interpretation*, we will turn our attention to the selection of adequate *comparison reference states*.

For example, *in elasticity* such *physically meaningful* comparison reference state makes *unloaded, undeformed configuration/state* of elastic body, i.e., state of (zero stress, zero strain).

Second order symmetric tensors, pure shears their properties and physical interpretations.

In search of physically meaningful, rational, *comparison reference state* to characterize shear stress mode used in parametrization of Cauchy stress, let us give some thought to a problem of *what is the most elementary (atom) non-trivial form of second order tensor?*

The first thought coming to ones mind is a tensor, which has *only single nonzero diagonal entry* in its matrix representation, e.g., $\mathrm{diag}(a,0,0)$. Such representation has for example the *uniaxial tension (extension)* and/or *uniaxial compression* tensors.

The option with single nonzero off-diagonal component is excluded due to symmetry requirement.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ?$$

Upon further reflection, it can be realized that *uniaxial tension* tensor is *not as simple as it seems*, and in fact several *elemental (atom) components* can be distilled from it along the lines of deviatoric decomposition of the second order symmetric tensors,

$$\boldsymbol{\sigma} = \sigma_m \mathbf{1} + \mathbf{s}, \qquad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix}$$

Second order symmetric tensors, pure shears their properties and physical interpretations.

The most elementary component of uniaxial tensor that can actually be identified as irreducible to more simple modes, is the spherical tensor, *spherical elementary mode*, having three identical in value diagonal components $\mathrm{diag}(\tfrac{1}{3}a, \tfrac{1}{3}a, \tfrac{1}{3}a)$.

The *spherical elementary mode* can be *physically interpreted* as describing the simplest *3D layout of action of forces* in physical space, i.e., forces operating uniformly in all physical directions, what corresponds to *action of pressure* only.
Alternatively it can be *physically interpreted* as describing *3D kinematics* of displacements *taking place uniformly* in physical space, corresponding to *change of volume* only.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}a & 0 & 0 \\ 0 & \frac{1}{3}a & 0 \\ 0 & 0 & \frac{1}{3}a \end{bmatrix} + \begin{bmatrix} \frac{2}{3}a & 0 & 0 \\ 0 & -\frac{1}{3}a & 0 \\ 0 & 0 & -\frac{1}{3}a \end{bmatrix}$$

Second order symmetric tensors, pure shears their properties and physical interpretations.

Any deviator of second order symmetric tensor proves to be always decomposable into *two pure shear modes*, in general in *infinitely many ways*.

Thus, *pure shears* prove to be *elementary (irreducible) deviatoric modes,* generators of space of deviators. For example, uniaxial tension tensor can be decomposed as follows with two pure shear tensors representing its deviatoric part,

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}a & 0 & 0 \\ 0 & \frac{1}{3}a & 0 \\ 0 & 0 & \frac{1}{3}a \end{bmatrix} + \begin{bmatrix} \frac{1}{3}a & 0 & 0 \\ 0 & -\frac{1}{3}a & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{3}a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3}a \end{bmatrix}$$

The *pure shear - elementary (atomic) deviatoric mode*, can be *physically interpreted* as the most simple 2D (plane) *layout of forces* generating homogeneous constant shear stress in planes having fixed common normal axis.

Alternatively, it can be physically interpreted as 2D (*plane) kinematics of displacements* of planes having fixed common normal axis with deformation gradient constant in direction of this normal axis, a deformation analogous to this taking place in the case of sliding deck of cards.

Blinowski A., Rychlewski J. Pure shears in the mechanics of materials, Math.Mech.Solids, 3,4, 1998, pp. 471-503.

Second order symmetric tensors,
pure shears their properties and physical interpretations.

Let us recall some more information about pure shear mode. In particular, precise *mathematical definition of pure shear mode*.

Definition 7.1 A second order tensor is called a *pure shear* when the following conditions are fulfilled, after Blinowski and Rychlewski, 1998,

$$I_1 = tr(\boldsymbol{\tau}) = 0, \quad I_3 = \det(\boldsymbol{\tau}) = 0 \quad \Rightarrow \quad J_3 = \tfrac{1}{3} tr(\boldsymbol{\tau}^3) = 0$$

Two orthonormal tensorial bases $\{\mathbf{e}_i\}$ and $\{\mathbf{n}_K\}$ are particularly convenient because pure shear tensor representations are in these bases especially simple (very characteristic),

$$t(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) = \boldsymbol{\tau} = t(\mathbf{n}_I \otimes \mathbf{n}_I - \mathbf{n}_{II} \otimes \mathbf{n}_{II}),$$

$$\mathbf{n}_I = \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2), \mathbf{n}_{II} = \tfrac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2), \mathbf{n}_{III} = \mathbf{e}_3,$$

$$\boldsymbol{\tau}\mathbf{e}_1 = t\,\mathbf{e}_2, \ \boldsymbol{\tau}\mathbf{e}_2 = t\,\mathbf{e}_1, \quad \boldsymbol{\tau}\mathbf{n}_I = t\,\mathbf{n}_I, \ \boldsymbol{\tau}\mathbf{n}_{II} = -t\,\mathbf{n}_{II},$$

A straight line along versor $\boldsymbol{e}_3$ ($\boldsymbol{n}_{\mathrm{III}}$) is called *shear axis* and a plane determined by versors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ or ($\boldsymbol{n}_{\mathrm{I}}$, $\boldsymbol{n}_{\mathrm{II}}$) is called the *plane of shear direction*. It is clear that pure shears are planar tensors.

Representation $\boldsymbol{\tau} = t(\mathbf{n}_I \otimes \mathbf{n}_I - \mathbf{n}_{II} \otimes \mathbf{n}_{II})$ actually stands for spectral decom-position of tensor $\boldsymbol{\tau}$ where versors $\mathbf{n}_K$ are principal directions.

Second order symmetric tensors, pure shears their properties and physical interpretations.

Graphical illustration of the same pure shear tensor $\boldsymbol{\tau}$ shown in two coordinate systems $\{\mathbf{e}_i\}$ and $\{\mathbf{n}_K\}$ rotated by 45 degrees, which result in two very characteristic for pure shear tensorial representations.

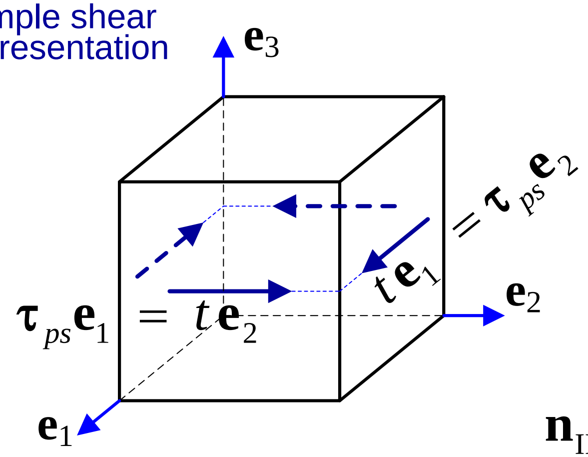
Pure Shear tensor

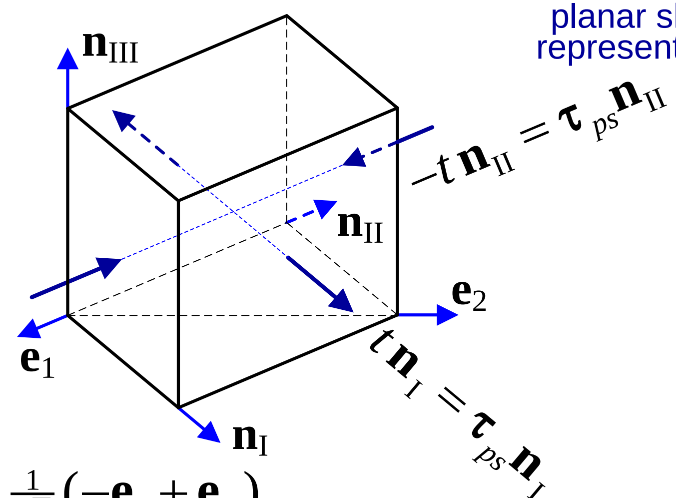
$$J_1 = tr(\boldsymbol{\tau}_{ps}) = 0, \quad J_3 = \det(\boldsymbol{\tau}_{ps}) = \tfrac{1}{3} tr(\boldsymbol{\tau}_{ps}{}^3) = 0$$

(a) (b)

$$\sim \begin{bmatrix} 0 & t & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad t(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) = \boldsymbol{\tau}_{ps} = t(\mathbf{n}_I \otimes \mathbf{n}_I - \mathbf{n}_{II} \otimes \mathbf{n}_{II}) \quad \sim \begin{bmatrix} t & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

simple shear representation | planar shear representation





$$\mathbf{n}_{III} = \mathbf{e}_3$$

$$\mathbf{n}_I = \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2), \quad \mathbf{n}_{II} = \tfrac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2)$$

$$(\lambda - \sqrt{J_2})(\lambda + \sqrt{J_2})\lambda = 0 \iff s_I = \sqrt{J_2},\; s_{II} = -\sqrt{J_2},\; s_{III} = 0; \quad J_2 \equiv \tfrac{1}{2} tr(\boldsymbol{\tau}_{ps}{}^2)$$

Blinowski A., Rychlewski J. Pure shears in the mechanics of materials, Math.Mech.Solids, 3,4, 1998, pp. 471-503.

Second order symmetric tensors, pure shears their properties and physical interpretations.

In accordance with the above nomenclature there can be distinguished two very useful classes of pure shears, namely the ones with *common shear direction* and the ones with *common shear axis*. Two parameter family of pure shears with common shear direction $\boldsymbol{\tau}^{(d)}$ and two parameter family of pure shears with common shear axis $\boldsymbol{\tau}^{(a)}$ can be expressed in the following mathematical form,

$$\boldsymbol{\tau}^{(d)} = (\boldsymbol{s}_{d1} + \boldsymbol{s}_{d2}) \otimes \boldsymbol{n}^{(d)} + \boldsymbol{n}^{(d)} \otimes (\boldsymbol{s}_{d1} + \boldsymbol{s}_{d2}), \quad \boldsymbol{\tau}^{(a)} = (\boldsymbol{s}_{a1} + \boldsymbol{s}_{a2}) \otimes (\boldsymbol{s}_{a1} - \boldsymbol{s}_{a2}) + (\boldsymbol{s}_{a1} - \boldsymbol{s}_{a2}) \otimes (\boldsymbol{s}_{a1} + \boldsymbol{s}_{a2}),$$

$$\boldsymbol{\tau}^{(d)} \sim \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_j, \quad \boldsymbol{n}^{(d)} = \mathbf{e}_3, \qquad \boldsymbol{\tau}^{(a)} \sim \begin{bmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_j, \quad a = r^2 - t^2, b = 2rt,$$

$$\mathbf{s}_{d1} \sim [a,0,0], \mathbf{s}_{d2} \sim [0,b,0], \quad \mathbf{s}_{d1} \cdot \mathbf{s}_{d2} = 0, \quad \mathbf{s}_{a1} \sim \tfrac{1}{\sqrt{2}}[r,t,0], \mathbf{s}_{a2} \sim \tfrac{1}{\sqrt{2}}[-t,r,0], \quad \mathbf{s}_{a1} \cdot \mathbf{s}_{a2} = 0,$$

$$\boldsymbol{n}^{(d)} = \mathbf{e}_3, \sim [0,0,1], \qquad \boldsymbol{n}^{(a)} = \mathbf{e}_3.$$

All possible pure shears having *common shear direction* $\boldsymbol{n}^{(d)}$ parallel to axis $\boldsymbol{e}_3$ can be generated with arbitrarily selected mutually orthogonal vectors $\boldsymbol{s}_{d1}$, $\boldsymbol{s}_{d2}$ orthogonal to direction $\boldsymbol{n}^{(d)}$. All pure shears with *common shear axis* $\boldsymbol{n}^{(a)}$ parallel to axis $\boldsymbol{e}_3$ can be generated with arbitrarily selected mutually orthogonal vectors $\boldsymbol{s}_{a1}$, $\boldsymbol{s}_{a2}$ both orthogonal to shear axis $\boldsymbol{n}^{(a)}$.

Second order symmetric tensors, pure shears their properties and physical interpretations.

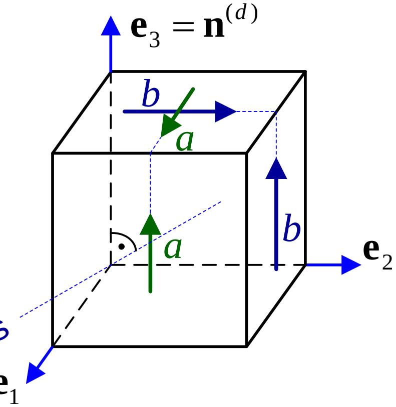
Pure shears with

a) common shear direction $\mathbf{n}^{(d)} = \mathbf{e}_3$

$$\boldsymbol{\tau} = (\mathbf{s}_{d1} + \mathbf{s}_{d2}) \otimes \mathbf{n}^{(d)} + \mathbf{n}^{(d)} \otimes (\mathbf{s}_{d1} + \mathbf{s}_{d2})$$

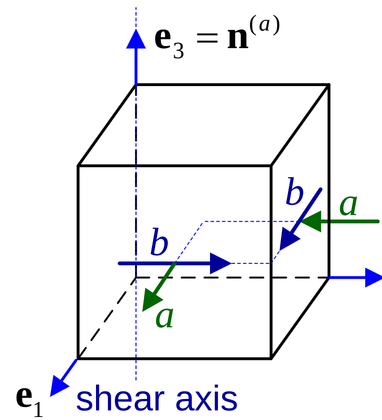
$$\mathbf{s}_{d1} \sim [a,0,0],\ \mathbf{s}_{d2} \sim [0,b,0],\ \mathbf{s}_{d1} \cdot \mathbf{s}_{d2} = 0$$

$$\boldsymbol{\tau} = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_j = \boldsymbol{\tau}_1^d + \boldsymbol{\tau}_2^d$$



b) common shear axis $\mathbf{n}^{(a)} = \mathbf{e}_3$

$$\boldsymbol{\tau} = (\mathbf{s}_{a1} + \mathbf{s}_{a2}) \otimes (\mathbf{s}_{a1} - \mathbf{s}_{a2}) + (\mathbf{s}_{a1} - \mathbf{s}_{a2}) \otimes (\mathbf{s}_{a1} + \mathbf{s}_{a2})$$



$$\mathbf{s}_{a1} \sim \tfrac{1}{\sqrt{2}}[r,t,0],\quad \mathbf{s}_{a2} \sim \tfrac{1}{\sqrt{2}}[-t,r,0],\quad \mathbf{s}_{a1} \cdot \mathbf{s}_{a2} = 0$$

$$a = r^2 - t^2,\ b = 2rt$$

$$\boldsymbol{\tau} = \begin{bmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_j = \boldsymbol{\tau}_1^a + \boldsymbol{\tau}_2^a$$

Experimental evidence proves the *pure shears*, interpreted as tensors modeling deformation, to be *excellent idealizations* of many commonly encountered in materials science physical situations. For example,

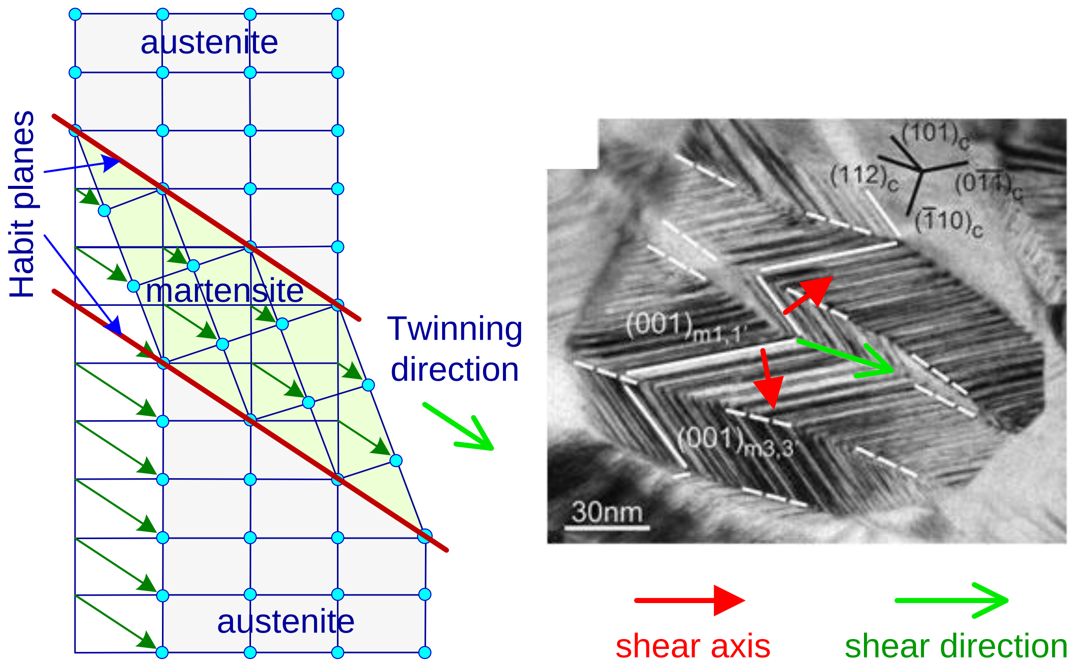
- *compound martensitic twin* deformation can be very well understood in modeling terms, *as a population of pure shears with common shear direction*.
- *plastic slip* deformation can be very well understood in modeling terms as as *a population of pure shears with common shear axis*.

Second order symmetric tensors, pure shears their properties and physical interpretations.

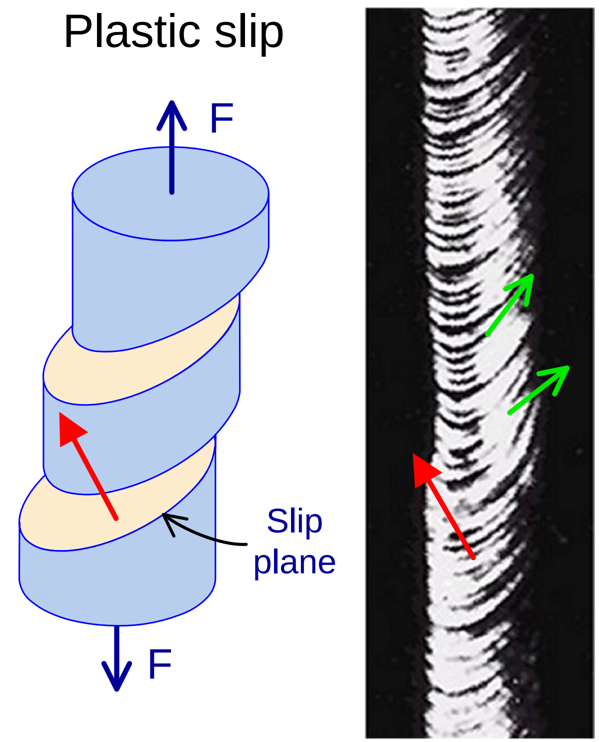
Compound martensitic twin formation can be understood as *a group of pure shears with common shear direction*.

Plastic slip deformation can be understood in modeling terms as *a group of pure shears with common shear axis.*

a) Martensitic twinning



b) Plastic slip



a) Schematic drawing of martensitic twinning composed of pure shears with common shear direction; photo of martensite twinning after P.Sittner, O.Molnarova, X.Bian, L.Heller &H. Seiner, Tensile deformation of B19′ martensite in nanocrystalline NiTi wires, Shape Memory and Superelasticity, 9, p.11–34 (2023), b) Schematic drawing of plastic slips composed of pure shears with common shear axis, photo of slip in a zinc single crystal after C. F. Elam, The Distortion of Metal Crystals, Oxford University Press, London, 1935. See also W.D. Callister, Materials Science and Engineering, Wiley, 1996.

Second order symmetric tensors, pure shears their properties and physical interpretations.

Experimental setups leading to pure shear stress or strain are very frequently used in experimental mechanics to determine, e.g., material properties.

It has been demonstrated by Blinowski and Rychlewski that population of all pure shears generates complete subspace of all deviators. This is so because any deviator can be decomposed into a sum of two pure shears, in particular orthogonal ones.

All pure shears *do not make a structure of linear space* because sum of two pure shears is not always a pure shear.

Pure shears can be regarded as elementary (atom) building blocks of space of deviators.

All pure shears have the same "shape" in this sense that any and all pure shears can be obtained from single fixed preselected pure shear by rotating it with all possible orthogonal tensors $\boldsymbol{Q}$ ($\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}=\mathbf{1}$) .

It is worth to recall that *sum of whatever number of pure shears will never result in spherical tensor*.

Lecture 8 – synopsis

The concept of isotropy angle θ_{iso}, the concept of shear stress mode (skewness) angle θ_{sk} and its statistical-physical interpretation.

8.1 New structural parametrization of Cauchy stress with isotropy angle θ_{iso} and skewness angle θ_{sk}.

8.2 Rychlewski's index of anisotropy based on normalized maximum diameter of a tensor.

8.3 Statistical interpretation of stress tensor deviator invariants, shear stress mode angle θ_{sk} as a measure of entropic part of anisotropy of Cauchy stress tensor

Second order symmetric tensors, new structural parametrization of Cauchy stress.

Let us introduce a *new set* of invariant parameters characterizing second order symmetric tensors. The set seems to be especially convenient for mechanical studies because it leads to simplification of many formulas expressing tensor properties and facilitates their physical interpretation.

