Parametrization of Cauchy stress tensor theory and applications

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Seminar Materials

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Session 1 – synopsis

What Science is about - definition, tasks, assumptions, methodology, results.

Science consists in activities leading to cognition and understanding of universe.

Science is based on a number of assumptions the main ones of which are as follows: the universe is at all cognizable with senses available to humans, the universe always provides true data, there exist general principles governing the universe that are invariable in space and time.

The main method of scientific activities is abductive reasoning (not deductive), i.e. operation in conditions of limited cognitiveness (irremovable uncertainty).

The main result of science are models of reality, logical, internally consistent, no more complex than it is necessary, clear and as accurate as possible, i.e. knowledge about the universe.

An important task of science is to document and disseminate knowledge.

Feliks Koneczny's Quincunx (the pentnomial of being), i.e. *Good* (*Ethics* finding its source in religion), *Truth* (*Knowledge* finding its source in science), *Health*, *Well-being* and *Beauty*, as a determinant of *different civilizations* (ways of collective life). Beauty being a Unifying Force between spiritual (Good, Truth) and material (Health, Well-being) determinants of civilization.

Role of science in humanity.

Science is not exercised for itself or for gaining knowledge about universe only but it is a necessary measure and intermediate step towards *acquiring wisdom* (personal and collective), i.e. using *knowledge* for attaining *good*.

Introduction, problem formulation.

The question arises what factors caused such broad widespread, use and popularity of *tensorial calculus* (*tensors*)?

When and how the notion of a tensor came to existence?

What tensors actually are, and/or how they can be understood/ interpreted?

What are the specific properties – eigenproperties, of tensors and what is the best manner to deal with them?

An effort is here undertaken to address listed above and some other issues taking *Cauchy stress tensor* (*second order symmetric tensor*) as a generic example.

While the problems are discussed here with the use of *Cauchy stress tensor*, all the obtained here results *mutatis mutandis* transfer to *all second order symmetric tensors*, having possibly miscellaneous interpretations in wide variety of pure science, engineering and/or other fields of application.

Introduction, science.

Our present examination belongs to the fundamental problems of science in this sense that modeling tool is developed useful and convenient, and precise for formulation of scientific problems, description of physical phenomena, documentation and proliferation of scientific results.

Plenty of misunderstandings exists regarding fundamental tasks and targets of scientific activities and the place and role of science in human life.

Due to that before we move on to discuss the main theme, i.e. tensors, let us outline and prepare the place of the drama and its scenery, i.e. discuss what are activities, problems, conditions and limitations involved with Science.

What is the motivation for running scientific activities, what is the target of science and what are its results?

Motivation	CURIOSITY of the universe
Target	COGNITION and UNDERSTANDING of the universe
Results	KNOWLEDGE about the universe

So, Science means acquiring knowledge about the *Universe*.
(from Latin *scientia*, meaning cognition, knowledge of things)
Science *is an effort to discover and to know* - and thus broaden the human understanding on *how the physical world works*.

Science no less important task is
documentation and *public dissemination*
of knowledge about universe
(and not keeping it secret or confidential).

Introduction, science.

Scientific activities *methodology* is based (out of necessity)

On *Abduction* (abductive reasoning), *Not Deduction* (deductive reasoning)!

Critical for abductive reasoning is operation in conditions of *uncertainty*, *limited information/knowledge*.

When examining any physical phenomenon there are *always present* consciously known and/or unknown factors that

a) do not affect the course of the phenomenon, and/or
b) are constant and due to that have a constant influence, (indistinguishable from the influence of other controlled factors).

There are many reasons for such situation, e.g. limitation of our senses.

Application sciences methodology, e.g. *Engineering* is based on *deduction*.

Deduction: allows you to *derive b* as a *consequence* of *a* : in other words, deduction is the process of deriving conclusions from what is already known.
Induction: allows you to derive *a* as a *precondition* of *b* : in other words, induction is the process of deriving causes from known effects in the *casual connection* "*a* implies *b*".
Abduction: allows you to pose a hypothesis (general rule) *a* as an explanation for *b (specific case/-es)*: in other word, abduction is the process of *formulating the best guess hypothesis* in the conditions of incomplete, limited information/knowledge.

Introduction, science.

The result of the abductive methodology is the *best result* (*assessment*) based on the knowledge and experience to which we have access.
All the time we assume so called good will in conduct of scientist.

A very good example of *abductive methodology result* taken from our daily life is *medical diagnosis*. The medical examination uses abductive methodology but has different target and rationale than scientific activity.

Employment of abductive methodology (*out of necessity*) in scientific work has *very strong consequence*, i.e. it can be reasonably regarded that *All Scientific Knowledge*, among others scientific theories (results of scientific activity) *represents models* of reality, and not *reality* itself.

ASSUMPTIONS underlying the SCIENTIFIC ACTIVITIES
The laws of the universe are *invariant* in time and space.
The laws of the universe, at all *can be recognized*.
The universe "*doesn't cheat*", when giving us the answers to our questions, e.g. in the form of results of experimental tests.

Introduction, science.

So, how actually scientists run research?

Scientific research process

Data -> Information -> Knowledge

(usually a long-term and cyclic process)

Using controlled methods (*research*), *scientists* use observable *physical evidence of natural phenomena* to collect *data*, next analyze and order it to create *information*, and next structure this last to build *predictive models* to gain *knowledge* that logically explains the processes ongoing in universe.

Practice of scientific methods is based on *properly designed real and/or thought experiments*, results of which are to enable revealing the actual causes and course of physical phenomena and their effects.

A very astute opinion on scientific research expressed professor Jan Rychlewski:

"… The goal of Natural Sciences (Science)*, … , is not to search for absolute truths, but to construct models describing reality in a way that is completely internally consistent, as clear and elegant as possible and sufficiently accurate. ..."*

after Jan Rychlewski, Dimensions and Similarity, PWN,1991; Appendix C: Newton's Mechanics.

Introduction, science.

In *ancient Greece*, where science was born, there were *a lot of philosophers*, *in modern times* there are *many scientists*, but philosophers are very hard to find.

Ancient great Greek thinkers practiced Natural Philosophy, under which they conducted activities aimed at penetrating the *secrets of nature*, not only to get to know them, *but to become wise people*.

Science was treated as an intermediate (partial) step in this process.

(lat. Philosophia < gr. φιλοσοφία < gr. φιλέω (love, cherish, adore) and gr. σοφία (wisdom); in verbatim translation "love of wisdom".)

Scientist: Data -> Information -> Knowledge
Philosopher: Data -> Information -> Knowledge -> Wisdom

Knowledge: the entirety of reliable information and understanding about real world along with the ability to use these assets.

Wisdom: is the ability and willingness to use knowledge to do G*ood*.
discernment about right and wrong (Socrates);
the ability to use reason accurately and do what is best (R. Descartes);

The source for recognition what is "Good" is *Religion* as the source of *Ethics.*

Contrasting and/or interchanging *Science* and *Religion* is a misunderstanding. These are two different categories (complementary to each other, not opposing/excluding each other).

Introduction, science.

Knowledge (together with education) is often equated with *wisdom*.
This is a *misjudgment*.
A person can be *well educated*, possess *deep knowledge* and still *can be not wise*.

Contemporary scientists extremely often terminate their activity on acquiring/gaining knowledge. It seems that it would be sensible to return to the original Greek thought and treat *science* as a step to the goal of *acquiring wisdom* (personal and collective).

The most eminent Polish philosopher, Feliks Koneczny, identified the so-called *Quincunx* (pentnomial of being)
as a determinant of *different civilizations* (ways of collective life), i.e.
Good (*Ethics* finding its source in religion), Truth (*Knowledge* finding its source in science), Health, Well-being and Beauty.

Good an Truth are considered spiritual elements, Health and Well-being are considered physical elements and Beauty is considered connecting link between spiritual and physical factors of collective (and individual) life.

Quincunx can be treated as a *measure* enabling determination of diversity of different civilizations (methods of the system of collective life).

Koneczny F., O wielości cywilizacji (in Polish), Kraków, 1935.
There exist English translation of this work by Toynbee A., On plurality of civilizations, London 1962.

Session 2 - synopsis

Tensors as lingua franca of all natural and engineering sciences. Algebraic-Geometric Dualism of tensors. This dualism proves to be very convenient and useful in formulating and documenting general laws of universe. Cauchy stress as precursor of tensor concept. Cauchy stress as macroscopic measure of internal forces interaction between particles in microscale of observation. Scales of observation. The idea of Representative Volume Element (RVE). Limitations of human perception/senses in cognition of universe. Tensors as excellent modeling objects of reality. Gregorio Ricci-Curbastro a father of tensorial calculus (1888-1892). Woldemar Voigt coining the term tensor in its contemporary meaning (1898).

Tensorial calculus, native language of natural sciences.

When contemplating on the place of *tensorial calculus* and *tensors* in human life. One can readily come to the conclusion that it makes a *native language* for *describing* and *documenting* in convenient and quantitatively exact manner knowledge about *natural sciences,* e.g. in physics or engineering sciences.

Care must be taken *not to entangle* the *phenomenon description* with the *phenomenon itself*. This is a methodological error.

Juliusz Słowacki „Beniowski", Pieśń 5

Chodzi mi o to, aby język giętki	*My point is that the gift of the gab*
Powiedział wszystko,	*Said everything,*
co pomyśli głowa:	*what the head will think:*
A czasem był jak piorun jasny, prędki,	And sometimes be like thunder bright, swift
A czasem smutny jako pieśń stepowa,	And sometimes sad as a steppe song
A czasem jako skarga nimfy miętki,	And sometimes as a nymph's complaint tender
A czasem piękny jak aniołów mowa...	And sometimes beautiful as Angels' speech..
Aby przeleciał wszystka ducha skrzydłem	To flash over all the soul on wing.
Strofa być winna taktem, nie wędzidłem.	Stanza should be tact, not the snaffle.

Algebraic-Geometric Dualism of tensors.

Pragmatically, science deals with phenomena *space-time relations* in their *cause and effect* aspect.

Such relations tell *what*, *where* and *when* is happening.

For centuries, also today, "*where"* has been expressed in the language of *geometry*. However, it turned out that there are great advantages when the quantitative methods and language of *algebra* are used for such a description.

Tensor calculus (tensors) is, in a way, the crowning achievement of the process of *algebraization of geometry*.

However, it turns out that the reverse geometrization of tensors, i.e. perceiving tensors as geometric objects proves to be very useful and convenient in building *models of universe*.

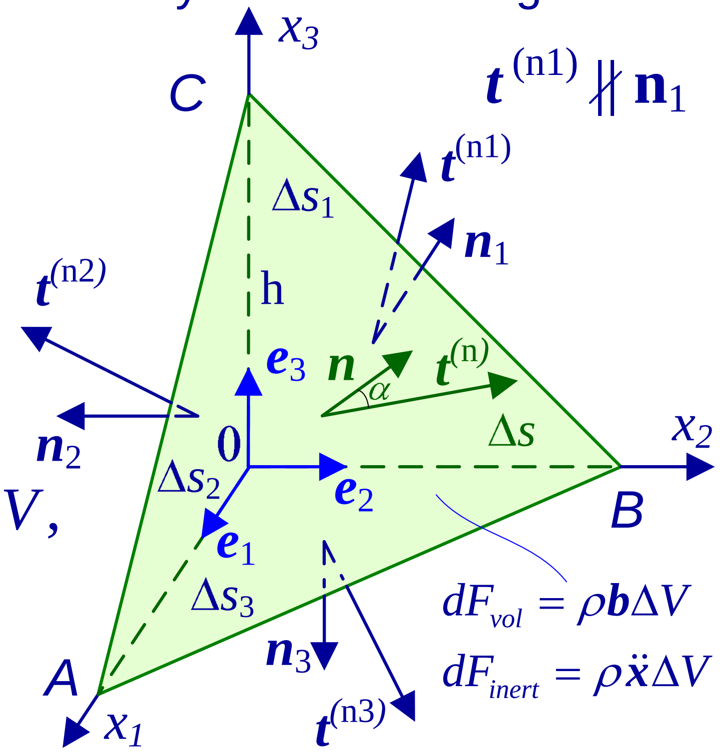
Algebraic-Geometric Dualism of tensors.

History, Cauchy's non perpendicular pressures.

Augustin Cauchy's landmark lecture before the Paris Académie in 1822, and publication of the abstracts from the lecture in 1823 deliver sound reason to judge that the concept of second order symmetric tensors were introduced into science and engineering by Cauchy himself.

Cauchy invented, came to the idea of a *tensor* upon realizing that classical Euler's *notion of pressure* can and should be extended to embrace "pressures" *not perpendicular to the planes on which they act*, and demonstrated this with his tetrahedron argument.

Cauchy tetrahedron argument



$$\int_{S=\partial V} \boldsymbol{t}\,ds + \int_V \rho \boldsymbol{b}\,dV = \int_V \rho \ddot{\boldsymbol{x}}\,dV, \quad \Delta V = \tfrac{1}{3} h\,\Delta s,$$

$$\boldsymbol{t}^{(n)}\Delta s + \boldsymbol{t}^{(n1)}\Delta s_1 + \boldsymbol{t}^{(n2)}\Delta s_2 + \boldsymbol{t}^{(n3)}\Delta s_3 = \rho(\ddot{\boldsymbol{x}} - \boldsymbol{b})\Delta V,$$

$$\Delta s_i = n_i\,\Delta s, \quad \boldsymbol{n} = n_i \boldsymbol{e}_i, \; \boldsymbol{n}_i = -\boldsymbol{e}_i \rightarrow \boldsymbol{t}^{(n_i)} = -\boldsymbol{t}^{(e_i)};$$

$\boldsymbol{t}(\boldsymbol{n}_i) = -\boldsymbol{t}(-\boldsymbol{n}_i)$ – Cauchy's lemma.

Cauchy A., Recherches sur l'equilibre et le mouvement interieur des corps solides ou uides, elastiques ou non-elastiques (in French), Bull Soc Filomat, Paris 913, 1823.
See, also e.g. Azadi E, Cauchy tetrahedron argument and the proofs of the existence of stress tensor, a comprehensive review, challenges, and improvements, 2017 pp. 1-34, arXiv:1706.08518v3

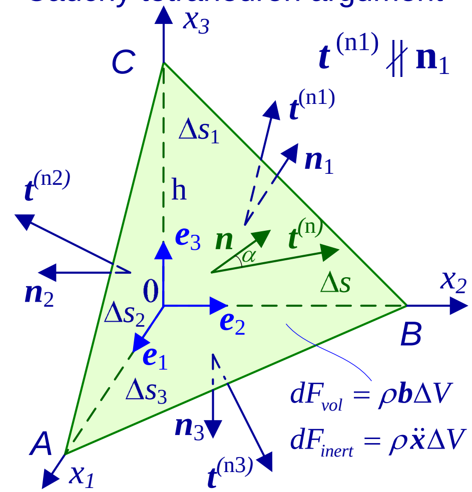
History, Cauchy's non perpendicular pressures.

Upon decreasing the volume of tetrahedron ($\Delta V \to 0$) the body and inertia forces become negligible in comparison to surface forces. Then, taking advantage of geometrical relations and *Cauchy lemma* simple transformations lead to the following balance of traction forces

$$\boldsymbol{t}^{(n)} = \boldsymbol{t}^{(e1)} + \boldsymbol{t}^{(e2)} + \boldsymbol{t}^{(e3)}; \quad \boldsymbol{t}^{(e1)} = T_{11}\boldsymbol{e}_1 + T_{12}\boldsymbol{e}_2 + T_{13}\boldsymbol{e}_3,$$

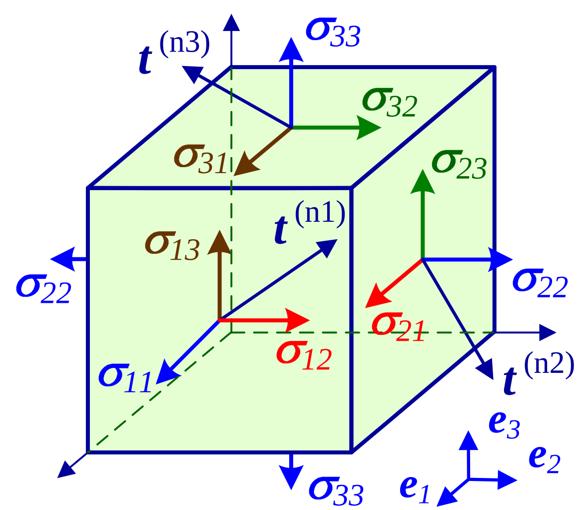
$$\boldsymbol{t}^{(e2)} = T_{21}\boldsymbol{e}_1 + T_{22}\boldsymbol{e}_2 + T_{23}\boldsymbol{e}_3, \ \boldsymbol{t}^{(e3)} = T_{31}\boldsymbol{e}_1 + T_{32}\boldsymbol{e}_2 + T_{33}\boldsymbol{e}_3,$$

Cauchy tetrahedron argument



and finally upon rearrangement of terms the notion of a stress tensor can be reached

$$\boldsymbol{t}^{(n)} = \begin{bmatrix} t_1^{(n)} \\ t_2^{(n)} \\ t_3^{(n)} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}^T \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}, \quad \boldsymbol{t}^{(n)} = \boldsymbol{\sigma}^T \boldsymbol{n}, \quad \boldsymbol{n} = n_i \boldsymbol{e}_i$$



Stress components shown in positive directions

where $\boldsymbol{n}$ can be any conceivable surface direction.

Balance of moments leads to the condition $\sigma_{ij} = \sigma_{ji}$.

See also, Chapter 4 in Truesdell C., Essays in the History of Mechanics. Springer-Verlag New York Inc, 1968

History, Cauchy's constitutive model of forces (material interactions).

It is worth reflection what actually Cauchy has done.

From *philosophical point of view* the Cauchy's contribution can be treated as a step towards development of *constitutive theory of forces*.

The motivation behind Cauchy's work was development of *theory of elasticity*, which he achieved by a very original move of generalizing the Euler's notion of *normal pressure* acting on surface element to introduce the idea of *force traction,* which can be *oblique to surface* on which it acts.

The Cauchy's tetrahedron argument can be understood as a certain *continuum model of force* describing in an averaged manner the microscopic interactions between molecules (through surfaces) – a transition from *molecular interactions* towards *continuous medium interactions*.

The model actually laid on the foundations of *continuum mechanics*.

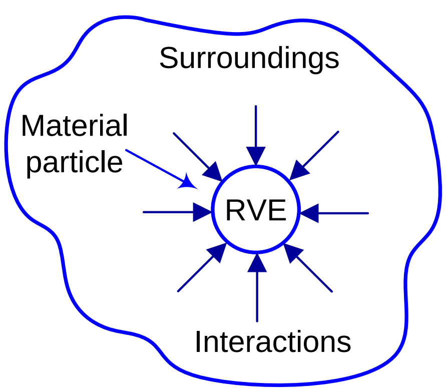
In his original presentation and publications Cauchy *was not talking about stress tensor* but about *pressures*, one of the reasons surely must have been that the *tensor notion* yet did not exist at that time.

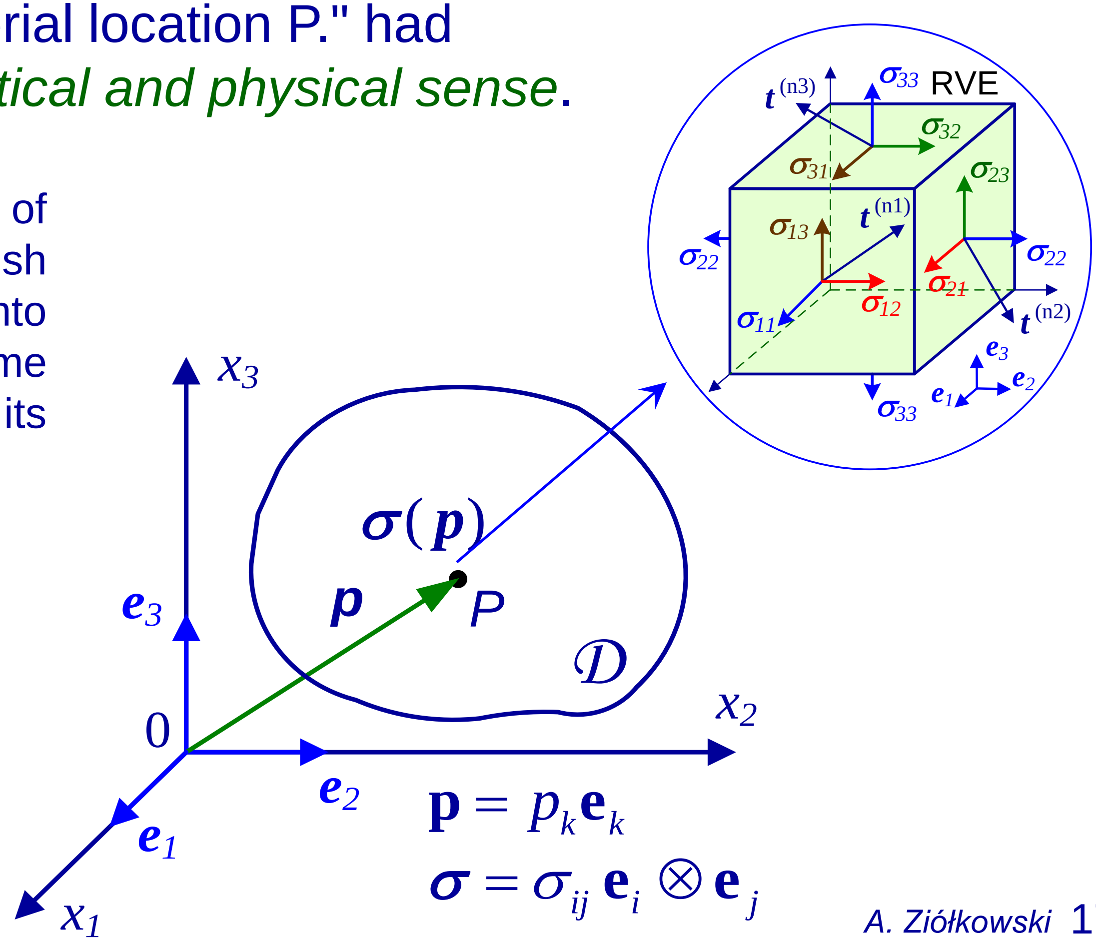
Cauchy stress tensor, representative volume element

Pragmatic justification for development and introduction of quite complex and abstract apparatus of *tensorial calculus* is creation of a tool for "*mathematization*" of real physical space and real physical phenomena.

For example in order that a sentence:
"Stress state $\boldsymbol{\sigma}$ exists at material location P." had
precise quantitative mathematical and physical sense.

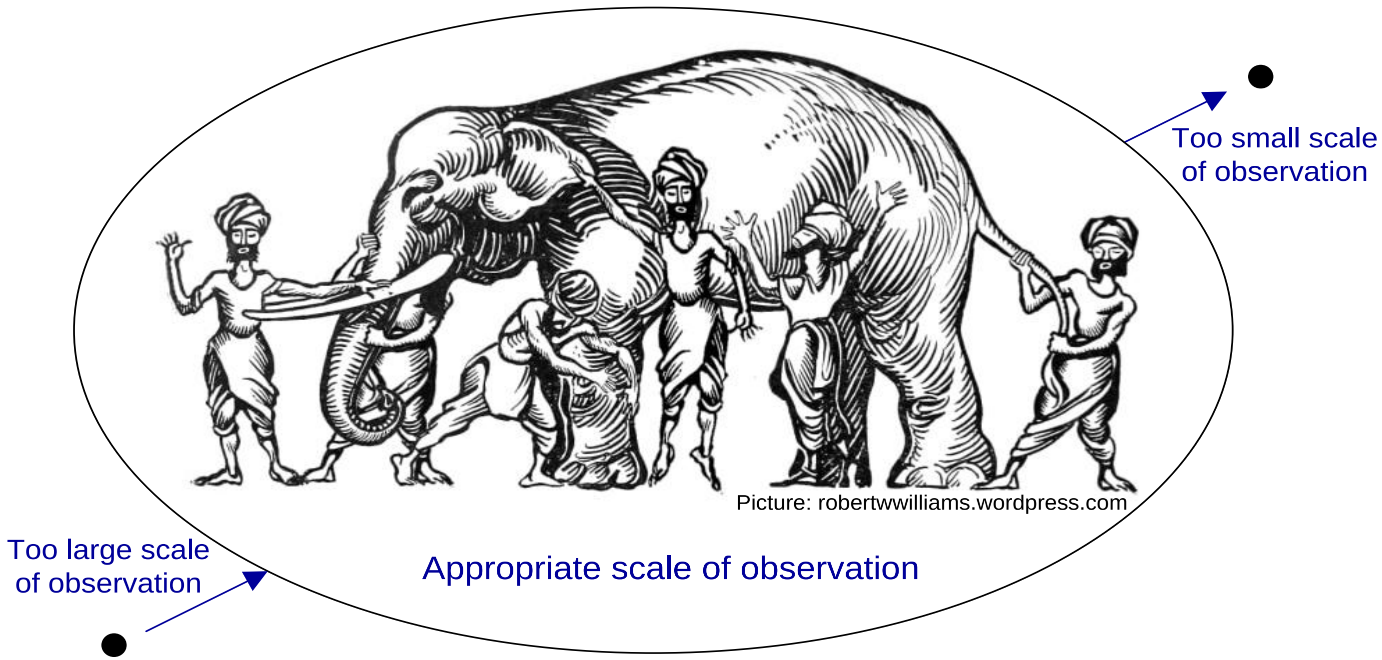
One of the most *ingenious idea of science* is ability to skillfully distinguish *isolated systems* and their division into *material particle* (Representative Volume Element - RVE) usually small, and its *surroundings*, which mutually interact.





So, what the tensors actually are, and how we can understand them from more pragmatic point of view?
Anecdotally, we can say that we are in a position of six blind man trying to find out what an elephant is ?
We must adopt a right scale of observation and right viewpoint to get information that we want.

Choosing scale and focus to find out what something is



Limitations or capabilities of human perception of physical phenomena?

Our senses have interesting limitations or maybe capabilities?,
that we *do see* (objectively) *different objects* depending on *how* we look.

Interesting limitations or maybe capabilities
of human senses

Picture: Fliegende Blatter, a German humor magazine
published in Munich, October 23, 1892, p.147

Depending on *how they look* humans objectively
do see different things.

Another interesting property of our senses is that we can have *only one specific interpretation* of what we see *at specific instant*.

What tensors are, and how they can be understood?, usefulness of tensors as objects modeling reality.

Let us return to proceed in general terms.

Tensors gained its today omnipresence in science and technology because they proved to be excellent *modeling objects*.

The same type of tensors enable description of, among the other:

- *state of real objects* (e.g. temperature, velocity, deformation, stress, strain, energy),
- properties *of real objects* (e.g. thermal expansion, piezoelectric effects, elastic stiffness or compliance)
- loading *of real objects* (e.g. force or displacement load)

Tensors of various orders proved to be very convenient and reliable modeling tools in description and/or prediction of broad range of various real phenomena. This delivers motivation to understand as best as possible what tensors actually are and what are their *eigenproperties* – represented by their various *invariants*. Fine comprehension of tensor objects themselves can deliver better insight and facilitate deep understanding of specific real physical situations modeled with their aid.

What tensors are, and how they can be understood?

There exist at least several definitions of a tensor notion.

Types of definitions of a tensor:

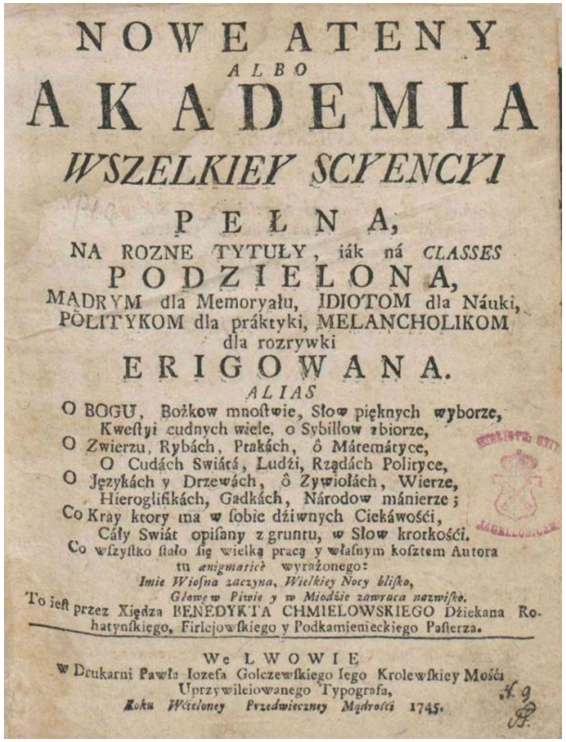
- *algebraic definition*, in which tensor is treated as *algebraic object* an element of advanced algebraic structure called tensorial linear space ($\mathcal{T}_q$).
- *operational definition*, in which tensor is understood as a *linear operator* transforming one tensorial object into another tensorial object linearly.
- *geometric definition*, in which tensor is envisioned as a *geometrical object* with specific *"shape"* and *orientation* with respect to some fixed reference/ coordinates frame.

Depending on *specific targets* of research and/or analysis one or the other approach to tensors can be more useful and/or convenient.

For example, for purposes of modeling of *real physical space* and *real physical phenomena* it is convenient to treat tensors as *geometrical objects* but in order to obtain some precise *quantitative results* using tensorial calculus, it is appropriate to treat tensors as *algebraic objects.*

Algebraic definition of a tensor.

Formal algebraic definition of tensors that: *Tensors are elements of tensorial linear spaces* $\boldsymbol{T}_q$ is actually very similar to the entry devoted to headword *horse* in the first Benedykt Chmielowski's Polish Encyclopedia entitled "New Athens" from 1745-1746, i.e.



NOWE ATENY
ALBO
AKADEMIA
WSZELKIEY SCYENCYI
PEŁNA,
NA ROZNE TYTUŁY, iák ná *CLASSES*
PODZIELONA,
MADRYM dla Memoryału, IDIOTOM dla Náuki,
POLITYKOM dla práktyki, MELANCHOLIKOM
dla rozrywki
ERIGOWANA.
ALIAS
O BOGU, Bożkow mnoſtwie, Słow pięknych wyborze,
Kweſtyi cudnych wiele, o Sybillow zbiorze,
O Zwierzu, Rybách, Ptakách, ô Mátemátyce,
O Cudách Swiátá, Ludźi, Rządách Polityce,
O Językách y Drzewách, ô Zywiołách, Wierze,
Hieroglifikách, Gadkách, Národow mánierze;
Co Kray ktory ma w ſobie dźiwnych Ciekáwośći,
Cáły Swiát opiſany z gruntu, w Słow krotkośći.
Co wſzyſtko ſtało ſię wielką pracą y właſnym koſztem Autora
tu *ænigmaticè* wyrażonego:
Imie Wioſna zaczyna, Wielkiey Nocy bliſko,
Głowę w Piwie y w Miodźie zawraca nazwiſko.
To ieſt przez Xiędza BENEDYKTA CHMIELOWSKIEGO Dźiekana Rohatynſkiego, Firlejowſkiego y Podkamienieckiego Paſterza.

We LWOWIE
w Drukarni Pawła Iozefa Golczewſkiego Iego Krolewſkiey Mośći
Uprzywileiowanego Typografa,
Roku Wćieloney Przedwieczney Mądrośći 1745.

"Horse, how it is everybody sees."

Both recalled above definitions are *practically meaningless* unless broad background information is already available to the readers of them. Here and attempt is undertaken to bring such background information regarding tensors so that one could say at the end of this work,

"Tensor, how it is everybody sees" ,

and this with profound understanding of the matter.

While not easily accessible and quite hermetic at first sight, actually the algebraic tools of tensor calculus are indispensable in obtaining any *quantitative results*.

Algebraic definition of a tensor, Gregorio Ricci-Curbastro's absolute differential calculus.

The mathematical grounds of *tensor calculus/analysis* with all the fundamental underlying formal apparatus was originally developed by Gregorio Ricci-Curbastro in the years 1888-1892.

The motivation behind this development was completely different from this of Cauchy, namely it was investigation on *invariance of quadratic forms* and Ricci-Curbastro called the technique *absolute differential calculus*.

Ricci-Curbastro can be regarded as father of *tensor notion* as it is *invariance feature* with respect to *coordinate systems*, which is the essence and profound sense of tensorial objects.

The term *tensor* in its contemporary meaning was coined by Woldemar Voigt in his work from 1898.

Tonolo A., Commemorazione di Gregorio Ricci-Curbastro nel primo centenario della nascita, Rendiconti del Seminario Matematico della Università di Padova (in Italian), tome 23 (1954), p. 1-24.
Voigt W., Die fundamentalen physikalischen Eigenschaften der Kristalle in elementarer Darstellung (in German), Verlag Von Veit & Comp. 1898, Leipzig.

Algebraic definition of a tensor,
Gregorio Ricci-Curbastro's absolute differential calculus.

Tonolo A., Commemorazione di Gregorio Ricci-Curbastro nel primo centenario della nascita, Rendiconti del Seminario Matematico della Università di Padova:

G. RICCI-CURBASTRO

1. *Delle derivazioni covarianti e controvarianti e del loro uso nella Analisi applicata*, «Studi editi dalla Università di Padova a commemorare l'ottavo centenario della origine della Università di Bologna», Vol. III, [1888].
2. *Sopra certi sistemi di funzioni*, «Rend. Acc. Lincei», Vol. V, [1889].
3. *Di un punto della teoria delle forme quadratiche ternarie*, «Ibidem». *Résumé de quelques travaux sur les systèmes variables de fonctions*, « Bull. Sc. Math.», T. XVI, [1892].

Label of wine produced to this day by Ricci-Curbastro family.
In vino veritas.

Session 3 - synopsis

Algebraic definition of a tensor. The following ideas/concepts can be identified to form preliminary steps leading to and enabling formulation of algebraic definition of tensors and quantitative (numerical) operations on them (tensor calculus):

Cartesian Coordinates System (Rene Descartes, 1637), Group (Evariste Galois, 1830), Abelian Group (Camile Jordan, 1870), Field of Real Numbers, Linear (Vector) Space (Peano, 1888), Euclidean Vector Space (Gibbs, 1881 and Heaviside, 1893), Euclidean Point (Affine) Space (*Affinitas,* Leonard Euler, 1748), Tensor Space (Gregorio Ricci-Curbastro, 1888-1992) (in bottom top order).

The Euclidean Point Space is at present commonly accepted model or real physical space. The most important information to acquire from algebraic definition of tensors is that tensor makes an inseparable integrity of a basis and its components (representation) in the basis. Expression of physical laws in tensorial form assures their invariance (symmetry) with respect to a group of transformations of changing coordinates system (translations and/or rotations). The physical laws have precisely the same tensorial form regardless of adopted coordinates system. This is an immense advantage and achievement of tensorial calculus.

Algebraic definition of a tensor.

A *Euclidean tensor* of order q and dimension n is and *algebraic structure*, an *element of linear tensorial space* $\mathcal{T}_q$, which is generated by the q-fold tensorial product of n dimensional *Euclidean vector spaces* E_n. When the same basis is accepted in all spaces E_n then any tensor $\mathbf{T}$ belonging to $\mathcal{T}_q$ can be presented in the form,

$$\mathbf{T} = \underbrace{T_{ij\ldots m}}_{components} \underbrace{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_m}_{basis\ (q\text{-}fold)} \quad i, j, \ldots, m = 1,..,n, \quad \mathbf{T} \in \mathcal{T}_q, \mathbf{e}_i \in E_n$$

where a set of q versors $\mathbf{e}_i \in E_n$ is a basis of Euclidean vector space E_n, a set of n^q simple tensors $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_m$ (q-fold tensorial product of versors $\mathbf{e}_i$) is a basis of space $\mathcal{T}_q \equiv E_n^{(1)} \otimes \ldots \otimes E_n^{(q)}$, and numbers $T_{ij\ldots m}$ are called *components* of tensor $\mathbf{T}$ – its *representation* in basis $\{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_m\}$.

By *algebraic structure*, it is understood a set composed of finite number of sets of *elements* and finite number of *mappings* of Cartesian products of these sets into these sets. The mappings are called *operations*.

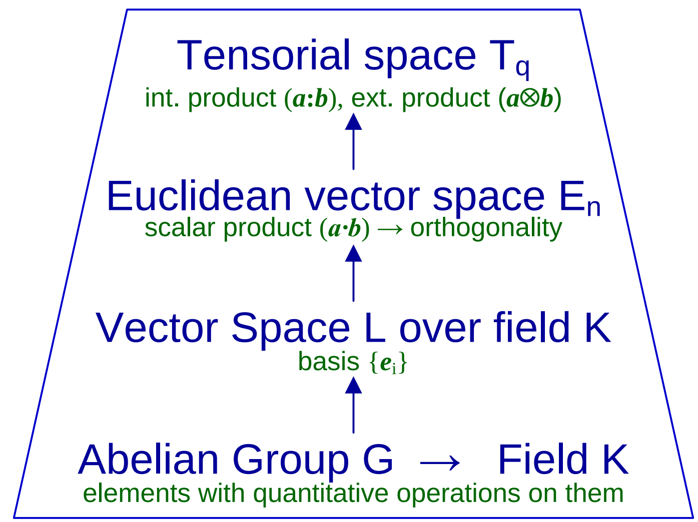
Tensorial linear space is higher level *(complex) algebraic structure* composed of lower level (simpler) algebraic structures.

Algebraic definition of a tensor.

The most important information to acquire from *algebraic definition* of tensors (elements of a linear space) is that *tensor makes integrity* of a *basis* and *components*, i.e. *representation* of the tensor in the *basis*.

When the basis is fixed, *isomorphism* exists between tensor and its representation (components) in this basis, i.e. the *tensor* can be identified with its *components*.

$$\mathbf{T} = \underbrace{T_{ij\ldots m}}_{Components} \underbrace{\mathbf{e}_i \otimes \mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_m}_{basis}$$



The components of a tensor transform in *linear manner* with change of the *coordinate system*.

For example

$$\mathbf{e}'_i = \mathbf{R}\mathbf{e}_i, \ \ \boldsymbol{\sigma} = \sigma_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j = R_{ki}\sigma_{ij}R_{jl}\, \mathbf{e}'_k \otimes \mathbf{e}'_l = \sigma'_{kl}\, \mathbf{e}'_k \otimes \mathbf{e}'_l \to \sigma'_{kl} = R_{ki}\sigma_{ij}R_{jl}$$

where $\mathbf{R}$ is so called rotation tensor $\mathcal{R} \equiv \{\, \mathbf{R} \in \mathcal{T}_2; \ \ \mathbf{R}\mathbf{R}^T = \mathbf{1}, \ \det(\mathbf{R}) = +1 \,\}$

Note: Care must be exercised because in general the *basis of a tensor* can be changed *in non-linear manner.*

Algebraic definition of a tensor.

The following algebraic structures of growing complexity make building blocks of *tensorial linear space* from the most simple to the most complex:

growing complexity ↑

	Elements, Operations	"Tools", e.g.
$\rightarrow$ *Tensorial linear space* $\mathcal{T}_q$	$\mathcal{T}_q \equiv (E_n; \otimes, :)$	contraction oper. $\mathrm{tr}(\mathbf{a}) \in R$
$\rightarrow$ *Euclidean point (affine) space* $\mathcal{E}_n$	$\mathcal{E}_n \equiv (\{P\}; \varphi_O), \varphi_O(O, X) \rightarrow \overrightarrow{OX} = \boldsymbol{x}$	affixed basis $(O, \{\mathbf{e}_i\})$
Gibbs (1881) Heaviside (1893) $\rightarrow$ *Euclidean vector (linear) space* E_n	$E_n \equiv (\mathcal{L}; \cdot)$	scalar and vector product $\mathbf{a} \cdot \mathbf{b}, \quad \mathbf{a} \times \mathbf{b} \in R$
Peano (1888) $\rightarrow$ Vector *space* $\mathcal{L}$ over field $\mathcal{K}$	$\mathcal{L} \equiv ((\{L\}; \oplus), \odot, (\{K\}; +, *))$	basis $\{\mathbf{e}_i\}$
$\rightarrow$ *Field* $\mathcal{K}$	$\mathcal{K} \equiv (\{K\}; +, *)$	
$\rightarrow$ Abelian Group $\mathcal{G}_{\mathrm{A}}$ Jordan (1870)	$g \Diamond h = h \Diamond g$ (*operation* "$\Diamond$" *is comutative*)	
$\rightarrow$ *Group* $\mathcal{G}$ Galois (1830)	$\mathcal{G} \equiv (\{G\}, \Diamond)$	
$\rightarrow$ Cartesian coordinates Descartes (1637)	$(x_1, \ldots, x_n)$ (*coordinates system*)	

Ogden R.W., Non-linear elastic deformations. Dover Publications, Inc. 1997.
Ostrowska-Maciejewska J., Podstawy i Zastosowania Rachunku Tensorowego, IPPT PAN, Reports, Warsaw, 2007.
Spencer A.J.M., Continuum Mechanics, Dover Publications Inc., 2004.
https://mathshistory.st-andrews.ac.uk/Miller/mathword/ Earliest Uses of Some Words of Mathematics

Algebraic definition of a tensor, Group (Abelian Group) $\mathcal{G}$.

Group $\mathcal{G}$

An algebraic structure $\mathcal{G}\equiv(\{G\},\Diamond)$ is called a Group $\mathcal{G}$ when $\{G\}$ is a non-empty set of *elements*, and $\Diamond$ is *operation* (mapping) assigning an element from $\{G\}$ to any pair of elements from $\{G\}$

$$\Diamond:\ (g,h)\in\{G\}\times\{G\}\Rightarrow g\Diamond h\in\{G\}$$

Operation must satisfy the following axioms:

$$\bigwedge_{g_1,g_2,g_3\in G} g_1\Diamond(g_2\Diamond g_3)=(g_1\Diamond g_2)\Diamond g_3$$ it is associative

$$\bigvee_{e\in G}\ \bigwedge_{g\in G} e\Diamond g=g\Diamond e=g$$ neutral element of the group exists

$$\bigwedge_{g\in G}\ \bigvee_{h\in G} g\Diamond h=h\Diamond g=e$$ inverse element exists

Abelian Group (commutative group) is a group, which operation is commutative

$$\bigwedge_{g,h\in G} g\Diamond h=h\Diamond g$$

For example, a set of all rotations of real space around fixed axis is a commutative group.

Algebraic definition of a tensor, Field $\mathcal{K}$.

Field $\mathcal{K}$

An algebraic structure $\mathcal{K} \equiv (\{K\}, +, *)$ is called a *Field* $\mathcal{K}$ when $\{K\}$ is a non-empty set of elements, and +, $*$ are operations (mappings) assigning an element from $\{K\}$ to any pair of elements from $\{K\}$

$$+: (\alpha,\beta) \in \{K\} \times \{K\} \rightarrow \alpha + \beta \in \{K\}, \qquad *: (\alpha,\beta) \in \{K\} \times \{K\} \rightarrow \alpha * \beta \in \{K\},$$

the following axioms must be true,

$(\{K\},+)$ set $\{K\}$ with operation "+" is Abelian group,

$(\{K\}-0,\cdot)$ set $\{K\}$ without neutral element of Abelian group $(\{K\},+)$, i.e. "0" is also Abelian group,

operation $*$ is distributive with respect to operation +,

$$\bigwedge_{\alpha,\beta,\gamma \in \{\mathcal{K}\}} \alpha * (\beta + \gamma) = \alpha * \beta + \alpha * \gamma \, .$$

For example, set of all real numbers with summation and multiplication operations is a field $(\{R\},+,*)$.

Algebraic definition of tensor, vector (linear) space $\mathcal{L}$.

Vector (Linear) space $\mathcal{L}$

Algebraic structure $\mathcal{L} \equiv ((\{L\},\oplus), \odot, (\{K\},+,*))$ is called linear space $\mathcal{L}$ over field $\mathcal{K}$ when the following axioms are true:

$(\{L\},+)$ is Abelian group

$(\{K\},+,*)$ is field

Operation $\odot$ of multiplication of elements from $\{L\}$ by elements from set $\{K\}$

$$\odot: \ (\alpha, A) \in \{K\} \times \{L\} \Rightarrow \alpha \odot A = \alpha A \in \{L\}$$

has the following properties

$\bigwedge_{\alpha\in\mathcal{K}} \bigwedge_{A,B\in\mathcal{L}} \alpha(A+B) = \alpha A + \alpha B$ + is distributive with respect to $\odot$

$\bigwedge_{\alpha,\beta\in\mathcal{K}} \bigwedge_{A\in\mathcal{L}} (\alpha+\beta)A = \alpha A + \beta A$ $\odot$ is distributive with respect to +

$\bigwedge_{\alpha,\beta\in\mathcal{K}} \bigwedge_{A\in\mathcal{L}} \alpha(\beta A) = (\alpha\beta)A$ $\odot$ is associative

$\bigvee_{1\in\mathcal{K}} \bigwedge_{A\in\mathcal{K}} 1A = A$ neutral element of field $\mathcal{K}$ exists

For example, set of all vectors in the plane with their summation and multiplication by real numbers is linear space.

Algebraic definition of tensor, vector (linear) space $\mathcal{L}$.

Basis and *dimension* of *vector (linear) space* $\mathcal{L}$

Each set of elements $A_1, A_2, \ldots, A_n \in \mathcal{L}$ such, that from equality

$$\alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_n A_n = 0$$

it results that

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

is called *linearly independent set of order n.*

Linear space $\mathcal{L}$ is called *n* – dimensional and denoted by $\mathcal{L}_n$, when it *does exists* linearly independent set of elements of order *n* in it, and *it does not exist* linearly independent set of elements of order greater than *n*.

Basis of space $\mathcal{L}_n$, it is called every *linearly independent set* of elements $A_1, A_2, \ldots, A_n \in \mathcal{L}_n$.

When the set $A_1, A_2, \ldots, A_n \in \mathcal{L}_n$ is a basis in $\mathcal{L}_n$ then each element A of space $\mathcal{L}_n$ can be expressed in the form

$$\bigwedge_{A \in \mathcal{L}_n} \quad \bigvee_{\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{K}} \quad A = \alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_n A_n$$

The set $A_1, A_2, \ldots, A_n$ allows for generation of all elements of space $\mathcal{L}_n$.

Algebraic definition of a tensor, Euclidean vector space E_n.

Euclidean vector space E_n

Linear space $\mathcal{L}_n$ (*n*-dimensional) over field of real numbers $\mathcal{R}$, equipped with *scalar product* operation defined on its elements, is called Euclidean vector space E_n.

Bilinear form is called *scalar product*

$$\cdot\ :\ (\boldsymbol{a},\boldsymbol{b}) \in \{L\} \times \{L\} \Rightarrow \boldsymbol{a} \cdot \boldsymbol{b} \in R$$

when it has the following properties: ◊ :

$$\bigwedge_{\boldsymbol{a},\boldsymbol{b} \in E_n} \quad \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

$$\bigwedge_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in E_n} \quad \boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}, \quad (\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}$$

$$\bigwedge_{\boldsymbol{a},\boldsymbol{b} \in E_n} \quad \bigwedge_{\alpha \in \mathcal{R}} \quad \alpha\, \boldsymbol{a} \cdot \boldsymbol{b} = \alpha (\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot (\alpha \boldsymbol{b})$$

$$\bigwedge_{\boldsymbol{a} \in E_n} \quad \boldsymbol{a} \cdot \boldsymbol{a} \geq 0 \quad \textit{and} \quad \boldsymbol{a} \cdot \boldsymbol{a} = 0 \quad \textit{for} \quad \boldsymbol{a} = 0$$

Scalar product enables introduction of the concepts of *vector norm (modulus)* and the concept of *angle* between vectors.

Algebraic definition of a tensor, Euclidean vector space E_{n}.

Norm and angle between vectors in Euclidean vector space E_n.

Norm (modulus) of vector $\boldsymbol{a}$, it is called a real number a satisfying the following conditions

$$a = |\boldsymbol{a}| \equiv (\boldsymbol{a}\cdot\boldsymbol{a})^{1/2}$$

$$|\boldsymbol{a}| \geq 0, \quad |\boldsymbol{a}| = 0 \quad \mathit{for} \quad \boldsymbol{a} = 0, \qquad |\alpha\,\boldsymbol{a}| = |\alpha|\,|\boldsymbol{a}|$$

$$\bigwedge_{\boldsymbol{a},\boldsymbol{b}\in E_n} \quad |\boldsymbol{a}+\boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}| \;\; - \text{ Schwartz inequality}$$

Vector $\boldsymbol{a}$, for which $|\boldsymbol{a}|=1$ is called a *versor*.

Angle φ between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be determined from the formula

$$\cos(\varphi) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}, \quad 0 \leq \varphi \leq \pi$$

Scalar product enables introduction of the notion of *orthonormal basis*.

Basis $\boldsymbol{e}_i$ is orthonormal when the following conditions are satisfied

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}, \quad |\boldsymbol{e}_i| = 1, \quad i, j = 1,\ldots,n$$

Vectors of orthonormal basis are mutually orthogonal and are versors.

Algebraic definition of a tensor, Euclidean point (affine) space $\mathcal{E}_n$.

Euclidean point (affine) space $\mathcal{E}_n$ (affine space) $\leftrightarrow$ *Euclidean vector space* E_n

Euclidean point space $\mathcal{E}_n$ of *dimension* n is an *algebraic structure* $\mathcal{E}_n \equiv (\{P\},\ E_n,\ \varphi)$, where $\{P\}$ is a set of elements (points), and φ is associated operation (mapping) uniquely (in one-to-one manner) assigning to each ordered pair of points $A, B \in \mathcal{E}_n$ a vector from vector space E_n

$$\varphi:\ (A,B) \in \{P\} \times \{P\} \Rightarrow \overrightarrow{AB} \in E_n$$

Mapping φ satisfies the following axioms,

$$\bigwedge_{A,B \in \mathcal{E}_n} \overrightarrow{AB} = -\overrightarrow{BA}, \qquad \bigwedge_{A,B,C \in \mathcal{E}_n} \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}, \qquad \bigwedge_{O \in \mathcal{E}_n} \bigwedge_{X \in \mathcal{E}_n} \bigvee_{\mathbf{x} \in E_n} \overrightarrow{OX} = \mathbf{x}$$

$\mathbf{x}$ *denotes vector radius* of point X with respect to selected, fixed point O.

Upon selection, fixing, point O from space $\mathcal{E}_n$, it can be defined one-to-one mapping between points from *Euclidean point space* $\mathcal{E}_n$ and vectors from *Euclidean vector space* E_n, i.e. arithmetization of space $\mathcal{E}_n$ can be attained

$$\varphi_O:\ \varphi_O(O, X) = \overrightarrow{OX} = \mathbf{x},\ \ \mathcal{E}_n \leftrightarrow E_n, \quad \varphi_O^{-1}:\ \varphi_O^{-1}(\mathbf{x}) = O \curlywedge \mathbf{x} = X$$

Affine space is *homogeneous*, no any special element is distinguished, like e.g. "zero" element.

Algebraic definition of tensor, Euclidean point space $\mathcal{E}_n$. Commonly accepted, convenient, model of real physical space.

Three dimensional Euclidean Point Space $\mathcal{E}_3$ with fixed coordinates frame (O, $\boldsymbol{e}_i$) ($i=1,2,3$) is commonly accepted as convenient *mathematical model of real physical space*.

Euclidean point space $\mathcal{E}_n$ *is not a vector (linear) space*, because it does not posses required structure of linear space, e.g. addition of its elements is not defined as required in definition of vector (linear) space.
Mapping φ , defining addition of vectors to points, *enables acquiring functionality of vector space* E_n - taking advantage of elements of its structure.

Each pair (O, $\boldsymbol{e}_i$), where $O \in \mathcal{E}_n$, $\boldsymbol{e}_i \in E_n$ is called *coordinates system* of point space $\mathcal{E}_n$ associated with vector space E_n. Point O is a hooking point of the coordinates system and $\boldsymbol{e}_i$ are basis versors of space E_n.

In continuum mechanics we are mainly working with *Euclidean (vector) space* E_3 when modeling real physical phenomena. This in view of existe-nce of *bijection* between spaces E_3 and $\mathcal{E}_3$ - upon fixing reference point O.

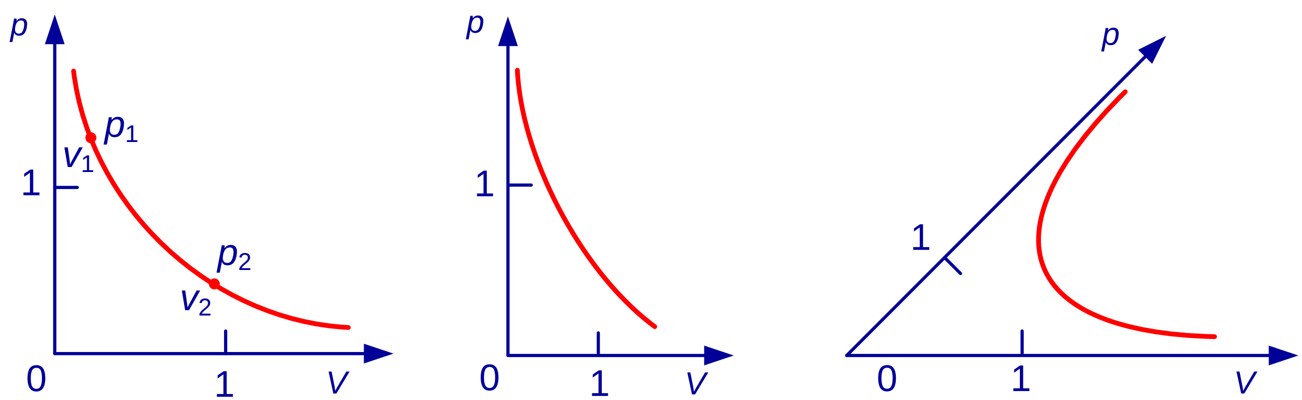
$$\bigwedge_{\boldsymbol{a} \in E_n} \quad \overset{1}{\bigvee_{\alpha_1, \alpha_2, \alpha_3 \in \mathcal{R}}} \quad \boldsymbol{a} = \alpha_1 \boldsymbol{e}_1 + \alpha_2 \boldsymbol{e}_2 + \alpha_3 \boldsymbol{e}_3 = \alpha_k \boldsymbol{e}_k$$

Algebraic definition of a tensor, Dimensional Affine Space.

Care must be exercised because there exists another concept/definition of affine space, in which affinity results from assigning different *dimensions* to the coordinate axes of the space.

Geometry in which properties of figures are examined *invariant* with respect to *units of measure* adopted on axes of *coordinate system* is called *affine geometry*. *Space* subject to *affine geometry* is called *affine vector space* (after Edmund Karaśkiewicz, p.355)

Operation of change of *physical units* of measure, and/or change of *length scale of line segments* and/or transition into *non-orthogonal coordinate systems* does not change the (depicted) *physical relation*.



Length of vector in such defined *(dimensional) affine space* does not have physical sense.

Karaśkiewicz E., Zarys teorii wektorów i tensorów, II Wyd., PWN, Warszawa, 1974.
Rychlewski J., Wymiary i Podobieństwo, PWN, Warszawa, 1991.

Algebraic definition of a tensor, tensorial linear spaces $\mathcal{M}_{nm}$.

Tensorial Product of linear spaces

Linear space $\mathcal{M}_{nm} = \mathcal{L}_n \oplus \mathcal{N}_m$ arising from Cartesian product $\mathcal{L}_n \otimes \mathcal{N}_m$ is called Tensorial Product of linear spaces $\mathcal{L}_n$ and $\mathcal{N}_m$ over field $\mathcal{K}$. Dimension of space $\mathcal{M}_{nm}$ is m·n.

Product operation $\otimes$ of elements from set $\{L\}$ by elements from set $\{N\}$

$$\otimes:\ (\boldsymbol{A},\boldsymbol{a}) \in \{L\} \times \{N\} \Rightarrow \boldsymbol{A} \otimes \boldsymbol{a} \in \mathcal{L}_n \otimes \mathcal{N}_m \quad (\mathcal{L}_n \times \mathcal{N}_m \rightarrow \mathcal{L}_n \otimes \mathcal{N}_m)$$

by conjecture has the following properties

$$\bigwedge_{A \in \mathcal{L}_n} \bigwedge_{a,b \in \mathcal{N}_m} A \otimes (a+b) = A \otimes a + A \otimes b$$

$+$ is distributive with respect to $\otimes$

$$\bigwedge_{A,B \in \mathcal{L}_n} \bigwedge_{a \in \mathcal{N}_m} (A+B) \otimes a = A \otimes a + B \otimes a$$

$\otimes$ is distributive with respect to $+$

$$\bigwedge_{A \in \mathcal{L}_n} \bigwedge_{b \in \mathcal{N}_m} \bigwedge_{\alpha \in \mathcal{K}} \alpha A \otimes b = \alpha (A \otimes b)$$

$\otimes$ is associative

Algebraic definition of a tensor, tensorial linear space $\mathcal{T}_q$.

Tensorial linear space $\mathcal{T}_q$ and Euclidean tensors $\boldsymbol{T}$

The q-tuple tensorial product of n-dimensional vector Euclidean spaces E_n *is called tensorial space $\mathcal{T}_q$ of Euclidean tensors with dimension n and order q.*

$$\mathbf{T}_q = \overset{(1)}{E}_n \otimes \overset{(2)}{E}_n \otimes \ldots \otimes \overset{(q)}{E}_n = \bigotimes_{i=1}^{q} \overset{(i)}{E}_n \in \mathcal{T}_q$$

Elements of Euclidean tensorial spaces are called *Euclidean Tensors*.

In continuum mechanics the most widespread use gained Euclidean tensors with dimension 3.