The new structural parametrization uses newly introduced concepts of

i) *isotropy angle* θ_{iso} ,

ii) *shear stress mode (skewness) angle* θ_{sk} .

The new generic structural parameterization is motivated by

a) *isotropic decomposition* of stress tensor

b) *pure shear states*, identified to be *atomic elements* of space of deviators, very convenient to be adopted as *comparison reference states* to characterize *shear stress mode*.

The new parametrization conveniently describes and transparently reveals a kind of internal structure of the second order symmetric tensors, when interpreted as modeling objects of mechanical phenomena.

New structural parametrization of Cauchy stress with isotropy angle θ_{iso} and skewness angle θ_{sk}.

Definition 8.1 Let us introduce the following definition of the *isotropy angle* θ_{iso},

$$\sin(\theta_{iso}) \equiv \frac{\sqrt{3}\sigma_m}{\|\boldsymbol{\sigma}\|} = \frac{z}{\|\boldsymbol{\sigma}\|} = sign(\sigma_m)\frac{\|\boldsymbol{\sigma}^{sph}\|}{\|\boldsymbol{\sigma}\|} \in <-1,1>,$$

$$\cos(\theta_{iso}) \equiv \frac{\|\boldsymbol{s}\|}{\|\boldsymbol{\sigma}\|} = \frac{\sqrt{2J_2}}{\|\boldsymbol{\sigma}\|} = \frac{r}{\|\boldsymbol{\sigma}\|} \in < 0,1>,$$

$$\theta_{iso} \in <-90^0, 90^0>, \quad \boldsymbol{\sigma}^{sph} = \sigma_m \mathbf{1}, \quad \|\boldsymbol{\sigma}\| = \sqrt{3\sigma_m{}^2 + 2J_2}.$$

The isotropy angle enables extraction of spherical (isotropic) part and deviatoric (anisotropic) part of the tensor in very straightforward and convenient manner.

The sine and cosine functions of isotropy angle can also be treated as convenient normalized indexes (factors) describing magnitude of spherical and/or deviatoric parts relative to overall magnitude (modulus) of the second order symmetric tensor.

New structural parametrization of Cauchy stress with isotropy angle θ_{iso} and skewness angle θ_{sk}.

Definition 8.2 Let us introduce the following definition of the *shear stress mode angle (skewness angle)* θ_{sk}

$$\sin(3\theta_{sk}) \equiv \frac{1}{2}\cdot\frac{3\sqrt{3}\,J_3}{J_2^{\,3/2}} = \sqrt{2}\cdot\frac{3\sqrt{3}\,J_3}{r^3} = \frac{27}{2}\cdot\frac{J_3}{\sigma_{ef}^{\,3}} = \bar{J}_3 \in <-1,1>,$$

$$\cos(3\theta_L) = \sin(3\theta_{sk}), \quad \theta_L = 30^0 - \theta_{sk}, \quad \theta_{sk} \in <-30^0, 30^0>.$$

The skewness angle in mathematical terms describes departure of the given tensor deviator from corresponding *pure shear* tensor (*comparison reference state*), i.e., deviator having the same modulus as the given tensor deviator, but which third invariant is equal to zero ($J_3=0$).

The *Skewness angle* is *linearly* connected with *Lode angle*. The following connections exists between so called Lode parameter μ_L and skewness angle

$$\mu_L \equiv \frac{2\sigma_{II} - \sigma_I - \sigma_{III}}{\sigma_I - \sigma_{III}} = \frac{3s_{II}}{s_I - s_{III}} = -\sqrt{3}\,tg(\theta_{sk}), \quad \theta_{sk} = 30^0 - \theta_L.$$

New structural parametrization of Cauchy stress with isotropy angle θ_{iso} and skewness angle θ_{sk}.

So, the new *generic structural parameterization* of second order symmetric tensor employs the following three invariants (parameters),

$$(\|\boldsymbol{\sigma}\|, \theta_{iso}, \theta_{sk}),$$

$$\|\boldsymbol{\sigma}\| \in <0, \infty), \quad \theta_{iso} \in <-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi>, \quad \theta_{sk} \in <-\tfrac{1}{6}\pi, \tfrac{1}{6}\pi>.$$

The newly proposed parameters can be conveniently used for many purposes in various areas of application. For example, Murzewski isomorphic coordinates can be express with their aid as follows,

$$(\|\boldsymbol{\sigma}\|, \theta_{iso}, \theta_{sk}) \leftrightarrow (z, r, \theta_L),$$

$$z = \sqrt{3}\,\sigma_m = \|\boldsymbol{\sigma}\| \sin(\theta_{iso}), \;\; r = \|\boldsymbol{s}\| = \sqrt{2J_2} = \|\boldsymbol{\sigma}\| \cos(\theta_{iso}), \;\; \theta_L = 30^0 - \theta_{sk}.$$

The stress principal values can be expressed with the following convenient expressions,

$$\sigma_I = \tfrac{1}{\sqrt{3}} z + \sqrt{\tfrac{2}{3}}\, r \sin(60^0 + \theta_{sk}), \quad \sigma_{II} = \tfrac{1}{\sqrt{3}} z - \sqrt{\tfrac{2}{3}}\, r \sin(\theta_{sk}),$$

$$\sigma_{III} = \tfrac{1}{\sqrt{3}} z - \sqrt{\tfrac{2}{3}}\, r \sin(60^0 - \theta_{sk}).$$

New structural parametrization of Cauchy stress with isotropy angle θ_{iso} and skewness angle θ_{sk}.

It seems that the most convenient parameters to be used for description of various critical surfaces of elastically isotropic materials are isomorphic coordinates (z,r,θ_{sk}). The advantage of these coordinates over Murzewski's coordinates (z,r,θ_L) is in rational physical interpretation of θ_{sk} as *shear stress mode angle*.

Ilustrative example of general critical surface of isotropic material in Haigh-Westergaard space of Cauchy principal stresses

Isomorphic coordinates

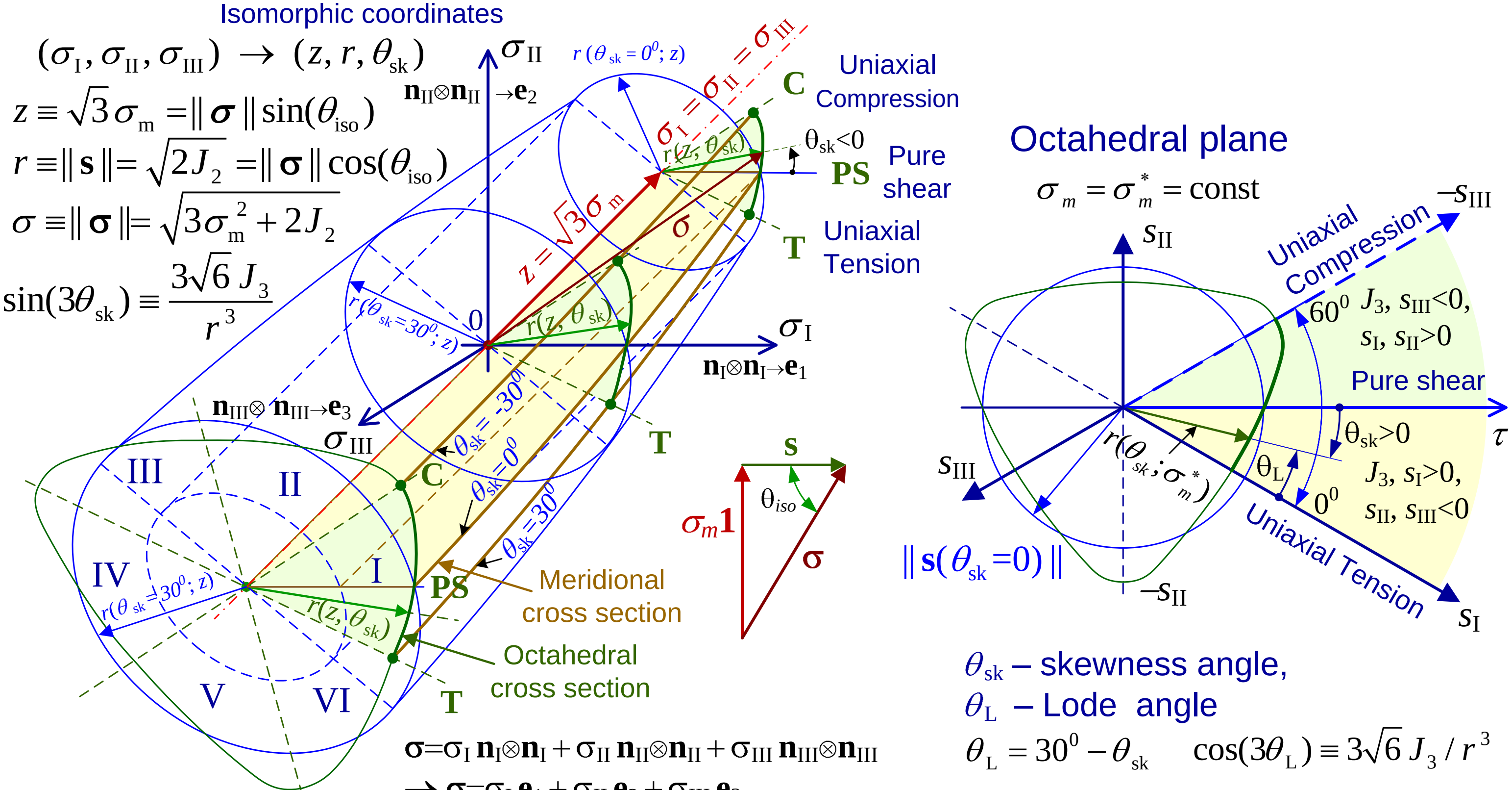
$$(\sigma_I, \sigma_{II}, \sigma_{III}) \rightarrow (z, r, \theta_{sk})$$

$$z \equiv \sqrt{3}\,\sigma_m = \|\boldsymbol{\sigma}\| \sin(\theta_{iso})$$

$$r \equiv \|\mathbf{s}\| = \sqrt{2J_2} = \|\boldsymbol{\sigma}\| \cos(\theta_{iso})$$

$$\sigma \equiv \|\boldsymbol{\sigma}\| = \sqrt{3\sigma_m^2 + 2J_2}$$

$$\sin(3\theta_{sk}) \equiv \frac{3\sqrt{6}\,J_3}{r^3}$$



$$\boldsymbol{\sigma} = \sigma_I\, \mathbf{n}_I \otimes \mathbf{n}_I + \sigma_{II}\, \mathbf{n}_{II} \otimes \mathbf{n}_{II} + \sigma_{III}\, \mathbf{n}_{III} \otimes \mathbf{n}_{III}$$

$$\rightarrow \boldsymbol{\sigma} = \sigma_I\, \mathbf{e}_1 + \sigma_{II}\, \mathbf{e}_2 + \sigma_{III}\, \mathbf{e}_3$$

θ_{sk} – skewness angle,

θ_L – Lode angle

$\theta_L = 30^0 - \theta_{sk}$ $\quad \cos(3\theta_L) \equiv 3\sqrt{6}\, J_3 / r^3$

Second order symmetric tensors, anisotropy index as normalized maximum diameter of tensor orbit.

One of the most important properties of materials is their *anisotropy*, which translates into anisotropy of tensors used for modeling materials behavior. The problem arises on how to measure tensor anisotropy ?

A very versatile quantitative *measure of tensor anisotropy* based on the concept of *tensor orbit*, applicable to tensors of any degree, was proposed by Jan Rychlewski in 1985.

The normalized measure of anisotropy of a tensor $\mathbf{T}$ called by Rychlewski *degree of anisotropy*, and here called *anisotropy index*, is defined as follows,

$$\eta_{ani}(\mathbf{T}) \equiv \frac{d(\mathbf{T})}{2\,\|\mathbf{T}\|}, \quad \mathbf{T} \neq 0, \quad \eta_{ani}(\mathbf{T}) \in <0,1>$$

where $d(\mathbf{T})$ denotes *diameter of tensor orbit* and $\|\mathbf{T}\|$ is its modulus.

Rychlewski J. *Zur Abschätzung der Anisotropie*, *ZAMM*, ***65***, *pp. 255-258,1985. Polish translation of the work exists,*
Rychlewski J., Ziółkowski A. (tłumacz), O szacowaniu Anizotropii, 2020.

Second order symmetric tensors, concept of orbit of a tensor.

Definition 8.3 *Orbit of a tensor* is a set $\{\mathbf{T}^Q\}$ of all tensors that can be obtained by rotation of preset tensor $\mathbf{T}$ with any orthogonal tensor $\mathbf{Q}\in\mathcal{R}$,

$$\mathbf{T}^Q = \mathbf{Q} * \mathbf{T} \quad - \quad \text{any rotated tensor } \mathbf{T}, \quad (\mathbf{Q}\mathbf{Q}^T = \mathbf{I},\ \det\mathbf{Q} > 0),$$

$$\{\mathbf{T}^Q\} = \{all\ \mathbf{T}^Q\} \quad - \quad \text{orbit of a tensor } \mathbf{T}.$$

Definition 8.4 *Diameter of tensor orbit* is defined as maximum distance between any two members in the orbit of the tensor,

$$d(\mathbf{T}) \equiv \max_{\mathbf{T}_1{}^Q,\mathbf{T}_2{}^Q\in\{\mathbf{T}^Q\}} \rho(\mathbf{T}_1{}^Q,\mathbf{T}_2{}^Q), \quad \rho(\boldsymbol{\alpha},\boldsymbol{\beta}) \equiv \|\boldsymbol{\alpha}-\boldsymbol{\beta}\|.$$

where $d(\cdot)$ denotes diameter of the tensor orbit, ρ is distance generated by usual tensorial norm $\|\cdot\|$, $\mathbf{T}_1{}^Q$, $\mathbf{T}_2{}^Q$ denote any two tensors in the tensor orbit, $\mathbf{Q}$ is any proper orthogonal (rotation) tensor.

Diameter of a tensor orbit makes convenient quantitative measure of the tensor sensitivity to the group of three dimensional rotations in space (SO_3 group).

In the case of second order tensors, e.g., stress tensor, these definitions are considerably simplified,

$$\boldsymbol{\sigma}^Q = \mathbf{Q}(\sigma_m\mathbf{1} + \boldsymbol{s})\mathbf{Q}^T = \sigma_m\mathbf{1} + \boldsymbol{s}^Q, \quad \boldsymbol{s}^Q = \mathbf{Q}\boldsymbol{s}\mathbf{Q}^T, \quad \mathbf{Q}\in\mathcal{R},$$

$$\{\boldsymbol{\sigma}^Q\} = \{\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T \mid \mathbf{Q}\in\mathcal{R}\} \ - \ \text{orbit of a tensor } \boldsymbol{\sigma},$$

$$d(\boldsymbol{\sigma}) = \max_{\boldsymbol{\sigma}_1{}^Q,\boldsymbol{\sigma}_2{}^Q\in\{\boldsymbol{\sigma}^Q\}} \rho(\boldsymbol{\sigma}_1{}^Q,\boldsymbol{\sigma}_2{}^Q), \quad \rho(\boldsymbol{\sigma}_1{}^Q,\boldsymbol{\sigma}_2{}^Q) \equiv \|\boldsymbol{\sigma}_1{}^Q - \boldsymbol{\sigma}_2{}^Q\|.$$

Second order symmetric tensors, anisotropy index as normalized maximum diameter of tensor orbit.

Jan Rychlewski has proved that diameter of the orbit of second order symmetric tensor, Cauchy stress tensor, is equal to $d = \sqrt{2}(\sigma_I - \sigma_{III})$ and next he showed that anisotropy index can be expressed in the following form,

$$\eta_{ani} \equiv \frac{d(\boldsymbol{\sigma})}{2\|\boldsymbol{\sigma}\|} = \frac{\sqrt{2}\,\tau_{\max}}{\|\boldsymbol{\sigma}\|} = \frac{\|\boldsymbol{s}\|}{\|\boldsymbol{\sigma}\|} \cdot \sin(\theta_L + 60^0)$$

$$d(\boldsymbol{\sigma}) = \sqrt{2} \cdot (\sigma_I - \sigma_{III}) = \sqrt{2} \cdot (s_I - s_{III}) = 2\sqrt{2} \cdot \tau_{\max}$$

$$\tau_{\max} \equiv \tfrac{1}{2}(\sigma_I - \sigma_{III}) = \tfrac{1}{\sqrt{2}} \|\boldsymbol{s}\| \sin(\theta_L + \tfrac{\pi}{3})$$

where $\tau_{\max}$ denotes maximum shear stress of the tensor $\boldsymbol{\sigma}$. It is clear that *anisotropy index* is still another *invariant* of tensor $\boldsymbol{\sigma}$, and taking it formally makes a fundamental measure of sensitivity of the tensor $\boldsymbol{\sigma}$ to rotations.

Note It is important and interesting *open scientific task* to create clear (lucid) *graphical illustration of the tensor orbit concept*.

Rychlewski J., On evaluation of anisotropy of properties described by symmetric second-order tensors, Czech. J. Phys. B34, pp. 499-506, 1984.

Second order symmetric tensors, anisotropy index as normalized maximum diameter of tensor orbit.

Taking advantage of new parametrization of second order tensor *anisotropy index* can be expressed in the following extremely simple and elucidating form,

$$\eta_{ani} = \cos(\theta_{iso}) \cdot \cos(\theta_{sk})$$

$$\cos(\theta_{iso}) \equiv \| \mathbf{s} \| / \| \boldsymbol{\sigma} \| = \sqrt{1/(1+\tfrac{3}{2}\sigma_m{}^2 / J_2)}, \quad \cos(\theta_{iso}) \in <0, 1>,$$

$$\sin(3\theta_{sk}) = \bar{J}_3 = 3\sqrt{3}\, J_3 / (2 J_2{}^{3/2}), \qquad \cos(\theta_{sk}) \in <1, \tfrac{1}{2}\sqrt{3}>.$$

The first term in the above formula clearly shows that anisotropy degree of second order symmetric tensor grows with growing fraction of its deviatoric part, reaching *maximum for pure deviators* ($\cos(\theta_{\mathrm{iso}}=0)=1$). The second term shows that *the most anisotropic deviators* are *pure shears* ($\cos(\theta_{\mathrm{sk}}=0)=1$).

The anisotropy index *decreases with* deviatoric part *departing* from respective *pure shear comparison reference state*. Proposition on how to explain the reasons for this rather puzzling behavior of anisotropy factor will be presented further in Lecture 9.

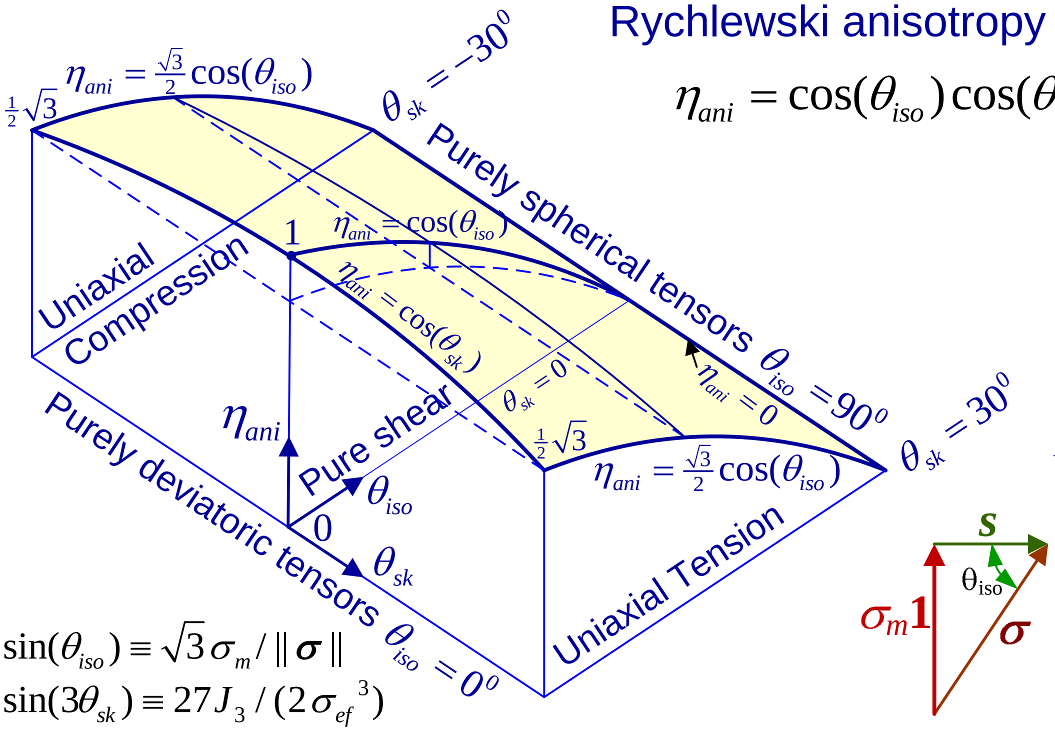
Second order symmetric tensors, anisotropy index as normalized maximum diameter of tensor orbit.

Graphical illustration of variation of *anisotropy index* in dependence on isotropy angle θ_{iso} $(\sim \| \boldsymbol{s} \|)$ and skewness angle θ_{sk} $(\sim \| \bar{J}_3 \|)$,
and illustration of anisotropy index variation in octahedral π plane (σ_{m}=0).

Variation of Cauchy stress anisotropy
with change of isotropy angle and skewness angle

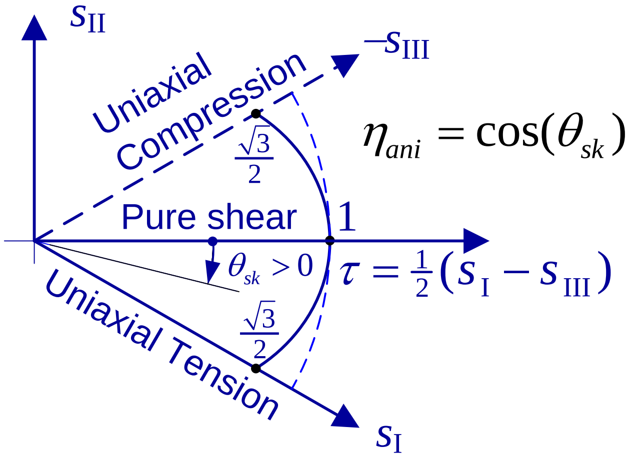
Rychlewski anisotropy factor

$$\eta_{ani} = \cos(\theta_{iso})\cos(\theta_{sk})$$



$\sin(\theta_{iso}) \equiv \sqrt{3}\,\sigma_m / \| \boldsymbol{\sigma} \|$
$\sin(3\theta_{sk}) \equiv 27 J_3 / (2\sigma_{ef}{}^3)$

Octahedral $\pi-$plane $(\sigma_m = 0)$



Statistical interpretation of stress tensor deviator invariants

It is interesting to note that very simple and straightforward connections can be assigned between principal invariants of stress tensor deviator J_α and quantities known as *statistical raw moments* (α_i) (moments about the zero value) and *statistical central moments* (μ_i) (moments about the mean of the population), namely,

$$\alpha_1(\mathbf{s}) \equiv \tfrac{1}{3}(s_I + s_{II} + s_{III}) \quad = \mu_1(\mathbf{s}) = \tfrac{1}{3} tr(\mathbf{s}) = \tfrac{1}{3} J_1 = 0,$$

$$\alpha_2(\mathbf{s}) \equiv \tfrac{1}{3}(s_I{}^2 + s_{II}{}^2 + s_{III}{}^2) = \mu_2(\mathbf{s}) = \mu_2(\boldsymbol{\sigma}) = \tfrac{1}{3}\|\mathbf{s}\|^2 = \tfrac{1}{3}(2J_2),$$

$$\alpha_3(\mathbf{s}) \equiv \tfrac{1}{3}(s_I{}^3 + s_{II}{}^3 + s_{III}{}^3) = \mu_3(\mathbf{s}) = \mu_3(\boldsymbol{\sigma}) = \tfrac{1}{3} tr(\mathbf{s}^3) = J_3 = s_I s_{II} s_{III};$$

$$\alpha_i \equiv \sum_{k=1,n} \tfrac{1}{n}(x_k)^i, \quad \mu_i \equiv \sum_{k=1,n} \tfrac{1}{n}(x_k - \bar{x})^i.$$

Substitution of the expressions $\mu_2(\mathbf{s}), \mu_3(\mathbf{s})$ for central moments of stress tensor deviator $\mathbf{s}$ into the formula for *Fischer-Pearson skewness coefficient* and comparing such obtained relation with formula for *shear stress mode angle* θ_{sk}, reveals existence of the following connection,

$$\left.\begin{aligned} g_1 &\equiv \frac{\mu_3}{\mu_2^{3/2}} = 3\sqrt{3}\frac{J_3}{(2J_2)^{3/2}} = 3\sqrt{3}\frac{s_I}{\|\mathbf{s}\|}\frac{s_{II}}{\|\mathbf{s}\|}\frac{s_{III}}{\|\mathbf{s}\|} = \frac{1}{\sqrt{2}}\bar{J}_3 \\ \bar{J}_3 &\equiv \sin(3\theta_{sk}) \in \langle -1, 1 \rangle \end{aligned}\right\} \Rightarrow \boxed{\sin(3\theta_{sk}) = \sqrt{2}\, g_1}$$

Polyanin A., Manzhirov A., Handbook of mathematics for engineers and scientists, Taylor & Francis, Boca Raton, 2007, formula (20.2.2.9)

Shear stress mode angle θ_{sk} as a measure of entropic part of anisotropy of Cauchy stress tensor

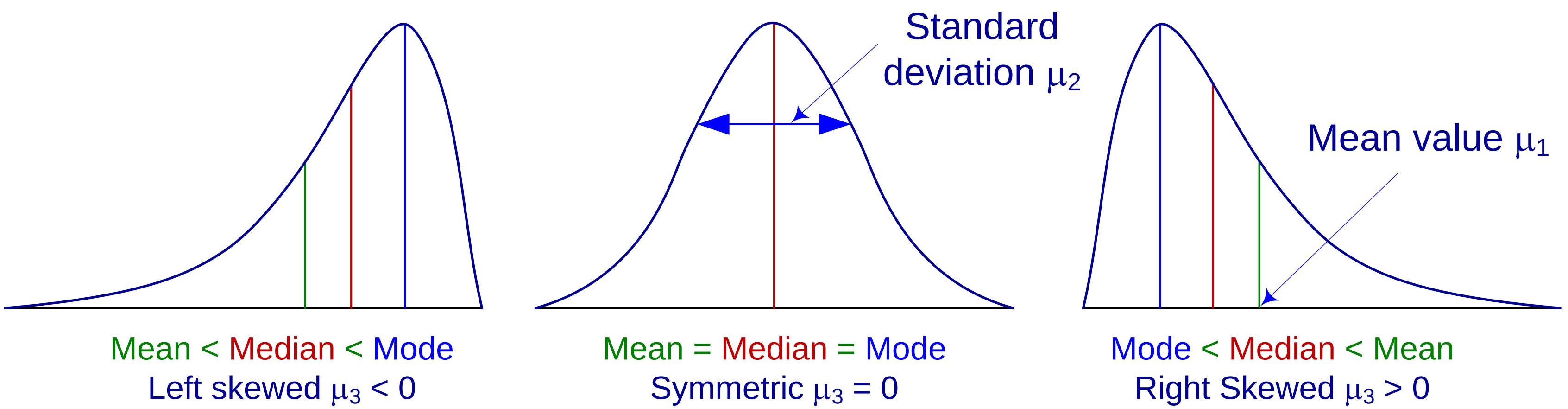
The above connection suggests assigning the *shear stress mode angle* θ_{sk} the name *skewness angle*. At the same time, this relation supplies lead to revealing one more unexpected physical interpretation of the shear stress mode (skewness) angle θ_{sk} of statistical character, which allows for an explanation of the mysterious, at first sight, reduction of *anisotropy index* of the stress tensor $\boldsymbol{\sigma}$ with increasing departure of its deviator $\boldsymbol{s}$ from *pure shear mode*.

In statistical literature there exist very well known interpretations of raw moments and central moments. The *first raw moment* (α_i) is understood and linked with the *mean value* of the feature described by the statistics of population of objects. For stress tensor $\boldsymbol{\sigma}$ it is equal to *mean stress* $\alpha_1(\boldsymbol{\sigma}) = \sigma_m$ but for stress tensor deviator $\boldsymbol{s}$ it is equal to zero $\alpha_1(\mathbf{s}) = 0$.
The *first central moment* $\mu_1(\boldsymbol{\sigma}) = \mu_1(\mathbf{s}) = 0$ is always zero.

Shear stress mode angle θ_{sk} as a measure of entropic part of anisotropy of Cauchy stress tensor

The square root of the *second central moment* ($\mu_2^{1/2}$) is called *standard deviation* and it is understood as describing the *magnitude of scatter* (or *magnitude of non-uniformity*) of the population statistics around its mean value.

The *third central moment* normalized with standard deviation ($\mu_3/\mu_2^{3/2}$) is understood as describing the *non-symmetry* or "*skewness*" of the population statistics towards the left or right wing of its statistical distribution.



Second order symmetric tensors, physical interpretation of shear mode angle of deviator θ_{sk}.

In order to reach some deeper understanding of *shear stress mode angle* θ_{sk} being function of stress tensor deviator $\boldsymbol{s}$ it is convenient and rational to adopt micromechanical approach and try to reach some *statistical-physical interpretations* of stress tensor $\boldsymbol{\sigma}$ an its derivative θ_{sk}. Adopting this line stress tensor deviator (shear stress) $\boldsymbol{s}$ can be treated as *macroscopic tensorial measure* (parameter) characterizing *population of micro pure shears* generating given macroscopic stress tensor deviator $\boldsymbol{s}$ in a certain neighborhood of a material point. The *micro pure shears*, being individual elements of such population, can in turn be treated as a kind of *directional dipoles* and as such their population is responsible for generating macroscopic *directional effects*.

Certain scalar parameters characterizing *population of micro pure shears* generating given macroscopic stress state will prove useful in our present effort to find rational physical interpretation of shear stress mode angle θ_{sk}.

Shear stress mode angle θ_{sk} as a measure of entropic part of anisotropy of Cauchy stress tensor

Novozhilov in his rather unknown work from 1952 demonstrated that *average shear stress* (τ_{av}) of stress tensor $\boldsymbol{\sigma}$ calculated over *all directions* on unit sphere is directly proportional to modulus of stress tensor deviator ($\|\mathbf{s}\| = \sqrt{2J_2}$),

$$\tau_{av}(\boldsymbol{\sigma}) \equiv \left((1/\Omega)\int \tau^2 d\Omega\right)^{1/2} = \tfrac{1}{\sqrt{5}}\|\mathbf{s}\| = 0.447\|\mathbf{s}\|,$$

$$(\tau^2 = \sigma_I{}^2 l_I{}^2 + \sigma_{II}{}^2 l_{II}{}^2 + \sigma_{III}{}^2 l_{III}{}^2 - (\sigma_I l_I{}^2 + \sigma_{II} l_{II}{}^2 + \sigma_{III} l_{III}{}^2)^2),$$

where τ_{av} is *average shear stress* over all possible directions on unit sphere, also called by Novozhilov *shear stress intensity*, τ is modulus of *shear stress traction* operating on elementary surface $\mathrm{d}\Omega$ of unit sphere, σ_{I}, σ_{II}, σ_{III} are principal stresses, l_{I}, l_{II}, l_{III} are direction cosines determining orientation of normal to surface $\mathrm{d}\Omega$ in relation to principal directions of tensor $\boldsymbol{\sigma}$.

Novozhilov V.V., О физическом смысле инвариантов напряжения, используемых в теории пластичности (in Russian), Prikl. Mat. Mekh. XVI, 1952, pp. 617-619

Shear stress mode angle θ_{sk} as a measure of entropic part of anisotropy of Cauchy stress tensor

Because the square root of *second central moment* of shear stress tensor ($\mu_2^{1/2}$) is also directly proportional to modulus of shear part of stress $\|\mathbf{s}\|$, we can obtain the following relation between τ_{av} and μ_2,

$$\{\tau_{av}(\boldsymbol{\sigma}) = \tfrac{1}{\sqrt{5}}\|\mathbf{s}\| \quad and \quad \sqrt{\mu_2} = \tfrac{1}{\sqrt{3}}\|\mathbf{s}\|\} \quad \Rightarrow \quad \tau_{av} = \sqrt{\tfrac{3}{5}}\sqrt{\mu_2} = 0.775\sqrt{\mu_2},$$

The second result of interest demonstrated in Novozhilov work from 1952, is the relation linking *maximum (macroscopic) shear stress* τ_{max} with *average (macroscopic) shear stress* τ_{av} and angle $\zeta = -\theta_{sk}$. Present in the original relation angle ζ has been here replaced with shear stress mode angle θ_{sk},

$$\tau_{\max} = \tfrac{1}{2}(\sigma_I - \sigma_{III}) = \tfrac{1}{\sqrt{2}}\cdot\|\mathbf{s}\|\cdot\cos(\theta_{sk}) = \sqrt{\tfrac{5}{2}}\cdot\tau_{av}\cdot\cos(\theta_{sk}) = \sqrt{\tfrac{3}{2}\mu_2}\cdot\cos(\theta_{sk}),$$

$$\theta_{sk} = \tfrac{1}{3}\sin^{-1}(\bar{J}_3) = \tfrac{1}{3}\sin^{-1}(\sqrt{2}\,\mu_3/\mu_2^{3/2}), \qquad \tfrac{\sqrt{3}}{2} \le \cos(\theta_{sk}) \le 1.$$

Introduced by Novozhilov quantities of *average shear stress* τ_{av} and *maximum shear stress* τ_{max} can be interpreted as scalar macroscopic measures (parameters) quantitatively characterizing overall (average) *orientational effects* resulting from the *action of a population of micro pure shears*, treated as a pool of (microscopic) *directional dipoles*.

Shear stress mode angle θ_{sk} as a measure of entropic part of anisotropy of Cauchy stress tensor

It is rational to interpret the value of parameter τ_{av} ($\sqrt{\mu_2(\mathbf{s})}$) to be a sort of a *mean modulus* proportional to the moduli of individual micro pure shears (directional dipoles) entering population generating examined macroscopic stress state $\boldsymbol{s}$.

In the case of parameter τ_{max} it can be noticed that at fixed strength of shearing τ_{av} its value depends only on shear stress mode angle θ_{sk} and that the greatest value of this parameter (greatest orientational effect) is attained when $\theta_{sk}=0$, i.e., at macroscopic pure shear stress, while the smallest value at $\theta_{sk}=\pm 30°$, i.e, at macroscopic uniaxial tension stress, compression stress, respectively,

$$1.37 = \frac{\sqrt{3}}{2} \cdot \sqrt{\frac{5}{2}} = \frac{\tau_{\max}(\theta_{sk} = \pm 30^0)}{\tau_{av}} \leq \frac{\tau_{\max}}{\tau_{av}} = \sqrt{\frac{5}{2}} \cdot \cos(\theta_{sk}) \leq \frac{\tau_{\max}(\theta_{sk} = 0^0)}{\tau_{av}} = \sqrt{\frac{5}{2}} = 1.58.$$

uniaxial tension / compr. *pure shear*

Shear stress mode angle θ_{sk} as a measure of entropic part of anisotropy of Cauchy stress tensor

In order to explain the above behavior of τ_{max} it is instrumental to turn again to micromechanics. Let us first indicate that when population of some directional entities causes some total (macroscopic) orientational effect, an analogy with magnetic and/or electric dipoles immediately comes to ones mind, then *the more ordered* are directional units in the population – here micro pure shears, the bigger is the the overall orientational effect. This gives grounds for accepting that the state of population of micro pure shears generating macroscopic pure shear stress mode must be *the most ordered directionally*, and the state of population of micro pure shears generating macroscopic uniaxial tension/compression stress mode is *the most disordered/scattered directionally*. The shear stress mode (skewness) angle θ_{sk} can thus be interpreted as macroscopic parameter describing the degree of ordering in population of micro pure shears.

Shear stress mode angle θ_{sk} as a measure of entropic part of anisotropy of Cauchy stress tensor

We can now deliver rational explanation of signaled earlier mysterious decrease of anisotropy index values η_{ani} of the stress tensor with its shear part $\boldsymbol{s}$ departing from macroscopic pure shear mode. Upon comparing formula expressing stress tensor anisotropy η_{ani} with formula expressing maximum shear stress τ_{max} we can notice that they are identical upon replacement of the terms $\cos(\theta_{iso}) \leftrightarrow \sqrt{\frac{5}{2}} \cdot \tau_{av}$,

$$\eta_{ani} = \cos(\theta_{iso}) \cdot \cos(\theta_{sk}) \leftrightarrow \tau_{\max} = \sqrt{\frac{5}{2}} \cdot \tau_{av} \cdot \cos(\theta_{sk})$$

This allows formulating the following Corollary.

Corollary 8.**1.** *Decrease of the value of anisotropy index* of the Cauchy stress tensor with the departure of its deviator from the *pure shear mode* – with growth of absolute magnitude of the shear stress mode (skewness) angle θ_{sk}, can be attributed to *increase of internal discorder (growth of orientational scatter)* in population of micro pure shears generating given Cauchy stress tensor deviator $\boldsymbol{s}$.

It is known from thermodynamics that good *measure of the degree of internal order (disorder)* of any system is *entropy.*

Second order symmetric tensors, notion of internal entropy of stress tensor.

Due to this it is reasonable to call the term $cos(\theta_{sk})$ *entropic part of stress tensor anisotropy* while $cos(\theta_{iso})$ can be called *deviator modulus part of stress tensor anisotropy*.

The present discussion can be concluded with the statement that the value of the shear stress mode (skewness) angle θ_{sk} delivers twofold information. Firstly, it informs about the magnitude of *directional disorder (internal entropy)* in the population of elementary micro pure shears (directional dipoles) generating given stress tensor. This order finds reflection in the values of the anisotropy index, i.e., the greater is microscopic order of the population of micro pure shears generating given stress tensor the bigger is the value of anisotropy index. Secondly, it informs about the *skewness* of the statistics of population of micro pure shears generating the specific macroscopic stress state ($J_3>0$ or $J_3<0$).

This in the sense that the direction of projection of stress tensor onto octahedral plane deviates from the direction of the projection onto octahedral plane of respective reference pure shear mode $0.5(\sigma_{\mathrm{I}}-\sigma_{\mathrm{III}})$ ($J_3=0$) towards either direction of the projection of the first (σ_{I}) or the third (σ_{III}) principal stress on the octahedral plane.

Lecture 9 – synopsis

Spectral decomposition of Hooke’s tensor and some of its consequences

9.1 Structure and some properties of Hooke's tensor of anisotropic materials resulting from its spectral decomposition.

9.2 Decomposition of Hooke's law into six uncoupled linear relations.

9.3 Decomposition of elastic energy of anisotropic materials into (maximum) six independent parts.

9.4 Strength criteria of anisotropic materials based on elastic energy.

9.5 Spectral decomposition of Hooke’s tensor of isotropic materials.

9.6 The concept of isometric tensorial bases.

Structure and some properties of Hooke's tensor of anisotropic materials resulting from its spectral decomposition.

In the previous discussion on the properties of *second order symmetric tensors* the Cauchy stress tensor was considered as an *autonomous object*. It appears that tensors similarly like people behave differently depending on an interaction with environment (external circumstances). It is interesting that taking into account *interaction of second order symmetric tensor with various other tensors* creates new situations and possibilities not accessible when the tensor is treated as self standing object.

We will examine the above issue on the example of interaction of *Cauchy stress tensor* $\boldsymbol{\sigma}\in\mathcal{T}_2^{sym}$ - interpreted as model of mechanical loading with *Hooke's tensor* $\mathbf{H}\in\mathcal{T}_4^{sym}$ - interpreted as model of anisotropic elastic material.

In our present examination very helpful will be information on mathematical structure and some properties of Hooke's tensor $\mathbf{H}$ resulting from its *spectral decomposition*, i.e., finding roots (*eigenvalues*) and *eigenstates* of *characteristic equation* of $\mathbf{H}$.

Rychlewski J., translator Ziółkowski A. (2033) CEIIINOSSSTTUV Mathematical structure of elastic bodies, pp. 1-131, 2023, IPPT PAN, Warsaw, Poland. (English translation of original work published in Russian in 1983.)
https://www.researchgate.net/publication/376594979_CEIIINOSSSTTUV_Mathematical_structure_of_elastic_bodies

Structure and some properties of Hooke's tensor of anisotropic materials resulting from its spectral decomposition.

The characteristic equation of fourth order symmetric tensors takes the form of six order polynomial equation with real coefficients,

$$\mathbf{H}\cdot\boldsymbol{\omega}=\lambda\,\boldsymbol{\omega},\;\; \mathbf{H}\in\mathcal{T}^{sym}_{2\,(n=6)} \;\rightarrow\; \det(\mathbf{H}-\lambda\mathbf{I}^{(4s)})=0 \;\rightarrow\; \lambda_K,\boldsymbol{\omega}_K\in\mathcal{T}_{1\,(n=6)},$$

$$(\boldsymbol{\omega}_K\cdot\boldsymbol{\omega}_L)=\delta_{KL},\quad (K,L=1,6).$$

In the most general case the solution of characteristic equation is composed of *maximum six* different *eigenvalues* (λ_K) and corresponding to them *elastic eigenstates* ($\boldsymbol{\omega}_K$).

The number of different in value eigenvalues λ_K of Hooke's tensor also called *Kelvin moduli* depends on external symmetry of Hooke's tensor. For example, isotropic materials have only two different in value Kelvin moduli.

Rychlewski in 1983 showed that elastic stiffness and compliance (Hooke's) tensors **S**, **C** can be expressed with the aid of *eigentensors* $\boldsymbol{\omega}_K$ corresponding to *all different* eigenvalues λ_K (*Kelvin moduli*) of characteristic equation, in the following form,

$$\mathbf{S}=\lambda_1\,\boldsymbol{\omega}_I\otimes\boldsymbol{\omega}_I+\ldots+\lambda_6\,\boldsymbol{\omega}_{VI}\otimes\boldsymbol{\omega}_{VI},\quad \mathbf{C}=(1/\lambda_1)\,\boldsymbol{\omega}_I\otimes\boldsymbol{\omega}_I+\ldots+(1/\lambda_6)\,\boldsymbol{\omega}_{VI}\otimes\boldsymbol{\omega}_{VI},$$

$$\mathbf{I}^{(4s)}=\boldsymbol{\omega}_I\otimes\boldsymbol{\omega}_I+\ldots+\boldsymbol{\omega}_{VI}\otimes\boldsymbol{\omega}_{VI},\quad \mathbf{I}^{(4s)}=\mathbf{P}_I+\ldots+\mathbf{P}_{VI},\;\; \mathbf{I}^{(4s)}\sim I^{(4s)}_{KL}=\delta_{KL},$$

$$\mathbf{P}_K\equiv\boldsymbol{\omega}_K\otimes\boldsymbol{\omega}_K,\;\;(K,L=1,6),\quad \mathbf{S}\circ\mathbf{C}=\mathbf{I}^{4s}\;(\sim S_{KL}C_{KL}=\delta_{KL}).$$

The above formula for tensor **S**, **C** ($\mathbf{S}=\mathbf{C}^{-1}$) respectively is called *fundamental structural formula of elastic body*.