Examples of tensorial spaces with different order (rank) are:

$\mathcal{T}_q = R$ space of real numbers,

$\mathcal{T}_1 = E_3$ space of 3D vectors,

$\mathcal{T}_2 = E_3 \otimes E_3$ space 3D second order tensors.

Any *second order symmetric tensor* can be expressed in the form

$$\mathbf{A} = A_{ij}\,\mathbf{e}_i \otimes \mathbf{e}_j \in \mathcal{T}_2^{\,sym} \equiv E_3^{(1)} \otimes E_3^{(2)}, \quad where \quad A_{ij} = A_{ji} \Leftrightarrow \mathbf{A} = \mathbf{A}^T$$

Ostrowska-Maciejewska J., Podstawy i Zastosowania Rachunku Tensorowego, IPPT PAN, Reports, Warsaw, 2007

Session 4 - synopsis

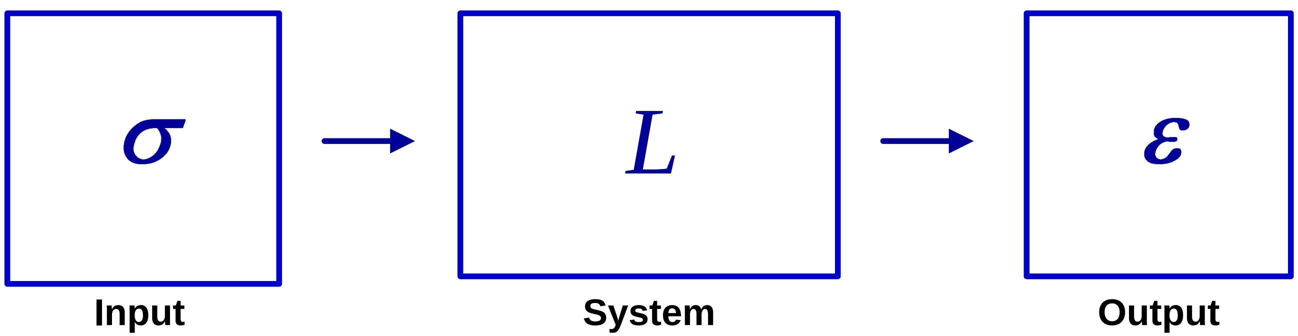
Tensors as linear transformations operators. Tensors as geometrical objects. Symmetry as universal philosophical category characterizing structures of organization of all systems in the universe. Permutation operation on tensors. Internal and external symmetries of tensors. Groups of orthogonal and proper orthogonal tensors as tools for distinguishing of isotropic and hemitropic tensors. Coordinates systems versus reference frames. Various convenient bases of second order symmetric tensors. Absolute notation and indicial notation of tensors. Hooke's tensor expressed in standard notation as fourth order tensor in three dimensional space and equivalently expressed in Kelvin notation as second order tensor in six dimensional space. Second order and fourth order unity tensors as generators of various projectors.

Operational definition of a tensor, constitutive models of materials are based on this interpretation.

Tensors can be treated as a *linear operators* transforming one tensorial object (space) into another tensorial object (space) linearly.

For example, fourth order tensor $\boldsymbol{L} \in \mathcal{T}_4$ upon its multiplication (contraction) with second order tensor $\boldsymbol{\sigma} \in \mathcal{T}_2$ transforms this tensor into some other second order tensor $\boldsymbol{\sigma} \rightarrow \boldsymbol{L\sigma} = \boldsymbol{\varepsilon} \in \mathcal{T}_2$.

$$\boldsymbol{\sigma} \Rightarrow \mathbf{L} \Rightarrow \boldsymbol{\varepsilon}; \quad \boldsymbol{\varepsilon} = \mathbf{L}\boldsymbol{\sigma}$$

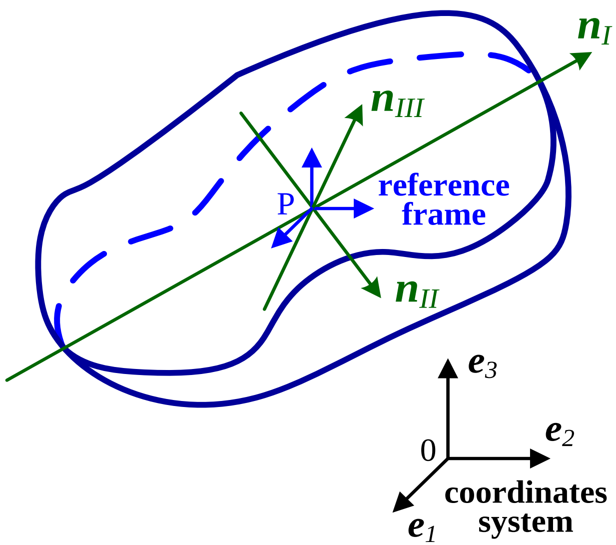


The *linear operators* were and are subject of broad and vivid research activities, documented in rich literature on the subject.
For example, actually such domains as *linear optimization*, *linear control theory* or *linear stability analysis* are varieties of linear operators analysis. The most common approach to tensors treated as linear operations takes the form of *matrix calculus*.

Geometrical definition of a tensor.

Tensors can be interpreted as *geometrical objects* possessing specific *orientation* in physical space (*reference frame*) and specific features (*eigenproperties*) - number of which depends e.g. on order of the tensor.

Tensors information content and complexity grows with their increasing order and dimension of vector space generating respective tensor space.



Scalars (zero order tensors) does not bear information on their orientation in reference frame (physical space).

Vectors (first order tensors) carry information on their orientation in reference frame, and on *one* feature (eigenproperty) expressed by their moduli.

Second order symmetric tensors carry information on their orientation in reference frame and on up to *three* linearly independent eigenproperties.

Fourth order symmetric tensors (with Hooke's tensor symmetries - in fixed basis taking the form $\boldsymbol{H}=\boldsymbol{H}^{<1234>}=\boldsymbol{H}^{<2134>}=\boldsymbol{H}^{<1243>} \sim H_{\mathrm{ijkl}}=H_{\mathrm{jikl}}=H_{\mathrm{klij}}$) can describe up to *eighteen* linearly independent eigenproperties.

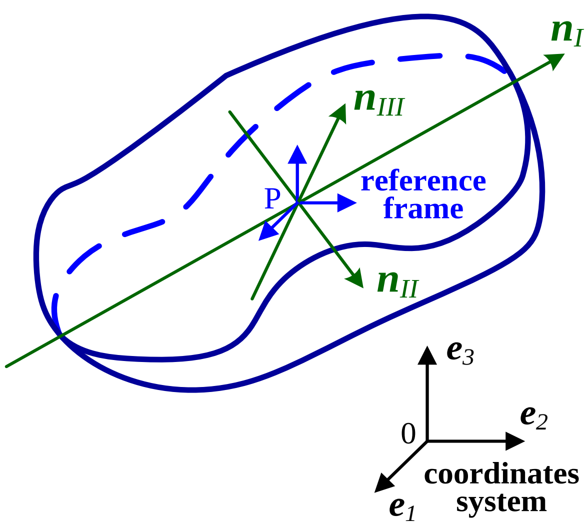
Coordinates system versus Reference frame.

A pair (affixed basis) composed of a point O belonging to *Euclidean point space* ($O \in \mathcal{E}_3$) and a set of basis versors $\{\mathbf{e}_\mathrm{i}\}$ of associated with it Euclidean vector space ($\mathbf{e}_\mathrm{i} \in E_3$) is called *coordinates system* (*coordinates frame*). Frequently, in shortcut, the coordinates system is denoted by the set of basis vectors only.

In continuum mechanics besides *coordinates system* it is used concept of *reference frame.* "Physically" both sets are composed of some anchoring point and a set of basis vectors, e.g. $\{O, \mathbf{e}_\mathrm{i}\}$.

The *difference* between *coordinates system* and *reference frame* is in their *functionality*. The coordinates system makes a reference for determination of vector (tensor) location and components, while the reference frame makes a reference for examination of e.g. motions (kinematics).

Depending on the *need* and *convenience* in examination of specific problem the *same* pair $\{O, \mathbf{e}_\mathrm{i}\}$ can be adopted for coordinates frame and reference frame or *different* pairs can be adopted.



Algebraic definition of a tensor, symmetries of tensors.

Symmetry concept plays *pivotal role* in tensorial calculus (*tensors*) and its applications.

A general, very capacious modern definition of symmetry can be formulated as follows (by the present author) :

Definition of Symmetry

Symmetry is the *invariance* (constancy, steadiness, stability) of some *feature* (geometric, physical, biological, information, etc.) of an *object* (an object can be a geometric system, a material object, a natural phenomenon, a physical law, a social relationship, a process in time, a physical field, etc.) after subjecting it to *action of transformations* from a certain *group* (transformations can be shifts, mirror images, rotations, changes of order, etc.) with respect to which symmetry is considered.

Symmetry can be perceived as a certain *universal philosophical category* (property) characterizing the *organization structure* of all systems existing in the universe (Weyl, *Symmetry*, 1952).

Weyl H., *Symmetry*, Princeton University Press, New Jersey, 1952.

Definition of internal symmetry of a tensor.

Before we proceed further Let us deliver more information on the concepts of internal and external symmetries of tensors. This properties are extensively used in modeling real phenomena with the aid of tensors.

Definition A *Permutation operation* σ on a *tensor* $\boldsymbol{T}$ $(\sigma\times\boldsymbol{T})$ p-th order is a linear mapping defined by the following formula

$$\sigma \times \mathbf{T}:\ \mathbf{T} = T_{12\ldots p}\,\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \ldots \otimes \mathbf{e}_p \rightarrow \sigma \times \mathbf{T} = T_{12\ldots p}\,\mathbf{e}_{\sigma(1)} \otimes \mathbf{e}_{\sigma(2)} \otimes \ldots \otimes \mathbf{e}_{\sigma(p)},$$

$$\sigma \equiv <\sigma(1),\sigma(2),\ldots,\sigma(p)>, \quad \mathbf{T},\sigma\times\mathbf{T} \in \mathcal{T}^p$$

where $\sigma(1),\ldots,\sigma(p)$ is a preset permutation of the first p natural numbers $(1,\ldots,p)$ and $\boldsymbol{T}_{12\ldots p}$ are components of tensor $\boldsymbol{T}$.

The permutation operation can be equivalently interpreted/treated as a permutation of the components of the tensor $\boldsymbol{T}$ written out in fixed basis.

$$\sigma \times \mathbf{T} \equiv <\sigma(1),\sigma(2),\ldots,\sigma(p)> \times \mathbf{T} = T_{\sigma(1)\,\sigma(2)\ldots\,\sigma(p)}\,\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \ldots \otimes \mathbf{e}_p \in \mathcal{T}^p$$

It is convenient to introduce the following more *compact notation* for permutation operation

$$\sigma \times \mathbf{T} \equiv <\sigma(1),\ldots,\sigma(p)> \times \mathbf{T} \equiv \mathbf{T}^{<(\sigma(1)\,\sigma(2)\ldots\,\sigma(p)>}$$

When it is known that only two indices permute it is convenient to use still more short denotation e.g. $\boldsymbol{T}^{<4,2>} \equiv \boldsymbol{T}^{<1432>}$.

Definitions of internal symmetry of a tensor.

The set of *all permutations operations* acting in the space of tensors of a fixed order (e.g. $\mathcal{T}^p$) constitutes the group $\mathcal{P}^\sigma$. This allows to introduce the concept of *internal symmetry of tensors*. The group $\mathcal{P}^\sigma$ is discreet, its size is finite and equals $p!$ elements, for example, for tensors of the 4-th order there are $4!=24$ elements of this group.

Definition An *internal symmetry group* of a tensor $\boldsymbol{T} \in \mathcal{T}_p$ is a subset of the permutation group $\mathcal{P}^\sigma$, whose elements satisfy the condition

$$\mathcal{P}_{\mathbf{T}}^{\sigma} \equiv \{\sigma \in \mathcal{P}^{\sigma}; \quad \sigma \times \mathbf{T} = \mathbf{T}\}; \quad \mathcal{P}_{\mathbf{T}}^{\sigma} \subset \mathcal{P}^{\sigma}$$

Tensors $\boldsymbol{T}$ satisfying the above condition are called (*internally*) *symmetric tensors* with respect to permutations of indices.

A tensor $\boldsymbol{T}$ is (*internally*) symmetric over a pair of indices (α,β), if equality holds, $\mathbf{T} = \mathbf{T}^{<\beta,\alpha>} \quad \leftrightarrow \quad T_{\ldots\alpha\ldots\ldots\beta\ldots} = T_{\ldots\beta\ldots\ldots\alpha\ldots}$, i.e., if the elements of the tensor $\boldsymbol{T}$ represen-ntation in any fixed basis when swapping the places of the indices (α,β) are the same. In the case of fourth-order tensors, the symmetry with respect to permutation operation $<1324>\times$ means that $\boldsymbol{T}=\boldsymbol{T}^{<1234>}=\boldsymbol{T}^{<1324>}$, tj. $T_{ijkl}=T_{ikjl}$ in any fixed basis.

Definition A tensor is *absolutely (internally) symmetric* when the group of its symmetries is the entire set of permutations $\mathcal{P}_{\mathbf{T}}^{\sigma} = \mathcal{P}^{\sigma}$.

Definition of external symmetry of a tensor.

Definition A set of second order tensors $\boldsymbol{Q}$ with properties,

$$\mathcal{O} = \{\mathbf{Q} \in \mathcal{T}^2;\ \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{1},\quad \det \mathbf{Q} = \pm 1\}$$

is a group and is called the *group of orthogonal tensors*.

Definition A subset of orthogonal tensors for which det ($\boldsymbol{Q}$)=+1

$$\mathcal{R} = \{\mathbf{Q} \in \mathcal{T}^2;\ \mathbf{Q}\mathbf{Q}^T = \mathbf{1},\ \det(\mathbf{Q}) = +1\},\quad \mathcal{R} \subset \mathcal{O}$$

is a group and is called the *special orthogonal group* or *rotation group (SO_n).*

Definition (□) A *group of external symmetry* of tensor $\boldsymbol{T} \in \mathcal{T}^p$ (p denotes order of the tensor) we call a subset of all orthogonal tensors $\boldsymbol{Q}$, which satisfy the following condition

$$\mathcal{O}_T = \{\mathbf{Q} \in \mathcal{O};\ \ \mathbf{Q} * \mathbf{T} = \mathbf{T}\},\quad \mathcal{O}_T \subseteq \mathcal{O};\qquad (\boldsymbol{Q} * \mathbf{T} \leftrightarrow Q_{ia} Q_{jb} \ldots Q_{kc} T_{ab\ldots c})$$

Tensors $\boldsymbol{T}$ that satisfy condition (□) are called (externally) *symmetric* with respect to orthogonal transformations $\boldsymbol{Q} \in \mathcal{O}_T$.

Definition Tensor is *isotropic* when the group of its external symmetry is the whole set of orthogonal tensors $\mathcal{O}_T = \mathcal{O}$.

Definition Tensor is *hemitropic* (also called *proper-isotropic*) when the group of its external symmetry is the whole set of proper orthogonal tensors $\mathcal{O}_T = \mathcal{R}$.

Second order symmetric Euclidean (Cartesian) tensors.

We will focus attention on *second order symmetric tensors* in *three dimensions* in the remainder of the work.

According to general representation theorem the basis of *second order Eulerian tensors* space $\mathcal{T}_2$ can be constructed from *nine* so called dyads $\{\mathbf{i}_i \otimes \mathbf{i}_j\}$, $(i, j=1,2,3)$, where $\mathbf{i}_\mathrm{i}$ are versors of basis of 3-dimensional linear vector space E_3 - generating tensor space $\mathcal{T}_2$.

For the versors $\mathbf{i}_\mathrm{i}$ usually there are adopted *orthonormal versors* $\{\mathbf{e}_\mathrm{i}\}$, $\mathbf{i}_i \otimes \mathbf{i}_j \to \mathbf{e}_i \otimes \mathbf{e}_j$. In such a case customarily *Eulerian tensors* are called *Cartesian tensors*. The second order tensors can be expressed with the aid of nine dyads $\mathbf{e}_i \otimes \mathbf{e}_j$ $(i, j = 1..3)$ as follows,

$$\boldsymbol{T} = \sum_{i,j=1,3} T_{ij}\,\mathbf{e}_i \otimes \mathbf{e}_j \in \mathcal{T}_2$$

In the case of so called *symmetric second order tensors* their components fulfill condition $\boldsymbol{T}=\boldsymbol{T}^T$ $(T_{ij}=T_{ji})$. This means that their 3x3 representation matrix has only 6 linearly independent components.

Symmetric second order tensors make *six-dimensional* subspace of general second order tensors space $(\mathcal{T}_2^{sym} \subset \mathcal{T}_2)$.

Second order symmetric tensors, notations and interpretations.

It is important to carefully distinguish between different *notations* used for tensors, as information may be differently distributed between the tensor *basis* and its *components* depending on the notation.

The following notations are very commonly used in the case of second order symmetric tensors (Cauchy tensor is used here as working example)

$$\boldsymbol{\sigma} = \sum_{i,j=1,3} \sigma_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j, \qquad \boldsymbol{\sigma} = \sum_{K=1,6} \sigma_K\, \mathbf{a}_K, \qquad \boldsymbol{\sigma} = \sigma_I \mathbf{N}_I + \sigma_{II} \mathbf{N}_{II} + \sigma_{III} \mathbf{N}_{III},$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}, \qquad [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6], \qquad [\sigma_I, \sigma_{II}, \sigma_{III}]. \qquad \begin{aligned} \mathbf{N}_J &\equiv \mathbf{n}_J \otimes \mathbf{n}_J \\ \mathbf{n}_J &= \mathbf{n}_J(\mathbf{e}_i) \end{aligned}$$

where $(\sigma_I, \sigma_{II}, \sigma_{III})$ denote principal values and $(\mathbf{n}_I, \mathbf{n}_{II}, \mathbf{n}_{III})$ are principal directions (eigenvectors) of the tensor $\boldsymbol{\sigma}$. The principal axes $\mathbf{n}_J$ are rotated with respect to laboratory frame axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by three (Euler) angles $(\theta_1, \theta_2, \theta_3)$.

$$\sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_3 = \sigma_{33}, \quad \sigma_4 = \sqrt{2}\,\sigma_{23}, \quad \sigma_5 = \sqrt{2}\,\sigma_{13}, \quad \sigma_6 = \sqrt{2}\,\sigma_{12},$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{a}_1 \equiv}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{a}_2 \equiv}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{a}_3 \equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\mathbf{a}_4 \equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{a}_5 \equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{a}_6 \equiv} \{\mathbf{e}_i \otimes \mathbf{e}_j\}$$

Absolute and indicial notation in tensorial calculus.

Absolute notation *and indicial notation*

A list below delivers a "recipe" for rewriting any tensorial formula written in *absolute notation* in the usual *Cartesian index notation*, i.e. in a fixed Cartesian basis.

$\boldsymbol{\alpha}\,\boldsymbol{\tau}$	$\leftrightarrow$	$\alpha_{ij}\tau_{jk}$
$\boldsymbol{\alpha}\cdot\boldsymbol{\beta}$	$\leftrightarrow$	$\alpha_{ij}\beta_{ij}$
$\mathbf{C}\cdot\boldsymbol{\omega}$	$\leftrightarrow$	$C_{ijkl}\omega_{kl}$
$\boldsymbol{\alpha}\cdot\mathbf{C}\cdot\boldsymbol{\beta}$	$\leftrightarrow$	$C_{ijkl}\alpha_{ij}\beta_{kl}$
$\mathbf{A}\cdot\mathbf{B}$	$\leftrightarrow$	$A_{ijkl}B_{ijkl}$
$\mathbf{C}\circ\mathbf{S}$	$\leftrightarrow$	$C_{ijkl}S_{klpq}$
$\mathbf{Q}*\boldsymbol{\omega}\equiv\mathbf{Q}\boldsymbol{\omega}\mathbf{Q}^T$	$\leftrightarrow$	$Q_{ij}Q_{kl}\omega_{jl}, \quad (\mathbf{Q}\,\mathbf{Q}^T=\mathbf{1})$
$\mathbf{Q}*\mathbf{C}$	$\leftrightarrow$	$Q_{ij}Q_{kl}Q_{pq}Q_{st}C_{jlqt}$
$\sigma\times\mathbf{T}$	$\leftrightarrow$	$T_{12\ldots p}\rightarrow T_{\sigma(1)\,\sigma(2)\ldots\,\sigma(p)}$

$\mathbf{n}\,,\boldsymbol{\omega}\,,\mathbf{C}$	$\leftrightarrow$	$n_i\;,\omega_{ij}\;,C_{ijkl}$
$\mathbf{1}$	$\leftrightarrow$	δ_{ij}
$\mathbf{n}\otimes\mathbf{m}\,,\mathbf{n}\otimes\boldsymbol{\omega}$	$\leftrightarrow$	$n_i m_j\;,n_i\omega_{jk}$
$\boldsymbol{\omega}\otimes\boldsymbol{\tau}$	$\leftrightarrow$	$\omega_{ij}\tau_{kl}$
$\boldsymbol{\omega}^2\,,\boldsymbol{\omega}^3$	$\leftrightarrow$	$\omega_{ij}\omega_{jk}\;,\omega_{ij}\omega_{jk}\omega_{kl}$
$\boldsymbol{\omega}\mathbf{n}\,,\mathbf{n}\boldsymbol{\omega}\mathbf{m}$	$\leftrightarrow$	$\omega_{ij}n_j\;,\omega_{ij}n_i m_j$
$\lvert\boldsymbol{\omega}\rvert=(\boldsymbol{\omega}\cdot\boldsymbol{\omega})^{1/2}$	$\leftrightarrow$	$(\omega_{ij}\omega_{ij})^{1/2}$
$\lvert\mathbf{C}\rvert$	$\leftrightarrow$	$(C_{ijkl}C_{ijkl})^{1/2}$

"   " - contraction over two indices, " $\cdot$ " – full contraction, " $\circ$ " – contraction over two indices (higher order tensors), " $*$ " – orthogonal transformation, " $\sigma\times$ " – permutation operation, " $\otimes$ " – tensorial product.

Absolute and indicial notation in tensorial calculus.

Example of *absolute* notation, *indicial* notation and *graphical illustration* of operation of *orthogonal* tensors on *second order tensors*.

$$\mathbf{Q}*\boldsymbol{\sigma} \equiv \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T \sim \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix}$$

$$\mathbf{Q}*\boldsymbol{\sigma} \equiv \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T \quad \leftrightarrow \quad Q_{ij}Q_{kl}\sigma_{jl}, \quad (\mathbf{Q}\mathbf{Q}^T = \mathbf{1})$$

$$\mathbf{Q}*\mathbf{C} \quad \leftrightarrow \quad Q_{ij}Q_{kl}Q_{pq}Q_{st}C_{jlqt}$$

The set of *orthogonal tensors* is used to define *external symmetry* of tensors.

$$\mathcal{O} = \{\mathbf{Q} \in \mathcal{T}^2;\ \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{1}, \quad \det\mathbf{Q} = \pm 1\}$$

$$\mathcal{R} = \{\mathbf{Q} \in \mathcal{T}^2;\ \mathbf{Q}\mathbf{Q}^T = \mathbf{1},\ \det(\mathbf{Q}) = +1\}, \quad \mathcal{R} \subset \mathcal{O}$$

$$\mathcal{O}_T = \{\mathbf{Q} \in \mathcal{O};\ \ \mathbf{Q}*\mathbf{T} = \mathbf{T} \in \mathcal{T}^p\}, \quad \mathcal{O}_T \subseteq \mathcal{O}; \quad (\mathbf{Q}*\mathbf{T} \leftrightarrow Q_{ia}Q_{jb}\ldots Q_{kc}T_{ab\ldots c})$$

Fourth order symmetric tensors, notations and interpretations.

Linear elasticity - Hooke's constitutive law

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\varepsilon} \sim \sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \quad i, j, k, l = 1,..,3; \quad \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathcal{T}^{2}_{(n=3)}, \ \mathbf{C} \in \mathcal{T}^{4}_{(n=3)}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1132} & C_{1113} & C_{1131} & C_{1112} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2232} & C_{2213} & C_{2231} & C_{2212} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3332} & C_{3313} & C_{3331} & C_{3312} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2313} & C_{2331} & C_{2312} & C_{2321} \\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3213} & C_{3231} & C_{3212} & C_{3221} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1313} & C_{1331} & C_{1312} & C_{1321} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3113} & C_{3131} & C_{3112} & C_{3121} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1213} & C_{1231} & C_{1212} & C_{1221} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2113} & C_{2131} & C_{2112} & C_{2121} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{13} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{bmatrix}$$

Hooke's tensor possess the following external symmetries

$$\boldsymbol{C}^{<1234>} = \boldsymbol{C}^{<2134>} = \boldsymbol{C}^{<1243>} = \boldsymbol{C}^{<3412>} \quad \sim \quad C_{jikl} = C_{ijlk} = C_{ijkl} = C_{klij}$$

what allows for treating it as second order tensor in six dimensional space.

Fourth order symmetric tensors, notations and interpretations.

It is *Kelvin notation*, which enables to achieve *full tensorial equivalence* of interpretation of symmetric tensors of the second and fourth order (Hooke's symmetries) from a 3-dimensional space, as vectors and second-order tensors from a 6-dimensional space, and the opposite,

$$\boldsymbol{\omega} \in \mathcal{S} \otimes \mathcal{S}_{(n=3)} \quad \leftrightarrow \quad \boldsymbol{\omega} \in \mathcal{S}_{(n=6)}, \qquad \omega_{ij} = \omega_{ji} \leftrightarrow \omega_K, \qquad i,j,k,l = 1,..,3, \;\; K,L = 1,..,6,$$

$$\mathbf{C} \in \mathcal{S} \otimes \mathcal{S} \otimes \mathcal{S} \otimes \mathcal{S}_{(n=3)} \;\leftrightarrow\; \mathbf{C} \in \mathcal{S} \otimes \mathcal{S}_{(n=6)}, \;\; C_{ijkl} = C_{jikl} = C_{klij} \leftrightarrow C_{KL} = C_{LK}$$

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\varepsilon} \sim \sigma_K^{Ke} = C_{KL}^{Ke}\, \varepsilon_L^{Ke}, \qquad K,L = 1,...,6, \qquad \mathbf{C} \in \mathcal{S} \otimes \mathcal{S}_{(n=6)}, \;\; \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathcal{S}_{(n=6)}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{13} \\ \sqrt{2}\sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & \sqrt{2}C_{1123} & \sqrt{2}C_{1113} & \sqrt{2}C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & \sqrt{2}C_{2223} & \sqrt{2}C_{2213} & \sqrt{2}C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & \sqrt{2}C_{3323} & \sqrt{2}C_{3313} & \sqrt{2}C_{3312} \\ \sqrt{2}C_{2311} & \sqrt{2}C_{2322} & \sqrt{2}C_{2333} & 2C_{2323} & 2C_{2313} & 2C_{2312} \\ \sqrt{2}C_{1311} & \sqrt{2}C_{1322} & \sqrt{2}C_{1333} & 2C_{1323} & 2C_{1313} & 2C_{1312} \\ \sqrt{2}C_{1211} & \sqrt{2}C_{1222} & \sqrt{2}C_{1233} & 2C_{1223} & 2C_{1213} & 2C_{1212} \end{bmatrix}^{Ke} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\varepsilon_{23} \\ \sqrt{2}\varepsilon_{13} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix} \qquad \mathbf{a}_K \otimes \mathbf{a}_L$$

$$\mathbf{a}_1 \equiv \mathbf{e}_1 \otimes \mathbf{e}_1, \;\; \mathbf{a}_2 \equiv \mathbf{e}_2 \otimes \mathbf{e}_2, \;\; \mathbf{a}_3 \equiv \mathbf{e}_3 \otimes \mathbf{e}_3, \;\; \mathbf{a}_4 \equiv \tfrac{1}{\sqrt{2}}[\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2],$$

$$\mathbf{a}_5 \equiv \tfrac{1}{\sqrt{2}}[\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1], \;\; \mathbf{a}_6 \equiv \tfrac{1}{\sqrt{2}}[\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1]; \quad \mathbf{a}_K \cdot \mathbf{a}_L = \delta_{KL}, \;\; \mathbf{a}_K \in \mathcal{S}_{(n=6)}, \; K,L = 1,..,6,$$

$$\boldsymbol{\omega} = \omega_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \omega_K^{Ke} \mathbf{a}_K = \omega_1 \mathbf{a}_1 + \omega_2 \mathbf{a}_2 + \omega_3 \mathbf{a}_3 + \sqrt{2}\,\omega_4 \mathbf{a}_4 + \sqrt{2}\,\omega_5 \mathbf{a}_5 + \sqrt{2}\,\omega_6 \mathbf{a}_6, \quad \boldsymbol{\omega} \in \mathcal{S}_{(n=6)}.$$

Fourth order symmetric tensors, notations and interpretations.

In *computational mechanics* very frequently there is used so called *Voigt notation*

$$\boldsymbol{\sigma} = \mathbf{C}^{Vo} \cdot \boldsymbol{\varepsilon} = \mathbf{C} \cdot \boldsymbol{\varepsilon}^{Vo} \sim \sigma_{ij} = C_{ijkl}\, \varepsilon^{Vo}_{kl} = C_{\alpha\beta}\, \gamma_{\beta} \quad (C_{\alpha\beta} = C_{\beta\alpha}) \quad \leftrightarrow \quad \sigma_K = C^{Vo}_{KL}\, \varepsilon_L \quad (C^{Vo}_{KL} \neq C^{Vo}_{LK}),$$

$$\varepsilon^{Vo}_{11} \equiv \varepsilon_{11} = \gamma_1, \qquad \varepsilon^{Vo}_{22} \equiv \varepsilon_{22} = \gamma_2, \quad \varepsilon^{Vo}_{33} \equiv \varepsilon_{33} = \gamma_3,$$

$$\varepsilon^{Vo}_{23} \equiv \gamma_4 = 2\varepsilon_{23}, \quad \varepsilon^{Vo}_{13} \equiv \gamma_5 = 2\varepsilon_{13}, \quad \varepsilon^{Vo}_{12} \equiv \gamma_6 = 2\varepsilon_{12}$$

$$\tfrac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}^{Vo} = \tfrac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} \sim \tfrac{1}{2}\sigma_K \gamma_K = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}^{Vo} \leftrightarrow \begin{bmatrix} \sigma_{1} \\ \sigma_{2} \\ \sigma_{3} \\ \sigma_{4} \\ \sigma_{5} \\ \sigma_{6} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 2C_{14} & 2C_{15} & 2C_{16} \\ C_{21} & C_{22} & C_{23} & 2C_{24} & 2C_{25} & 2C_{26} \\ C_{31} & C_{32} & C_{33} & 2C_{34} & 2C_{35} & 2C_{36} \\ C_{41} & C_{42} & C_{43} & 2C_{44} & 2C_{45} & 2C_{46} \\ C_{51} & C_{52} & C_{53} & 2C_{54} & 2C_{55} & 2C_{56} \\ C_{61} & C_{62} & C_{63} & 2C_{64} & 2C_{65} & 2C_{66} \end{bmatrix}^{Vo} \begin{bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{3} \\ \varepsilon_{4} \\ \varepsilon_{5} \\ \varepsilon_{6} \end{bmatrix}$$

Note: Care must be exercised because *Voigt notation is not consistent* with principles of tensorial calculus.

$$\boldsymbol{\sigma} \sim \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{23} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \to \| \boldsymbol{\sigma} \|^2 = \| \sigma_{ij} \|^2 = \sigma_{11}{}^2 + \sigma_{22}{}^2 + \sigma_{33}{}^2 + 2(\sigma_{23}{}^2 + \sigma_{13}{}^2 + \sigma_{12}{}^2),$$

$$\boldsymbol{\sigma} \sim \sigma_{\alpha} = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6]^T \to \| \sigma^{Vo}_K \|^2 = \sigma_{11}{}^2 + \sigma_{22}{}^2 + \sigma_{33}{}^2 + \sigma_{23}{}^2 + \sigma_{13}{}^2 + \sigma_{12}{}^2 \neq \| \sigma_{ij} \|^2,$$

$$\boldsymbol{\varepsilon} \sim \varepsilon_{\alpha} = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6]^T \to \| \varepsilon^{Vo}_K \|^2 = \varepsilon_{11}{}^2 + \varepsilon_{22}{}^2 + \varepsilon_{33}{}^2 + \varepsilon_{23}{}^2 + \varepsilon_{13}{}^2 + \varepsilon_{12}{}^2 \neq \| \varepsilon_{ij} \|^2,$$

$$\boldsymbol{\varepsilon}^{Vo} \sim \gamma_{\alpha} = [\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6]^T \to \| \gamma_K \|^2 = \varepsilon_{11}{}^2 + \varepsilon_{22}{}^2 + \varepsilon_{33}{}^2 + 4\varepsilon_{23}{}^2 + 4\varepsilon_{13}{}^2 + 4\varepsilon_{12}{}^2 \neq \| \varepsilon_{ij} \|^2 .$$

$$\| C^{Vo}_{\alpha\beta} \|^2 \neq \| C_{ijkl} \|^2$$

Symmetric tensors, unit tensors 0,1, 2 and 4-th order.

Unit tensors of zero, first, second and fourth order

scalar, vector, 2 - order tensor, 4 - order symmetric tensors

$$1 \sim [1], \ \mathbf{1} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{1} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{1} \otimes \mathbf{1} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{I}^{(4s)} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{1} = \mathbf{1}^{(2)} \ (\delta_{ij}), \quad \mathbf{1} \otimes \mathbf{1} \ (\delta_{ij}\delta_{kl}), \qquad \mathbf{a}_K \otimes \mathbf{a}_L \qquad \mathbf{a}_K \otimes \mathbf{a}_L$$

$$\boldsymbol{I}^{(4s)} \equiv \tfrac{1}{2}[\mathbf{1} \otimes \mathbf{1}^{<3,2>} + \mathbf{1} \otimes \mathbf{1}^{<4,2>}] \sim \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}) \sim \delta_{KL}$$

$$(i, j, k, l = 1, 2, 3, \ K, L = 1, \ldots, 6)$$

Symmetric tensors, unit tensors as projectors.

Projectors constructed on the base of unit tensors

$\boldsymbol{J} = \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1}\ \ (\tfrac{1}{3}\delta_{ij}\delta_{kl})$ – isotropic projector, $\boldsymbol{J}\boldsymbol{J} = \boldsymbol{J}$, $\sigma_m\mathbf{1} = \boldsymbol{J}\boldsymbol{\sigma}$, $\mathbf{I}^{(4s)} \equiv \boldsymbol{J} + \boldsymbol{K}$,

$\boldsymbol{K} = \mathbf{I}^{(4s)} - \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1}$ – deviatoric projector, $\boldsymbol{K}\boldsymbol{K} = \boldsymbol{K}$, $\boldsymbol{J}\boldsymbol{K} = 0$, $\boldsymbol{s} = \boldsymbol{\sigma} - \sigma_m\mathbf{1} = \boldsymbol{K}\boldsymbol{\sigma}$

$$\begin{bmatrix}\sigma_m\\ \sigma_m\\ \sigma_m\\ 0\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}\tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})\\ \tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})\\ \tfrac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})\\ 0\\ 0\\ 0\end{bmatrix} = \overbrace{\frac{1}{3}\begin{bmatrix}1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\end{bmatrix}}^{\boldsymbol{J}}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}$$

$$\begin{bmatrix}s_{11}\\ s_{22}\\ s_{33}\\ s_{23}\\ s_{13}\\ s_{12}\end{bmatrix} = \begin{bmatrix}\tfrac{1}{3}(2\sigma_{11}-\sigma_{22}-\sigma_{33})\\ \tfrac{1}{3}(-\sigma_{11}+2\sigma_{22}-\sigma_{33})\\ \tfrac{1}{3}(-\sigma_{11}-\sigma_{22}+2\sigma_{33})\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \overbrace{\begin{bmatrix}\tfrac{2}{3}&-\tfrac{1}{3}&-\tfrac{1}{3}&0&0&0\\ -\tfrac{1}{3}&\tfrac{2}{3}&-\tfrac{1}{3}&0&0&0\\ -\tfrac{1}{3}&-\tfrac{1}{3}&\tfrac{2}{3}&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\end{bmatrix}}^{\boldsymbol{K}}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}$$

$$\boldsymbol{I}^{(4s)} \equiv \tfrac{1}{2}[\mathbf{1}\otimes\mathbf{1}^{<3,2>} + \mathbf{1}\otimes\mathbf{1}^{<4,2>}] \sim \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$$

$$\boldsymbol{I}^{(4skew)} \equiv \tfrac{1}{2}[\mathbf{1}\otimes\mathbf{1}^{<3,2>} - \mathbf{1}\otimes\mathbf{1}^{<4,2>}] \sim \tfrac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kj}),$$

$$\boldsymbol{A} = \mathbf{1}\otimes\mathbf{1}^{<2,3>}\circ\boldsymbol{A},\quad \boldsymbol{A}^T = \mathbf{1}\otimes\mathbf{1}^{<2,4>}\circ\boldsymbol{A},\quad tr(\boldsymbol{A})\mathbf{1} = \mathbf{1}\otimes\mathbf{1}\circ\boldsymbol{A},$$

$$\boldsymbol{A}^{(s)} = \mathbf{I}^{(4s)}\circ\boldsymbol{A},\quad \boldsymbol{A}^{(skew)} = \mathbf{I}^{(4skew)}\circ\boldsymbol{A}$$

Session 5 - synopsis

Various tensorial bases. Standard Cartesian basis, Kelvin basis for symmetric tensors, Principal directions basis (finding source in spectral decomposition of second order symmetric tensors), Eigenstrains basis (finding source in spectral decomposition of fourth order symmetric tensors), Symmetry basis (finding source in material symmetry), Laboratory frame basis. Various sets of invariants of second order symmetric tensors e.g. main (basic) invariants, principal invariants, principal invariants of deviator. Some useful physical interpretations of invariants. Various decompositions of tensors, e.g. decomposition into symmetric and antisymmetric part, decomposition into spherical and deviatoric part, decomposition into isotropic and anisotropic part, spectral decomposition, polar decomposition. Possibility of parametrization of Cauchy stress tensor with some set of three linearly independent invariants and Euler angles. Characteristic equation of full second order symmetric tensor and characteristic equation of its deviator. Principal values of second order symmetric tensor. It turns out that decomposition of Cauchy stress into spherical and deviatoric part is equivalent to decomposition into isotropic and anisotropic part. Explicit relations between principal invariants of second order symmetric tensor and principal invariants of its deviator.

Second order symmetric tensors, bases of tensor space.

1. Standard (Cartesian) basis

$$\mathbf{e}_i \otimes \mathbf{e}_j, \quad i, j = 1,3$$

2. Kelvin basis

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{\boldsymbol{a}_1\equiv}, \underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{\boldsymbol{a}_2\equiv}, \underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{\boldsymbol{a}_3\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{\boldsymbol{a}_4\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}}_{\boldsymbol{a}_5\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{\boldsymbol{a}_6\equiv} \{\mathbf{e}_i \otimes \mathbf{e}_j\}$$

3. Principal directions basis

$$\mathbf{N}_J \equiv \mathbf{n}_J \otimes \mathbf{n}_J, \quad \mathbf{n}_J = \mathbf{n}_J(\mathbf{e}_i), \quad J = I, II, II$$

4. Hooke's tensor eigenstresses basis

$$\boldsymbol{\omega}_K \in \mathcal{S}_{(n=6)}(\mathcal{T}_2^{sym}); \;\; \boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_K = \delta_{KL}, \;\; \mathbf{C} \in \mathcal{S} \otimes \mathcal{S}_{(n=6)}(\mathcal{T}_4^{sym}), \;\; K, L = 1,..,6,$$

$$\mathbf{C} \cdot \boldsymbol{\omega} = \lambda \boldsymbol{\omega} \;\rightarrow\; \det(\mathbf{C} - \lambda \mathbf{I}^{(4s)}) = 0 \;\rightarrow\; \lambda_K, \boldsymbol{\omega}_K, \quad \mathbf{I}^{(4s)} \in \mathcal{T}_4^{sym} \sim diag[1,1,1,1,1,1]$$

For example for isotropic materials

$$\{\mathbf{C}^{iso} \cdot \mathbf{h} = \lambda \mathbf{h} \sim C^{iso}_{\alpha\beta} h_\beta = \lambda h_\alpha\} \Leftrightarrow \det(\mathbf{C}^{iso} - \lambda \mathbf{I}^{(4s)}) = 0 \;\rightarrow\; \lambda_K, \mathbf{h}_K$$

$$\underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{\boldsymbol{h}_1\equiv}, \underbrace{\frac{1}{\sqrt{6}}\begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}}_{\boldsymbol{h}_2\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}}_{\boldsymbol{h}_3\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{\boldsymbol{h}_4\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}}_{\boldsymbol{h}_5\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{\boldsymbol{h}_6\equiv} \{\mathbf{e}_i \otimes \mathbf{e}_j\}$$

Ziółkowski A. G. Parametrization of Cauchy Stress Tensor Treated as Autonomous Object Using Isotropy Angle and Skewness Angle, *Engineering Transactions*, **70**, 3, pp. 239–286, 2022, doi: 10.24423/EngTrans.2210.20220809. Full text available freely at the link: https://et.ippt.gov.pl/index.php/et/article/view/2210

Bases of second order symmetric tensors space.

It was already mentioned that *basis of tensor space* can be freely selected. However, depending on the target of the analysis some bases are more convenient than the other, e.g. suitable selection of the basis makes analytical and/or numerical computations more simple and/or effective.

Let us list some convenient bases of second order symmetric tensors space useful in the further discussion:

i) *Laboratory frame basis*. Such a basis is selected for example in view of convenient expression of imposed boundary conditions, e.g. loadings and/or constraints.
ii) *Symmetries oriented basis*. Such basis is selected to be collinear with axes of some kind of material symmetry or geometrical shape/layout of examined engineering structure/device. For example, it is chosen to be collinear with natural axes of symmetry of (anisotropic) material.
iii) *Principal axes basis*. Such a basis is selected when it is subject matter justified or convenient to work with principal values of second order symmetric tensor only.
iv) *Eigenstates (Eigenstresses) basis*. Such a basis might be convenient in formulation of strength of materials criteria.

Features of second order tensors.

The *second order symmetric tensor* is fully characterized (defined) by *six components/parameters* (linearly independent) being its representation in some *fixed coordinates system* (*basis*). The components of a tensor change in *linear manner* with rotation of coordinates system.

From six components of second order symmetric tensor, there can be:

- constructed *infinite number of sets*, each consisting of *three invariants of the tensor* (linearly independent). Such invariants *do not change* when basis (coordinates system) of the tensor is changed.

- extracted a set of complementary parameters, three *Euler angles*, characterizing *orientation of the tensor object*, treated as *geometric entity*, with respect to axes $\{\mathbf{e}_i\}$ of the specific coordinates system (usually collinear with some convenient laboratory/reference frame) and generating basis of tensor space $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$. *Euler angles do change (are not invariants)* with change of coordinates system (tensor space basis).

Note: The notion of *invariants of a tensor*, the *tensor itself being invariant* with respect to a change of coordinates system sounds like "butterfish butter" – tautology, but actually it rather delivers a hint that the idea of a tensor is quite complex.

Second order symmetric tensors, various useful sets of invariants (eigenproperties).

Three linearly independent tensor invariants constructed from a second order symmetric tensor components, invariant under change of coordinates system (basis), can be treated as *characteristic features (eigenproperties)* of the specific tensor. They deliver convenient specification (description) of the tensor when it is treated as a *geometrical object.*

Actually, *infinite number of triad sets of second order symmetric tensor invariants* can be constructed $\{I_1^{\sigma}, I_2^{\sigma}, I_3^{\sigma}\}$.

Construction/Selection of specific set of invariants and their usefulness depends on the area of study and/or specific examined problem.

For example this might be ("Length", Width", "Height") or ("Hue", "Brightness", "Saturation").

Second order symmetric tensors, various useful sets of invariants (eigenproperties).

Tensorial modeling description *Reality*

Growing information content with growing tensor order:

Scalar description (1 datum), e.g. $a \sim [a_1]$

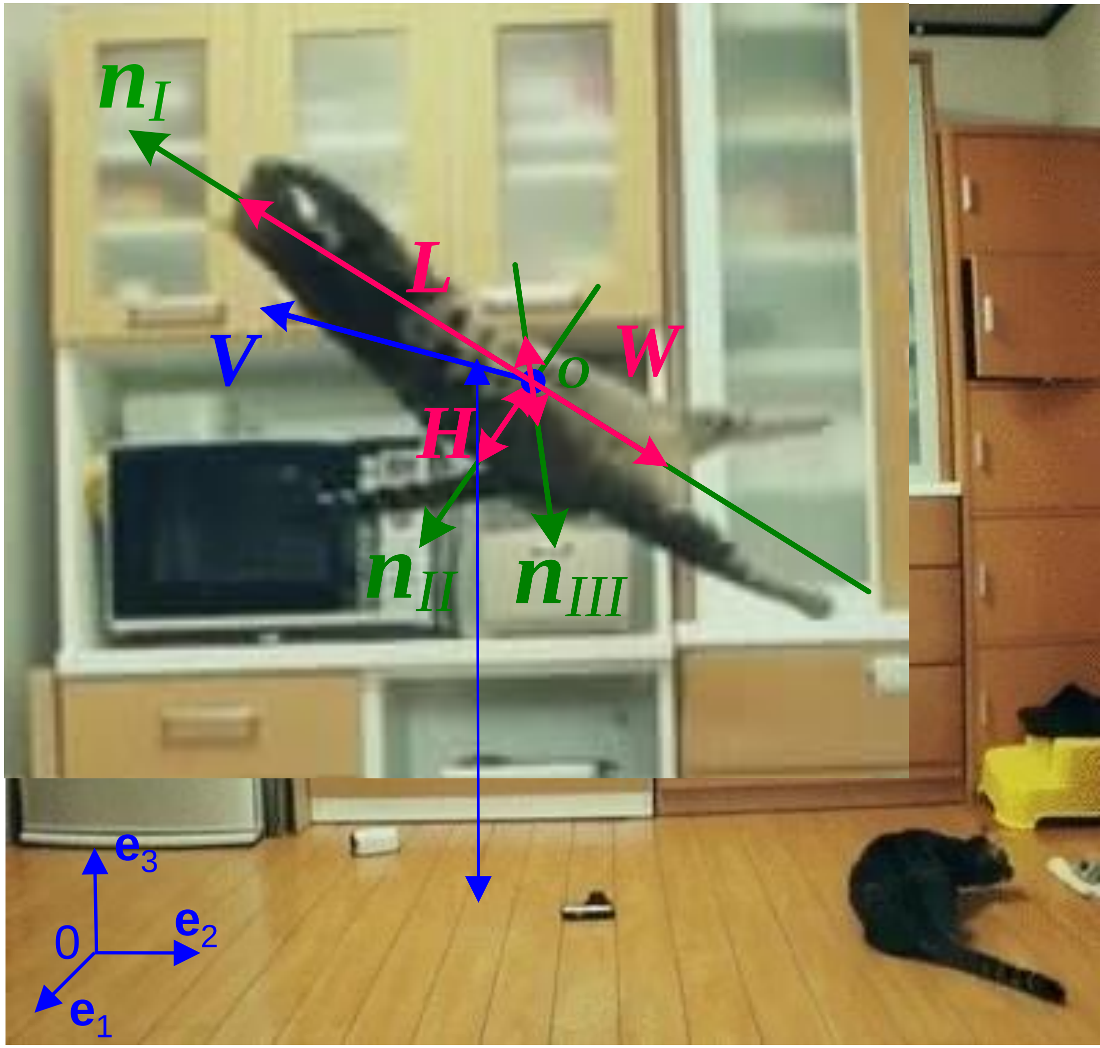
Vector description (3 data), e.g. $\mathbf{v} \sim v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$

2nd order symmetric tensor description (6 data), e.g.

$$\mathbf{A} = \begin{bmatrix} L & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & W \end{bmatrix} \quad \mathbf{n}_i \otimes \mathbf{n}_j$$

three characteristic directions

three invariants Length,Height,Width



Second order symmetric tensors, various useful sets of tensor invariants (eigenproperties)

Various invariants of second order symmetric tensors find useful physical interpretations and applications in different scientific and/or technological research areas. For example,

trace of *stress tensor* has very important physical interpretation of pressure

$$\tfrac{1}{3}tr(\boldsymbol{\sigma}) = \tfrac{1}{3}\sigma_{ii} = -p,$$

trace of *strain tensor* only approximately describes volumetric changes of the material

$$\tfrac{1}{3}tr(\boldsymbol{\varepsilon}) = \tfrac{1}{3}\varepsilon_{ii} \approx dV/dV_0 \ ,$$

determinant of *deformation gradient tensor* delivers exact measure of volumetric changes

$$det(\boldsymbol{F}) = dV/dV_0.$$

Different *sets of tensor invariants* deliver, can be treated, as *various parameterizations* of tensor *eigenproperties*.

Second order symmetric tensors, set of basic (main) invariants.

The common set of second order symmetric tensor invariants $\{I_{b1}, I_{b2}, I_{b3}\}$ most frequently encountered in mathematical studies are so called *basic invariants,* in some publications also called *main invariants*,

$$I_{b1} \equiv tr(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot \mathbf{1}, \qquad I_{b2} \equiv tr(\boldsymbol{\sigma}^2) = \|\boldsymbol{\sigma}\|^2 = \boldsymbol{\sigma}^2 \cdot \mathbf{1}, \qquad I_{b3} \equiv tr(\boldsymbol{\sigma}^3) = \boldsymbol{\sigma}^3 \cdot \mathbf{1}$$

where $||\boldsymbol{\sigma}|| = (\sigma_{ij}\sigma_{ij})^{1/2}$ denotes *norm of a tensor*, $\mathbf{1}$ (δ_{ij}) denotes *unit tensor* in second order tensors space. The dot symbol denotes full (double) contraction of second order tensors $\boldsymbol{a} \cdot \boldsymbol{b}$ $(a_{ij}b_{ij})$.

The popularity of basic invariants comes from *computational effectiveness* of their determination, which requires only multiplication of tensor matrix representation, for which very effective numerical algorithms exist.

In continuum mechanics alternative set of linearly independent invariants so called *principal values* $\{\sigma_{I}, \sigma_{II}, \sigma_{III}\}$ of second order symmetric tensor gained popularity and is in widespread use. The reason for that is their physical interpretation, e.g. in the case of stress tensor they very well characterize the *effort state* of a medium under specific loading.

Second order symmetric tensors, various decompositions.

A number of various *decompositions of tensors* exist, for example:

- decomposition into symmetric and antisymmetric part $\boldsymbol{A} = \boldsymbol{A}^{sym} + \boldsymbol{A}^{asym}$
- decomposition into spherical and deviatoric part $\boldsymbol{A} = \boldsymbol{A}^{sph} + \boldsymbol{A}^{dev}$
- isotropic decomposition $\boldsymbol{A} = \boldsymbol{A}^{iso} + \boldsymbol{A}^{aniso}$
- spectral decomposition *e.g.* $\boldsymbol{A} = \Sigma_{i=1,3}\, \lambda_i\, \mathbf{n}_i \otimes \mathbf{n}_i$
- polar decomposition $\boldsymbol{A} = \boldsymbol{RU} = \boldsymbol{VR}$

They are found to be useful for various purposes.

Second order symmetric tensors, stress tensor characteristic equation (spectral decomposition of second order symmetric tensor).

The principal values are determined as roots of so called *characteristic equation* for principal values

$$\boldsymbol{\sigma}\,\mathbf{n} = \sigma\,\mathbf{n} \rightarrow (\boldsymbol{\sigma} - \sigma\,\mathbf{1})\,\mathbf{n} = 0 \rightarrow \quad \det(\boldsymbol{\sigma} - \sigma\,\mathbf{1}) = 0 \rightarrow$$

$$\sigma^3 - I_1\,\sigma^2 + I_2\,\sigma - I_3 = 0 \rightarrow$$

$$\sigma_I, \sigma_{II}, \sigma_{III}, \;\; \boldsymbol{\sigma} = \sigma_I\,\mathbf{n}_I \otimes \mathbf{n}_I + \sigma_{II}\,\mathbf{n}_{II} \otimes \mathbf{n}_{II} + \sigma_{III}\,\mathbf{n}_{III} \otimes \mathbf{n}_{III},$$

$$\boldsymbol{\sigma}\,\mathbf{n}_J = \sigma_J\,\mathbf{n}_J\;(!J) \rightarrow \boldsymbol{\sigma}^n\mathbf{n}_J = \sigma_J{}^n\mathbf{n}_J\;(!J), \;\; J = I, II, III$$

where $\mathbf{n}_{\mathrm{I}}$, $\mathbf{n}_{\mathrm{II}}$, $\mathbf{n}_{\mathrm{III}}$ are *principal directions* (eigenvectors) of a tensor $\boldsymbol{\sigma}$. It is adopted naming convention here that $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$.

The following denotation was introduced for principal invariants

$$I_1 \equiv tr(\boldsymbol{\sigma}) \equiv 3\sigma_m \;\; (\sigma_{ii}),\, I_2 \equiv \tfrac{1}{2}[(tr\,\boldsymbol{\sigma})^2 - tr(\boldsymbol{\sigma}^2)]\,(\tfrac{1}{2}\sigma_{ij}\sigma_{ij}),\, I_3 \equiv \det(\boldsymbol{\sigma})\,(\tfrac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr})$$

symbol $\det(\cdot)$ denotes determinant operation, σ_m is mean value of principal values, and ε_{ijk} is permutation symbol.

The set of three coefficients $\{I_1, I_2, I_3\}$ appearing in characteristic equation for determination of principal values of second order symmetric tensor are called *principal invariants*.

Second order symmetric tensors, decomposition into spherical (isotropic) and deviatoric (anisotropic) parts.

Any second order tensor can be decomposed into *direct sum* of *spherical part* and *deviatoric part*

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{sph} + \boldsymbol{s}, \quad \boldsymbol{\sigma}_{sph} \equiv \sigma_m \mathbf{1} \;\; (\sigma_m = \tfrac{1}{3}\sigma_{ii} = \tfrac{1}{3} I_1), \quad \boldsymbol{s} \equiv \boldsymbol{\sigma} - \sigma_m \mathbf{1} \;\; (s_{ij} \equiv \sigma_{ij} - \sigma_m),$$

$$tr(\boldsymbol{s}) = 0, \quad \boldsymbol{\sigma}_{sph} \cdot \boldsymbol{s} = 0 \Leftrightarrow \boldsymbol{\sigma}_{sph} \perp \boldsymbol{s}, \quad \| \boldsymbol{\sigma}_{sph} \| \equiv [\boldsymbol{\sigma}_{sph}^2 \cdot \mathbf{1}]^{1/2} = \sqrt{3} | \sigma_m | = | I_1 | / \sqrt{3}$$

$\boldsymbol{\sigma}_{sph}$ denotes spherical part, $\boldsymbol{s}$ is deviator of the tensor.

The *direct sum* decomposition means that sum of any two spherical tensors ($a\mathbf{1}$, $b\mathbf{1}$) always gives spherical tensor $(a+b)\mathbf{1}$, and that sum of any two deviatoric tensors ($\boldsymbol{s}_a$,$\boldsymbol{s}_b$) always gives deviatoric tensor $\boldsymbol{s}_{ab}$.

Thus, deviatoric decomposition leads to division of the space of second order symmetric tensors into two separate, complementary (orthogonal) subspaces $\mathcal{T}_2^{sym} = \mathcal{P} \oplus \mathcal{D}$.

Second order symmetric tensors, decomposition into spherical (isotropic) and deviatoric (anisotropic) parts.

Decomposition of *second order symmetric tensor* into

spherical part $\sigma_m \mathbf{1}$ and *deviatoric part* $\mathbf{s}$,

is equivalent with its decomposition into

isotropic part and *anisotropic part.*

The *spherical part* of the tensor ($\sigma_m \mathbf{1}$) is *isotropic* in conventional sense, that is it does no change under application of any proper orthogonal (rotation) tensor $\mathbf{Q} \in \mathcal{R}$, where $\mathcal{R}$ is a *group of all proper orthogonal tensors*. So, *deviatoric part* is *anisotropic*.

Second order symmetric tensors, stress tensor deviator characteristic equation.

In analogy to characteristic equation for principal values of full stress tensor $\boldsymbol{\sigma}$, there can be formulated characteristic equation for eigenvalues of tensor deviator $\boldsymbol{s}$, coefficients of which makes a set of *principal invariants of tensor deviator* $\{J_1, J_2, J_3\}$ defined as follows

$$s^3 - J_1 s^2 - J_2 s - J_3 = 0 \rightarrow s_I, s_{II}, s_{III} \rightarrow \boldsymbol{s}^3 - J_2 \boldsymbol{s} - J_3 \mathbf{1} = 0,$$

$$J_1 \equiv tr(\boldsymbol{s}) = 0, \quad J_2 \equiv \tfrac{1}{2} tr(\boldsymbol{s}^2), \quad J_3 \equiv \det(\boldsymbol{s}) = \tfrac{1}{3} tr(\boldsymbol{s}^3),$$

$$J_1 = 0, \; J_2 = \tfrac{1}{2} \| \boldsymbol{s} \|^2 \geq 0 \;\; (\tfrac{1}{2} s_{ij} s_{ji}), \; J_3 = \det(\boldsymbol{s}) \;\; (\tfrac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} s_{ip} s_{jq} s_{kr})$$

The opposite sign in definition of second invariant of deviator ($J_2 \geq 0$) in comparison to definition of second invariant of full tensor (I_2) assures that it is always nonnegative. The J_2 invariant gained widespread use due to its physical interpretation of *shear stress intensity* measure.