Decomposition of Hooke's law into six uncoupled linear relations.

The *sets of second order eigentensors* $\{\boldsymbol{\omega}_{\mathrm{K}}\}$, K=1,6 make an *orthonormal bases* of second order symmetric tensors ($\mathcal{T}_{1\,(n=6)}$ space).

The sets of fourth order *material tensors* $\mathbf{P}_{\mathrm{K}} \equiv \boldsymbol{\omega}_{\mathrm{K}} \otimes \boldsymbol{\omega}_{\mathrm{K}}$, K=1,6 make *principal orthonormal bases* of fourth order symmetric tensors ($\mathcal{T}_2^{\,sym}{}_{(n=6)}$ space), i.e., in these bases only diagonal elements of Hooke's tensor representation have non zero values. The *material tensors* $\mathbf{P}_{\mathrm{K}}$ characterizing specific elastic body $\mathbf{S}$, $\mathbf{C}$ are *projectors*, i.e., they have the following properties,

$$\mathbf{P}_K \circ \mathbf{P}_K = \mathbf{P}_K \;\; \sim (P_K)_{MN}(P_K)_{NL} = (P_K)_{ML},$$

$$\mathbf{P}_K \cdot \mathbf{P}_L = 0 \;\; K \neq L \;\; (\mathbf{P}_K \perp \mathbf{P}_L), \quad (K, L, M, N = 1,...,6).$$

For the *specific elastic body,* characterized with stiffness and/or corresponding compliance tensors $\mathbf{S}$, $\mathbf{C}$ ($\mathbf{S}$=$\mathbf{C}^{-1}$), taking advantage of the *fundamental structural formulas of elastic body*, the following decompositions of Cauchy stress tensor and small strains tensor can be obtained (K=1,..,6),

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \ldots + \boldsymbol{\sigma}_6 = \sigma_{(1)}\boldsymbol{\omega}_I + \ldots + \sigma_{(6)}\boldsymbol{\omega}_{VI}, \quad \boldsymbol{\sigma}_K \equiv \mathbf{P}_K \cdot \boldsymbol{\sigma} = \sigma_{(K)}\boldsymbol{\omega}_K, \;\; \sigma_{(K)} \equiv \boldsymbol{\sigma} \cdot \boldsymbol{\omega}_K,$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1 + \ldots + \boldsymbol{\varepsilon}_6 = \varepsilon_{(1)}\boldsymbol{\omega}_I + \ldots + \varepsilon_{(6)}\boldsymbol{\omega}_{VI}, \quad \boldsymbol{\varepsilon}_K \equiv \mathbf{P}_K \cdot \boldsymbol{\varepsilon} = \varepsilon_{(K)}\boldsymbol{\omega}_K, \quad \varepsilon_{(K)} \equiv \boldsymbol{\varepsilon} \cdot \boldsymbol{\omega}_K.$$

The relations above show that stresses and strains spaces can be decomposed into *maximum six* mutually orthogonal subspaces $\mathcal{P}_1 \oplus ... \oplus \mathcal{P}_6 \subset \mathcal{T}_2^{\,s}{}_{(n=6)}$.

Rychlewski J., On Hooke's Law, J. Appl. Math.&Mech. 1984, 48, 3, pp. 303-314, 0021-8928/84 Pergamon Press Ltd.
(English translation from Russian original published in PMM, U.S.S.R.)

Decomposition of Hooke's law into six uncoupled linear relations.

When some of the Kelvin moduli λ_K have the same value the number of subspaces can be reduced by merging some of the subspaces together $\mathcal{P}_1 \oplus ... \oplus \mathcal{P}_\rho \subset \mathcal{T}_2^{\,s}$, $(\rho \le 6)$. In such a case subspaces corresponding to multiple Kelvin moduli have dimension higher than one. Then, also the terms present in *fundamental structural formula of elastic body* belonging to merged subspaces can be grouped together and the number of individual terms appearing in it can be lowered to $\rho < 6$.

The decompositions $\boldsymbol{\sigma} = \sigma_{(1)}\boldsymbol{\omega}_I + \ldots + \sigma_{(6)}\boldsymbol{\omega}_{VI}, \quad \boldsymbol{\varepsilon} = \varepsilon_{(1)}\boldsymbol{\omega}_I + \ldots + \varepsilon_{(6)}\boldsymbol{\omega}_{VI}$ are called *energy-orthogonal decompositions* because *projections* $\boldsymbol{\sigma}_K$ of Cauchy stress tensor $\boldsymbol{\sigma}$ and *projections* $\boldsymbol{\varepsilon}_K$ of strain tensor $\boldsymbol{\varepsilon}$ satisfy the following relations,

$$\boldsymbol{\sigma}_K \cdot \mathbf{C}\boldsymbol{\sigma}_L = \delta_{KL} \;\; \boldsymbol{\sigma}_K \overset{E}{\perp} \boldsymbol{\sigma}_L, \quad \boldsymbol{\varepsilon}_K \cdot \mathbf{S}\boldsymbol{\varepsilon}_L = \delta_{KL} \;\; \boldsymbol{\varepsilon}_K \overset{E}{\perp} \boldsymbol{\varepsilon}_L,$$

$$\boldsymbol{\sigma}_K \cdot \boldsymbol{\sigma}_L = \delta_{KL} \quad \boldsymbol{\sigma}_K \perp \boldsymbol{\sigma}_L, \quad \boldsymbol{\varepsilon}_K \cdot \boldsymbol{\varepsilon}_L = \delta_{KL} \quad \boldsymbol{\varepsilon}_K \perp \boldsymbol{\varepsilon}_L, \quad (K, L = 1, ..., 6).$$

The above means that stresses $\boldsymbol{\sigma}_K$ from subspace $\mathcal{P}_K$ do not execute work on strains from other subspaces $\mathcal{P}_L$ $L \neq K$ and due to that comes the name *energy-orthogonal*. *Energy-orthogonal decompositions* $\boldsymbol{\sigma}_K$, $\boldsymbol{\varepsilon}_K$ of stress and strain tensors for a *given elastic body* $\mathbf{S}$, $\mathbf{C}$ $(\mathbf{S}=\mathbf{C}^{-1})$ are *unique.*

Decomposition of Hooke's law into six uncoupled linear relations.

For specific elastic body characterized by elasticity tensors $\mathbf{S}$, $\mathbf{C}$ ($\mathbf{S}=\mathbf{C}^{-1}$) the stress representation components (scalars) $\sigma_{(K)}$ make *six linearly independent invariants* of stress tensor $\boldsymbol{\sigma}$. This in a standard sense, i.e., they do not change when coordinate system is changed (upon change of the basis $\mathbf{e}_i$ only representation components of tensors $\boldsymbol{\omega}_K$ change). It may be said that $\sigma_{(K)}$ are *invariants of* $\boldsymbol{\sigma}$ *in interaction with elasticity tensor* $\mathbf{C}$ ($\sigma_{(K)}^{\sigma\leftrightarrow\mathbf{C}}$).

The statement about *six linearly independent invariants* of Cauchy stress may seem contradicting an earlier statement that only *three linearly independent invariants* of stress tensor can be generated. This can be explained as follows.

Ricci-Curbastro motivated by the idea of *quadratic forms invariance* devised the whole mathematical apparatus – tensorial calculus, which predicts that in the case of second order symmetric tensor *six components* of its representation *transform in linear manner with change of coordinate system*.

Next, it was identified that from these six components there can always be formed various sets of maximum *three linearly independent invariants* remaining constant upon change of coordinate system and a *set of another three parameters changing with change of coordinate system* (Euler angles). This, when *tensor* is considered an *autonomous object* – an analogy with a free vector comes to mind.

Decomposition of Hooke's law into six uncoupled linear relations.

When the tensor is considered in some environment, *in interaction with other tensors then* it turns out that *six invariants can be formed* out of its components – and analogy with anchoring the vector to a fixed reference frame (point) comes to mind.

Let us consider the following situation in order to better understand in what sense anchoring of the tensor takes place. Take two autonomous (free) tensors for example stress tensor and not coaxial with it strain tensor, a typical situation for non-isotropic materials. Each of these tensors is fully described by *three invariants* and *three Euler angles*. Respective Euler angles characterize orientation of each tensor with respect to any conceivable coordinate system (laboratory reference frame). While these angles change with change of coordinate system (reference frame) the *relative orientation* of specific stress tensor with respect to specific non-coaxial strain tensor does not change. Upon the tensors interaction, for example, taking their scalar product, *only their relative orientation is important* what manifests itself in reported possibility of *generating six invariants*.

So, *anchoring of the tensor* means that orientation of the *principal axes* of the first or the second tensor take over the role of *reference frame* and no other reference frame is needed, does not play any role.

Decomposition of Hooke's law into six uncoupled linear relations.

Upon substitution of relations for *energy-orthogonal decomposition of stresses and strains* into the general Hooke's law for anisotropic materials $\boldsymbol{\sigma} = \mathbf{S}\boldsymbol{\varepsilon}$ and taking advantage of the *fundamental structural formula of elastic body* there can be obtained maximum six proportionality relations,

$$\boldsymbol{\sigma}_1 = \lambda_1 \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\sigma}_6 = \lambda_6 \boldsymbol{\varepsilon}_6 \quad (*)$$

Depending on the *external symmetry of Hooke's* tensor (number of different in value Kelvin moduli) the number of the relations in can be reduced to number $\rho \leq 6$. In the case of isotropic materials two such relations are usually specified because there are only two different in value Kelvin moduli. Detailed derivation of discussed here issues can be found in chapters 3 and 5 of Rychlewski’s Report from 1984.

In the present author opinion formula (*) makes a *Crowning Achievement of 300 years of development of the theory of elasticity*. It delivers the deepest philosophical knowledge and full understanding of the behavior of anisotropic linear elastic materials.

Decomposition of Hooke's law into six uncoupled linear relations.

In 1676, Robert Hooke announced the publication of a rule for the behavior of elastic materials, pronouncing it in the hidden form of the Latin anagram CEIIINOSSSTTUU. Hooke deciphered this anagram two years later, in 1678, as: *ut tensio, sic vis*, "*as the stretch, so the force*". The rule formulated by Robert Hooke, valid for uniaxial tension/compression, mathematically takes an extremely simple form of linear proportionality $F=k\cdot x$, where F is the force, k is the proportionality coefficient, and x is the stretch. Jan Rychlewski was the first researcher who in 1984, using *internal symmetries* and the *spectral decomposition* of the Hooke tensor, showed that in the most general case of a 3D anisotropic elastic materials, the Hooke's law can be *uncoupled* into maximum *six linear proportionality relations*

$$\boldsymbol{\sigma}_1 = \lambda_1 \boldsymbol{\varepsilon}_1,\ \boldsymbol{\sigma}_2 = \lambda_2 \boldsymbol{\varepsilon}_2,\ \boldsymbol{\sigma}_3 = \lambda_3 \boldsymbol{\varepsilon}_3,\ \boldsymbol{\sigma}_4 = \lambda_4 \boldsymbol{\varepsilon}_4,\ \boldsymbol{\sigma}_5 = \lambda_5 \boldsymbol{\varepsilon}_5,\ \boldsymbol{\sigma}_6 = \lambda_6 \boldsymbol{\varepsilon}_6.$$

In this manner Jan Rychlewski turned the circle and returned to the very first formulation of Robert Hooke but this time embracing description of behavior of the *most general anisotropic linear elastic materials*.

Development of strength of anisotropic materials criteria based on elastic energy.

Another very interesting and useful consequence of the *energy-orthogonal decomposition of the stress and strain tensors* is the possibility of decomposition of elastic energy into maximum six uncoupled components. Upon substitution of the relations for material decomposition of stress and strain tensor

$$\boldsymbol{\sigma}_1 = \lambda_1 \boldsymbol{\varepsilon}_1,\ \boldsymbol{\sigma}_2 = \lambda_2 \boldsymbol{\varepsilon}_2,\ \boldsymbol{\sigma}_3 = \lambda_3 \boldsymbol{\varepsilon}_3,\ \boldsymbol{\sigma}_4 = \lambda_4 \boldsymbol{\varepsilon}_4,\ \boldsymbol{\sigma}_5 = \lambda_5 \boldsymbol{\varepsilon}_5,\ \boldsymbol{\sigma}_6 = \lambda_6 \boldsymbol{\varepsilon}_6.$$

$$\boldsymbol{\sigma} = \sigma_{(1)}\boldsymbol{\omega}_I + \ldots + \sigma_{(6)}\boldsymbol{\omega}_{VI}, \quad \boldsymbol{\varepsilon} = \varepsilon_{(1)}\boldsymbol{\omega}_I + \ldots + \varepsilon_{(6)}\boldsymbol{\omega}_{VI}$$

into the formula for elastic energy ($\Phi=\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}$) there can be obtained the following expressions for elastic energy of any elastic body,

$$\Phi(\boldsymbol{\sigma}) = \frac{1}{2\lambda_1}\sigma_1{}^2 + \ldots + \frac{1}{2\lambda_\rho}\sigma_\rho{}^2, \quad \Phi(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\lambda_1\varepsilon_1{}^2 + \ldots + \tfrac{1}{2}\lambda_\rho\varepsilon_\rho{}^2, \quad \rho \le 6.$$

The above formula is sometimes called *main distribution of elastic energy*.

The elastic energy decomposition delivers very inspiring and prolific hint in formulation of various *strength hypotheses of anisotropic materials*.

It demonstrates that when with some physical phenomenon (start of plastic yield flow, cracking, damage, phase transition, etc) there can be associated elastic energy stored in the anisotropic material then loadings inducing the occurrence of the phenomenon can be divided/decomposed, in the most general case, into maximum 6 *classes of loadings*, depending on the external symmetry of the elasticity (Hooke's) tensors $\mathbf{S}$, $\mathbf{C}$ ($\mathbf{S}=\mathbf{C}^{-1}$).

Development of strength of anisotropic materials criteria based on elastic energy.

When safety of some structure is analyzed in view of some phenomenon depending on elastic energy stored in anisotropic material then *six different coefficients of safety* will in general be applicable, in order to assure secure operation of the structure, depending on the specific class of loading.

Taking into account knowledge about structure of Hooke's tensor gives grounds for introduction of *weighted effective stress* notion $\bar{\boldsymbol{\sigma}}_{ef}$, for example, in the form of a quadratic involving Hooke's tensor $\mathbf{C}$ and stress tensor $\boldsymbol{\sigma}$ ($\sim\boldsymbol{\sigma}\ \mathbf{C}\boldsymbol{\sigma}$). Such quantity will enable more precise evaluation of effort of materials (also other utility features) then classical *effective stress* notion σ_{ef} depending only on stress tensor $\boldsymbol{\sigma}$.

In the literature there exist attempts for further generalizations of anisotropic materials limit strength criteria by introduction of the criteria based on the quadratic form of so called *tensor of limit states* $\mathbf{M}$. The quadratic $\mathrm{m}\ (\boldsymbol{\sigma},\mathbf{M}) \equiv \boldsymbol{\sigma}\mathbf{M}\boldsymbol{\sigma}$ is sometimes called *Mises stress intensity*. The 4th order tensor $\mathbf{M}$ is assumed to possess the same structure – external symmetries, as Hooke's tensors but it is assigned different than elastic energy physical interpretations. Actually, this makes a return to an original idea of von Mises from 1928, but now much better developed.

Rychlewski (1984) J. *Elastic energy decompositions and limit criteria* [in Russian], Uspekhi Mekh., 7, 3, 1984.

Importance of distinguishing classes of loadings.

video: Railroad tank car vacuum implosion, Tom Brattain,
https://www.youtube.com/watch?v=Zz95_VvTxZM&t=2s

While in majority of situations taking into account prevailing type/class of loadings (design loads) lead to safe exploitation of engineering structure. In some situations neglected (in design) *specific class* of loadings can lead to *catastrophic situations*.

Spectral decomposition of Hooke's tensor of isotropic materials.

The classical Hooke's law describing elastic properties of isotropic linear elastic material leads to the following constitutive relations between stress and strain tensors,

$$\boldsymbol{\sigma} = \mathbf{S}^{iso} \cdot \boldsymbol{\varepsilon} = \lambda(tr\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon},$$

$$\boldsymbol{\varepsilon} = \mathbf{C}^{iso} \cdot \boldsymbol{\sigma} = (1/9K)(tr\boldsymbol{\sigma})\mathbf{1} + (1/2\mu)\mathbf{s},$$

$$\sigma_m \equiv \tfrac{1}{3}tr(\boldsymbol{\sigma}) = K\varepsilon_v, \quad \mathbf{s} = 2\mu\boldsymbol{\varepsilon}^d,$$

$$\varepsilon_v \equiv tr(\boldsymbol{\varepsilon}) = \varepsilon_{ii}, \quad \boldsymbol{\varepsilon}^d \equiv \boldsymbol{\varepsilon} - \tfrac{1}{3}\varepsilon_v\mathbf{1},$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sqrt{2}\sigma_4 \\ \sqrt{2}\sigma_5 \\ \sqrt{2}\sigma_6 \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \sqrt{2}\varepsilon_4 \\ \sqrt{2}\varepsilon_5 \\ \sqrt{2}\varepsilon_6 \end{bmatrix},$$

$$\boldsymbol{\sigma} = \mathbf{C}^{iso} \cdot \boldsymbol{\varepsilon} \Leftrightarrow \sigma_K^{Ke}\mathbf{a}_K = C_{KL}^{iso}\mathbf{a}_K \otimes \mathbf{a}_L \cdot \varepsilon_L^{Ke}\mathbf{a}_L, \quad K, L = 1, \ldots, 6.$$

$$K = \lambda + \tfrac{2}{3}\mu, \quad E = \mu(3\lambda + 2\mu)/(\lambda + \mu),$$

$$\nu = \tfrac{1}{2}\lambda/(\lambda + \mu), \quad 2\mu = E/(1+\nu),$$

$$\mathbf{S}^{iso} = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}^{(4s)} \quad (\sim S_{ijkl}^{iso} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})), \quad \mathbf{S}^{iso} \circ \mathbf{C}^{iso} = \mathbf{I}^{(4s)},$$

$$\mathbf{1}(\sim \delta_{ij}), \ \mathbf{1}\otimes\mathbf{1}(\sim \delta_{ij}\delta_{kl}), \ \mathbf{I}^{(4s)}(\sim \delta_{KL} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})), \ K, L = 1..6, \ i, j, k, l = 1..3.$$

where $\mathbf{S}^{iso}$, $\mathbf{C}^{iso}$ denote linear elastic, isotropic stiffness and compliance tensors, λ, μ denote Lame constants, $\mu = G$ is shear modulus, E, K are Young and Bulk modules, ν denotes Poisson's ratio, $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^d$ are strain and strain deviator, respectively, $\mathbf{I}^{(4s)}$ is fourth order, symmetric unit tensor.

Spectral decomposition of Hooke's tensor of isotropic materials.

Thus the most general form of *Hooke's law* valid for *isotropic materials* is reduced to the following form (Kelvin notation),

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sqrt{2}\sigma_4 \\ \sqrt{2}\sigma_5 \\ \sqrt{2}\sigma_6 \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \sqrt{2}\varepsilon_4 \\ \sqrt{2}\varepsilon_5 \\ \sqrt{2}\varepsilon_6 \end{bmatrix},$$

$$\boldsymbol{\sigma} = \mathbf{S}^{iso} \cdot \boldsymbol{\varepsilon} \Leftrightarrow \sigma_K^{Ke}\,\mathbf{a}_K = S_{KL}^{iso}\,\mathbf{a}_K \otimes \mathbf{a}_L \cdot \varepsilon_L^{Ke}\,\mathbf{a}_L, \quad K, L = 1,\ldots,6.$$

In many academic textbooks the following set of tensors $\boldsymbol{h}_k$ is presented and used as an orthonormal basis for space of second order symmetric tensors,

$$\{\mathbf{S}^{iso} \cdot \mathbf{h} = \lambda \mathbf{h} \sim S_{\alpha\beta}^{iso} h_\beta = \lambda h_\alpha\} \quad \Leftrightarrow \quad \det(\mathbf{S}^{iso} - \lambda \mathbf{I}^{(4s)}) = 0 \rightarrow \lambda_I = 3K \leftrightarrow \mathbf{h}_I, \lambda_K = 2\mu \leftrightarrow \mathbf{h}_K, K = 2,..,6,$$

$$\overbrace{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{h}_1=}_{\mathbf{e}_i\otimes\mathbf{e}_j}, \overbrace{\frac{1}{\sqrt{6}}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}^{\mathbf{h}_2=}_{\mathbf{e}_i\otimes\mathbf{e}_j}, \overbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}^{\mathbf{h}_3=}_{\mathbf{e}_i\otimes\mathbf{e}_j}, \overbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}^{\mathbf{h}_4=}_{\mathbf{e}_i\otimes\mathbf{e}_j}, \overbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}^{\mathbf{h}_5=}_{\mathbf{e}_i\otimes\mathbf{e}_j}, \overbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{\mathbf{h}_6=}_{\mathbf{e}_i\otimes\mathbf{e}_j}.$$

It is not so common knowledge that the set of tensors $\{\mathbf{h}_K\}$ are *elastic eigenstates* of Hooke's tensors describing *isotropic elastic materials*. This can be straightforwardly found by direct calculation. Actually, *any* and *each* second order symmetric deviator is *elastic eigenstate of isotropic Hooke's* tensor.