Note: Care must be exercised because the sign of second invariant of deviator is changed to opposite in comparison to second invariant of the "full" tensor.

$$(s^3 - J_2 s - J_3 = 0 \leftrightarrow \sigma^3 - \cancel{I_1 \sigma^2} + I_2 \sigma - I_3 = 0)$$

Second order symmetric tensors, deviator characteristic equation.

The characteristic equation for principal values of deviator can be solved upon substitution of $s=(2/3)^{1/2}(2J_2)^{1/2}\cos(\theta)$ to obtain explicit formulas for stress deviator principal values $\{s_I, s_{II}, s_{III}\}$

$$s_I = \tfrac{2}{3}\sigma_{ef}\cos(\theta_L),\quad s_{II} = \tfrac{2}{3}\sigma_{ef}\cos(\theta_L - 120^0),\quad s_{III} = \tfrac{2}{3}\sigma_{ef}\cos(\theta_L + 120^0)$$

The full stress tensor and its principal values can be expressed as follows

$$\sigma_I = \sigma_m + s_I,\quad \sigma_{II} = \sigma_m + s_{II},\quad \sigma_{III} = \sigma_m + s_{III},$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\sigma_m, \sigma_{ef}, \theta_L) = \sigma_m \mathbf{1} + s_I\, \mathbf{n}_I \otimes \mathbf{n}_I + s_{II}\, \mathbf{n}_{II} \otimes \mathbf{n}_{II} + s_{III}\, \mathbf{n}_{III} \otimes \mathbf{n}_{III},$$

$$\|\boldsymbol{\sigma}\|^2 = \|\sigma_m \mathbf{1}\|^2 + \|\mathbf{s}\|^2 = 3\sigma_m{}^2 + 2J_2,\quad r \equiv \sqrt{2J_2},\quad \sigma_{ef} \equiv \sqrt{3J_2} = \sqrt{\tfrac{3}{2} s_{ij} s_{ij}}$$

where σ_{ef} denotes so called *effective stress*

$$\cos(3\theta_L) \equiv \bar{J}_3,\ \theta_L \in < 0, \pi/3 >,\quad \bar{J}_3 \equiv \frac{3\sqrt{3}}{2}\frac{J_3}{J_2{}^{3/2}} = \frac{3\sqrt{6}J_3}{r^{3/2}} \in < -1, 1 >$$

θ_L is called *Lode angle*, and $\bar{J}_3$ is called *normalized third invariant of deviator.* The following identities prove useful

$$\cos^3(\theta) - \tfrac{3}{4}\cdot\cos(\theta) - \tfrac{1}{4}\cos(3\theta) = 0,\quad \cos(\theta)\cos(\theta + 120^0)\cos(\theta - 120^0) = \tfrac{1}{4}\cos(3\theta)$$

Second order symmetric tensors, relations between invariants of full tensor and invariants of its deviator.

The invariants I_α, J_α, $\alpha=1,2,3$ can be expressed in terms of general components of stress tensor and in terms of its principal values as follows

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}, \quad I_1 = \sigma_I + \sigma_{II} + \sigma_{III}, \quad J_1 = s_I + s_{II} + s_{III} = 0,$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}\sigma_{21} - \sigma_{23}\sigma_{32} - \sigma_{13}\sigma_{31} = \sigma_I\sigma_{II} + \sigma_I\sigma_{III} + \sigma_{II}\sigma_{III},$$

$$J_2 = \tfrac{1}{6}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2] + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2$$

$$J_2 = -(s_I s_{II} + s_{II} s_{III} + s_I s_{III}) = \tfrac{1}{2}(s_I{}^2 + s_{II}{}^2 + s_{III}{}^2) \ge 0, \qquad I_3 = \sigma_I\,\sigma_{II}\,\sigma_{III},$$

$$I_3 = \tfrac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr} = \sigma_{11}\sigma_{22}\sigma_{33} + \sigma_{12}\sigma_{23}\sigma_{31} + \sigma_{32}\sigma_{21}\sigma_{13} - \sigma_{12}\sigma_{33}\sigma_{21} - \sigma_{13}\sigma_{22}\sigma_{31} - \sigma_{23}\sigma_{11}\sigma_{32}$$

$$J_3 = \tfrac{1}{3}s_{ij}s_{jk}s_{ki} = \tfrac{1}{3}(s_I{}^3 + s_{II}{}^3 + s_{III}{}^3) = s_I s_{II} s_{III} = (\sigma_I - \sigma_m)(\sigma_{II} - \sigma_m)(\sigma_{III} - \sigma_m).$$

The following relations are valid for basic (main), principal and deviator principal invariants $I_{b\alpha}$, I_α, J_α

$$J_2(\mathbf{s}) = -I_2(\boldsymbol{\sigma}) = \tfrac{1}{2}tr(\boldsymbol{\sigma}^2) - \tfrac{1}{6}(tr\,\boldsymbol{\sigma})^2 = 3\sigma_m{}^2 - I_2(\boldsymbol{\sigma}) = \tfrac{1}{3}I_1{}^2(\boldsymbol{\sigma}) - I_2(\boldsymbol{\sigma}),$$

$$I_3(\boldsymbol{\sigma}) = \det(\boldsymbol{\sigma}) = \tfrac{1}{3}tr(\boldsymbol{\sigma}^3) - \tfrac{1}{2}tr(\boldsymbol{\sigma}^2)tr(\boldsymbol{\sigma}) + \tfrac{1}{6}(tr\,\boldsymbol{\sigma})^3 = J_3 - J_2\cdot\sigma_m + \sigma_m{}^3,$$

$$J_3(\mathbf{s}) = \quad I_3(\boldsymbol{\sigma}) = \tfrac{1}{3}tr(\boldsymbol{\sigma}^3) - \tfrac{1}{3}tr(\boldsymbol{\sigma}^2)tr(\boldsymbol{\sigma}) + \tfrac{2}{27}(tr\,\boldsymbol{\sigma})^3 = I_3(\boldsymbol{\sigma}) - \tfrac{1}{3}I_2(\boldsymbol{\sigma})I_1(\boldsymbol{\sigma}) + \tfrac{2}{27}I_1{}^3(\boldsymbol{\sigma}).$$

$$\boldsymbol{\sigma}^3 - I_1\boldsymbol{\sigma}^2 + I_2\boldsymbol{\sigma} - I_3\mathbf{1} = 0, \quad \Rightarrow \quad \mathbf{s}^3 = J_3\mathbf{1} + J_2\,\mathbf{s} \quad \Rightarrow \quad tr(\mathbf{s}^3) = 3J_3$$

Session 6 - synopsis

The concept of Haigh-Westergaard (H-W) space – the space of Cauchy stress principal values. The H-W space does not posses standard structure of vector (linear) space - summing of two vectors in this space is physically meaningless. One point in H-W space represents infinitely many stress tensors having the same principal values but different principal directions. The H-W space can be interpreted as space of numerical markers of Cauchy stress tensor orbits. The concept of octahedral plane, motivated by stress tensors decomposition into spherical and deviatoric part. Graphical representation of critical surface, e.g. plastic yield flow of isotropic materials in H-W space. Three-fold (mirror) symmetry of isotropic critical surface in H-W-space, an excellent example of symmetry Principle of Ornament. The concept of octahedral and meridional cross sections of isotropic critical surface. The cylindrical set of coordinates in H-W space, i.e. pressure, effective stress, Lode (mode) angle. The Murzewski's isomorphic coordinates in H-W space (1958), i.e. modulus of spherical part, modulus of shear part and mode (Lode) angle. The very useful advantage of isomorphic coordinates is that they preserve correct shapes (distances and angles) of critical surfaces in octahedral and meridional cross sections. Tensorial decomposition of stress tensor into spherical (isotropic) and deviatoric (anisotropic) parts versus vectorial decomposition of octahedral traction into octahedral normal stress and octahedral shear stress.

Second order symmetric tensors,
Haigh-Westergaard Cauchy stress principal values space.

Haigh and independently *Westergaard* in 1920 tried to establish some *criteria of material strength*, i.e. evaluate when the material will start to yield plastically when submitted to multiaxial loading. They intuitively conjectured that for isotropic elastic materials Euler angles of stress tensor, describing orientation of principal axes in laboratory frame, should not influence material strength, and can be neglected.

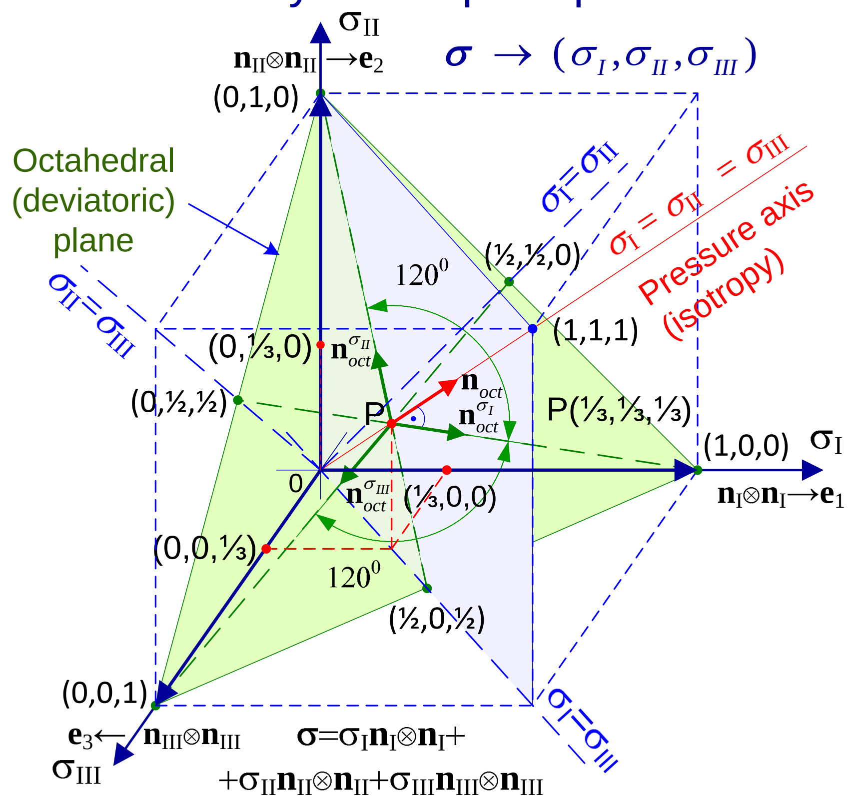
Basing on this conjecture, they proposed to introduce three dimensional *principal values vector space* with orthogonal coordinates frame composed of principal directions of stress tensor ($\sigma_I, \sigma_{II}, \sigma_{III}$). The principal values of stress tensor are Cartesian coordinates of points in this space.

The *principal values space* was coined the name *Haigh-Westergaard space*, according to *Maugin*.

Haigh B. P., The strain-energy function and the elastic limit, Engineering (London), Jan. 30, 1920, pp. 158-160.
Westergaard H. M., On the resistance of ductile materials to combined stresses in two or three directions perpendicular to one another, J.F.I., May 1920, pp. 627-640.
see page 14, Maugin G., Thermomechanics of plasticity and Fracture, Cambridge University Press, 1992.

Second order symmetric tensors, Haigh-Westergaard (H-W) Cauchy stress principal values space.

Graphical illustration of Haigh-Westergaard space together with elements frequently used in modeling of materials, *octahedral plane(-s)* and *isotropy axis*.



It is worth pointing out that an *infinite number of non-coaxial stress tensors* having the *same principal values* $(\sigma_I,\sigma_{II},\sigma_{III})$ but *different principal directions/orientations* $(\mathbf{n}_I, \mathbf{n}_{II}, \mathbf{n}_{III})$ reduce to a *single point representation* in the H-W space.

Second order symmetric tensors, critical surfaces for isotropic materials in Haigh-Westergaard space.

Graphical illustration of hypothetical critical surface (e.g. plastic yield flow, phase transition, etc,) for some isotropic material in Haigh-Westergaard (H-W) space.

Alternative coordinate systems:

1. $(\sigma_I, \sigma_{II}, \sigma_{III})$
2. (p, r, θ_L)

$$p = -\sigma_m = -\tfrac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III}),$$

$$r = \| \mathbf{s} \|^2 = \sqrt{s_I^{\ 2} + s_{II}^{\ 2} + s_{III}^{\ 2}},$$

$$\theta_L = 30^0 + tg^{-1}\left(\frac{\sqrt{3} s_{II}}{s_I - s_{III}}\right)$$

Ilustrative example of general critical surface of isotropic material in Haigh-Westergaard space

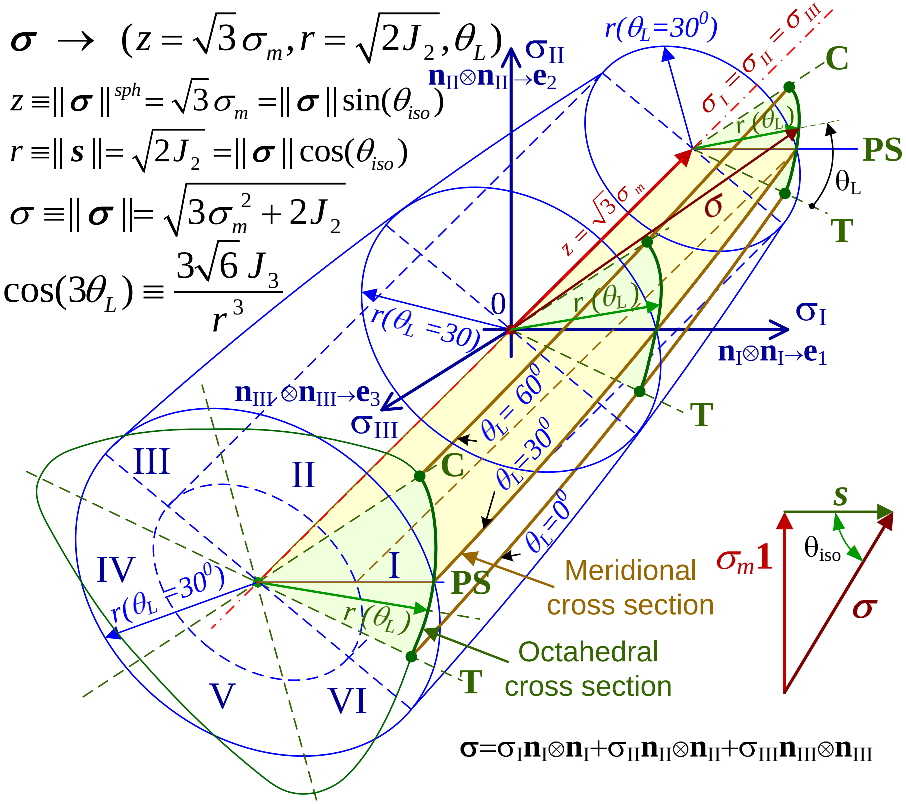
$$\boldsymbol{\sigma} \rightarrow (z = \sqrt{3}\sigma_m, r = \sqrt{2J_2}, \theta_L)$$

$$z \equiv \|\boldsymbol{\sigma}\|^{sph} = \sqrt{3}\sigma_m = \|\boldsymbol{\sigma}\| \sin(\theta_{iso})$$

$$r \equiv \|\mathbf{s}\| = \sqrt{2J_2} = \|\boldsymbol{\sigma}\| \cos(\theta_{iso})$$

$$\sigma \equiv \|\boldsymbol{\sigma}\| = \sqrt{3\sigma_m^{\ 2} + 2J_2}$$

$$\cos(3\theta_L) \equiv \frac{3\sqrt{6}\, J_3}{r^3}$$



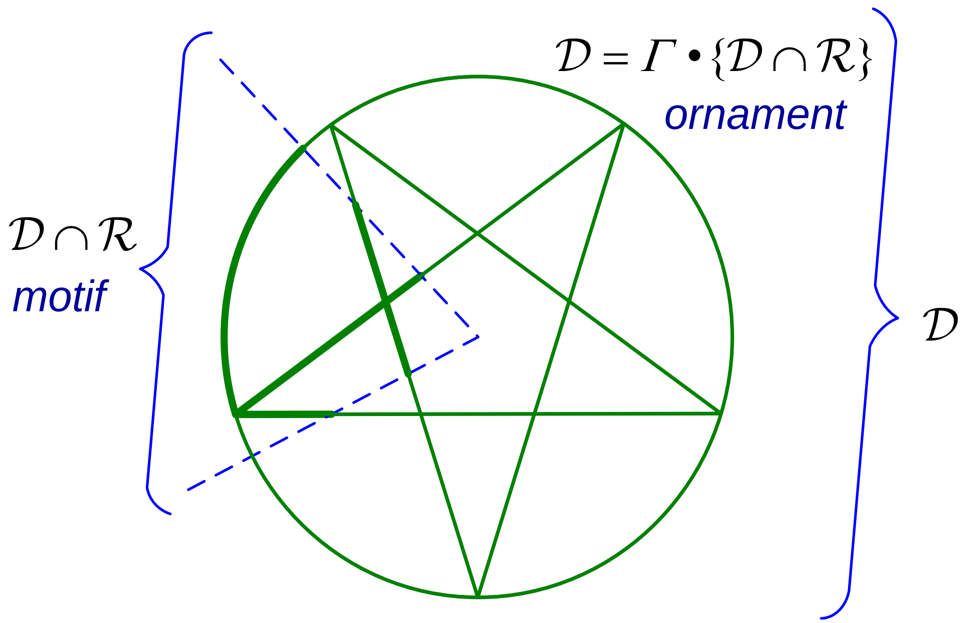
$$\boldsymbol{\sigma} = \sigma_I \mathbf{n}_I \otimes \mathbf{n}_I + \sigma_{II} \mathbf{n}_{II} \otimes \mathbf{n}_{II} + \sigma_{III} \mathbf{n}_{III} \otimes \mathbf{n}_{III}$$

Sometimes in the literature it is raised a problem of ambiguity resulting from different *ordering of principal values*. Please note that this problem actually disappears upon adopting, e.g., coordinates (p, r, θ_L). Then, the source of the "problem" can be immediately identified to be in six-fold ("permutation") symmetry exhibited by any/all *isotropic critical surfaces*.

Second order symmetric tensors, Principle of Ornament.

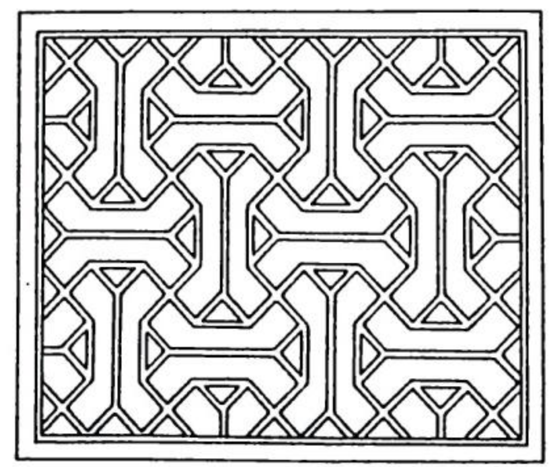
Isotropic critical surfaces in Haigh-Westergaard space make an excellent example of the symmetry *Principle of Ornament*.

Ilustrative example of Principle of ornament



$\mathcal{D}$: ornament
$\mathcal{D} \cap \mathcal{R}$: motif

Γ: repetition operation (rotation by 72^{o})



Example of ornament borrowed from Hermann Weyl's book, Symmetry, 1952.

Illustration of the *Principle of Ornament,* an *ornament is generated* using specific *motif* and specific *repetition operation*.

Rychlewski J. *Symmetry of Causes and Effects*, PWN, 1991.

Second order symmetric tensors, different coordinate systems in Haigh-Westergaard space.

In Haigh-Westergaard space in place of *stress principal values* $(\sigma_I, \sigma_{II}, \sigma_{III})$ *coordinates system* any set of *three linearly independent stress tensor invariants* $(I^*_I, I^*_{II}, I^*_{III})$ may be adopted as a *system of coordinates*.

A very popular set of this kind are (cylindrical) coordinates − *pressure*, *effective stress*, *Lode angle*

$$(p \equiv -\tfrac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III}), \sigma_{ef} \equiv \sqrt{\tfrac{3}{2}(s_I{}^2 + s_{II}{}^2 + s_{III}{}^2)}, \theta_L \equiv 30^0 + tg^{-1}(\sqrt{3}s_{II} / (s_I - s_{III}))\ ).$$

They are used for example to present *plastic flow yield*, *damage*, *failure* or *phase transition* critical surfaces for different materials in

- *octahedral* (p=const, σ_{ef}, θ_L) and/or
- *meridional* (p, σ_{ef}, θ_L=const) two dimensional cross sections.

In the continuum mechanics and materials literature often the coordinates $(p, r=(s_I{}^2+s_{II}{}^2+s_{III}{}^2)^{1/2}, \theta_L)$ or $(3p, r, \theta_L)$ are encountered.

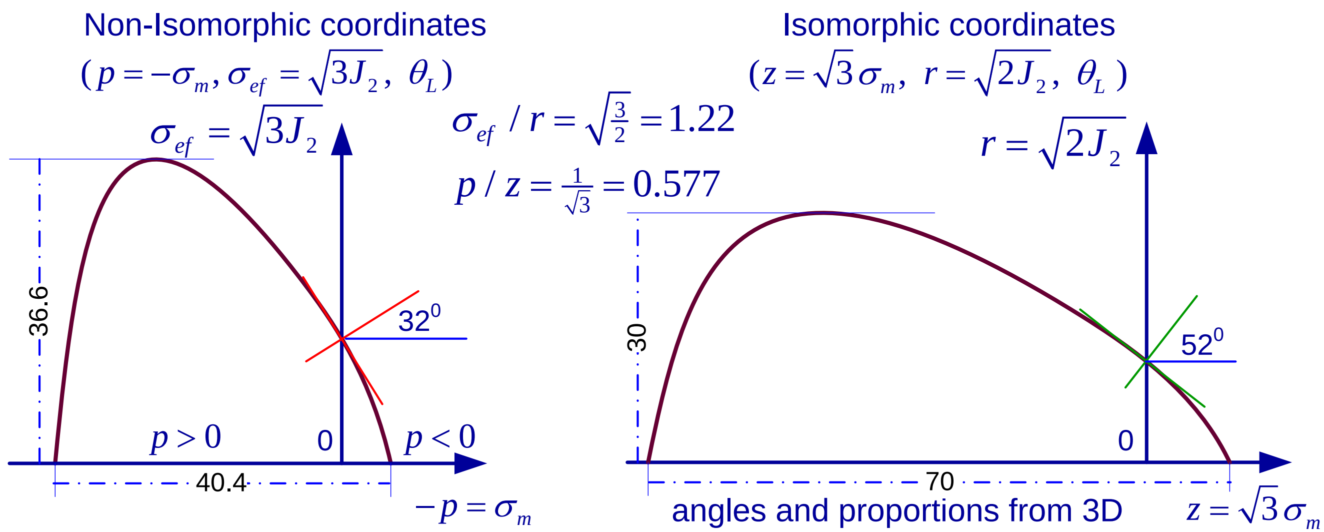
The problem with all listed above sets of coordinates is that they *distort actual 3D shape* of the critical surfaces *in 2D meridional projections*.

$$\sigma_I = \sigma_m + \sqrt{\tfrac{2}{3}}\, r\cos(\theta_L),\ \ \sigma_{II} = \sigma_m + \sqrt{\tfrac{2}{3}}\, r\cos(\theta_L - 120^0),\ \ \sigma_{III} = \sigma_m + \sqrt{\tfrac{2}{3}}\, r\cos(\theta_L - 240^0)$$

Second order symmetric tensors, non-isomorphic and isomorphic coordinates in H-W space.

Graphical illustration of distortion of 2D projections of 3D critical surface upon using *non-isomorphic coordinates* $(p=-\sigma_m, \sigma_{ef}=\sqrt{3J_2},\ \theta_L)$ in comparison to using *isomorphic coordinates* $(z=\sqrt{3}\sigma_m,\ r=\sqrt{2J_2},\ \theta_L)$ in Haigh-Westergaard space.

Meridional cross section of critical surface of isotropic material
in Haigh-Westergaard principal stresses space



angles and proportions from 3D critical surface are retained

Janusz Murzewski was the first researcher who in 1958 consciously introduced *isomorphic coordinates* in *Haigh-Westergaard space*, i.e. those preserving correct shapes (distances and angles) of critical surfaces in respective cross sections, according to the present author literature survey.

Murzewski J., A probabilistic theory of plastic and brittle behaviour of quasi-homogeneous materials, Archives of Mechanics, 1960 (Received 1st, December, 1958), 2, 12, pp. 203-227.

Second order symmetric tensors, isomorphic coordinates in H-W space.

Murzewski isomorphic cylindrical coordinates are as follows

$$(z \equiv \| \sigma_m \| = \sqrt{3}\,\sigma_m,\ \ r \equiv \| \mathbf{s} \| = \sqrt{2J_2},\ \ \theta_L \equiv \tfrac{1}{3}\cos^{-1}(3\sqrt{6}\,J_3/(2J_2)^{3/2}),$$

$$(p = -\sigma_m, \sigma_{ef} = \sqrt{3J_2},\ \theta_L) \quad - \text{non-isomorphic coordinates,}$$

$$|z| \equiv \| \sigma^{sph} \| = (\boldsymbol{\sigma}\cdot\mathbf{1})^{1/2} = \sqrt{3}|\sigma_m|, \quad \sigma_m = -p = \tfrac{1}{3}I_1, \quad r \equiv \| \mathbf{s} \| = \sqrt{2J_2}, \quad \| \boldsymbol{\sigma} \| = \sqrt{3\sigma_m{}^2 + 2J_2}.$$

Stress tensor and its principal values expressed in Murzewski's coordinates take the following form

$$\sigma_I = \tfrac{1}{\sqrt{3}}z + \sqrt{\tfrac{2}{3}}\,r\cos(\theta_L),\ \ \sigma_{II} = \tfrac{1}{\sqrt{3}}z + \sqrt{\tfrac{2}{3}}\,r\cos(\theta_L - 120^0),$$

$$\sigma_{III} = \tfrac{1}{\sqrt{3}}z + \sqrt{\tfrac{2}{3}}\,r\cos(\theta_L - 240^0),$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(z,r,\theta_L) = z\,\mathbf{N}_{oct} + r\cos\theta_L\,\mathbf{N}_{oct}^{\sigma_I} + r\sin\theta_L\,\mathbf{N}_{oct}^{\sigma_I\perp}, \quad \boldsymbol{\sigma}_{sph} = z\,\mathbf{N}_{oct},$$

$$\mathbf{N}_{oct} = \tfrac{1}{\sqrt{3}}(\mathbf{N}_I + \mathbf{N}_{II} + \mathbf{N}_{III}) = \tfrac{1}{\sqrt{3}}\mathbf{1}, \qquad \| \mathbf{N}_{oct} \| = \| \mathbf{N}_{oct}^{\sigma_I} \| = \| \mathbf{N}_{oct}^{\sigma_I\perp} \| = 1,$$

$$\mathbf{N}_{oct}^{\sigma_I} = \tfrac{1}{\sqrt{6}}(2\mathbf{N}_I - \mathbf{N}_{II} - \mathbf{N}_{III}),\ \ \mathbf{N}_{oct}^{\sigma_I\perp} = \tfrac{1}{\sqrt{2}}(\mathbf{N}_{II} - \mathbf{N}_{III}); \quad \mathbf{N}_J \equiv \mathbf{n}_J \otimes \mathbf{n}_J$$

An alternative to ($\sqrt{3}\sigma_m$, $r=\sqrt{(2J_2)}$, θ_L) isomorphic coordinates for meridional cross sections in Haigh-Westergaard space is pair of variables ($\sigma_{oct}=\sigma_m$, $\tau_{oct}=r/\sqrt{3}$). These last coordinates, with denotation ($\omega_1 \leftrightarrow \sigma_{oct}$, $\omega_2 \leftrightarrow \tau_{oct}$), were already used in 1929 by Burzyński in proposed by him extended plastic yield strength criterion for linearly elastic, isotropic solids.

Second order symmetric tensors, vectorial decomposition of octahedral traction vector.