The concept of isometric tensorial bases.

Definition 9.1 Two *orthonormal bases* are *isometric*, with respect to a *proper orthogonal group*, when a rotation tensor $\mathbf{Q}\in\mathcal{T}_2$ exists ($\mathbf{Q}\mathbf{Q}^{\mathrm{T}}=\mathbf{1}$), such that

$$\mathbf{p}_\alpha = \delta_\alpha^{\ i}\mathbf{Q}\mathbf{e}_i, \quad (\mathbf{p}_\alpha \otimes \mathbf{p}_\beta = \delta_\alpha^{\ i}\mathbf{Q}\mathbf{e}_i \otimes \delta_\beta^{\ j}\mathbf{Q}\mathbf{e}_j, \ldots, etc), \quad \mathbf{e}_i, \mathbf{p}_\alpha, \in E_3$$

cf., e.g., chapter 4 in Ostrowska–Maciejewska textbook.

Not all orthonormal tensorial bases are *isometric* with respect to the *proper orthogonal group*. For example, Kelvin basis $\{\mathbf{a}_K\}$ is *not isometric* with the basis $\{\mathbf{h}_K\}$, resulting from spectral decomposition of isotropic Hooke's tensor,

$$\mathbf{e}_i \otimes \mathbf{e}_j \to \mathbf{a}_K, \; \mathbf{a}_K^{\ Q} = \mathbf{Q}\mathbf{e}_i \otimes \mathbf{Q}\mathbf{e}_j \;\to\; \{\mathbf{a}_K^{\ Q}\} \neq \{\mathbf{h}_K\}.$$

Isotropic tensors have *identical* representation components in all *mutually isometric orthonormal* bases, but *isotropic tensors* in general have *different* representation components in *orthonormal* bases, which are not mutually isometric.

Note. It is worth noting that all (single-handed) orthonormal bases in three-dimensional Euclidean space are isometric.

Ostrowska-Maciejewska J., Podstawy i Zastosowania Rachunku Tensorowego, IPPT PAN, Warsaw, 2007.
http://prace.ippt.gov.pl/IFTR Reports 1 2007.pdf

Second order symmetric tensors, isometric and non-isometric bases.

The *Kelvin basis* $\{\mathbf{a}_K\}$ and *isotropic Hooke's tensor eigenstrains basis* $\{\mathbf{h}_K\}$ of second order symmetric tensors are *non-isometric* orthogonal bases.

$$\boldsymbol{\sigma} = \sigma_1 \mathbf{a}_1 + \sigma_2 \mathbf{a}_2 + \sigma_3 \mathbf{a}_3 + \sigma_4 \mathbf{a}_4 + \sigma_5 \mathbf{a}_5 + \sigma_6 \mathbf{a}_6$$

$$\boldsymbol{\sigma} = \tfrac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3)\mathbf{h}_1 + \tfrac{1}{\sqrt{6}}(2\sigma_1 - \sigma_2 - \sigma_3)\mathbf{h}_2 + \tfrac{1}{\sqrt{2}}(\sigma_2 - \sigma_3)\mathbf{h}_3 + \sigma_4 \mathbf{h}_4 + \sigma_5 \mathbf{h}_5 + \sigma_6 \mathbf{h}_6$$

$$\sigma_1 = \sigma_{11}, \sigma_2 = \sigma_{22}, \sigma_3 = \sigma_{33}, \sigma_4 = \sqrt{2}\,\sigma_{23}, \sigma_5 = \sqrt{2}\,\sigma_{13}, \sigma_6 = \sqrt{2}\,\sigma_{12},$$

$$\mathbf{a}_1 \equiv \mathbf{e}_1 \otimes \mathbf{e}_1 = \tfrac{1}{\sqrt{3}}\mathbf{h}_1 + \tfrac{2}{\sqrt{6}}\mathbf{h}_2,\ \ \mathbf{a}_2 \equiv \mathbf{e}_2 \otimes \mathbf{e}_2 = \tfrac{1}{\sqrt{3}}\mathbf{h}_1 - \tfrac{1}{\sqrt{6}}\mathbf{h}_2 + \tfrac{1}{\sqrt{2}}\mathbf{h}_3,\ \ \mathbf{a}_3 \equiv \mathbf{e}_3 \otimes \mathbf{e}_3 = \tfrac{1}{\sqrt{3}}\mathbf{h}_1 - \tfrac{1}{\sqrt{6}}\mathbf{h}_2 - \tfrac{1}{\sqrt{2}}\mathbf{h}_3,$$

$$\mathbf{h}_1 \equiv \tfrac{1}{\sqrt{3}}\mathbf{1} = \tfrac{1}{\sqrt{3}}[\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3],\ \ \mathbf{h}_2 \equiv \tfrac{1}{\sqrt{6}}[2\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3],\ \ \mathbf{h}_3 \equiv \tfrac{1}{\sqrt{2}}[\mathbf{a}_2 - \mathbf{a}_3],$$

$$\mathbf{a}_4 \equiv \tfrac{1}{\sqrt{2}}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) = \mathbf{h}_4,\ \ \mathbf{a}_5 \equiv \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1) = \mathbf{h}_5,\ \ \mathbf{a}_6 \equiv \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) = \mathbf{h}_6,$$

$$\mathbf{a}_K \cdot \mathbf{a}_L = \delta_{KL} = \mathbf{h}_K \cdot \mathbf{h}_L,\quad \mathbf{a}_K, \mathbf{h}_K \in \mathcal{T}^{s}_{2(n=3)},\ \ K, L = 1,..,6; \qquad \mathbf{1} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \sqrt{3}\,\mathbf{h}_1.$$

$$\underbrace{\boldsymbol{a}_1=}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}_{\mathbf{e}_i\otimes\mathbf{e}_j},\quad \underbrace{\boldsymbol{a}_2=}\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}_{\mathbf{e}_i\otimes\mathbf{e}_j},\quad \underbrace{\boldsymbol{a}_3=}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}_{\mathbf{e}_i\otimes\mathbf{e}_j},$$

$$\mathbf{h}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}_{\mathbf{e}_i\otimes\mathbf{e}_j},\quad \mathbf{h}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}_{\mathbf{e}_i\otimes\mathbf{e}_j},\quad \mathbf{h}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}_{\mathbf{e}_i\otimes\mathbf{e}_j},$$

$$\boldsymbol{a}_4=\boldsymbol{h}_4= \frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}_{\mathbf{e}_i\otimes\mathbf{e}_j},\quad \boldsymbol{a}_5=\boldsymbol{h}_5= \frac{1}{\sqrt{2}}\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}_{\mathbf{e}_i\otimes\mathbf{e}_j},\quad \boldsymbol{a}_6=\boldsymbol{h}_6= \frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}_{\mathbf{e}_i\otimes\mathbf{e}_j}.$$

The concept of isometric tensorial bases.

Representations of the fourth order *isotropic tensors* in two *non-isometric orthonormal bases* $\{\mathbf{a}_K \otimes \mathbf{a}_L\}$ and $\{\mathbf{h}_K \otimes \mathbf{h}_L\}$ *may be different*, for example,

$$\mathbf{I}^{(4s)} = I^{(4s)}{}_{KL}\mathbf{a}_K \otimes \mathbf{a}_L, \qquad \mathbf{I}_{\mathcal{P}} = \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1} = I_{\mathcal{P}\,KL}\mathbf{a}_K \otimes \mathbf{a}_L, \qquad \mathbf{I}_{\mathcal{D}} \equiv \mathbf{I}^{(4s)} - \mathbf{I}_{\mathcal{P}} = I_{\mathcal{D}\,KL}\mathbf{a}_K \otimes \mathbf{a}_L,$$

$$I^{(4s)}_{KL} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_{\mathcal{P}\,KL} \sim \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad I_{\mathcal{D}\,KL} \sim \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{I}^{(4s)} = I^{(4s)}{}_{KL}\mathbf{h}_K \otimes \mathbf{h}_L, \quad \mathbf{I}_{\mathcal{P}} = \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1} = I_{\mathcal{P}\,KL}\mathbf{h}_K \otimes \mathbf{h}_L, \qquad \mathbf{I}_{\mathcal{D}} \equiv \mathbf{I}^{(4s)} - \mathbf{I}_{\mathcal{P}} = I_{\mathcal{D}\,KL}\mathbf{h}_K \otimes \mathbf{h}_L,$$

$$I^{(4s)}_{KL} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_{\mathcal{P}\,KL} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad I_{\mathcal{D}\,KL} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Lecture 10 – synopsis

Interaction of Cauchy stress and Hooke's tensor - the case of elastic isotropy.

10.1 Decomposition of elastic energy of isotropic materials into two uncoupled parts, volumetric and distortional.

10.2 Some remarks on strength of materials criteria based on stored elastic energy for linear elastic isotropic materials.

10.3 Simple shear (SS) and planar shear (PLS) as experimental layouts realizing pure shear (PS) states of stress and/or strain.

Decomposition of elastic energy of isotropic materials.

In the case of linear elastic isotropic materials elastic energy stored in the material fully decouples into only two independent parts one connected with pressure and second connected with shearing forces. This elastic energy partition corresponds to decomposition of stress tensor into spherical (pressure) and deviatoric (shearing) parts and/or decomposition of strain tensor into volumetric and distortional parts. It can be expressed as follows,

$$\Phi(\boldsymbol{\sigma}) \equiv \tfrac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} = \tfrac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{C}^{iso}\cdot\boldsymbol{\sigma} = \frac{1}{2K}\sigma_m{}^2 + \frac{1}{4\mu}\boldsymbol{s}\cdot\boldsymbol{s} = \Phi(\sigma_m\mathbf{1}) + \Phi(\boldsymbol{s}),$$

$$\Phi(\boldsymbol{\varepsilon}) = \qquad = \tfrac{1}{2}\boldsymbol{\varepsilon}\cdot\mathbf{S}^{iso}\cdot\boldsymbol{\varepsilon} = \tfrac{1}{2}K\varepsilon_v{}^2 + \mu\boldsymbol{\varepsilon}^d\cdot\boldsymbol{\varepsilon}^d = \Phi(\varepsilon_v\mathbf{1}) + \Phi(\boldsymbol{\varepsilon}^d).$$

in view of the properties $\sigma_m\mathbf{1}\cdot\boldsymbol{s} = 0, \quad \varepsilon_v\mathbf{1}\cdot\boldsymbol{\varepsilon}^d = 0.$

Immediate conclusion from the above formula is that elastic energy of linear elastic, isotropic material does not depend on *shear stress mode*, i.e., on shear stress mode angle (skewness) angle θ_{sk} (Lode angle θ_L).

Decomposition of elastic energy of isotropic materials.

It is interesting to try to find out why the formula for *elastic energy of linear elastic, isotropic materials does not depend on the shear stress mode angle* θ_{sk} *(third stress invariant)*. A clue for the reason can be grasped upon execution of the following sequence of computations,

$$\Phi(\boldsymbol{\sigma}) = \tfrac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{S}\cdot\boldsymbol{\sigma} = \tfrac{1}{2}(tr(\boldsymbol{\sigma})\mathbf{1}+\mathbf{s})\cdot(\tfrac{1}{9K}tr(\boldsymbol{\sigma})\mathbf{1}+\tfrac{1}{2\mu}\mathbf{s}) = \frac{1}{2K}\sigma_m^{\ 2} + \frac{1}{4\mu}\|\mathbf{s}\|^2$$

$$\sigma_I = \sigma_m + \tfrac{2}{3}\sigma_{ef}\cos(\theta_L),\ \ \sigma_{II} = \sigma_m + \tfrac{2}{3}\sigma_{ef}\cos(\theta_L - 120^0),\ \ \sigma_{III} = \sigma_m + \tfrac{2}{3}\sigma_{ef}\cos(\theta_L + 120^0),$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\sigma_m, \sigma_{ef}, \theta_L) = \sigma_I\,\mathbf{n}_I + \sigma_{II}\,\mathbf{n}_{II} + \sigma_{III}\,\mathbf{n}_{III} = \sigma_m\mathbf{1} + \mathbf{s}$$

$$\|\mathbf{s}\|^2 = \mathbf{s}\cdot\mathbf{s} = \tfrac{2}{3}\sigma_{ef}[\cos(\theta_L)\mathbf{n}_I + \cos(\theta_L - 120^0)\mathbf{n}_{II} + \cos(\theta_L + 120^0)\mathbf{n}_{III}]\cdot$$

$$\tfrac{2}{3}\sigma_{ef}[\cos(\theta_L)\mathbf{n}_I + \cos(\theta_L - 120^0)\mathbf{n}_{II} + \cos(\theta_L + 120^0)\mathbf{n}_{III}] =$$

$$= (\tfrac{2}{3}\sigma_{ef})^2[\cos^2(\theta_L) + \cos^2(\theta_L - 120^0) + \cos^2(\theta_L + 120^0)] = (\tfrac{2}{3}\sigma_{ef})^2\,\tfrac{3}{2} = \tfrac{2}{3}\sigma_{ef}^{\ 2} = r^2.$$

The above computations gives grounds to judge that independence of elastic energy of isotropic materials from shear stress mode (skewness) angle θ_{sk} finds its source in *collinearity of stress and strain* tensors and their *decoupling into pressure (volumetric) and shearing (distortional) parts*.

Some remarks on strength of materials criteria for isotropic materials based on elastic energy.

In numerous works devoted to more advanced materials research, *linear elastic, isotropic constitutive relation* is accepted for the description of the behavior of investigated material. It is next frequently adopted *criterion of material effort* (e.g., plastic flow criterion) in the shape of quadratic form explicitly depending on Lode angle – which is physically interpreted as *elastic energy* stored in the material.

In view of formula $\Phi(\boldsymbol{\sigma}) = \sigma_m^{\,2} / 2K + \|\boldsymbol{s}\|^2 / 4\mu$ such proceedings make methodological inconsistency. Removal of such inconsistency requires in each specific case clearly formulated and well justified additional studies in order to identify the factors actually responsible for very often encountered in experimental works dependence of critical stress of plastic yielding on shear stress mode angle θ_{sk} (Lode angle θ_L), besides the fact that at the same time material is exhibiting with good approximation isotropic elastic deformation behavior.

A very elucidating argument why such situations may occur provides Pierre Curie's symmetry *Principle of Causes and Effects*, which says,

Some remarks on strength of materials criteria for isotropic materials based on elastic energy.

"…Since certain causes produce certain effects, the elements of symmetry of the causes must find reflection in the elements of symmetry of the caused effects.

When certain effects exhibit a certain dissymmetry, this dissymmetry must manifest itself in the causes that generated these effects. …"

$\boldsymbol{\sigma}$ → $\mathbf{M}$ → $f^{cr}(\boldsymbol{\sigma},\mathbf{M})=0$

Input
anisotropic

System
isotropic

Output
can be isotropic or anisotropic

In view of Pierre Curie's symmetry principle somehow mysterious fact that plastic flow condition of some materials while exhibiting linear, isotropic deformation in elastic range of their behavior is non-isotropic – deviates from Huber-Mises condition, can be straightforwardly explained.

Curie P., Ziółkowski A. translator, On Symmetry in Physical Phenomena, Symmetry of an Electric Field and of a Magnetic Field. *Studia Historiae Scientiarum*, **23**, pp. 23–67, 2024 (English translation of Sur la symétrie dans les phénomènes physiques, symétrie d'un champ électrique et d'un champ magnétique. Journal de Physique Théorique et Appliquée, 3e série, 3(1), pp. 393–415, 1894. DOI: 10.1051/jphystap:018940030039300;

Some remarks on strength of materials criteria for isotropic materials based on elastic energy.

This can be explained as follows: the cause (stress $\boldsymbol{\sigma}$) is *anisotropic* in general, the system (Hooke's stiffness tensor $\mathbf{C}$ describing elastic properties of material) is *isotropic*, but the result, plastic flow function $f^{cr}(\boldsymbol{\sigma}, \mathbf{C})$ can be *isotropic* or *anisotropic* depending on the "symmetrization strength" of the system (here material).

The frequently adopted simple and straightforward modeling approach to deal with such situations is to accept as the material yielding criterion the quadratic form $f^{cr}(\boldsymbol{\sigma}, \mathbf{M})=\boldsymbol{\sigma}\mathbf{M}\boldsymbol{\sigma}$ - sometimes called *Mises stress intensity*, which cannot be physically interpreted as elastic energy stored in the material. The tensor $\mathbf{M}$ present in these kinds of strength criteria is called *limit tensor*.

There can be identified other factors creating modeling difficulties. For example in the case of polymeric, rubberlike materials their elastic behavior is physically generated by change of internal entropy of these materials and not internal energy, cf., e.g., Ingo Müller or A. Ziółkowski.

Müller I., History of Thermodynamics, Springer, Berlin, 2007. pp. 111-112
Ziółkowski A., Methodology and micromechanical estimation of macroscopic elastic energy, coherence energy, and phase transition strains for SMA materials, Researchgate 10.13140/RG.2.2.30420.50561, 2018 pp. 31-33.

Some remarks on strength of materials criteria for isotropic materials based on elastic energy.

Due to that material symmetry of polymeric materials changes as a result of application of standard engineering loads. Even when material is isotropic at zero loading its symmetry of internal structure changes, usually into transversely isotropic one, when loaded to moderate strains. The material internal structure returns to original symmetry (isotropy) upon removal of the loading. The other reason for anisotropic plastic yielding might be existence of so called, *internal constraints* operating in the material (of force or kinematic character, of known or unknown physical origins).

Considerable attention has been devoted in the present work to the theoretical issues connected with pure shear mode/state. Let us discuss at present so called *planar shear* and *simple shear,* i.e., two major *experimental testing layouts* leading to *actual physical realization* of the *pure shear*.

A lot of misunderstandings exist in the literature regarding difference between *simple shear* versus *planar shear* testing. In mechanics the "pure shear mode" is considered in terms of not only stress but also in terms of strain. These later interpretation, i.e. *pure shear strain*, is more convenient for the present discussion. No conceptual difference between stress and strain interpretation of pure shear mode exists when tested material is isotropic because in such a case straightforward equivalence exists between stress and strain tensors due to their coaxiality.

Simple shear and pure shear as experimental layouts realizing pure shear states of stress and/or strain.

The *simple shear* and *planar shear* testing layouts belong to the class of *biaxial tests*. In order to clarify the issues, *simple shear* and *planar shear are both pure shear modes* of deformation because in both cases trace and determinant of strain tensor during testing are equal to zero,

$$tr(\boldsymbol{\varepsilon}) = 0, \det(\boldsymbol{\varepsilon}) = 0 \Rightarrow tr(\boldsymbol{\varepsilon}^3) = 0.$$

The difference in kinematics, i.e., physical motion of material points, *exists between these two testing layouts*. While different kinematics means different deformation gradients, the principal values of stretch tensor $\mathbf{U}$ ($\mathbf{F}=\mathbf{RU}$) are exactly the same in both layouts, though differently situated in laboratory frame.

Here, only the most important characteristics of simple and planar shear are succinctly and explicitly recalled in order to possibly facilitate taking decision on the selection of one or the other experimental layout for attaining specific experimental research tasks.

More detailed discussion of pure shear and simple shear interested reader can find for example in Ogden and/or Ziółkowski.

Ogden R.W., Non-linear elastic deformations. Dover Publications, Inc. 1997.
Ziółkowski A., Simple shear test in identification of constitutive behavior of materials submitted to large deformations – hyperelastic materials case, Engng. Trans., 54, 4, 2006, pp. 251-269.

Simple shear and pure shear as experimental layouts realizing pure shear states of stress and/or strain.

The deformation gradients $\mathbf{F}$ and right stretch tensors $\mathbf{U}$ for simple shear and planar shear can be expressed as follows

$$\mathbf{F}^{ss} = \mathbf{R}^{ss}\mathbf{U}^{ss} = \underset{\mathbf{e}_i \otimes \mathbf{e}_j}{\begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Leftrightarrow \mathbf{U}^{ss} = \underset{\mathbf{u}_i \otimes \mathbf{u}_j}{\begin{bmatrix} \lambda^{ss} & 0 & 0 \\ 0 & 1/\lambda^{ss} & 0 \\ 0 & 0 & 1 \end{bmatrix}}, \quad \mathbf{F}^{pls} = \mathbf{U}^{pls} = \underset{\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{u}_i \otimes \mathbf{u}_j}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^{pls} & 0 \\ 0 & 0 & 1/\lambda^{pls} \end{bmatrix}}$$

where denotation symbol "ss" denotes simple shear and "pls" denotes planar shear.

The *major matching feature* of *simple shear* and *planar shear* testing layouts is that - in both layouts *identical strain pattern* is formed what finds reflection in *identical values of principal stretches*.

In the case of *simple shear* principal stretches take the form,

$$\lambda_I^{ss} = \lambda^{ss}, \lambda_{II}^{ss} = 1/\lambda^{ss}, \lambda_{III}^{ss} = 1, \quad \lambda^{ss} = \tfrac{1}{2}\gamma + \sqrt{1+(\tfrac{1}{2}\gamma)^2},$$

where γ denotes so called *shear parameter*.

In the case of *planar shear* principal stretches take the form,

$$\lambda_I^{ps} = \lambda^{ps}, \lambda_{II}^{ps} = 1, \lambda_{III}^{ps} = 1/\lambda^{ps}, \quad \lambda^{ps} = \Delta L / L_0.$$

where $\Delta L = L - L_0$ denotes elongation of the specimen.

Simple shear and pure shear as experimental layouts realizing pure shear states of stress and/or strain.

When λ^{ss} is equated to λ^{ps} one to one correspondence can be immediately found between γ and ΔL.
In both experimental testing layouts volume is preserved,

$$\det(\mathbf{F}) = \lambda_I^{ss}\lambda_{II}^{ss}\lambda_{III}^{ss} = \lambda_I^{ps}\lambda_{II}^{ps}\lambda_{III}^{ps} = dv / dV = 1$$

where dv denotes elementary volume in actual configuration and dV is elementary volume in initial configuration.

The *major distinctive feature* differing *simple shear* from *planar shear* is that

- in *simple shear* layout *principal axes constantly rotate* ($\mathbf{R}^{ss} \neq \mathbf{1}$), while
- in *planar shear* layout *principal axes remain fixed* ($\mathbf{R}^{ps} \equiv \mathbf{1}$) relative to fixed laboratory frame at all times during advancement of shear loading.

Simple shear and pure shear as experimental layouts realizing pure shear states of stress and/or strain.

Pure Shear experimental layouts

$$\mathbf{E}^{(0)} \equiv ln(\mathbf{U}),\ \ tr(\mathbf{E}^{(0)}) = 0$$

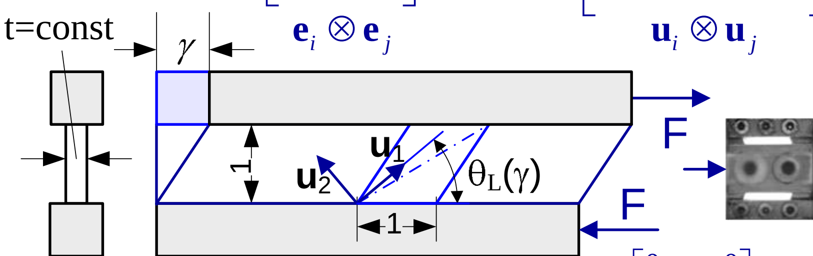
$$\det(\mathbf{E}^{(0)}) = tr((\mathbf{E}^{(0)})^3) = 0$$

$$\det(\mathbf{F}) = V/V_0 = \lambda_{\mathrm{I}}\lambda_{\mathrm{II}}\lambda_{\mathrm{III}} = 1$$

Simple Shear

(= Pure Shear with constantly rotating principal axes)

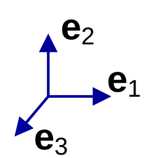
$$\mathbf{F}^{ss} = \mathbf{R}^{ss}\mathbf{U}^{ss} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i \otimes \mathbf{e}_j} \Leftrightarrow \mathbf{U}^{ss} = \begin{bmatrix} \lambda^{ss} & 0 & 0 \\ 0 & 1/\lambda^{ss} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{u}_i \otimes \mathbf{u}_j}$$



$$\lambda_{\mathrm{I}}^{ss} = \lambda^{ss},\ \lambda_{\mathrm{II}}^{ss} = 1/\lambda^{ss},\ \lambda_{\mathrm{III}}^{ss} = 1 \qquad \boldsymbol{\sigma} \sim \begin{bmatrix} 0 & t & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda^{ss} = \tfrac{1}{2}\gamma + \sqrt{1 + (\tfrac{1}{2}\gamma)^2},\ \gamma = \lambda^{ss} - 1/\lambda^{ss}$$

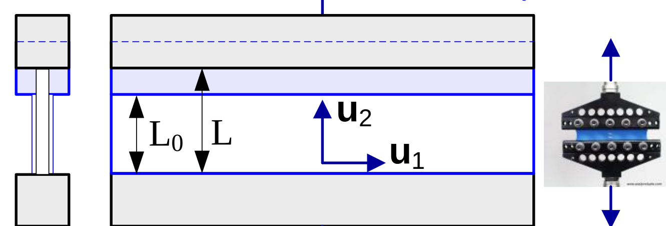
$$\tan(2\theta_L) = -2/\gamma,\ \ \theta_L \in <\pi/4 \nearrow \pi/2>$$



Planar Shear

(= Pure Shear with fixed principal axes)

$$\mathbf{F}^{pls} = \mathbf{U}^{pls} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^{pls} & 0 \\ 0 & 0 & 1/\lambda^{pls} \end{bmatrix}_{\mathbf{e}_i \otimes \mathbf{e}_j}$$



$$\boldsymbol{\sigma} \sim \begin{bmatrix} t & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda^{pls} = (L - L_0)/L_0$$

$$\lambda_{\mathrm{I}}^{pls} = 1,\ \ \lambda_{\mathrm{II}}^{pls} = \lambda^{pls},\ \ \lambda_{\mathrm{III}}^{pls} = 1/\lambda^{pls}$$

The $\mathbf{F}$ denotes deformation gradient, $\mathbf{U}$ is right stretch tensor, $\mathbf{E}^{(0)}$ denotes logarithmic Lagrangian strain measure, λ_J are principal stretches, $\mathbf{u}_{\mathrm{i}}$ denote Lagrangian principal axes, γ denotes amount of shear parameter, θ_L denotes orientation angle of Lagrangian principal axes with respect to fixed laboratory frame;

Photo of double simple shear grip after Fig. 4 in Nowacki et al. Effect of strain rate on ductile fracture. A new methodology. (https://www.ippt.pan.pl/_repository/open/o119.pdf); Photo of planar shear grip after Axel Physical Testing Services (https://www.axelproducts.com).

Simple shear and pure shear as experimental layouts realizing pure shear states of stress and/or strain.

The *simple shear* testing layout is very popular (standard) in experimental testing of behavior and/or properties of *metallic materials*.

The *planar shear* testing layout is very often used (standard) in examination of *polymeric materials*.

Many additional factors, besides strain pattern, may have influence on choosing one layout or the other. For example stiffness of metallic samples prevents early warping of the sample during simple shear testing. On the other hand testing metallic sheets in planar shear scheme might require considerably larger forces in comparison to simple shear scheme of testing.

It is worth to indicate that *loadings* used in testing *of metallic samples* as a standard *does not involve change of symmetry of the material*.

In the case of testing *polymeric materials* used in their testing *loadings* as a standard *do induce change of their internal structure symmetry* – due to entropic origin of polymeric elasticity, e.g., initially isotropic polymeric material changes its symmetry to transversely isotropic under testing load.

From the above discussion it can be concluded that execution of simple shear and planar shear tests on the same material allows to evaluate the influence of principal axes rotation on the behavior of the material.

Lecture 11 – synopsis

Biaxial (planar) tests in experimental examination of materials behavior.

11.1 Specific features of biaxial tests.

11.2 Relations between triaxiality factor and shear stress mode angle θ_{sk} (θ_L) in biaxial tests.

Specific features of biaxial tests

The class of *biaxial tests* is defined by the condition that *always one of the principal values of stress tensor is equal to zero*.

According to the ordering convention of principal values ($\sigma_{\mathrm{I}} \geq \sigma_{\mathrm{II}} \geq \sigma_{\mathrm{III}}$) this should be denoted as middle principal value ($\sigma_{\mathrm{II}}=0$) but usually, conventionally, it is written that the third principal value is zero ($\sigma_{\mathrm{III}}=0$) regardless of the standard ordering convention.

In view of $\sigma_{\mathrm{III}} \equiv 0$ two control *parameters* only, for example, *two principal values* $\{\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}\}$, *uniquely determine* any set of *three principal stress invariants* fully characterizing stress tensor treated as sovereign object, e.g., $\{\sigma_m, J_2, J_3\}$. Instead of stress principal values some other convenient pair of control parameters can be selected for example $\{\sigma_{\mathrm{m}}, \Delta\sigma\}$.

The following relations are valid in the case of *biaxial tests*,

$$\sigma_{III} = 0 \;\Rightarrow\; \sigma_m = \tfrac{1}{3}(\sigma_I + \sigma_{II}),\;\; \Delta\sigma = (\sigma_I - \sigma_{II}),$$

$$s_I = \sigma_I - \sigma_m,\; s_{II} = \sigma_{II} - \sigma_m,\; s_{III} = -\sigma_m,$$

$$J_2 = s_{III}{}^2 - s_I s_{II} = \tfrac{1}{3}[\sigma_I{}^2 + \sigma_{II}{}^2 - \sigma_I \sigma_{II}] = \tfrac{1}{4}[3\sigma_m{}^2 + \Delta\sigma^2],$$

$$J_3 = s_I s_{II} s_{III} = -\tfrac{1}{27}(\sigma_I + \sigma_{II})(2\sigma_{II} - \sigma_I)(2\sigma_I - \sigma_{II}) = \tfrac{1}{4}\sigma_m[\Delta\sigma^2 - \sigma_m{}^2] = \sigma_m[J_2 - \sigma_m{}^2]$$

Specific features of biaxial tests

Thus, for biaxial tests ($\sigma_{\mathrm{III}}=0$) the following inequalities are valid,

$$\|\mathbf{s}\|=\sqrt{2J_2}=\sqrt{\tfrac{3}{2}[\sigma_m{}^2+\tfrac{1}{3}\Delta\sigma^2]}\geq\sqrt{\tfrac{3}{2}}\,|\sigma_m|\geq 0,\quad \sigma_{ef}=\sqrt{3J_2}\geq\tfrac{3}{2}|\sigma_m|$$

The above reveals interesting information that *modulus of deviatoric* part and *effective stress* – measure of shearing stresses intensity, of any non-zero planar (2D) stress tensor is always greater than *absolute value of its mean (pressure)*.

However, *modulus of deviator* and *effective stress* of any non-zero planar (2D) stress tensor *is not always* greater that the norm of its hydrostatic part - spherical part $(\|\boldsymbol{\sigma}^{sph}\|=\sqrt{3}|\sigma_m|)$.

Specific features of biaxial tests

Theorem 11.1

The *radial lines (rays) coming out from the origin* ($\sigma_I=0, \sigma_{II}=0$) of coordinate frame of biaxial tests domain, i.e., half plane $\sigma_I \geq \sigma_{II}$ are lines of constant values of triaxiality factor $\eta=const$ and at the same time lines of constant values of shear stress mode (skewness) angle $\theta_{sk}=const$ (Lode angle $\theta_L=const$) .

Proof

The radial lines running from the origin can be described as follows,

$$\sigma_{II} = a\,\sigma_I \;(a = const) \quad \Rightarrow \quad \sigma_m = \tfrac{1}{3}(\sigma_I + \sigma_{II}) = \tfrac{1}{3}(1+a)\sigma_I,$$

$$\sigma_{ef} = [\sigma_I^{\,2} + \sigma_{II}^{\,2} - \sigma_I\sigma_{II}]^{1/2} = [1 - a + a^2]^{1/2}\,|\sigma_I| \Rightarrow \eta = \frac{\sigma_m}{\sigma_{ef}} = \frac{1}{3}\frac{(1+a)}{[1-a+a^2]^{1/2}}\,sign(\sigma_I),$$

$$a = const \Leftrightarrow \eta = const \Leftrightarrow \theta_{sk} = const.$$

The equivalence relation $\eta=const <=> \theta_{sk}=const$ results from formulas (*) given in slide 165.

In the case when $\sigma_I=0$, σ_{II} can take any value, and then it is $\eta=-⅓=const$, $\theta_{sk}=-30^0= const$.   q.e.d.

Theorem 11.2

The relations $\sigma_m(\sigma_{ef}, \theta_{sk})$, $\sigma_{ef}(\sigma_m, \theta_{sk})$, $\theta_{sk}(\sigma_m, \sigma_{ef})$ are *bijections (one to one relations)* in *three complementary areas* of the whole domain of biaxial tests domain parameterized with ($\sigma_I \geq \sigma_{II}$; $\sigma_{III}=0$), except on the line $\sigma_m=(\sigma_I+\sigma_{II})/3=0$, on which $\theta_{sk}=\eta=0$ for any value of $\sigma_{ef}=\sqrt{3}\sigma_I$.

Proof

It is straightforward to show that,

- *skewness angle* θ_{sk} maintains *constant* value on *radial lines* running from the origin ($\sigma_I=0, \sigma_{II}=0$) of biaxial tests domain coordinates frame,
- *mean value of stress* σ_m maintains *constant* value on *45 degrees slanted lines* in the biaxial tests domain,
- *effective stress* σ_{ef} maintains *constant* value on *ellipsoids with centers in the origin* ($\sigma_I=0, \sigma_{II}=0$) of biaxial tests domain.

In view of the above, at any specific point of three subdomains of biaxial tests domain (mutually separate), the value of any variable chosen from triple set $\{\sigma_m, \sigma_{ef}, \theta_{sk}\}$ can be uniquely determined by the values of two remaining ones.

On line $\sigma_m=\sigma_I+\sigma_{II}=0$ it is

$(\sigma_m=(\sigma_I+\sigma_{II})/3=0) \Rightarrow (J_3=0,\ \sigma_{ef}=\sqrt{3}\sigma_I) \Rightarrow \theta_{sk}=\eta=0.$ q.e.d.

Specific features of biaxial tests

An important open problem of *experimental mechanics of materials* is determination of *critical stress states surfaces* conditioning initiation of some physical processes in materials, for example plastic yield flow, damage, cracking or start of phase transition. In that respect the following observation can be formulated.

Corollary 11.1

In the case of convex critical surface with the aid of whatever type of biaxial test, for any fixed value of mean stress (pressure) σ_m^*, critical effective stress σ_{ef}^* can be determined for only single value of skewness (Lode) angle θ_{sk}^*.

Corollary 11.2

In the case of convex critical surface with the aid of whatever type of biaxial test, for any fixed value of skewness (Lode) angle θ_{sk}^*, critical effective stresses σ_{ef}^* can be determined for only three values of mean stress (pressure) σ_m^*.

The above Corollaries 11.1 and 11.2 are consequences of Theorems 11.1 and 11.2.

Specific features of biaxial tests

The direct conclusion from the Corollaries 11.1 and 11.2 is that planar (biaxial) tests, among them very common tension (compression)-torsion tests on tubular samples (also the ones with internal pressure), are *not suitable* for executing methodologically correct experimental examination of the influence of *shear stress mode (skewness) angle (Lode angle)* on materials behavior.

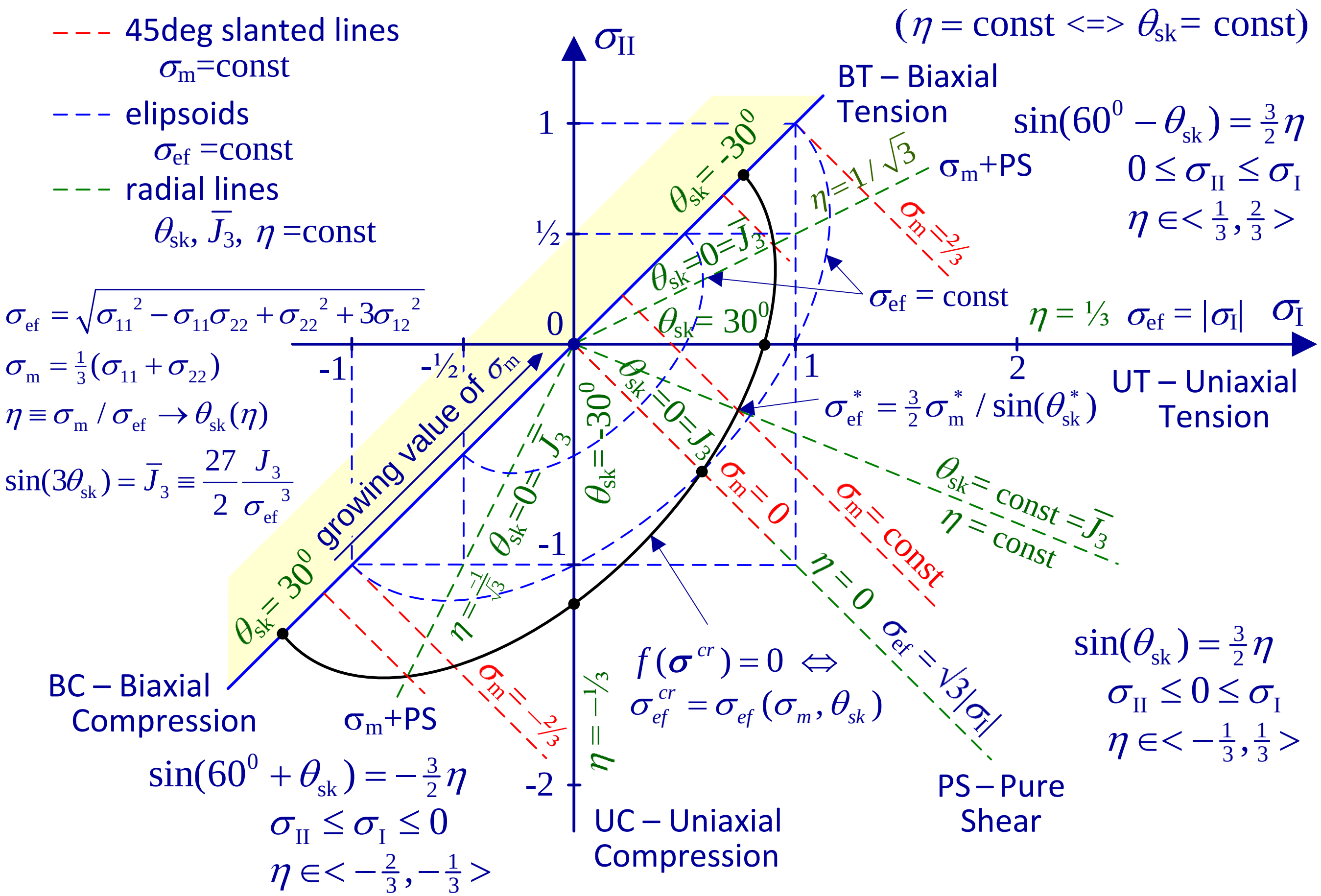
This is so because using biaxial tests only, *no sufficient experimental data can be collected to reliably separate the influence of mean stress and/or skewness angle* on the possible variations of critical effective stresses. One value for any fixed pressure and/or three values for any fixed skewness angle are rather insufficient for such purpose.

This observation delivers a clear incentive for development and use of experimental techniques in which all three parameters characterizing stress state can be independently controlled to induce in the specimen not only 2D planar stress state but fully 3D complex stress state loadings.

They should make possible determination of critical effective stresses, or other parameters, e.g., example effective fracture strains in the whole domain of shear stress mode (skewness) angle values at freely prescribed, fixed mean stress.

Specific features of biaxial tests

Biaxial tests domain of stress loadings $(\sigma_I \geq \sigma_{II};\ \sigma_{III} \equiv 0)$



Graphical illustration of biaxial tests domain of possible stress loadings showing paths of constant values of parameters σ_m=*const*, σ_{ef}=*const*, θ_{sk}=*const* (η=*const*). The $f(\boldsymbol{\sigma}^{cr})=0$ illustrates hypothetical convex critical surface, e.g., plastic yield stresses, of a certain isotropic material for which critical values of effective stress depend on pressure and shear stress mode angle $\sigma_{ef}^{cr} = \sigma_{ef}(\sigma_m, \theta_{sk})$.

Relations between triaxiality factor and shear stress mode angle θ_{sk} (θ_L Lode angle) in biaxial tests.

In 1959 Davies and Connelly introduced the concept of *triaxiality factor*, defined as quotient of Cauchy stress first principal invariant divided by effective stress,

$$\eta_{DC} \equiv I_1 / \sqrt{3J_2} = 3\sigma_m / \sigma_{ef}, \quad \sigma_{ef} \neq 0$$

They were motivated in this proposal by supposition, correct in view of their own and later research, that spherical tension σ_m (*pressure*) called by them rather exotically *triaxial tension* has strong influence on the *loss of ductility of metals*, and the need to have some parameter to describe this effect.

The name *triaxiality factor* for the parameter is rather *unfortunate* because it gives false impression that *some general 3D multiaxial stress states* are subject of description with this parameter, while actually it describes magnitude of *pressure forces in relation to shear forces*.

The triaxiality factor gained considerable attention and use when Wierzbicki and his collaborators pointed out that not only *spherical tension* (*negative pressure*) but also Lode angle can considerably *influence ductility and other properties of metals*.

Davies E.A., Connelly F.M., Stress Distribution and Plastic Deformation in Rotating Cylinders of Strain-Hardening Material, J. Applied Mech., March 1959, pp. 25-30.

Relations between triaxiality factor and shear stress mode angle θ_{sk} (θ_L Lode angle) in biaxial tests.

Wierzbicki and Xue in 2005 found that in the case of *biaxial tests* unique relation exists between *Lode angle* (normalized principal third invariant of deviator) and *triaxiality factor*, formula (8) in Bai and Wierzbicki,

$$\bar{J}_3 \equiv \cos(3\theta_L) = -\tfrac{27}{2}\eta(\eta^2 - \tfrac{1}{3}).$$

Wierzbicki et. al. adopted slightly modified definition of the triaxiality factor than the original one,

$$\eta \equiv \sigma_m / \sigma_{ef} = \tfrac{1}{3}\eta_{DC}.$$

Since that time *triaxiality factor* started to be very frequently used in charts as *governing parameter* to present experimental results obtained in *biaxial tests* in order to present *influence of the value of Lode angle* on various properties of metals and other materials.

Wierzbicki and Xue constraint relation valid for *biaxial tests* can be expres-sed, in the equivalent form, as classical third power polynomial equation,

$$\eta^3 - \tfrac{1}{3}\eta + \tfrac{2}{27}\sin(3\theta_{sk}) = 0, \qquad \leftrightarrow \qquad \bar{J}_3 = \sin(3\theta_{sk}) = \tfrac{27}{2}[\tfrac{1}{3}\eta - \eta^3].$$

in the above in place of relation for Lode angle θ_L, used originally by Wierzbicki and Xue, the relation for shear stress mode angle is used θ_{sk}.