Tensorial decomposition of tensor $\boldsymbol{\sigma}$

into spherical (isotropic) and deviatoric (anisotropic) parts

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(z, r, \theta_L) = z\,\mathbf{N}_{oct} + r\cos\theta_L\,\mathbf{N}_{oct}^{\sigma_I} + r\sin\theta_L\,\mathbf{N}_{oct}^{\sigma_I\perp}, \quad \boldsymbol{\sigma}_{sph} = z\,\mathbf{N}_{oct},$$

$$\mathbf{N}_{oct} = \tfrac{1}{\sqrt{3}}(\mathbf{N}_I + \mathbf{N}_{II} + \mathbf{N}_{III}) = \tfrac{1}{\sqrt{3}}\mathbf{1}, \qquad \|\mathbf{N}_{oct}\| = \|\mathbf{N}_{oct}^{\sigma_I}\| = \|\mathbf{N}_{oct}^{\sigma_I\perp}\| = 1,$$

$$\mathbf{N}_{oct}^{\sigma_I} = \tfrac{1}{\sqrt{6}}(2\,\mathbf{N}_I - \mathbf{N}_{II} - \mathbf{N}_{III}), \; \mathbf{N}_{oct}^{\sigma_I\perp} = \tfrac{1}{\sqrt{2}}(\mathbf{N}_{II} - \mathbf{N}_{III}); \quad \mathbf{N}_J \equiv \mathbf{n}_J \otimes \mathbf{n}_J$$

should be carefully distinguished from
Vectorial decomposition of *octahedral traction* (stress vector)

into *octahedral normal stress* and *octahedral shear stress*, which values can be expressed in terms of principal values as follows,

$$\boldsymbol{t}_{oct} \equiv \boldsymbol{\sigma}\,\mathbf{n}_{oct} = \tfrac{1}{\sqrt{3}}(\sigma_I\mathbf{n}_I + \sigma_{II}\mathbf{n}_{II} + \sigma_{III}\mathbf{n}_{III}), \quad \mathbf{n}_{oct} = \tfrac{1}{\sqrt{3}}\mathbf{1}, \quad \mathbf{n}_s = \mathbf{s}/\|\mathbf{s}\|,$$

$$\boldsymbol{t}_{oct} = \mathrm{t}_{oct}\,\mathbf{n}_t = \sigma_{oct}\,\mathbf{n}_{oct} + \tau_{oct}\,\mathbf{n}_s, \qquad \mathrm{t}_{oct} = \|\mathbf{t}_{oct}\| = \sqrt{\sigma_{oct}{}^2 + \tau_{oct}{}^2} = \tfrac{1}{\sqrt{3}}\|\boldsymbol{\sigma}\|,$$

$$\sigma_{oct} = \boldsymbol{t}_{oct}\cdot\mathbf{n}_{oct} = \mathbf{n}_{oct}\boldsymbol{\sigma}\cdot\mathbf{n}_{oct} = (1/3)(\sigma_I\,\mathbf{n}_I + \sigma_{II}\,\mathbf{n}_{II} + \sigma_{III}\,\mathbf{n}_{III})\cdot(\mathbf{n}_I + \mathbf{n}_{II} + \mathbf{n}_{III}) = \sigma_m,$$

$$\tau_{oct} = [\|\boldsymbol{t}_{oct}\|^2 - \sigma_{oct}{}^2]^{1/2} = \sqrt{2J_2/3} = r/\sqrt{3}.$$

Second order symmetric tensors, vectorial decomposition of octahedral traction vector.

Tensorial decomposition of *stress tensor* $\boldsymbol{\sigma}$ into

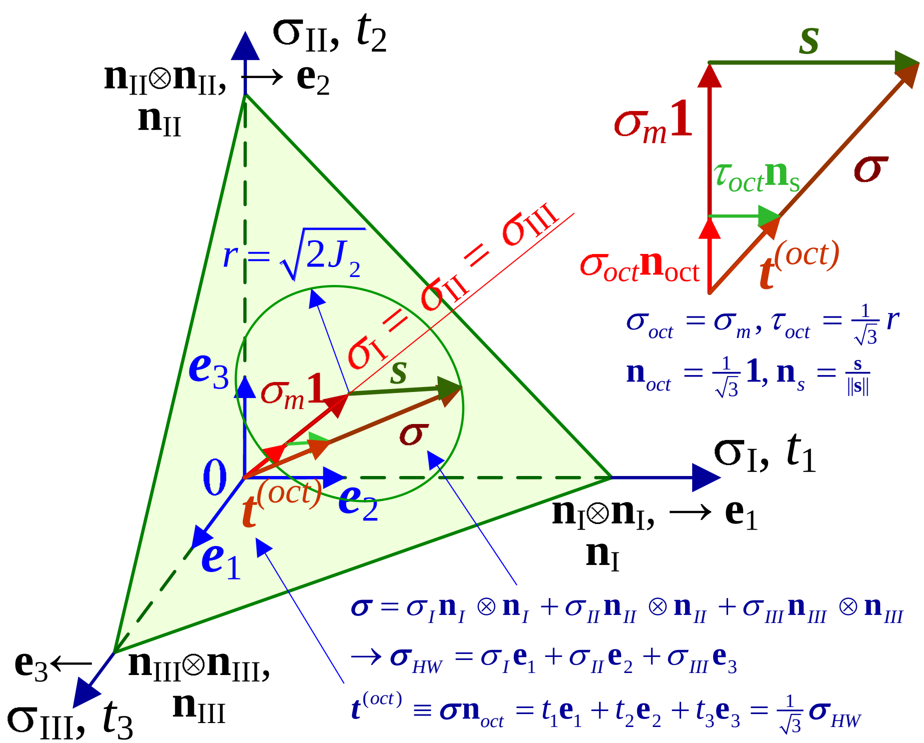
spherical (isotropic) part $\sigma_m\,\mathbf{1}$ and *deviatoric (anisotropic)* part $r\cdot\mathbf{s}/\|\mathbf{s}\|$,

versus

Vectorial decomposition of *octahedral traction* $\boldsymbol{t}^{(n)}$ into

octahedral normal stress $\sigma_{oct}\frac{1}{\sqrt{3}}\mathbf{1}$ and *octahedral shear stress* $\tau_{oct}\cdot\mathbf{s}/\|\mathbf{s}\|$.

Tensorial and vectorial decomposition of stress tensor and traction vector



$$\boldsymbol{\sigma}=\sigma_I\mathbf{n}_I\otimes\mathbf{n}_I+\sigma_{II}\mathbf{n}_{II}\otimes\mathbf{n}_{II}+\sigma_{III}\mathbf{n}_{III}\otimes\mathbf{n}_{III}$$

$$\|\boldsymbol{\sigma}\|^2=\ 3\sigma_m{}^2+r^2,\ \ \sigma_m=\tfrac{1}{3}I_1,\ \ r=\sqrt{2J_2}$$

$$\boldsymbol{t}_{oct}=\left(\tfrac{\sigma_I}{\sqrt{3}}\mathbf{n}_I+\tfrac{\sigma_{II}}{\sqrt{3}}\mathbf{n}_{II}+\tfrac{\sigma_{III}}{\sqrt{3}}\mathbf{n}_{III}\right)=\boldsymbol{\sigma}\mathbf{n}_{oct}$$

$$\boldsymbol{t}_{oct}=\sigma_m\mathbf{n}_{oct}+\tfrac{r}{\sqrt{3}}\mathbf{n}_s,\qquad \boldsymbol{\sigma}=\sqrt{3}\sigma_m\mathbf{n}_{oct}+r\,\mathbf{n}_s$$

$$\mathbf{n}_{oct}=\tfrac{1}{\sqrt{3}}\mathbf{1},\ \ \mathbf{n}_s=\mathbf{s}/\|\mathbf{s}\|$$

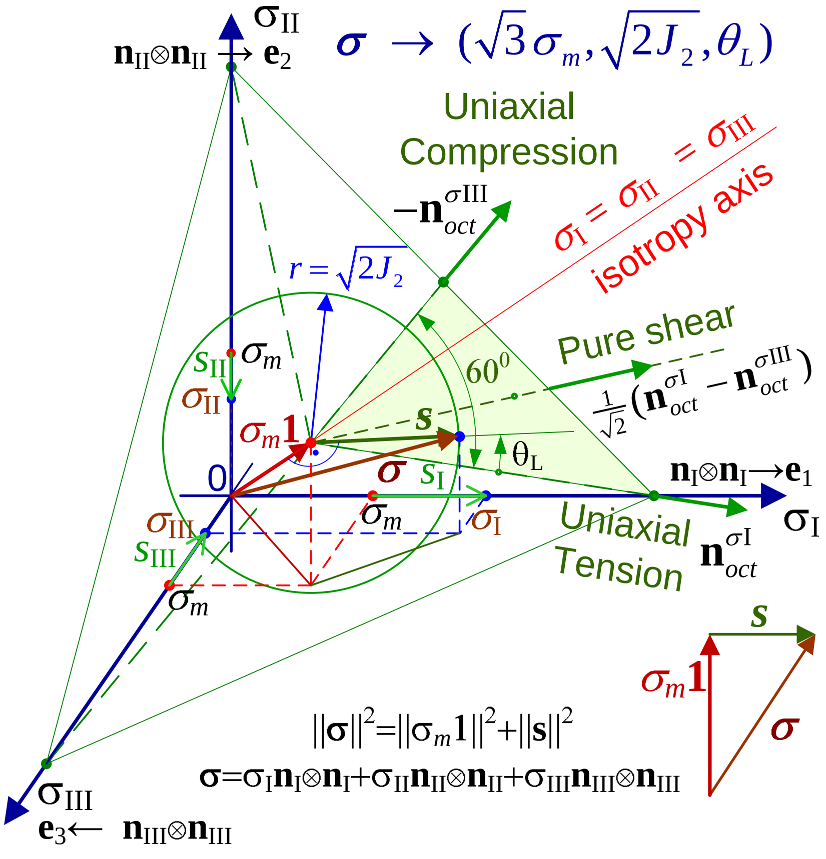
$$\|\mathbf{t}_{oct}\|^2=\sigma_{oct}{}^2+\tau_{oct}{}^2=\tfrac{1}{3}\|\boldsymbol{\sigma}\|^2$$

$$\sigma_{oct}=\sigma_m,\qquad \tau_{oct}=\tfrac{1}{\sqrt{3}}r$$

Second order symmetric tensors, isomorphic coordinates in H-W space.

It can be noticed that behind the idea of *isomorphic coordinates* stands the (orthogonal) *decomposition* of second order symmetric tensor into (*spherical* + *deviatoric*) ↔ (*isotropic* + *anisotropic*) parts.

Haigh-Westergaard space
isomorphic coordinates



Session 7 - synopsis

The concept and definition of Lode angle. Valentin Novozhilov to be the first (1951) to express Lode angle (Cauchy stress shear mode angle) in terms of stress principal invariants. Open scientific problem: lack of lucid (clear) physical interpretation of Lode angle. Pure shears as convenient elementary (atomic) elements of stress deviators space. Pure shears as excellent reference comparison states for any stress deviator (shear stress). Precise mathematical definition of pure shear (plane shear). Two classes of pure shears: pure shears with common direction and pure shears with common axis. Group of pure shears with common axis makes an excellent modeling idealization of uniform macroscopic plastic slip. Group of pure shears with common shear direction makes an excellent modeling idealization of macroscopic compound martensitic twin.

Second order symmetric tensors, Lode angle expressed in terms of principal values and principal invariants.

Originally Lode expressed *shear mode angle (today called Lode angle) in terms of stress tensor principal values*

$$\mu_L \equiv \frac{2\sigma_{II} - \sigma_I - \sigma_{III}}{\sigma_I - \sigma_{III}} = \sqrt{3}\, tg(\theta_L - 30^0)$$

This formula is very *inconvenient* (inefficient) from numerical standpoint because *first there must be calculated* principal values of stress tensor (with the aid of principal invariants) and only later the value of Lode angle can be computed. Upon some reflection it is clear that *Lode angle can be calculated directly from stress tensor principal invariants* (without computing principal stresses). This gives large savings in computational effort.

The present author, upon historical survey of the literature, found that the first researcher who explicitly expressed Lode angle in terms of second and third stress deviator invariants was Valentin Novozhilov, in a paper from 1951. Actually he published relevant formulas for *mode angle* ζ defined by him with sinus function, whereas Lode angle θ_L is defined with cosine function.

Given by him relation is as follows

$$\sin(\zeta) \equiv -\bar{J}_3, \quad \bar{J}_3 = 3\sqrt{6}\, J_3 / (2J_2)^{3/2} \in <-1, 1>; \quad (\cos(\theta_L) \equiv \bar{J}_3)$$

Novozhilov V.V., О связи между напряжениями и деформациями в нелинейно упругой среде (in Russian) On relations between stresses and strains in non-linear elastic media, Prikl. Mat. Mekh. XV, 1951, pp. 183-194.

Second order symmetric tensors,
Lode angle lack of lucid physical interpretation.

In this place one can get impression that everything what could be done regarding parametrization of Cauchy stress tensor Has already been done. However, more careful analysis proves that it is not the case.

The σ_m and $r = \sqrt{(2J_2)}$ coordinates *have clear physical interpretations* of pressure (with negative sign), and modulus (norm) of deviatoric (shear) part of stress tensor. However, *Lode angle* θ_L coordinate *does not have clear physical interpretation*.

From *mathematical standpoint Lode angle describes* angle between *projection* of specific stress tensor $\boldsymbol{\sigma}$ and *projection* of corresponding (having the same modulus) *uniaxial tension* tensor, on *octahedral plane*.

In a search for better *parameter* describing *mode of shear stress*, possessing lucid and meaningful *physical interpretation*, we will turn our attention to the selection of adequate *reference comparison stress state*.

For example, *in elasticity* such *physically meaningful* reference comparison state makes *unloaded, undeformed configuration/state* of elastic body, i.e., state of (zero stress, zero strain).

Second order symmetric tensors, pure shear as an atomic element of any deviator.

In search of physically meaningful, rational, shear stress reference comparison state to be used in parametrization of Cauchy stress, let us give some thought to a problem of *what is the most elementary (atom) non-trivial form of second order tensor?*

The first thought coming to ones mind is a tensor, which has *only single nonzero diagonal entry* in its matrix representation, e.g. $\mathrm{diag}(a,0,0)$. Such representation has for example the *uniaxial tension (extension)* and/or *uniaxial compression* tensors.

The option with single nonzero off-diagonal component is excluded due to symmetry requirement.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ?$$

Upon further reflection, it can be realized that *uniaxial tension* tensor is *not as simple as it seems*, and in fact several *elemental (atom) components* can be distilled from it along the lines of deviatoric decomposition of the second order symmetric tensors $(\boldsymbol{\sigma} = \sigma_m \mathbf{1} + \boldsymbol{s})$.

$$\boldsymbol{\sigma} = \sigma_m \mathbf{1} + \boldsymbol{s}, \qquad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix}$$

Second order symmetric tensors, pure shear as an atomic element of any deviator.

The most elementary component of uniaxial tensor that can actually be identified as irreducible to more simple modes, is the spherical tensor, *spherical elementary mode*, having three identical in value diagonal components $\mathrm{diag}(\tfrac{1}{3}a, \tfrac{1}{3}a, \tfrac{1}{3}a)$.

The *spherical elementary mode* can be *physically interpreted* as describing the simplest *3D layout of action of forces* in physical space, i.e. forces operating uniformly in all three physical directions, what corresponds to *action of pressure* only.
Alternatively it can be *physically interpreted* as describing *3D kinematics* of displacements *taking place uniformly* in physical space, corresponding to *change of volume* only.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}a & 0 & 0 \\ 0 & \frac{1}{3}a & 0 \\ 0 & 0 & \frac{1}{3}a \end{bmatrix} + \begin{bmatrix} \frac{2}{3}a & 0 & 0 \\ 0 & -\frac{1}{3}a & 0 \\ 0 & 0 & -\frac{1}{3}a \end{bmatrix}$$

Second order symmetric tensors, pure shear as an atomic element of any deviator.

Any deviator of second order symmetric tensor proves to be always decomposable into *two pure shear modes*, in general in *infinitely many ways,* see section 3.4 in Blinowski, Rychlewski, 1998.

Thus, *pure shear* modes prove to be *the most elementary irreducible deviator modes,* generators of space of deviators.

The *pure shear - deviatoric elementary (atomic) mode*, can be *physically interpreted* as the most simple 2D (plane) *layout of action of forces* in physical space, i.e. forces operating uniformly in all parallel planes having fixed common normal axis

Alternatively *2D kinematics* of displacements *taking place uniformly* in planes with common normal axis, and proportional to the distance from some fixed plane, similarly like it is in the case of sliding tile of cards.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}a & 0 & 0 \\ 0 & \frac{1}{3}a & 0 \\ 0 & 0 & \frac{1}{3}a \end{bmatrix} + \begin{bmatrix} \frac{1}{3}a & 0 & 0 \\ 0 & -\frac{1}{3}a & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{3}a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3}a \end{bmatrix}$$

Blinowski A., Rychlewski J. Pure shears in the mechanics of materials, Math.Mech.Solids, 3,4, 1998, pp. 471-503.

Second order symmetric tensors, some more information on very special characteristics of pure shear mode.

Let us recall some more information about pure shear mode. In particular, precise *mathematical definition of pure shear mode*.

A second order tensor is called a *pure shear* when the following conditions are fulfilled, after Blinowski and Rychlewski 1984,

$$I_1 = tr(\boldsymbol{\tau}) = 0, \quad I_3 = \det(\boldsymbol{\tau}) = 0 \quad \Rightarrow \quad J_3 = \tfrac{1}{3} tr(\boldsymbol{\tau}^3) = 0$$

several other equivalent definitions can be found in original publication

Depending on the selection of coordinates system the following two very characteristic, easily recognizable tensor representations of pure shear can be specified

$$\boldsymbol{\tau} = t(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \qquad \boldsymbol{\tau} = t(\mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_2 \otimes \mathbf{n}_2),$$

$$\boldsymbol{\tau}\mathbf{e}_1 = t\,\mathbf{e}_2, \;\; \boldsymbol{\tau}\mathbf{e}_2 = t\,\mathbf{e}_1, \qquad \boldsymbol{\tau}\mathbf{n}_1 = t\,\mathbf{n}_1, \;\; \boldsymbol{\tau}\mathbf{n}_2 = -t\,\mathbf{n}_2,$$

$$\boldsymbol{n}_1 = \tfrac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \boldsymbol{e}_2), \quad \boldsymbol{n}_2 = \tfrac{1}{\sqrt{2}}(-\boldsymbol{e}_1 + \boldsymbol{e}_2), \quad \boldsymbol{n}_3 = \boldsymbol{e}_3$$

where versors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are called *shear directions*; a plane determined by the pairs ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$) or ($\boldsymbol{n}_1$, $\boldsymbol{n}_2$) is called *shear plane*; straight line along versor $\boldsymbol{e}_3$ ($\boldsymbol{n}_1$) is called *shear axis*. It is clear that pure shears are planar tensors.

Blinowski A., Rychlewski J. Pure shears in the mechanics of materials, Math.Mech.Solids, 3,4, 1998, pp. 471-503.

Second order symmetric tensors, some more information on very special characteristics of pure shear mode.

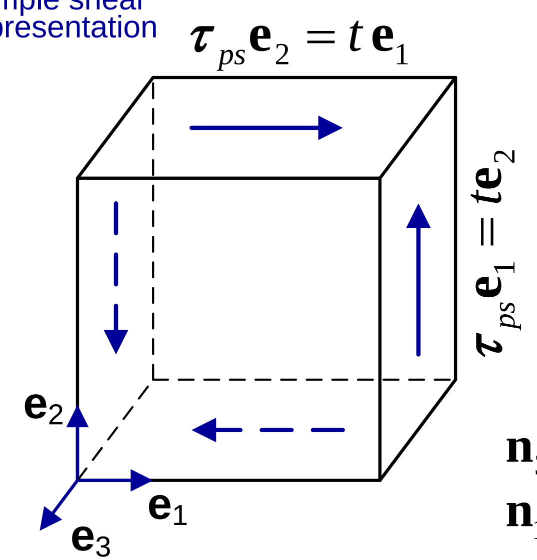
Graphical illustration of the same pure shear tensor $\boldsymbol{\tau}$ shown in two coordinate systems rotated by 45 degrees, which result in two very characteristic for pure shear tensorial representations.

Pure Shear tensor

$$tr(\boldsymbol{\tau}_{ps}) = 0, \quad \det(\boldsymbol{\tau}_{ps}) = \tfrac{1}{3} tr(\boldsymbol{\tau}_{ps}^{\;3}) = 0$$

$$\begin{bmatrix} 0 & t & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad t(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) = \boldsymbol{\tau}_{ps} = t(\mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_2 \otimes \mathbf{n}_2) \quad \begin{bmatrix} t & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

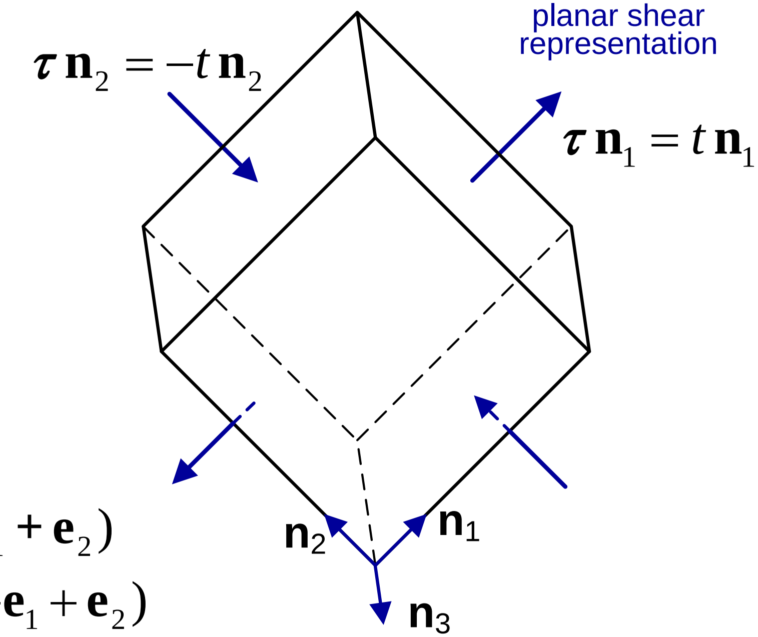
simple shear representation — planar shear representation



$$\mathbf{n}_3 = \mathbf{e}_3$$

$$\mathbf{n}_1 = \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$$

$$\mathbf{n}_2 = \tfrac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2)$$



Second order symmetric tensors, some more information on very special characteristics of pure shear mode.

In accordance with the above nomenclature there can be distinguished two very useful classes of pure shears, namely the ones with *common shear direction* and the ones with *common shear axis*. Two parameter family of pure shears with common shear direction $\boldsymbol{\tau}^{(n)}$ and two parameter family of pure shears with common shear axis $\boldsymbol{\tau}^{(k)}$ can be expressed in the following mathematical form,

$$\boldsymbol{\tau}^{(n)} = (\mathbf{s}_{n1} + \mathbf{s}_{n2}) \otimes \boldsymbol{n} + \boldsymbol{n} \otimes (\mathbf{s}_{n1} + \mathbf{s}_{n2}), \qquad \boldsymbol{\tau}^{(k)} = \mathbf{s}_{k1} \otimes \mathbf{s}_{k2} + \mathbf{s}_{k2} \otimes \mathbf{s}_{k1}, \quad \mathbf{s}_{k1} \cdot \mathbf{s}_{k2} = 0$$

$$\boldsymbol{\tau}^{(n)} \sim \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{bmatrix}, \ \boldsymbol{n} = \mathbf{e}_3, \qquad \boldsymbol{\tau}^{(k)} \sim \begin{bmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{bmatrix} \ \boldsymbol{k} = \mathbf{e}_3,$$

$$\mathbf{s}_{n1} \sim [a,0,0], \mathbf{s}_{n1} \sim [0,b,0], \qquad \mathbf{s}_{k1} \sim [s_1, s_2, 0], \mathbf{s}_{k2} \sim [s_2, -s_1, 0],$$

$$\boldsymbol{n} \sim [0,0,1] \qquad s_1 = \tfrac{1}{\sqrt{2}}(r-s), \ s_2 = \tfrac{1}{\sqrt{2}}(r+s)$$

All possible pure shears having *common shear direction* $\boldsymbol{n}$ parallel to axis $\boldsymbol{e}_3$ can be generated with arbitrarily selected mutually ortogonal vectors $\boldsymbol{s}_{n1}$, $\boldsymbol{s}_{n2}$ orthogonal to direction $\boldsymbol{n}$. All pure shears with *common shear axis* $\boldsymbol{k}$ parallel to axis $\boldsymbol{e}_3$ can be generated with arbitrarily selected mutually orthogonal vectors $\boldsymbol{s}_{k1}$, $\boldsymbol{s}_{k2}$ both orthogonal to shear axis $\boldsymbol{k}$.

Blinowski A., Rychlewski J. Pure shears in the mechanics of materials, Math.Mech.Solids, 3,4, 1998, pp. 471-503.

Second order symmetric tensors, some more information on very special characteristics of pure shear mode.

The *pure shears* prove to be *excellent modeling idealizations* of many commonly encountered, actual physical situations.

For example

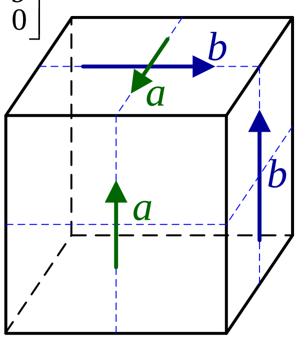
- *uniform plastic slip* deformation can be understood in modeling terms as *a group of pure shears with common axis*,
- *compound martensitic twin* formation can be understood as *a pair of two pure shears with common shear direction*.

Experimental setups leading to pure shear stress or strain are very frequently used in experimental mechanics to determine, e.g., material properties.

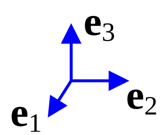
Two pure shears with

common shear direction $\boldsymbol{n} = \boldsymbol{e}_3$

$$\boldsymbol{\tau}^{(n)} = (\boldsymbol{s}_{n1} + \boldsymbol{s}_{n2}) \otimes \boldsymbol{n} + \boldsymbol{n} \otimes (\boldsymbol{s}_{n1} + \boldsymbol{s}_{n2}) \sim \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{bmatrix}$$

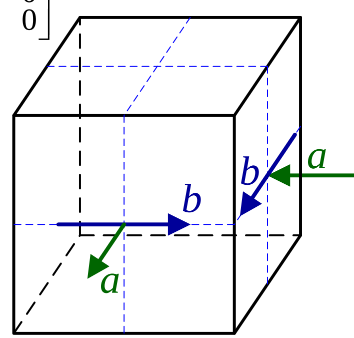


$$\boldsymbol{s}_{n1} \sim [a,0,0], \quad \boldsymbol{s}_{n2} \sim [0,b,0], \quad \boldsymbol{n} \sim [0,0,1]$$



common shear axis $\boldsymbol{k} = \mathbf{e}_3$

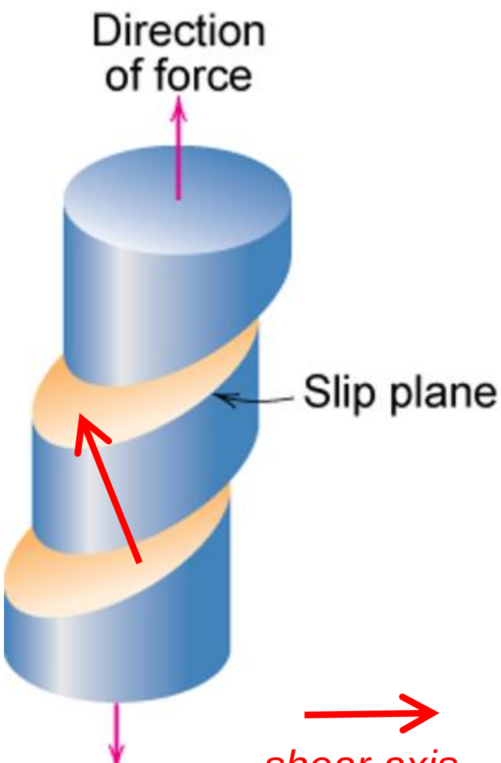
$$\boldsymbol{\tau}^{(k)} = \boldsymbol{s}_{k1} \otimes \boldsymbol{s}_{k2} + \boldsymbol{s}_{k2} \otimes \boldsymbol{s}_{k1} \sim \begin{bmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

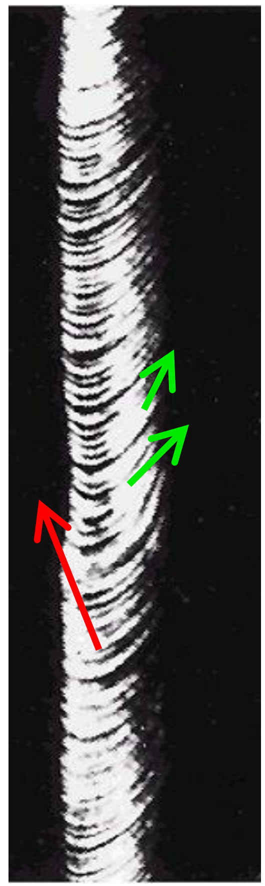


$$\boldsymbol{s}_{k1} \sim [s_1, s_2, 0], \quad \boldsymbol{s}_{k2} \sim [s_2, -s_1, 0],$$
$$s_1 = \tfrac{1}{\sqrt{2}}(r - s), \quad s_2 = \tfrac{1}{\sqrt{2}}(r + s),$$
$$a = r^2 - s^2, \quad b = 2rs$$

Second order symmetric tensors, some more information on very special characteristics of pure shear mode.

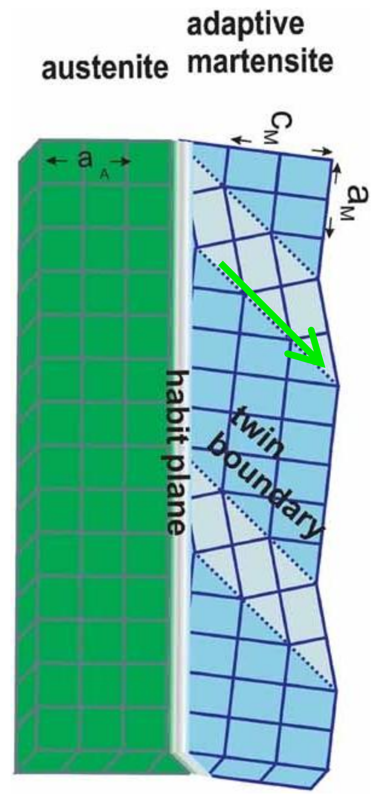
Plastic slip deformation can be understood in modeling terms as *a group of pure shears with common axis.*

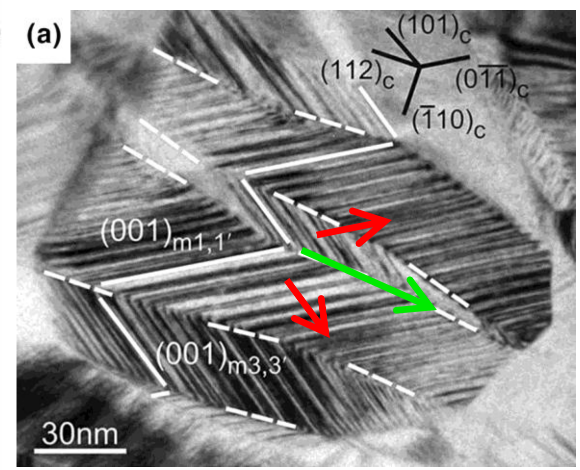




shear axis

Compound martensitic twin formation can be understood as *a pair of two pure shears with common shear direction*.





shear direction

Calister W.D. Materials Science and Engineering: An Introduction, Willey, 1996.
P. Sittner, O. Molnarova, X. Bian, L. Heller & H. Seiner , Tensile Deformation of B19′ Martensite in Nanocrystalline NiTi Wires, Shape Memory and Superelasticity, 9, p.11–34 (2023)

Second order symmetric tensors, some more information on very special characteristics of pure shear mode.

It has been demonstrated by Blinowski and Rychlewski that population of all pure shears generates complete subspace of all deviators. This is so because any deviator can be decomposed into a sum of two pure shears, in particular orthogonal ones.

Pure shears themselves *do not create a linear subspace* because what is obvious from previous statement *sum of two pure shears is not always a pure shear*.

Pure shears can be regarded as elementary (atom) building blocks of deviators subspace.

All pure shears have the same "shape" in this sense that any and all pure shears can be obtained from single fixed preselected pure shear by rotating it with all possible orthogonal tensors $\boldsymbol{Q}$ ($\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}=\mathbf{1}$) .

It is worth noting that *sum of whatever number of pure shears will never result in spherical tensor*.

Session 8 - synopsis

New structural parametrization of stress tensor with: i) stress modulus, ii) isotropy angle and iii) skewness angle (linearly connected with Lode angle). Concept of orbit of a tensor. Diameter of tensor orbit as quantitative measure of a tensor sensitivity to group of three dimensional rotations. Concept of Rychlewski's anisotropy factor based on diameter of tensor orbit as universal quantitative measure of tensor anisotropy degree. Anisotropy factor of stress tensor expressed, in a very simple and elucidating manner, with explicit formula involving isotropy angle and skewness angle. Identification of open scientific problem: why anisotropy factor decreases with deviatoric part of stress tensor departing from comparison pure shear mode.

Second order symmetric tensors,
new structural parametrization of Cauchy stress.

Let us introduce a *new set* of invariant parameters characterizing second order symmetric tensors. The set seems to be especially convenient for mechanical studies because it leads to simplification of many formulas expressing tensor properties and facilitates their physical interpretation.

The new structural parametrization uses newly introduced concepts of

i) *isotropy angle* θ_{iso} and

ii) *skewness angle* θ_{sk}.

The new generic structural parameterization is motivated by

a) *isotropic decomposition* of stress tensor

b) *pure shear states*, identified to be *atomic elements* of space of deviators, adopted as *reference comparison shear stress state*.

The new parametrization conveniently describes and transparently reveals a kind of internal structure of the second order symmetric tensors, when interpreted as modeling objects of mechanical phenomena.

Second order symmetric tensors, new structural parametrization of Cauchy stress.

The concept of *isotropy angle* θ_{iso} and *skewness angle* θ_{sk}.

Ilustrative example of general critical surface
of isotropic material in Haigh-Westergaard space

Isomorphic coordinates

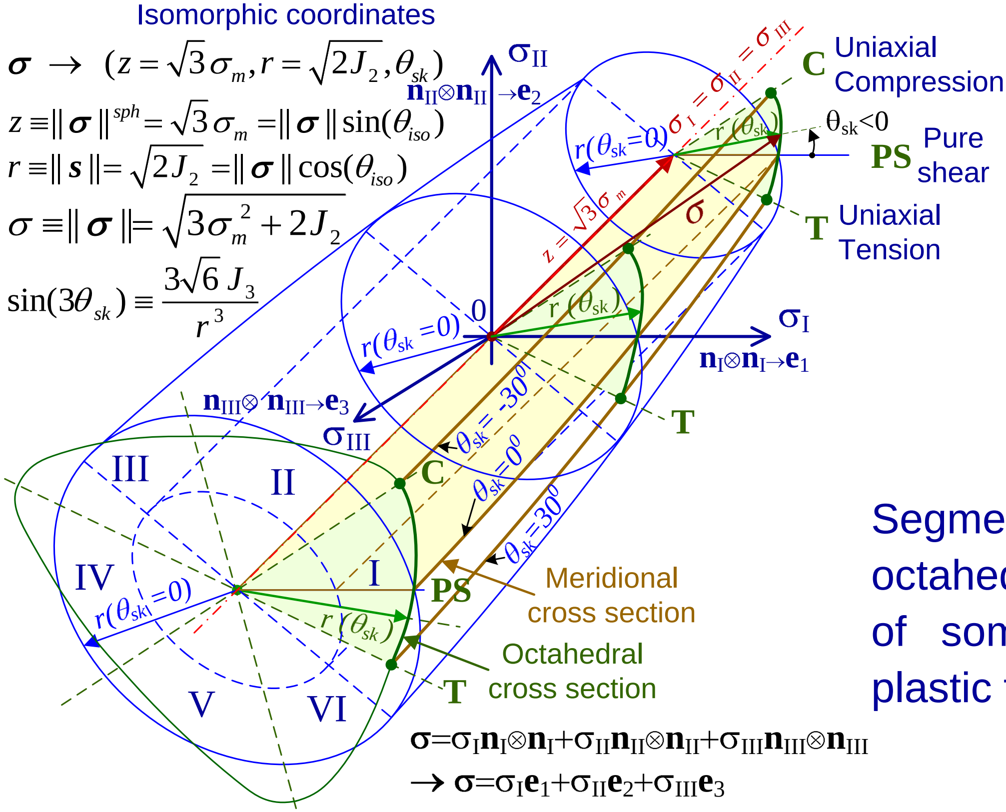
$\boldsymbol{\sigma} \rightarrow (z = \sqrt{3}\sigma_m, r = \sqrt{2J_2}, \theta_{sk})$

$z \equiv \|\boldsymbol{\sigma}\|^{sph} = \sqrt{3}\sigma_m = \|\boldsymbol{\sigma}\|\sin(\theta_{iso})$

$r \equiv \|\mathbf{s}\| = \sqrt{2J_2} = \|\boldsymbol{\sigma}\|\cos(\theta_{iso})$

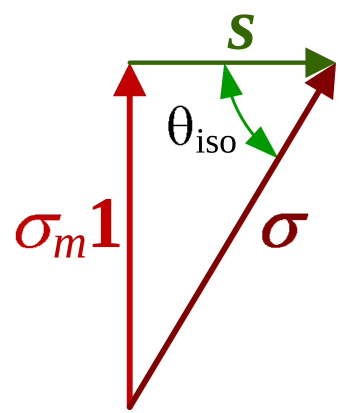
$\sigma \equiv \|\boldsymbol{\sigma}\| = \sqrt{3\sigma_m^2 + 2J_2}$

$\sin(3\theta_{sk}) \equiv \dfrac{3\sqrt{6}\,J_3}{r^3}$



$\sigma = \sigma_I \mathbf{n}_I \otimes \mathbf{n}_I + \sigma_{II} \mathbf{n}_{II} \otimes \mathbf{n}_{II} + \sigma_{III} \mathbf{n}_{III} \otimes \mathbf{n}_{III}$

$\rightarrow \sigma = \sigma_I \mathbf{e}_1 + \sigma_{II} \mathbf{e}_2 + \sigma_{III} \mathbf{e}_3$



$\|\mathbf{s}\| = \sqrt{2J_2} =$
$= \|\boldsymbol{\sigma}\|\cos(\theta_{iso})$

$\|\boldsymbol{\sigma}\| = \sqrt{\tfrac{1}{3}I_1^{\,2} + 2J_2}$

$\|\sigma_m \mathbf{1}\| = \sqrt{3}|\sigma_m| = \|\boldsymbol{\sigma}\|\sin(|\theta_{iso}|)$

Segments drawn with green color in octahedral planes mark respective projections of some hypothetical critical surface, e.g. plastic flow yield surface $\sigma_{0.2}(\sigma_m, \mathbf{s}, \theta_{sk})$.

Graphical illustration of direct sum decomposition of second order symmetric tensor into spherical (isotropic) and deviatoric (anisotropic) parts in Haigh-Westergaard (H-W) principal values space $\boldsymbol{\sigma} = \sigma_m \mathbf{1} + \mathbf{s}$, $\sigma_m \perp \mathbf{s}$).

Second order symmetric tensors, new structural parametrization of Cauchy stress.

The concept of *isotropy angle* θ_{iso} and *skewness angle* θ_{sk}.

Ilustrative example of general critical surface of isotropic material
in Haigh-Westergaard space of Cauchy principal stresses

Isomorphic coordinates

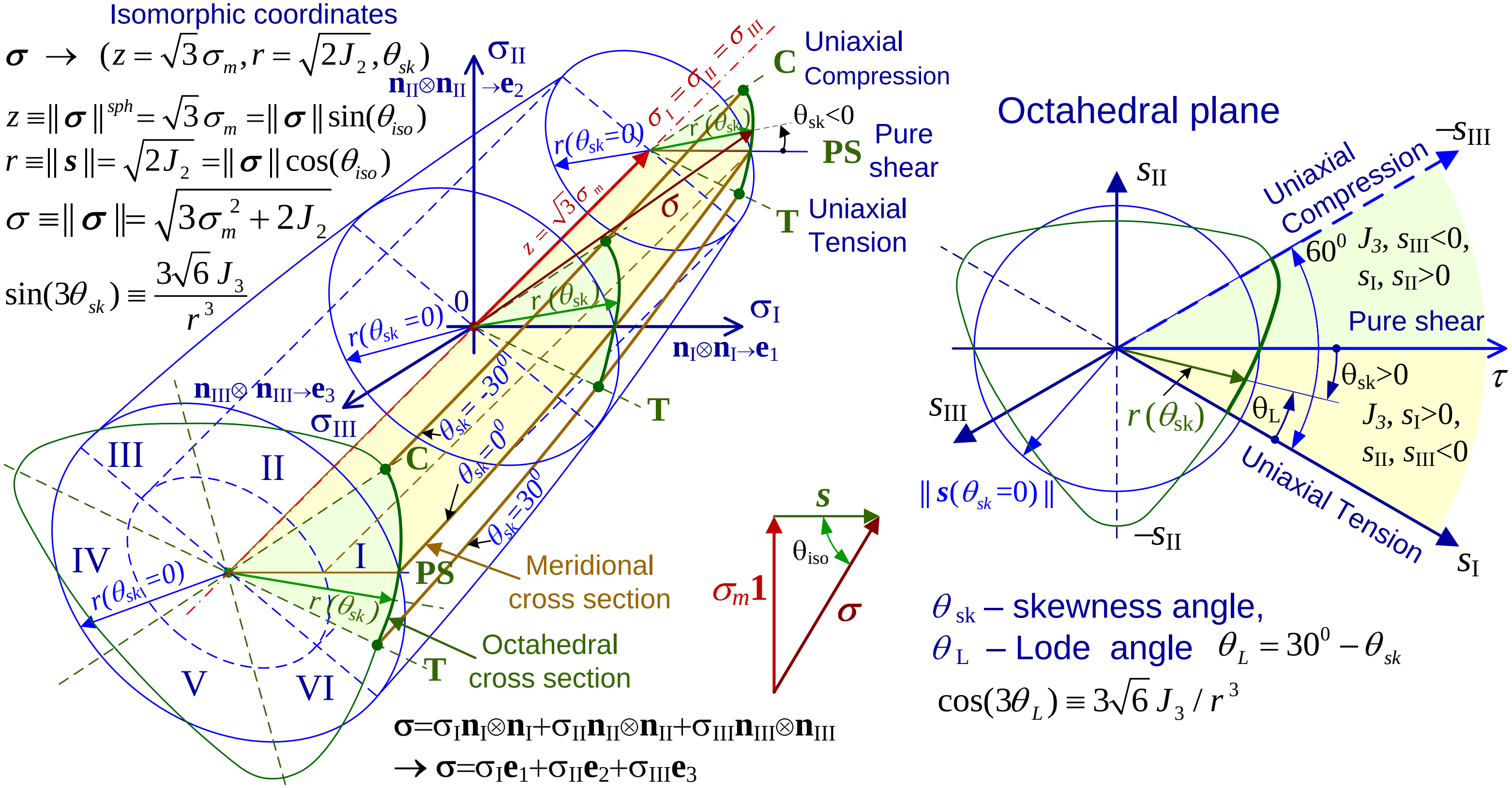
$\boldsymbol{\sigma} \rightarrow (z = \sqrt{3}\sigma_m, r = \sqrt{2J_2}, \theta_{sk})$

$z \equiv \|\boldsymbol{\sigma}\|^{sph} = \sqrt{3}\sigma_m = \|\boldsymbol{\sigma}\|\sin(\theta_{iso})$

$r \equiv \|\boldsymbol{s}\| = \sqrt{2J_2} = \|\boldsymbol{\sigma}\|\cos(\theta_{iso})$

$\sigma \equiv \|\boldsymbol{\sigma}\| = \sqrt{3\sigma_m^2 + 2J_2}$

$\sin(3\theta_{sk}) \equiv \dfrac{3\sqrt{6}\,J_3}{r^3}$



θ_{sk} – skewness angle,

θ_L – Lode angle $\theta_L = 30^0 - \theta_{sk}$

$\cos(3\theta_L) \equiv 3\sqrt{6}\,J_3 / r^3$

Graphical illustration of elements of isomorphic cylindrical coordinates description of tensor deviator in octahedral plane; θ_{sk} is skewness angle, θ_L denotes Lode angle.

Second order symmetric tensors,
new structural parametrization of Cauchy stress.

Let us introduce the following definition of the *isotropy angle* θ_{iso}

$$\sin(\theta_{iso}) \equiv \frac{\sqrt{3}\,\sigma_m}{\|\boldsymbol{\sigma}\|} = \frac{z}{\|\boldsymbol{\sigma}\|} = sign(\sigma_m)\frac{\|\boldsymbol{\sigma}_{sph}\|}{\|\boldsymbol{\sigma}\|} \in <-1,1>,$$

$$\cos(\theta_{iso}) \equiv \frac{\|\mathbf{s}\|}{\|\boldsymbol{\sigma}\|} = \frac{\sqrt{2J_2}}{\|\boldsymbol{\sigma}\|} \in <0,1>,$$

$$\theta_{iso} \in <-90^0, 90^0>; \qquad \|\boldsymbol{\sigma}\| = \sqrt{3\sigma_m{}^2 + 2J_2}$$

The isotropy angle enables extraction of spherical (isotropic) part and deviatoric (anisotropic) part of the tensor in very straightforward and convenient manner.

The sine and cosine functions of isotropy angle can also be treated as convenient normalized factors (indexes) describing magnitude of spherical and/or deviatoric parts relative to overall magnitude (modulus) of the second order symmetric tensor.

Second order symmetric tensors, new structural parametrization of Cauchy stress.

Let us introduce the following definition of the *skewness angle* θ_{sk}

$$\sin(3\theta_{sk}) \equiv \frac{1}{2}\cdot\frac{3\sqrt{3}\,J_3}{J_2^{\;3/2}} = \sqrt{2}\cdot\frac{3\sqrt{3}\,J_3}{r^3} = \frac{27}{2}\cdot\frac{J_3}{\sigma_{ef}^{\;3}} = \bar{J}_3 \;\in\; <-1,1>,$$

$$\theta_{sk} \in <-30^0, 30^0>; \quad \cos(3\theta_L) = \sin(3\theta_{sk}), \quad \theta_L = 30^0 - \theta_{sk}$$

The skewness angle in mathematical terms describes departure of the actual tensor deviator from corresponding *pure shear* (*reference comparison mode*), i.e. deviator having the modulus equal to the modulus of the original tensor deviator, but which third invariant is equal to zero (J_3=0). The *Skewness angle* is *linearly* connected with *Lode angle*.

The following connections exists between so called Lode parameter μ_L and skewness angle

$$\mu_L \equiv \frac{2\sigma_{II} - \sigma_I - \sigma_{III}}{\sigma_I - \sigma_{III}} = \frac{3s_{II}}{s_I - s_{III}} = -\sqrt{3}\,tg(\theta_{sk}), \quad \cos(3\theta_L) = \sin(3\theta_{sk}), \quad \theta_L = 30^0 - \theta_{sk}$$

Second order symmetric tensors, new structural parametrization of Cauchy stress.

So, the new *generic structural parameterization* of second order symmetric tensor employs the following three invariants

$$(\|\boldsymbol{\sigma}\|, \theta_{iso}, \theta_{sk})$$

$$\|\boldsymbol{\sigma}\| \in <0, \infty), \quad \theta_{iso} \in <-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi>, \quad \theta_{sk} \in <-\tfrac{1}{6}\pi, \tfrac{1}{6}\pi>$$

$$\|\boldsymbol{\sigma}_{sph}\| = \|\boldsymbol{\sigma}\| \sin(|\theta_{iso}|) \quad \Leftrightarrow \quad z = \sqrt{3}\,\sigma_m$$

$$\|\boldsymbol{s}\| = \|\boldsymbol{\sigma}\| \cos(\theta_{iso}) \quad \Leftrightarrow \quad r = \sqrt{2J_2}$$

$$\text{mode of shearing} \quad \Leftrightarrow \quad \theta_{sk}$$

The newly proposed parameterization can be conveniently adapted for many purposes in specific areas of application. For example to express Murzewski isomorphic coordinates, or principal values of stress tensor

$$(\ z = \sqrt{3}\,\sigma_m = \|\boldsymbol{\sigma}\| \sin(\theta_{iso}), \ \ r = \|\boldsymbol{s}\| = \sqrt{2J_2} = \|\boldsymbol{\sigma}\| \cos(\theta_{iso}), \ \ \theta_L = 30^0 - \theta_{sk} \ )$$

$$\sigma_I = \tfrac{1}{\sqrt{3}} z + \sqrt{\tfrac{2}{3}}\, r \sin(60^0 + \theta_{sk}), \sigma_{II} = \tfrac{1}{\sqrt{3}} z + \sqrt{\tfrac{2}{3}}\, r \sin(-\theta_{sk}), \sigma_{III} = \tfrac{1}{\sqrt{3}} z - \sqrt{\tfrac{2}{3}}\, r \sin(60^0 - \theta_{sk})$$

Second order symmetric tensors, concept of orbit of a tensor.

A set $\{\boldsymbol{\sigma}^Q\}$ of all tensors that can be obtained by rotation of stress tensor $\boldsymbol{\sigma}$ with any orthogonal tensor $\mathbf{Q}\in\mathcal{R}$ is called an *orbit of a tensor*

$$\boldsymbol{\sigma}^Q = \mathbf{Q}(\sigma_m \mathbf{1} + \boldsymbol{s})\mathbf{Q}^T = \sigma_m \mathbf{1} + \boldsymbol{s}^Q, \quad \boldsymbol{s}^Q = \mathbf{Q}\,\boldsymbol{s}\,\mathbf{Q}^T; \quad \mathbf{Q}\in\mathcal{R},$$

$$\{\boldsymbol{\sigma}^Q\} = \{\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T \mid \mathbf{Q}\in\mathcal{R}\} \quad - \quad \text{orbit of a tensor } \boldsymbol{\sigma} \;\; (\mathbf{Q}\mathbf{Q}^T = \mathbf{1},\; \det\mathbf{Q} > 0)$$

The notion of orbit will be used in definition of quantitative measure of tensors anisotropy, i.e. *anisotropy factor*.

Actually the concept of tensor orbit and based on it measure of tensor anisotropy degree - anisotropy factor, is applicable to tensors of any degree, see the original works by Rychlewski.

Rychlewski J. *Zur Abschätzung der Anisotropie*, *ZAMM, 65, 255-258, 1985. Exists Polish translation of the work,*
Rychlewski J., Ziółkowski A. (tłumacz), O szacowaniu Anizotropii, 2020 *available under the link:*
Rychlewski J., *On evaluation of anisotropy of properties described by symmetric second-order tensors, Czech. J. Phys. B34, pp. 499-506, 1984.*

Second order symmetric tensors, anisotropy factor as normalized maximum diameter of tensor orbit.

The problem arises how to measure tensor anisotropy ?

Rychlewski proposed the following measure of tensor anisotropy, which he called *degree of anisotropy*, which is here called *anisotropy factor*

$$\eta_{ani}(\boldsymbol{\sigma}) \equiv \frac{d(\boldsymbol{\sigma})}{2\,\|\boldsymbol{\sigma}\|}, \quad \boldsymbol{\sigma} \neq 0, \quad \eta_{ani}(\boldsymbol{\sigma}) \in < 0,1 >$$

$d(\boldsymbol{\sigma})$ denotes *diameter of tensor orbit* defined as maximum distance between any two members in the orbit of tensor

$$d(\boldsymbol{\sigma}) \equiv \max_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \{\boldsymbol{\sigma}^Q\}} \rho(\boldsymbol{\alpha},\boldsymbol{\beta}) = \max_{\mathbf{Q}\in\mathcal{R}} \rho(\boldsymbol{\sigma}, \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T), \quad \rho(\boldsymbol{\alpha},\boldsymbol{\beta}) \equiv \|\boldsymbol{\alpha}-\boldsymbol{\beta}\|, \ \mathbf{Q}\mathbf{Q}^T = 1$$

ρ denotes diameter of the tensor orbit, d is distance generated by usual tensorial norm ||.||, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ denote any two tensors in the tensor orbit, $\mathbf{Q}$ is any proper orthogonal (rotation) tensor.

Rychlewski J., On evaluation of anisotropy of properties described by symmetric second-order tensors, Czech. J. Phys. B34, pp. 499-506, 1984.

Second order symmetric tensors, anisotropy factor as normalized maximum diameter of tensor orbit.

Rychlewski has proved that diameter of the orbit of second order symmetric tensor is equal to $d = \sqrt{2}(\sigma_I - \sigma_{III})$ and next he showed that anisotropy factor can be expressed in the following form

$$\eta_{ani} \equiv \frac{d(\boldsymbol{\sigma})}{2\|\boldsymbol{\sigma}\|} = \frac{\sqrt{2}\,\tau_{\max}}{\|\boldsymbol{\sigma}\|} = \frac{\|\boldsymbol{s}\|}{\|\boldsymbol{\sigma}\|} \cdot \sin(\theta_L + 60^0)$$

$$d(\boldsymbol{\sigma}) = \sqrt{2} \cdot (\sigma_I - \sigma_{III}) = \sqrt{2} \cdot (s_I - s_{III}) = 2\sqrt{2} \cdot \tau_{\max}$$

$$\tau_{\max} \equiv \tfrac{1}{2}(\sigma_I - \sigma_{III}) = \tfrac{1}{\sqrt{2}} \|\boldsymbol{s}\| \sin(\theta_L + \tfrac{\pi}{3})$$

where $\tau_{\max}$ denotes maximum shear stress of the tensor $\boldsymbol{\sigma}$. It is clear that *anisotropy factor* is still another *invariant* of tensor $\boldsymbol{\sigma}$, and taking it formally makes a fundamental measure of sensitivity of the tensor $\boldsymbol{\sigma}$ to rotations.

Note: It is important and interesting *open scientific task* to create clear (lucid) *graphical illustration of the tensor orbit concept*.

Rychlewski J., On evaluation of anisotropy of properties described by symmetric second-order tensors, Czech. J. Phys. B34, pp. 499-506, 1984.

Second order symmetric tensors, anisotropy factor as normalized maximum diameter of tensor orbit.

Taking advantage of new parametrization of second order tensor *anisotropy factor* can be expressed in the following extremely simple and elucidating form,

$$\eta_{ani} = \cos(\theta_{iso}) \cdot \cos(\theta_{sk})$$

$$\cos(\theta_{iso}) \in < 0, 1 >, \ \cos(\theta_{sk}) \in < 1, \tfrac{1}{2}\sqrt{3} >$$

$$\cos(\theta_{iso}) \equiv \| \boldsymbol{s} \| / \| \boldsymbol{\sigma} \| = \sqrt{1/(1+\tfrac{3}{2}\sigma_m{}^2 / J_2)}, \quad \sin(3\theta_{sk}) = \bar{J}_3 = 3\sqrt{3}\, J_3 / (2 J_2{}^{3/2})$$

The first term in the above formula clearly shows that anisotropy degree of second order symmetric tensor grows with growing fraction of its deviatoric part, reaching *maximum for pure deviators* ($\cos(\theta_{\mathrm{iso}}=0)=1$). The second term shows that *the most anisotropic deviators* are *pure shears* ($\cos(\theta_{\mathrm{sk}}=0)=1$).

The anisotropy factor *decreases with* deviatoric part *departing from* respective comparison *pure shear mode*. Proposition on how to explain the reasons for this rather puzzling behavior of anisotropy factor will be presented further in the sequel.

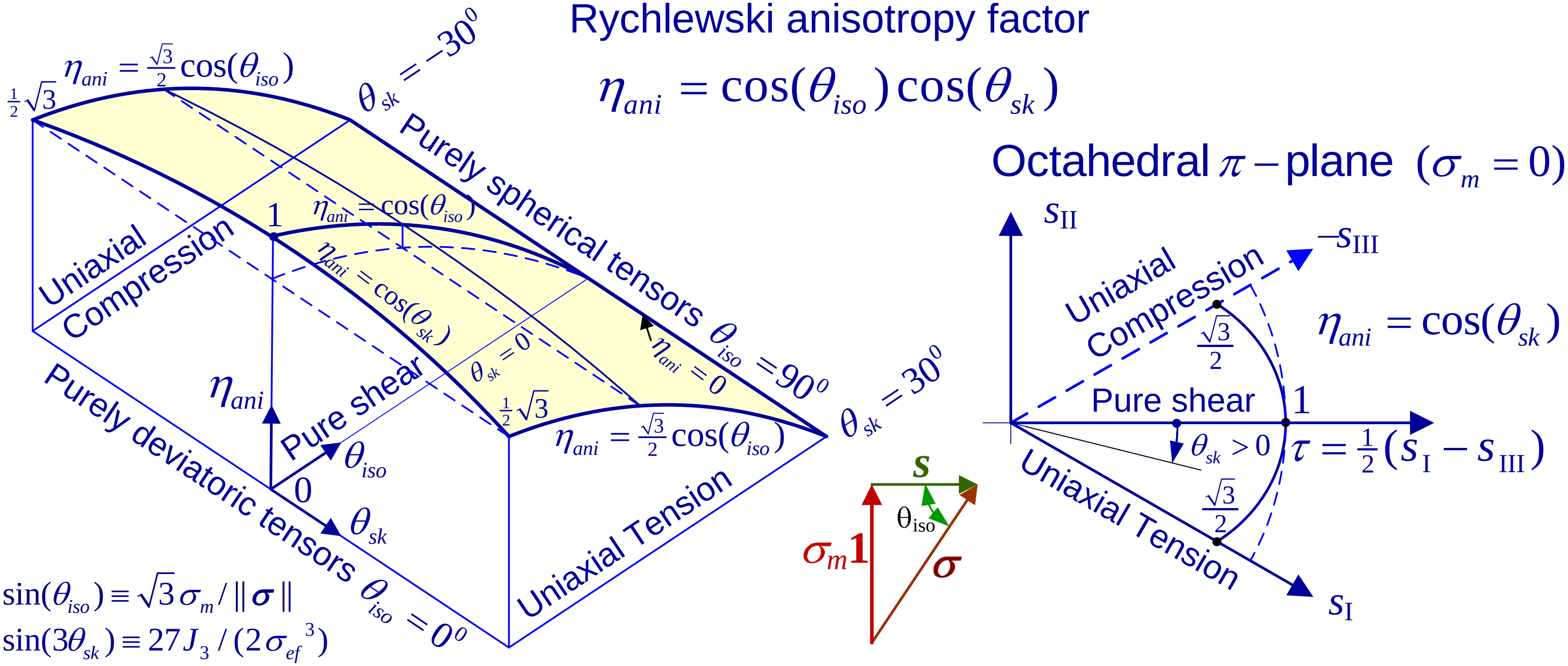
Second order symmetric tensors, anisotropy factor as normalized maximum diameter of tensor orbit.