Bai, Y., Wierzbicki, T., A new model of metal plasticity and fracture with pressure and Lode dependence, Int. J. Plast. 24, 6, 2008, pp. 1071-1096.

Relations between triaxiality factor and shear stress mode angle θ_{sk} (Lode angle θ_L) in biaxial tests.

This equation $\eta^3 - \frac{1}{3}\eta + \frac{2}{27}\sin(3\theta_{sk}) = 0$ can be solved with the same method as the one used for finding stress principal values from characteristic equation. The solution can be written in the following form,

$$\sin(60^0 - \theta_{sk}) = \tfrac{3}{2}\eta \quad when \quad \sigma_{III} = 0 \le \sigma_{II} < \sigma_I, \quad \eta \in < \tfrac{2}{3}, \tfrac{1}{3} >, \quad \theta_{sk} \in < -30^0, 30^0 >,$$

$$\sin(\theta_{sk}) = \tfrac{3}{2}\eta \quad when \quad \sigma_{III} \le \sigma_{II} = 0 \le \sigma_I, \quad \eta \in < -\tfrac{1}{3}, \tfrac{1}{3} >, \quad \theta_{sk} \in < -30^0, 30^0 >,$$

$$\sin(60^0 + \theta_{sk}) = -\tfrac{3}{2}\eta \quad when \quad \sigma_{III} < \sigma_{II} \le \sigma_I = 0, \quad \eta \in < -\tfrac{1}{3}, -\tfrac{2}{3} > \quad \theta_{sk} \in < -30^0, 30^0 >.$$

$$sign(\eta) = sign(\sigma_m), \quad sign(\theta_{sk}) = sign(\bar{J}_3),$$

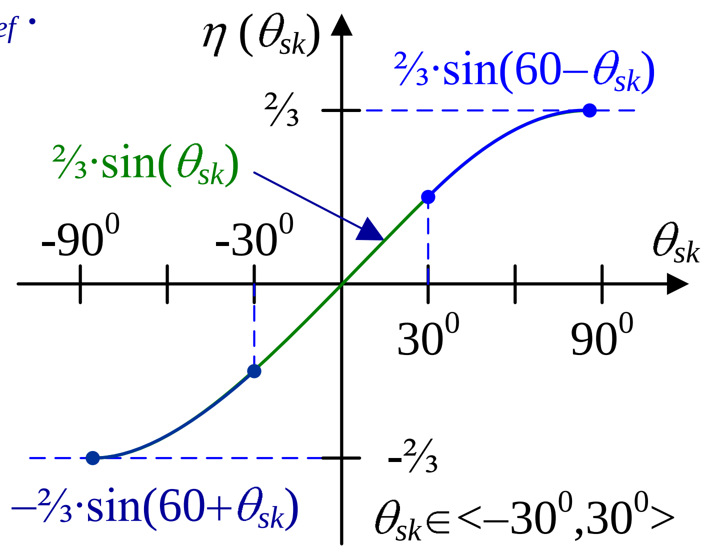
$$\sigma_{ij} \Rightarrow \sigma_m = \tfrac{1}{3}\sigma_{ii}, \sigma_{ef} = \sqrt{3J_2} = \sqrt{\tfrac{3}{2} s_{ij} s_{ij}} \Rightarrow \eta \equiv \sigma_m / \sigma_{ef}.$$

In expressing the above formulas standard denotation convention of principal stresses ordering ($\sigma_I \ge \sigma_{II} \ge \sigma_{III}$) and the following identities were employed,

$$4\sin^3(\theta_{sk}) - 3\sin(\theta_{sk}) + \sin(3\theta_{sk}) = 0,$$

$$\sin(\theta - 120^0) = -\sin(\theta + 60^0),$$

$$\sin(\theta + 120^0) = \sin(60^0 - \theta).$$



Relations between triaxiality factor and shear stress mode angle θ_{sk} (Lode angle θ_L) in biaxial tests.

Valid for biaxial tests explicit relations linking triaxiality factor and skewness angle $\eta \leftrightarrow \theta_{sk}$ are three bijections (one to one relations) in three sharing edges but otherwise separate areas. Together they embrace entire *two parameters* domain (half-plane) of biaxial tests stress loadings ($\sigma_I \geq \sigma_{II}; \sigma_{III}=0$),

$$\eta = \tfrac{2}{3}\sin(60^0 - \theta_{sk}) \;\; when \;\; 0 \leq \sigma_{II}, \sigma_I, \quad \eta = \tfrac{2}{3}\sin(\theta_{sk}) \;\; when \;\; \sigma_{II} \leq 0 \leq \sigma_I, \quad (*)$$

$$\eta = -\tfrac{2}{3}\sin(60^0 + \theta_{sk}) \;\; when \;\; \sigma_{II}, \sigma_I \leq 0.$$

Formulas for triaxiality factor valid for *biaxial tests* show that in such a case the values of triaxiality factor must always remain in the range $\eta \in < -⅔, ⅔ >$, while in general case of *unconditioned 3D multiaxial tests* the triaxiality factor can take any value from the range $\eta \in < -\infty, \infty >$.

In many *experimental mechanics* publications, in which results from *biaxial tests* are presented, there can be noticed values of triaxiality factor exceeding the two third value $⅔ \leq \eta$, which may seem to be *incorrect*.

However, experimental observation of *triaxiality factor greater than ⅔* rather indicates that *conditions of biaxiality were lost*, and in the sample true general triaxial stress state started to exist.

Hint This delivers a hint to develop experimental methodologies, in which *triaxiality factor* is used as an effective *indicator of passing from plane (2D) state of stress to three dimensional (3D) state of stress*.

Relations between triaxiality factor and shear stress mode angle θ_{sk} (Lode angle θ_L) in biaxial tests.

Hint

The explicit relations linking triaxiality factor and shear stress mode (skewness) angle $\eta \leftrightarrow \theta_{sk}$ are *convenient for numerical computations*. This is so because they enable efficient determination of the value of skewness angle θ_{sk} (Lode angle θ_L) from the value of triaxiality factor η,

$$\sigma_{ij} \;\Rightarrow\; \sigma_m = \tfrac{1}{3}\sigma_{ii}, \sigma_{ef} = \sqrt{3J_2} = \sqrt{\tfrac{3}{2} s_{ij} s_{ij}} \;\Rightarrow\; \eta \equiv \sigma_m / \sigma_{ef} \;\Rightarrow$$

$$\eta \in < \tfrac{2}{3}, \tfrac{1}{3} > \quad (0 \le \sigma_{II} \le \sigma_I) \Rightarrow \quad \theta_{sk} = -\arcsin(\tfrac{3}{2}\eta) + 60^0, \quad \theta_{sk} \in < -30^0, 30^0 >,$$

$$\eta \in < -\tfrac{1}{3}, \tfrac{1}{3} > \quad (\sigma_{II} \le 0 \le \sigma_I) \Rightarrow \quad \theta_{sk} = \arcsin(\tfrac{3}{2}\eta), \quad \theta_{sk} \in < -30^0, 30^0 >,$$

$$\eta \in < -\tfrac{1}{3}, -\tfrac{2}{3} > (\sigma_{II} \le \sigma_I \le 0) \Rightarrow \quad \theta_{sk} = -\arcsin(\tfrac{3}{2}\eta) - 60^0, \quad \theta_{sk} \in < -30^0, 30^0 >.$$

Selection of the *proper formula* for calculation of skweness angle does not require computation of principal values of stress tensor because it can be decided upon *the value of* η falling into specific range of values of η.

For example, when $\eta^* = 0.51 \in < ⅔, ⅓ >$ then $\theta_{sk}^* = -\sin^{-1}(1.5 \cdot \eta^*) + 60^0$.

The use of above formulas is *more efficient numerically* then using the Wierzbicki, Xue formula $\theta_{sk} = \tfrac{1}{3}\sin^{-1}(\tfrac{27}{2}(\tfrac{1}{3}\eta - \eta^3))$.

Relations between triaxiality factor and shear stress mode angle θ_{sk} (θ_L) in biaxial tests.

Hint

Triaxiality factor η *is not convenient operand*, in general, to be used *for presentation of experimental biaxial tests results*.

This is so because when taken at its face value it contains tangled together information on two in principle linearly independent parameters characterizing stress tensor (loading), i.e., σ_m and σ_{ef}. This entanglement projects to the presented results making them somehow blurred.

In the case of *biaxial tests* in view of the existence of one to one relation between triaxiality factor and shear stress mode (skewness) angle θ_{sk} (Lode angle θ_L), actually *constant value of triaxiality factor corresponds to constant value of shear stress mode (skewness) angle*. It is advisable to directly use *shear stress mode angle* θ_{sk} *as governing parameter in charts* presenting experimental biaxial tests results. Possibly, with information indicating the mode of loading: tensioning ($0<\sigma_I,\sigma_{II}$), mixed ($\sigma_{II}<0<\sigma_I$) or compressive ($\sigma_{II},\sigma_I < 0$).

In this manner specific information presented in the chart will be delivered in transparent, methodologically unambiguous manner.

Lecture 12 – synopsis

Closing remarks and summary.

Other than stress physical interpretations of second order symmetric tensors, Novozhilov (material) tensors.

The *second order symmetric tensors* interpreted in this work as *Cauchy stress*, i.e., characterizing force interactions can have and have other physical interpretations.

An interesting and useful *physical interpretation* of second order symmetric tensors was delivered by V.V. Novozhilov who introduced two tensors characterizing elastic properties of materials defined as follows,

$$\boldsymbol{\mu} \equiv \mathbf{H} \cdot \mathbf{1} \qquad \sim \mu_{ij} \equiv H_{ijkk} \qquad \Rightarrow \quad tr(\boldsymbol{\mu}) = \mathbf{1} \cdot \mathbf{H} \cdot \mathbf{1},$$

$$\boldsymbol{\nu} \equiv \mathbf{H}^{<3,2>} \cdot \mathbf{1} \quad \sim \nu_{ij} \equiv H_{ikkj} = H_{ikjk} \quad \Rightarrow \quad tr(\boldsymbol{\nu}) = \boldsymbol{\nu} \cdot \mathbf{1} = tr(\mathbf{H})$$

where $\mathbf{H}$ denotes some Hooke's tensor.

The *Novozhilov tensors* are linear isotropic functions of Hooke's tensor, i.e., they are tensors *characterizing elastic properties* of materials.

Tensor $\boldsymbol{\mu}$ calculated for elastic stiffness tensor $\mathbf{S}$ describes the body's reaction to spherical deformation, i.e., when $\boldsymbol{\varepsilon} = \mathbf{1}$ then $\boldsymbol{\sigma} = \mathbf{S} \cdot \mathbf{1} = \boldsymbol{\mu}$.. Taking advantage of *spectral decomposition* of Hooke's tensor, it can be expressed as follows,

$$\boldsymbol{\mu} = \mathbf{S} \cdot \mathbf{1} = \lambda_I \, tr(\boldsymbol{\omega}_I)\boldsymbol{\omega}_I + \ldots + \lambda_{VI} \, tr(\boldsymbol{\omega}_{VI})\boldsymbol{\omega}_{VI}.$$

Tensor $\boldsymbol{\nu}$ calculated for elastic stiffness tensor plays important role in dynamics of elastic waves, $\boldsymbol{\nu} = \mathbf{S}^{<3,2>} \cdot \mathbf{1} = \lambda_I \, \boldsymbol{\omega}_I^{\,2} + \ldots + \lambda_{VI} \, \boldsymbol{\omega}_{VI}^{\,2}$.

Novozhilov V.V. Theory of elasticity, Sudpromgiz, Leningrad, 1958.
Rychlewski J. Mathematical structure of elastic bodies, IPPT PAN Reports, Warszawa 2023

Other than stress physical interpretations of second order symmetric tensors, Christoffel (acoustic) tensor.

Christoffel E.S. in 1877 published work in which he formulated the problem of propagation of plane waves in anisotropic elastic media,

$$(\rho\frac{\partial^2 u_i}{\partial t^2}=\frac{\partial\sigma_{ij}}{\partial x_j},\ \boldsymbol{\sigma}=\mathbf{S}\boldsymbol{\varepsilon}\ \sim\sigma_{ij}=S_{ijkl}\varepsilon_{kl}\ )\ \Rightarrow\ (\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=\mathbf{S}\frac{\partial^2\mathbf{u}}{\partial\mathbf{x}^2}\ \sim\rho\frac{\partial^2 u_i}{\partial t^2}=S_{ijkl}\frac{\partial^2 u_k}{\partial x_j\partial x_l})$$

Christoffel showed that the problem of *waves propagation* in *anisotropic elastic media* can be reduced to the *eigenvalue problem* of second order symmetric tensor,

$$\mathbf{u}(\boldsymbol{x},t)=\mathbf{p}e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)},\quad \boldsymbol{k}=(\omega/V)\boldsymbol{n}\quad\Rightarrow\quad(\boldsymbol{\chi}-\rho V_M^2\mathbf{1})\boldsymbol{p}_M=\mathbf{0}$$

The tensor $\boldsymbol{\chi}$ is called *acoustic (Christoffel) tensor* $\boldsymbol{\chi}(\boldsymbol{n})\equiv\boldsymbol{n}\mathbf{S}\boldsymbol{n}$ ($\chi_{ik}=S_{ijkl}n_j n_l$), where $\mathbf{S}$ denotes elastic stiffness tensor and $\boldsymbol{n}$ is versor of wave propagation direction.

Polarization versor $\boldsymbol{p}$ is parallel to versor of wave propagation $\boldsymbol{n}$ for *longitudinal waves*, and it is orthogonal to $\boldsymbol{n}$ for *transverse waves*.

Christoffel wave equation reveals that generally *three plane waves* can propagate in any direction $\boldsymbol{n}$ in a *solid anisotropic elastic medium*. They have mutually perpendicular polarizations.

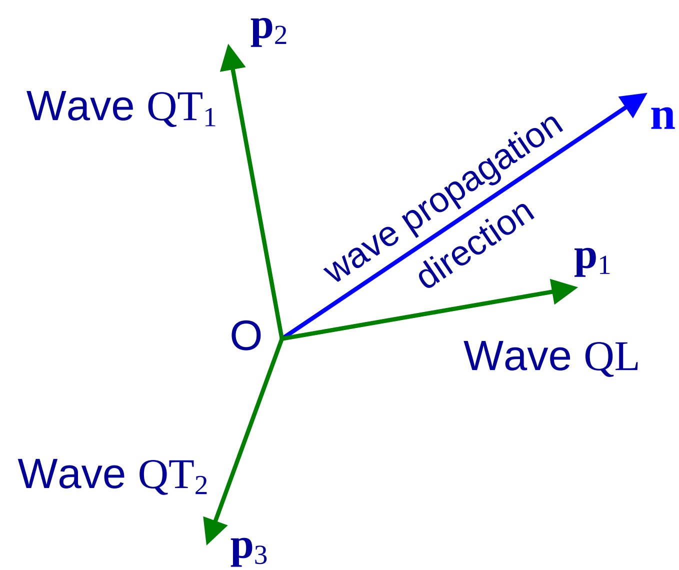
Christoffel E B. Ueber die Fortpflanzung von Stossen durch elastische feste Korper, Annali di Matematica, VIII, 1877.

Other than stress physical interpretations of second order symmetric tensors, Christoffel (acoustic) tensor.

The wave with polarization $\mathbf{p}$ closest to the propagation direction versor $\mathbf{n}$ is called *quasi-longitudinal* (QL) wave.

The two remaining waves are called *quasi-transversal* (QT).

Experiments show that in metals the QL wave propagates approximately two times more quickly than QT waves.



Taking advantage of *spectral decomposition* formula of Hooke's tensor, the *Christoffel (acoustic) tensor* can be expressed as follows,

$$\boldsymbol{\chi}(\boldsymbol{n}) \equiv \boldsymbol{n}\mathbf{S}\boldsymbol{n} = \mathbf{S}\cdot(\boldsymbol{n}\otimes\boldsymbol{n}) = \lambda_I(\boldsymbol{\omega}_I\boldsymbol{n})\otimes(\boldsymbol{\omega}_I\boldsymbol{n}) + \ldots + \lambda_{VI}(\boldsymbol{\omega}_{VI}\boldsymbol{n})\otimes(\boldsymbol{\omega}_{VI}\boldsymbol{n}).$$

Hint. The above formula suggests that *acoustic (elastic) waves* in *anisotropic media* can be divided into finite set of *classes of waves*. Similarly like all symmetries of *crystallographic materials* are structured into *32 classes of symmetry*. Structuralization of elastic waves in anisotropic materials makes an interesting open scientific problem.

Royer D., Valier-Brassier T., Elastic waves in Solids 1, Propagation, Iste, Wiley, 2022.
Rychlewski J., Elastic waves under unusual anisotropy, J. Mech. and Phys. of Solids, 49, 2001, pp. 2651–2666

Other than stress physical interpretations of second order symmetric tensors, Christoffel (acoustic) tensor.

For example, materials with *cubic symmetry* have three distinct Kelvin moduli of single, double and triple multiplicity. The eigenstates of cubic materials Hooke's tensor presented in parametric form are as follows,

$$\boldsymbol{\omega}_I = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\omega}_{II,III} = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos(\theta) & 0 & 0 \\ 0 & \cos(\theta + \frac{2}{3}\pi) & 0 \\ 0 & 0 & \cos(\theta - \frac{2}{3}\pi) \end{bmatrix},$$

$$\boldsymbol{\omega}_{IV,V,VI} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \sin\varphi\cos\psi & \sin\varphi\sin\psi \\ \sin\varphi\cos\psi & 0 & \cos\varphi \\ \sin\varphi\sin\psi & \cos\varphi & 0 \end{bmatrix}.$$

where θ,φ,ψ are parameters which can take values from $<0, \pi/2>$.
Various specific sets of parameters θ, φ, ψ can be adopted, e.g.,

$$(\theta = 0 \; or \; \theta = \tfrac{\pi}{2}) \; and \; (\varphi = 0 \; or \; \varphi = \tfrac{\pi}{2} \wedge \psi = 0 \; or \; \varphi = \tfrac{\pi}{2} \wedge \psi = \tfrac{\pi}{2}).$$

Thus, in view of spectral decomposition of Hooke's tensor acoustic tensor $\boldsymbol{\chi}(\boldsymbol{n})$ for materials with cubic symmetry can be presented in the form,

$$\mathbf{S}^{cub} = \lambda_I \boldsymbol{\omega}_I \otimes \boldsymbol{\omega}_I + \lambda_{II}[\boldsymbol{\omega}_{II} \otimes \boldsymbol{\omega}_{II} + \boldsymbol{\omega}_{III} \otimes \boldsymbol{\omega}_{III}] +$$
$$+ \lambda_{IV}[\boldsymbol{\omega}_{IV} \otimes \boldsymbol{\omega}_{IV} + \boldsymbol{\omega}_V \otimes \boldsymbol{\omega}_V + \boldsymbol{\omega}_{VI} \otimes \boldsymbol{\omega}_{VI}]; \quad \lambda_{II} = \lambda_{III}, \quad \lambda_{IV} = \lambda_V = \lambda_{VI}$$

$$\boldsymbol{\chi}^{cub}(\boldsymbol{n}) \equiv \boldsymbol{n}\mathbf{S}^{cub}\boldsymbol{n} = \lambda_I \boldsymbol{\omega}_I\boldsymbol{n} \otimes \boldsymbol{\omega}_I\boldsymbol{n} + \lambda_{II}(\boldsymbol{\omega}_{II}\boldsymbol{n} \otimes \boldsymbol{\omega}_{II}\boldsymbol{n} + \boldsymbol{\omega}_{III}\boldsymbol{n} \otimes \boldsymbol{\omega}_{III}\boldsymbol{n}) +$$
$$+ \lambda_{IV}(\boldsymbol{\omega}_{IV}\boldsymbol{n} \otimes \boldsymbol{\omega}_{IV}\boldsymbol{n} + \boldsymbol{\omega}_V\boldsymbol{n} \otimes \boldsymbol{\omega}_V\boldsymbol{n} + \boldsymbol{\omega}_{VI}\boldsymbol{n} \otimes \boldsymbol{\omega}_{VI}\boldsymbol{n})$$

Other than stress physical interpretations of second order symmetric tensors, 2D materials Hooke's tensor.

A very interesting physical interpretation of *second order symmetric tensors* makes *two dimensional (2D) Hooke's tensor,*

$$\boldsymbol{\sigma} = \mathbf{S}\cdot\boldsymbol{\varepsilon} \sim \sigma_K^{Ke} = S_{KL}^{Ke}\,\varepsilon_L^{Ke}, \quad (K,L=1,3), \quad \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\sigma_{12} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & \sqrt{2}S_{1112} \\ S_{2211} & S_{2222} & \sqrt{2}S_{2212} \\ \sqrt{2}S_{1211} & \sqrt{2}S_{1222} & 2S_{1212} \end{bmatrix}^{Ke} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix}$$

The following invariants prove to be of importance for 2D Hooke's tensor,

$$J_1 \equiv H_1 = \frac{1}{2}(S_{11}+S_{22}+2S_{12}) = \lambda_{\mathcal{P}},$$

$$J_2 \equiv H_2 = \frac{1}{2\sqrt{2}}(S_{11}+S_{22}-2S_{12}+2S_{33}) = \sqrt{2}\,\lambda_{\mathcal{D}},$$

$$J_3 \equiv R_1^{\,2} = H_3^{\,2} + H_4^{\,2} = \frac{1}{2}(S_{11}-S_{22})^2 + (S_{13}+S_{23})^2 = |\boldsymbol{\tau}|^2,$$

$$J_4 \equiv R_2^{\,2} = H_5^{\,2} + H_6^{\,2} = \frac{1}{8}(S_{11}+S_{22}-2S_{12}-2S_{13})^2 + (S_{13}-S_{23})^2 = |\mathbf{D}|^2$$

$$J_5 = (H_3^{\,2} - H_4^{\,2})H_6 - 2H_3H_4H_5 = 0$$ condition for orthotropic symmetry

$$H_1 = \tfrac{1}{2}[S_{11}+S_{22}+2S_{12}],$$

$$H_2 = \tfrac{1}{2\sqrt{2}}[S_{11}+S_{22}-2S_{12}+2S_{33}],$$

$$H_3 = \tfrac{1}{\sqrt{2}}[S_{11}-S_{22}],$$

$$H_4 = S_{13}+S_{23},$$

$$H_5 = \frac{S_{11}+S_{22}-2S_{12}-2S_{33}}{2\sqrt{2}},$$

$$H_6 = S_{13}-S_{23}$$

The Novozhilov's tensors, Christoffel tensor, 2D-Hooke's tensor, being second order symmetric tensors, require and deserve separate works analogous to this presented here for Cauchy stress tensor for identification of physically meaningful and useful sets of their invariants.

Blinowski A. , Ostrowska-Maciejewska J., Rychlewski J., Two-dimensional Hooke's tensors − isotropic decomposition, effective symmetry criteria, Arch. Mech., 48, 2, pp. 325-345, Warszawa 1996.

Some open scientific problems

In connection with elaboration of new parametrization of Cauchy stress tensor several interesting open scientific problems could be identified:

a) Development of lucid graphical illustration of the tensor orbit concept.

b) Development of classification (structuring) of elastic waves in anisotropic materials taking advantage of spectral decomposition of Hooke's tensor.

c) Development of strength of elastic material hypotheses taking advantage of the information that energy stored in the elastic material can always be decomposed into six mutually independent parts.

d) Examination of Novozhilov's tensors, Christoffel tensor, 2D materials Hooke's tensor and identification of their physically meaningful and important sets of invariants.

Summary

1. The work is focused on the idea of tensors their role in science and mechanics in particular. The *Cauchy stress tensor*, generic instant of symmetric second order tensor and *Hooke's tensor* generic instant of symmetric fourth order tensor, are taken as primary subjects of the discussion due to obvious for mechanical community reasons.

2. Elements of philosophy and science are recalled to deliver broader perspective of the presented discussions. *Historical survey* is delivered on how, why, when and by whom the key steps in development of *tensor notion* and *tensor calculus* were achieved. The survey gives grounds to the view that the prime *philosophical and practical reasons* why tensors became the *natural language* of all advanced engineering and the other sciences are their properties of *invariance – upon changing coordinate system,* and *linearity*.

3. It is indicated that *tensors* can be viewed and/or understood from many different perspectives as: *algebraic objects, linear transformations* and/or *geometrical objects* making them very rich and comprehensive notion.

4. Historical survey of the development of *algebraic structures*, together with their mathematically precise definitions, leading to the algebraic definition of tensors is presented. This to show what is the complete building structure *necessary* and *sufficient* to reach the notion of tensor in its quantitative technical side, i.e., enabling execution of precise mathematically quantitative analyses. This section also excellently illustrates that mere craftsmanship of technical algebraic calculations does not allow fully grasping all the *richness, flavors and beauty* of tensor notion.

It is indicated interesting property of *algebraic-geometric duality* of tensors. It is much more convenient to understand tensors as *geometrical objects* when building physical models but it is practically necessary to treat them as algebraic objects when running actual quantitative computations. For example second order symmetric tensor treated as geometrical object is characterized by *three features* (set of three invariants) and *specific orientation in space* (set of three *Euler angles*).

Summary

5. There are investigated several as it would seem at first sight paradoxes of tensorial calculus. For example, how comes that second order symmetric *tensor,* an object *invariant with respect to change of coordinate system* – the very philosophical idea underlying tensor concept, which is *characterized by six linearly independent components* has only *three linearly independent invariants* of its components? It is shown that actually *six invariants* of the second order symmetric tensor *can be uniquely constructed* when the tensor is considered in interaction with other tensorial objects (some environment) and it is explained why it is so. The other issue is as follows. It is common belief or at least it is commonly perceived that *isotropic tensors* have *identical* representation components in *all orthonormal bases*. It is not true. *Isotropic tensors* have *identical* representation components in *mutually isometric orthonormal* bases, but *isotropic tensors* in general have *different* representation components in *orthonormal* bases, which are not mutually *isometric*. The *non-isometric orthonormal tensorial bases* are generated in different manner than tensorial product of rotated basis (triple orthogonal unit vectors) of 3D Euclidean space generating higher order tensor spaces.

Summary

6. Continuum mechanics literature is plenty of discussions on different *sets of invariants* of second order symmetric tensors, e.g., interpreted as stress or strain tensor. However, it is not adequately underlined that actually *infinite number of triplets of invariants* can be distinguished for any second order symmetric tensor. There has been explicitly specified, with their unique naming and mutual mathematical relations, the most popular sets of triplets of invariants of second order symmetric tensor, i.e., *basic (main) invariants*, *set of principal values*, *principal invariants*. While the relevant formulas are known, they are scattered among many different publications (books and papers) with many different denotations. Gathering them in one place and their specification in consistent notation makes very convenient and *handy reference resource*.

Summary

7. *Historical survey* is delivered and *state of the art* presented regarding parametric description of Cauchy stress tensor up to the so called *isomorphic orthogonal cylindrical coordinates* ($z=\sqrt{3}\cdot\sigma_m$, $r=(2J_2)^{1/2}$, θ_L) where σ_m is *mean value of stress*, r is *modulus of deviator* $\mathbf{s}$ and θ_L is the *Lode angle*.

In many continuum mechanics publications since several years information is widely disseminated that cylindrical orthogonal coordinates ($p=-\sigma_m$, $r=(2J_2)^{1/2}$, θ_L) make *Haigh-Westergaard (H-W) space*. This is *incorrect information*. Haigh and independently Westergaard, nearly at the same time in 1920, introduced the concept of 3D space in which principal values of stress (σ_I, σ_{II}, σ_{III}) were proposed as independent coordinates (parameters), and this is the *actual definition of Haigh-Westergaard space*.

It was found out by the present author that the set of *isomorphic, orthogonal, cylindrical coordinates* ($\sqrt{3}\cdot\sigma_m$, $r=(2J_2)^{1/2}$, θ_L) based on invariants of stress tensor was for the first time introduced by J. Murzewski in 1958.

8. A *completely new generic parametrization* of the Cauchy stress tensor It is proposed employing *new notions* of so called *isotropy angle* θ_{iso} and *shear stress mode (skewness) angle* θ_{sk}, this last to replace the concept of *Lode angle* θ_L. The definition of *skewness angle* θ_{sk} is based on accepting *pure shears* as convenient *comparison reference states*.

9. The physical interpretation is delivered of *pure shears* to be *elementary (atomic) elements* of any *deviator of second order symmetric tensor*. This is so because any deviator can be always decomposed into two pure shears. The *pure shears* can be identified to be generators of deviatoric space of second order symmetric tensors.

10. It is derived and explicitly specified a completely new formula for *index of anisotropy degree* of second order symmetric tensors employing isotropy angle and shear mode angle, this being further advancement of the original ideas of Jan Rychlewski. The Rychlewski's index of anisotropy degree based on *tensor orbit* is much more precise and subtle measure than the index based on the size of *modulus of tensor deviator* only. The formula for *anisotropy index* valid for second order symmetric tensors becomes extremely simple when it is expressed in terms of *isotropy angle* θ_{iso} and *shear stress mode angle* θ_{sk}.

Summary

11. It is delivered statistical-physical interpretation of principal invariants of deviatoric (shear) part of Cauchy stress. The investigation delivered grounds for interpretation of shear stress mode (skewness) angle θ_{sk}, as quantity describing the amount of directional disorder of micromechanical population of pure shears generating given macroscopic Cauchy stress state $\boldsymbol{\sigma}$. Results of this precursory approach gives motivation and justification for naming the *shear stress mode angle* θ_{sk} of Cauchy shear stress, which defining *comparison reference state* is *pure shear*, the *"skewness" angle*. Statistical-physical interpretation of Cauchy stress deviatoric part allowed for finding out the reason and explaining why the Rychlewski's *anisotropy index* of Cauchy stress *decreases* with departure of its deviatoric part from *pure shear* mode. It also delivers grounds for introduction of the concept of *internal entropy of the Cauchy stress* and thus elucidating a very interesting *link* between *continuum mechanics* and *thermodynamics*.

Summary

12. It is expounded that *elastic energy of linear elastic isotropic material does not and cannot depend on shear stress mode (skewness) angle* θ_{sk}. This is in contrast with the conjectures made in many papers that the material is assumed to be linear elastic and at the same time strength criteria for the same material, based on stored in it elastic energy, is assumed to be dependent on Lode angle. Such proceedings make methodological inconsistency, or rather error.

13. It is indicated that *interaction* of second order symmetric tensors with other tensorial objects, e.g. fourth order (Hooke's) tensors creates qualitatively new situation which enables construction of *not only three* but *six invariants for second order symmetric tensors* with respect to change of coordinate system. It is explained the reason for such situation. It is indicated that this gives premises for introduction and development of a *notion of weighted effective stress*, for example in the shape of stress quadratic form, which takes into account interaction of stress tensor with other tensorial object characterizing material. Such approach can result in improvement of the classical *effective stress* notion.

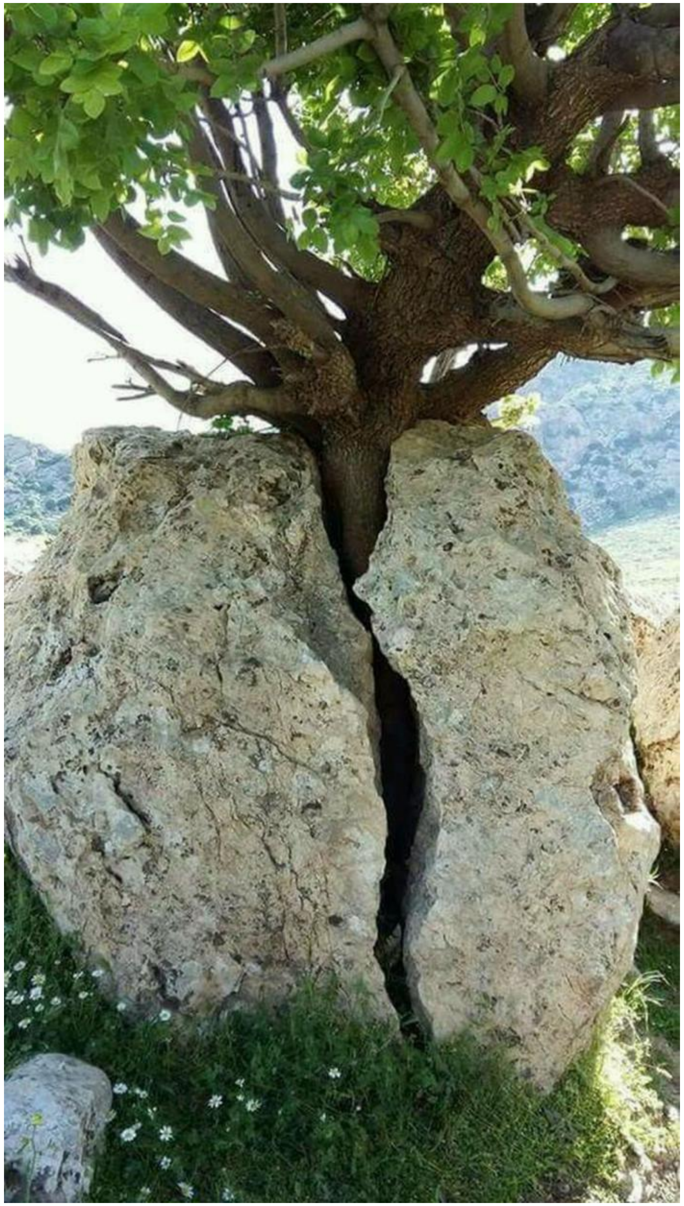
14. It is explained essential difference between very popular in experimental mechanics, testing layouts of so called *simple shear* and *planar shear* tests, both being practical implementations of *pure shear* stress (strain) states. In simple shear *principal axes rotate constantly* with increasing loading while in planar shear orientation of *principal axes remain all the time stationary* with respect to laboratory reference frame. The kinematics in simple shear and planar shear testing layouts, described by deformation gradient $\boldsymbol{F}$, are different but strains are exactly the same in the both layouts.

15. History of introduction of *triaxiality factor* into mechanical literature is presented very concisely. New, original, *explicit formulas linking triaxiality factor* and *shear stress mode (skewness) angle* θ_{sk} valid in the case of *biaxial tests* are derived, basing on Wierzbicki and Xiao constraint relation. It is indicated that with the use of *biaxial tests only* it is *impossible in correct methodologically manner* to precisely separate out the influence of mean stress σ_m and shear stress mode (skewness) angle θ_{sk} on strength of material, a factor very important when formulating, e.g., criteria of plastic yielding, phase transition start or initiation of fracture. The finding delivers strong argument for experimental mechanics researchers to develop 3D multiaxial tests adequate for the purpose.

Summary

16. A very strong need and demand can be contemporary noticed for development of efficient methods of *computer visualization of second order (and higher order) tensorial fields*. The classical visualization approaches e.g., in the form of *principal axes ellipsoid* can be evaluated as not insufficient but rather a completely unsatisfactory. The present study indicates that it is practically impossible to deliver efficient and lucid graphical representation of second order symmetric tensor fields without its prior *structuralization* (construction of adequate set of invariants). The set of invariants must be constructed in such a way to relevantly describe the features of interest. Even second order symmetric tensors structure is too reach to show graphically all of its properties simultaneously. Without structuring the visualization results usually prove to be very obscure, incomprehensible and intricate. Proposed here new parametrization of the Cauchy stress eigenproperties delivers good example of such structuralization for visualization purposes.

Summary



vis vitalis

On wisdom

Jan Kochanowski z "Wykładu cnoty”; from the "Lecture on Virtues",

"Dwie tedy rzeczy człowieka szlachcią: obyczaje a rozum,
obyczaje z cnót pochodzą, a rozum z nauki,
obiedwie rzeczy w sobie mieć rzecz nieprzepłacona człowiekowi.
Ale jeśli tylko przy jednej masz zostać,
raczej przy cnocie niż przy nauce zostań,
bo NAUKA BEZ CNOTY, jako miecz u szalonego,
I SOBIE, I LUDZIOM SZKODZI,
cnota, choć dobrze sama będzie, chwalebna jest i pożyteczna".

"Two things then give a man nobility: morals and reason,
morals come from virtue, and reason from science,
to have both attributes, a thing priceless to man.
But if you only have to stay with one,
stay with virtue rather than with science,
because SCIENCE WITHOUT A VIRTUE, as a sword at the crazyman,
HARMS ITSELF AND PEOPLE,
virtue, even if it is alone, is glorious and profitable."

Some supplementary information, linear elasticity (Hooke's) constitutive law.

Robert Hooke initially (1676) announced his law of elastic materials behavior, linking force with deformation, in the form of Latin anagram

ceiiinosssttuv

He decoded his anagram two years later (1678) to read

ut tensio sic vis $(F=k\cdot x)$

or in the form that we know it today

$$\boldsymbol{\sigma} = \mathbf{L}\boldsymbol{\varepsilon}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ denote stress and strain tensors, $\mathbf{L}$ denotes Hooke's (stiffness) tensor.

Rychlewski J., translator Ziółkowski A . (2023) CEIIINOSSSTTUV Mathematical structure of elastic bodies, pp. 1-131, 2023, IPPT PAN, Warsaw, Poland.. (English translation of original work in Russian published in 1983.)
https://www.researchgate.net/publication/376594979_CEIIINOSSSTTUV_Mathematical_structure_of_elastic_bodies
Rychlewski J. (1995) Unconventional approach to linear elasticity, Arch. Mech.

Some supplementary information, linear elasticity (Hooke's) constitutive law.

A motivating question arises *execution of how many and what kind of experimental tests* is necessary and effective to uniquely determine elastic properties of the most general elastic, anisotropic material, or speaking otherwise all components of elastic stiffness (compliance) tensor?

Experimental answer to this question as a first delivered Woldemar Voigt in 1887 in favor of 21 constants. Enlightening structure of these constants was for the first time revealed by Jan Rychlewski in 1983, where he proved that any symmetric (Hooke's) fourth order tensor can be *spectrally decomposed* into 6 mutually orthogonal subspaces. Each subspace is characterized by *stiffness (Kelvin) modulus* λ_K – scalar, and *elastic eigenstate* $\boldsymbol{\omega}_K$ - symmetric second order tensor ($K=1,..,6$). The elastic eigenstates can be generated by 12 so called *stiffness distributors* $\aleph_\alpha$ ($\alpha=1,..,12$).

Some supplementary information, spectral decomposition of elastic stiffness (Hooke's) tensor.

In summary set of 21 components (elastic constants) present in representation of any Hooke's tensor can be divided into 3 classes

$$6 + 12 + 3 = 21$$

1. The first group consists of *6 Kelvin moduli* $\lambda_I, .., \lambda_{VI}$,

2. The second group consists of *12 stiffness distributors* $\aleph_1, \dots, \aleph_{12}$ generators of *6 elastic eigenstates* $\omega_I, .., \omega_{VI}$,

3. The third group consists of *3 Euler angles* ϕ_I, ϕ_2, ϕ_3.

The *18 parameters* from the first and the second group are *invariants* of elastic stiffness tensor and they fully characterize elastic properties of a material.

See also Kowalczyk–Gajewska K., Ostrowska –Maciejewska J. (2009) Review on spectral decomposition of Hooke's tensor for all symmetry groups of linear elastic material, Engng. Trans.

Some supplementary information, spectral decomposition of elastic stiffness (Hooke's) tensor.

The information on mathematical structure of Hooke's tensors given above resulting from their spectral decomposition straightforwardly helps to resolve the following interesting problem:

When two sets of 21 components of elastic stiffness tensor determined in experimental program for respective specimens of two otherwise unknown materials show to have the same corresponding values,

Does that mean that the respective specimens were made of the same material?

The answer to this question is

No.

This is so because only 18 parameters (invariants of Hooke's tensor) characterizes elastic properties of specific material and 3 parameters characterizes material orientation in laboratory frame (coordinate system). They are present in 21 elastic constants in an involved manner. Due to that it may happen that 21 elastic constants determined by testing specimens of *two different* materials may show to have the same respective values but this due to different orientation of the specimens in fixed laboratory frame. In order to unambiguously find out whether two sets of 21 elastic constants represent the same material 18 Hooke's tensor invariants must be extracted.

Some supplementary information, qualitatively different types of (micro) ordering patterns in materials.

Abeyaratne presented nice schematic diagram illustrating different types of possible ordering patterns characterizing different materials, see Figure 13.1 on page 372 of his Lecture notes.

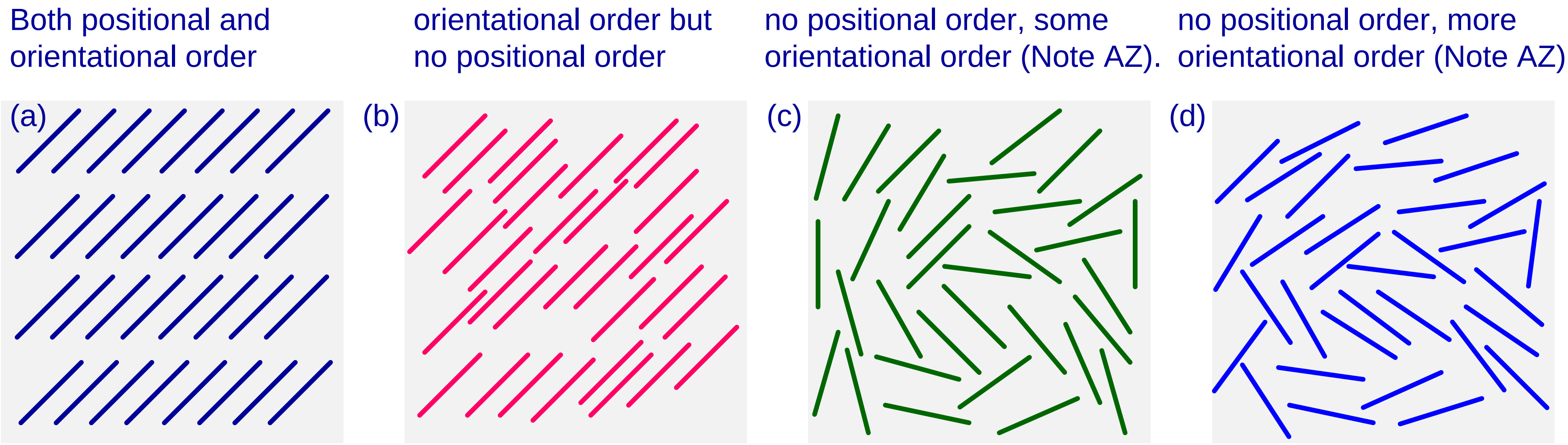


Fig. Different possible ordering patterns of orientational dipoles populations, after Abeyaratne.

Abeyaratne , Continuum Mechanics, Vol. 2, Lecture Notes on the mechanics of elastic solids, 2006.
http://web.mit.edu/abeyaratne/lecture notes.html

Notes