Graphical illustration of variation of *anisotropy factor* in dependence on isotropy angle θ_{iso} $(\sim \| \boldsymbol{s} \|)$ and skewness angle θ_{sk} $(\sim \| \bar{J}_3 \|)$,
and illustration of anisotropy factor variation in octahedral π plane (σ_{m}=0).

Variation of Cauchy stress anisotropy
with change of isotropy angle and skewness angle

Rychlewski anisotropy factor

$$\eta_{ani} = \cos(\theta_{iso})\cos(\theta_{sk})$$



Session 9 - synopsis

Statistical-physical interpretation of principal invariants of shear (deviator) stress tensor. Correspondence between shear stress principal invariants and statistical central moments. Formal identity of formulas expressing skewness (shear stress mode) angle of Cauchy stress tensor and Fischer-Pearson skewness coefficient. Micro pure shears generating macroscopic shear stress interpreted as population of directional dipoles. Novozhilov's formula connecting maximum (macroscopic) shear stress – reached when population of micro pure shears is ordered in the same direction, with average (macroscopic) shear stress obtained for arbitrary population of micro pure shears (directional dipoles) population. Physical interpretation of skewness angle θ_{sk} as orientational standard deviation (directional disorder) of population of micro pure shears. Interpretation of skewness angle as macroscopic measure of entropy of population of micro pure shears generating specific macroscopic stress state – measure of entropy of stress tensor. Decrease of anisotropy factor with departure of deviatoric part of stress tensor from comparison pure shear mode identified to be effect of growing directional disorder (entropy) of micro pure shears population.

Second order symmetric tensors, statistical interpretation of mode angle of deviator.

In order to identify some *physical interpretation of skewness angle* let us note that very simple and straightforward connections exist between principal invariants of deviator J_α and statistical central moments μ_i, namely

$$\mu_2 \equiv \tfrac{1}{3}(s_I{}^2 + s_{II}{}^2 + s_{III}{}^2) = \tfrac{1}{3} tr(\mathbf{s}^2) = \tfrac{1}{3}(2J_2), \quad \mu_3 \equiv \tfrac{1}{3}(s_I{}^3 + s_{II}{}^3 + s_{III}{}^3) = \tfrac{1}{3} tr(\mathbf{s}^3) = J_3,$$

$$\mu_1 = \bar{\mu} = \tfrac{1}{3}(s_I + s_{II} + s_{III}) = \tfrac{1}{3} J_1 = 0, \qquad \mu_i \equiv \sum_{k=1,n} \tfrac{1}{n}(x_k - \bar{x})^i$$

It is worth noting that also in statistical sense there exists "orthogonal" decomposition of the tensor $\boldsymbol{\sigma}$ into spherical and deviatoric parts.
It takes place in this sense that for spherical part only the first central moment is different from zero and all the remaining central moments are equal to zero

$$\mu_1(\sigma_m \mathbf{1}) = \sigma_m, \qquad \mu_i(\sigma_m \mathbf{1}) = 0, i = 2, \ldots$$

while for the deviator first central moment is different from zero and in general all part remaining central moments can be not equal to zero

$$\mu_1(\mathbf{s}) = 0, \qquad \mu_i(\mathbf{s}) \neq 0, i = 2, \ldots$$

Polyanin A., Manzhirov A., Handbook of mathematics for engineers and scientists, Taylor & Francis, Boca Raton, 2007

Second order symmetric tensors, statistical interpretation of mode angle of deviator.

Substitution of the expressions obtained for the central moments of stress tensor deviator into the formula for *Fischer-Pearson skewness coefficient,* and comparing such obtained relation with formula defining shear mode *angle* θ_{sk} reveals existence of the following connection

$$g_1 \equiv \frac{\mu_3}{\mu_2^{3/2}} = 3\sqrt{3}\frac{J_3}{(2J_2)^{3/2}} = 3\sqrt{3}\frac{s_I}{r}\frac{s_{II}}{r}\frac{s_{III}}{r} = \frac{1}{\sqrt{2}}\bar{J}_3, \qquad \sin(3\theta_{sk}) = \bar{J}_3 \Rightarrow$$

$$\boxed{\sin(3\theta_{sk}) = \sqrt{2}\, g_1 \quad \in <-1,1>}$$

The above elucidating relation reveals that *angle* θ_{sk} can be interpreted as *a measure* of some kind *of second order tensor "skewness"* and justifies assigning the deviator mode angle θ_{sk} the name *skewness angle*.

The question arises, *what kind of skewness* and *in what sense* the angle θ_{sk} describes?

Polyanin A., Manzhirov A., Handbook of mathematics for engineers and scientists, Taylor & Francis, Boca Raton, 2007

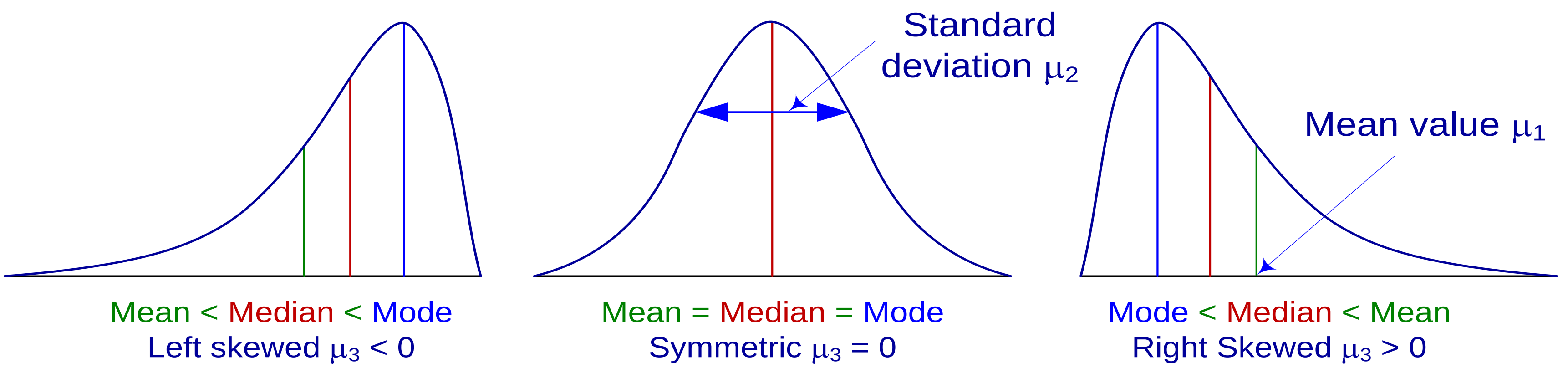
Second order symmetric tensors, statistical interpretation of mode angle of deviator.

In statistical literature there exist very well known interpretations of central moments.

The *first moment* (μ_1) describes the *mean value* of some feature in the population of objects.

The *square root of the second central moment* ($\mu_2^{1/2}$) is called *standard deviation*. It describes the *magnitude of scatter, or non-uniformity, or disorder* of the population statistics around its mean.

The *third central moment* (μ_3) normalized with standard deviation describes the *non-symmetry or "skewness"* of the population statistics towards the left or right wing of its distribution.



Second order symmetric tensors, physical interpretation of skewness angle.

When trying to assign some reasonable *physical interpretation to skewness angle* – characterizing shear stress mode, the specific situation must be taken into account that it is defined, with *quantity which first central moment is always equal to zero* ($\mu_1(\mathbf{s})=⅓J_1\equiv 0$), i.e. deviator of a tensor.

This apparently requires *reinterpretation* of the understanding of the meaning of *central moments of stress tensor*.

The clue for such reinterpretation can be found in rather little known work of *V.V. Novozhilov from 1952* entitled "On the physical meaning of stress invariants".

Second order symmetric tensors, physical interpretation of skewness angle.

Novozhilov demonstrated that second principal invariant of deviator, which is linearly proportional to *second central moment* ($\mu_2 = \tfrac{2}{3} J_2$), is proportional to *average shear stress* (τ_{av}) of tensor $\boldsymbol{\sigma}$ calculated over *all directions* on unit sphere

$$\tau_{av} = \sqrt{\tfrac{3}{5}}\sqrt{\mu_2} = \tfrac{1}{\sqrt{5}}\sqrt{2J_2} = \tfrac{1}{\sqrt{5}}\|\mathbf{s}\|$$

$$\tau_{av} \equiv \left((1/\Omega)\int \tau^2 d\Omega\right)^{1/2} = \tfrac{1}{\sqrt{5}}\sqrt{2J_2}, \quad \sqrt{\mu_2} = \tfrac{1}{\sqrt{3}}\sqrt{2J_2} \quad \Rightarrow$$

$$\tau^2 = \sigma_I^{\,2} l_I^{\,2} + \sigma_{II}^{\,2} l_{II}^{\,2} + \sigma_{III}^{\,2} l_{III}^{\,2} - (\sigma_I l_I^{\,2} + \sigma_{II} l_{II}^{\,2} + \sigma_{III} l_{III}^{\,2})^2$$

where τ_{av} is *average shear stress* over all possible directions on unit sphere, also called by Novozhilov *shear stress intensity*, $\boldsymbol{\tau}$ is shear stress traction operating on elementary surface $\mathrm{d}\Omega$ of unit sphere, σ_{I}, σ_{II}, σ_{III} are principal stresses, l_{I}, l_{II}, l_{III} are direction cosines determining orientation of normal to surface $\mathrm{d}\Omega$ in relation to principal directions of tensor $\boldsymbol{\sigma}$. Please note that τ_{av} /$\|\mathbf{s}\|=1/\sqrt{5}=0.447=const$

Novozhilov V.V., О физическом смысле инвариантов напряжения, используемых в теории пластичности (in Russian), Prikl. Mat. Mekh. XVI, 1952, pp. 617-619

Second order symmetric tensors, physical interpretation of skewness angle.

The Novozhilov's result from 1952 combined with statistical interpretation of stress tensor invariants delivers a *hint* that tensor deviator $\boldsymbol{s}$ can be interpreted as a *macroscopic tensorial measure* describing *orientational effect* resulting from the *action of a population of micro pure shears*, treated as a pool of (microscopic) *directional dipoles*.

Then, second central moment of deviator $\mu_2(\boldsymbol{s})$, i.e. shear stress intensity τ_{av} (linearly proportional to J_2) gains interpretation of *scalar macroscopic measure* describing quantitatively in average manner *orientational effect* originating from action of *directional dipoles* (population of micro pure shears).

We have identified the *qualitative physical feature* described by *shear (deviator) stress*, i.e. that it can be interpreted as (macroscopic) *directional dipole*.

Second order symmetric tensors, physical interpretation of skewness angle.

If some macroscopic resultant *directional effect* occurs, then what are the possible extreme values for such orientational effect,
and in what situations such extrema appear?

In order to address the question of *why variation of orientational effect appears* (e.g. anisotropy factor) and in *what situations the extrema appear*, an analogy with magnetic and/or electric dipoles comes to ones mind.

Namely, whenever state of some type of directional entities (orientational dipoles population) determines the macroscopic (average) value of resultant orientational property then
the more ordered (directionally) is the state of directional entities population
the larger is the resultant (overall) directional effect
(the value of parameter describing this effect).

Second order symmetric tensors, physical interpretation of skewness angle.

As parameter well describing quantitatively the *resultant (total) directional effect*, the *maximum shear stress* τ_{max} can be identified.

Novozhilov, in his work from 1952, presented the following relation linking maximum shear stress $\boldsymbol{\tau}_{max}$ with average shear stress $\boldsymbol{\tau}_{av}$

$$\tau_{\max} = \tfrac{1}{2}(\sigma_I - \sigma_{III}) = \tfrac{1}{\sqrt{2}} \cdot \| \mathbf{s} \| \cdot \cos(\theta_{sk}) = \tfrac{1}{\sqrt{2}} (\sqrt{5} \cdot \tau_{av}) \cdot \cos(\theta_{sk}),$$

$$\sqrt{3}/2 \le \cos(\theta_{sk}) \le 1,$$

$$1.39 = \tfrac{\sqrt{3}}{2} \cdot \sqrt{\tfrac{5}{2}} = \frac{\tau_{\max}(\theta_{sk} = \pm 30^0)}{\tau_{av}} \le \frac{\tau_{\max}(\theta_s)}{\tau_{av}} = \sqrt{\tfrac{5}{2}} \cdot \cos(\theta_{sk}) \le \frac{\tau_{\max}(\theta_{sk} = 0^0)}{\tau_{av}} = \sqrt{\tfrac{5}{2}} = 1.58$$

uniaxial tension / compr. *pure shear*

It can be noticed that at fixed strength of shearing characterized by modulus of stress deviator $||\mathbf{s}||=const$ ($\tau_{av}=const$), the largest directional effect, i.e. greatest value of parameter τ_{max} takes place at (macroscopic) pure shear mode, i.e., when $\theta_{sk}=0$, and the smallest directional effect occurs at uniaxial tension/compression mode when $\theta_{sk}=\pm 30^0$.

Second order symmetric tensors, physical interpretation of skewness angle.

Now we can identify statistical interpretation of deviator principal invariants. *Second principal invariant of tensor deviator* can be assigned physical interpretation of *directional mean value* characterizing size of directional effect $\tau_{av}=\sqrt{\tfrac{3}{5}}\sqrt{\mu_2}=\tfrac{1}{\sqrt{5}}\sqrt{2J_2}=\tfrac{1}{\sqrt{5}}\|\mathbf{s}\|=0.447\|\mathbf{s}\|$.

Third principal invariant of tensor deviator $\bar{J}_3=\sin(3\theta_{sk})=\sqrt{2}\,g_1$, can be assigned physical interpretation of *directional standard deviation* characterizing *directional (orientational) disorder, variance* of population of elementary pure shears (directional dipoles) around their average orientation.

The same physical interpretation of *directional standard deviation* can be attributed to the *skewness angle* θ_{sk} treated as quantity derivative from normalized third principal invariant of deviator.

The statistical interpretation of skewness angle θ_{sk} enables *rational explanation of* signaled earlier mysterious *reduction of anisotropy degree* of the stress tensor with its deviator *departure from pure shear mode*.

Second order symmetric tensors, notion of internal entropy of stress tensor.

This can be achieved upon recognizing that the state of population of elemental micro pure shears, generating macroscopic *pure shear mode*, is *the most ordered directionally state*, and the state of elemental micro pure shears generating macroscopic *uniaxial tension/compression mode* is *the most disordered/ scattered directionally state*.

It is known from thermodynamics that good *measure of the degree of internal order (disorder)* of any system is *entropy*.

Thus, it can be conjectured that *decrease of the value of anisotropy factor* of second order tensor with departure of its deviator from *pure shear mode*, i.e. with growth of absolute magnitude of skewness angle, can be attributed to *increase of internal entropy* of the tensor understood as growth of orientational scatter in population of micro pure shears generating specific *mode of tensor deviator.*

The above gives grounds to call the term $cos(\theta_{sk})$ *entropic part of tensor anisotropy*, $cos(\theta_{iso})$ can be called *deviator modulus part of tensor anisotropy*.

Second order symmetric tensors, physical interpretation of skewness angle.

The presented discussion can be concluded with the evaluation that the value of *skewness angle* describes, delivers information on:

- magnitude of *internal entropy* of the stress tensor,

  [The change of internal entropy finds reflection in changing value of *anisotropy factor;* the greater is internal order (smaller internal entropy) the bigger is the value of anisotropy factor.]

- magnitude of *skewness of the population* of micro pure shears.

  [The negative skewness ($J_3<0$) shifts the direction of the projection of tensor deviator on the octahedral plane towards the direction of the projection of the first (greatest) principal value,]

  and

  the positive skewness ($J_3>0$) shifts the direction of the projection of tensor deviator on the octahedral plane towards the direction of the projection of the third (smallest) principal value from the direction of pure shear mode ($J_3=0$).

Session 10 - synopsis

Constitutive (Hooke's) law for anisotropic linear elastic materials. Uncoupling of Cauchy stress of anisotropic linear elastic material into six linearly independent homogeneous, proportionality laws, i.e. energy-orthogonal decomposition of Cauchy stress tensors in a basis of elastic eigenstresses. True (Kelvin) moduli of elasticity. Hooke's tensor expressed in Kelvin notation. Isometric and non-isometric tensorial bases. Components (representations) of isotropic tensors are not invariant in all orthonormal bases, but they are invariant in all isometric bases. eigenstresses of Hooke's tensor (solutions of its characteristic equation) make up orthogonal isometric bases. Fourth order isotropic unit tensors and their representations in different bases. Christoffel (acoustic) second order symmetric tensor. Open scientific problem identified, i.e. development of very useful classification (structuring) of elastic waves in anisotropic materials characterized by Hooke's tensor with different external symmetries taking advantage of spectral decomposition of Hooke's tensor. Rychlewski's classification of elastic materials delivers hint that such a structure of elastic waves in anisotropic elastic materials exists. Open scientific problem identified, i.e. development of strength of elastic material hypotheses taking advantage of the information that elastic energy stored in the elastic material can always be decomposed into six mutually independent parts.

Second order symmetric tensors,
interaction with external system (environment).

In the previous discussion on the properties of *second order symmetric tensors* the Cauchy stress tensor was considered as an *autonomous object*.

It appears that tensors similarly like people behave differently depending on an interaction with environment (external circumstances).

It is interesting that taking into account *interaction of second order symmetric tensor with some external objects* creates possibility for introduction of still another bases (notation) of second order tensors.

Let us assume that it is prescribed fourth order symmetric tensor $\boldsymbol{S} \in \mathcal{T}_4^{sym}$, which may be interpreted physically to represent elastic properties of some material. In such a case it is called *Hooke's tensor*.

Then the problem for eigenvalues (eigenstresses) of this 4-th order tensor can be posed and solved, i.e. roots can be found of so called *characteristic equation* of $\boldsymbol{S}$. The characteristic equation of fourth order symmetric tensor takes the form of six order polynomial equation with real coefficients. In the most general case the solution of characteristic equation is composed of 6 different eigenvalues (λ_α),

$$\mathbf{S} \cdot \boldsymbol{\omega} = \lambda\, \boldsymbol{\omega},\;\; \mathbf{S} \in T_4^{sym} \rightarrow \det(\mathbf{S} - \lambda \mathbf{I}^{(4s)}) = 0 \;\rightarrow\; \lambda_K, \boldsymbol{\omega}_K \in T_2^{sym};\;\; K = 1,..,6.$$

Rychlewski J., On Hooke's Law, J. Appl. Math.&Mech. 1984, 48, 3, pp. 303-314, 0021-8928/84 Pergamon Press Ltd. (English translation from Russian original published in PMM, U.S.S.R.)

Interaction of tensors, spectral decomposition of fourth order symmetric tensors.

Rychlewski in 1983 showed that Hooke's tensor $\boldsymbol{S}$ can be expressed with the aid of *eigentensors* $\boldsymbol{\omega}_\alpha$ corresponding to *all different* eigenvalues λ_α (*Kelvin moduli*) of characteristic equation, in the following form,

$$\mathbf{S}\cdot\boldsymbol{\omega}=\lambda\boldsymbol{\omega} \;\rightarrow\; \det(\mathbf{S}-\lambda\mathbf{I}^{(4s)})=0 \;\rightarrow\; \lambda_\alpha,\boldsymbol{\omega}_\alpha\in T_2^{sym};\; \boldsymbol{\omega}_\alpha\cdot\boldsymbol{\omega}_\beta=\delta_{\alpha\beta},\;\; \alpha,\beta=1,..,6,$$

$$\mathbf{S}=\lambda_1\boldsymbol{\omega}_I\otimes\boldsymbol{\omega}_I+\ldots+\lambda_6\boldsymbol{\omega}_{VI}\otimes\boldsymbol{\omega}_{VI},\quad \mathbf{I}^{(4s)}=\boldsymbol{\omega}_I\otimes\boldsymbol{\omega}_I+\ldots+\boldsymbol{\omega}_{VI}\otimes\boldsymbol{\omega}_{VI},$$

$$\mathbf{I}^{(4s)}\sim I_{ijkl}^{(4s)}=\delta_{\alpha\beta}=\tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk});\;\; \alpha,\beta=1,6,\;\; i,j,k,l=1\ldots3,\quad \sim diag[1,1,1,1,1,1].$$

Naturally Kelvin moduli λ_α and corresponding to them 4-th order eigentensors $\boldsymbol{\omega}_\alpha\otimes\boldsymbol{\omega}_\alpha$ generate *spectral decomposition* of the whole space of fourth order symmetric (Hooke's) tensors.

The *set of second order eigentensors* $\{\boldsymbol{\omega}_\alpha\}$, $\alpha=1,6$ may be accepted as

- an *orthogonal basis* of second order symmetric tensors ($\mathcal{T}_2^{sym}$ space),

and the set $\{\boldsymbol{\omega}_\alpha\otimes\boldsymbol{\omega}_\alpha\}$, $\alpha=1,6$ can be adopted as

- an *orthogonal basis* of fourth order symmetric tensors ($\mathcal{T}_4^{sym}$ space).

Rychlewski J., (translator Ziółkowski A.), CEIIINOSSSTTUV Mathematical structure of elastic bodies, pp. 1-131, 2023, IPPT PAN, Warsaw, Poland.. (English translation from Russian original published in 1983.)
https://www.researchgate.net/publication/376594979_CEIIINOSSSTTUV_Mathematical_structure_of_elastic_bodies

Interaction of tensors, energy decomposition of stress tensor.

Taking advantage of the *spectral decomposition of Hooke's* tensor, the following decomposition of Cauchy stress tensor can be obtained

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_4 + \boldsymbol{\sigma}_5 + \boldsymbol{\sigma}_6, \qquad \boldsymbol{\sigma}_\alpha \cdot \mathbf{S}\boldsymbol{\sigma}_\beta = \delta_{\alpha\beta},$$

$$\boldsymbol{\sigma} = \mathbf{I}^{(4s)} \cdot \boldsymbol{\sigma} = \sigma_{(1)}\,\boldsymbol{\omega}_I + \quad \dots \quad + \sigma_{(6)}\,\boldsymbol{\omega}_{VI}, \quad \sigma_{(\alpha)} \equiv \boldsymbol{\sigma} \cdot \boldsymbol{\omega}_\alpha, \quad (\ \alpha, \beta = 1,\dots,6).$$

The above decomposition (notation) is called *energy-orthogonal decomposition* of stress tensors (space) *for a given elastic body* $\boldsymbol{S}$.

Energy-orthogonal decomposition of stress tensors with respect to set of eigenstresses $\{\boldsymbol{\omega}_\alpha\}$, $\alpha=1,6$ is *unique* when all the Kelvin moduli λ_α have different values.

For fixed tensor $\boldsymbol{S}$, the *six components* $\sigma_{(\alpha)}$ (parts $\boldsymbol{\sigma}_\alpha$) are *invariants* of stress tensor $\boldsymbol{\sigma}$, in standard sense, i.e. they do not change when the coordinates system (basis $\{\mathbf{e}_i\}$) is changed.

It may be said that $\sigma_{(\alpha)}$ are *invariants of* $\boldsymbol{\sigma}$ *in interaction with* $\boldsymbol{S}$ $(\sigma_{(\alpha)}{}^{\boldsymbol{\sigma}\leftrightarrow\boldsymbol{S}})$.

Interaction of tensors, Hooke's law for isotropic materials

Example of energy decomposition (isotropic elastic materials)

Hooke's law for an isotropic material in Kelvin (notation) has the form

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sqrt{2}\sigma_4 \\ \sqrt{2}\sigma_5 \\ \sqrt{2}\sigma_6 \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \sqrt{2}\varepsilon_4 \\ \sqrt{2}\varepsilon_5 \\ \sqrt{2}\varepsilon_6 \end{bmatrix},$$

$$\boldsymbol{\sigma} = \mathbf{C}^{iso} \cdot \boldsymbol{\varepsilon} \Leftrightarrow \sigma_K^{Ke}\, \mathbf{a}_K = C_{KL}^{iso}\, \mathbf{a}_K \otimes \mathbf{a}_L \cdot \varepsilon_L^{Ke}\, \mathbf{a}_L, \quad K, L = 1,\ldots,6.$$

Isotropic elastic material has the following set of elastic eigenstates, (this can be verified by direct calculation)

$$\{\mathbf{C}^{iso} \cdot \mathbf{h} = \lambda\, \mathbf{h} \sim C_{\alpha\beta}^{iso}\, h_\beta = \lambda\, h_\alpha\} \Leftrightarrow \det(\mathbf{C}^{iso} - \lambda \mathbf{I}^{(4s)}) = 0 \;\rightarrow\; \lambda_K, \mathbf{h}_K$$

$$\underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{h}_1\equiv}, \underbrace{\frac{1}{\sqrt{6}}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{\boldsymbol{h}_2\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{\boldsymbol{h}_3\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\boldsymbol{h}_4\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\boldsymbol{h}_5\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{h}_6\equiv} \{\mathbf{e}_i \otimes \mathbf{e}_j\}$$

The set of tensor $\mathbf{h}_K$ makes an orthonormal basis for second order symmetric tensors. This set is very frequently encountered in the literature without explaining its *special property* of being solution to (isotropic) characteristic equation.

Definition of isometric tensorial bases.

Definition Two *orthonormal bases* are *isometric*, with respect to a *proper orthogonal group*, when a rotation tensor $\mathbf{Q}\in\mathcal{T}_2$ exists ($\mathbf{QQ}^{\mathrm{T}}=1$), such that

$$\mathbf{p}_\alpha = \delta_\alpha^i \mathbf{Q}\mathbf{e}_i, \quad (\mathbf{p}_\alpha \otimes \mathbf{p}_\beta = \delta_\alpha^i \mathbf{Q}\mathbf{e}_i \otimes \delta_\beta^j \mathbf{Q}\mathbf{e}_j, \dots, etc), \quad \mathbf{e}_i, \mathbf{p}_\alpha, \in E_3$$

cf., e.g., chapter 4 in Ostrowska–Maciejewska textbook.

Not all orthonormal tensorial bases are *isometric* with respect to the *proper orthogonal group*. For example, Kelvin basis $\{\mathbf{a}_K\}$ is *not isometric* with the $\{\mathbf{h}_\alpha\}$ basis, resulting from spectral decomposition of isotropic Hooke's tensor,

$$\mathbf{e}_i \otimes \mathbf{e}_j \rightarrow \mathbf{a}_K, \ \mathbf{a}_K{}^Q = \mathbf{Q}\mathbf{e}_i \otimes \mathbf{Q}\mathbf{e}_j \ \rightarrow \ \mathbf{a}_K{}^Q \neq \{\mathbf{h}_\alpha\}.$$

Isotropic tensors have *identical* representation components in all *mutually isometric orthonormal* bases, but *isotropic tensors* in general have *different* representation components in *orthonormal* bases, which are not mutually isometric.

Note. It is worth noting that all (single-handed) orthonormal bases in three-dimensional Euclidean space are isometric.

Ostrowska-Maciejewska J., Podstawy i Zastosowania Rachunku Tensorowego, IPPT PAN, Warsaw, 2007.
http://prace.ippt.gov.pl/IFTR Reports 1 2007.pdf

Second order symmetric tensors, isometric and non-isometric bases.

Two bases of second order symmetric tensors $\mathbf{a}_K$ and $\mathbf{h}_K$ are *non-isometric*

$$\boldsymbol{\sigma} = \sigma_1 \mathbf{a}_1 + \sigma_2 \mathbf{a}_2 + \sigma_3 \mathbf{a}_3 + \sigma_4 \mathbf{a}_4 + \sigma_5 \mathbf{a}_5 + \sigma_6 \mathbf{a}_6$$

$$\sigma_1 = \sigma_{11}, \sigma_2 = \sigma_{22}, \sigma_3 = \sigma_{33}, \sigma_4 = \sqrt{2}\,\sigma_{23}, \sigma_5 = \sqrt{2}\,\sigma_{13}, \sigma_6 = \sqrt{2}\,\sigma_{12},$$

$$\mathbf{a}_1 \equiv \mathbf{e}_1 \otimes \mathbf{e}_1 = \tfrac{1}{\sqrt{3}}\mathbf{h}_1 + \tfrac{2}{\sqrt{6}}\mathbf{h}_2, \quad \mathbf{a}_2 \equiv \mathbf{e}_2 \otimes \mathbf{e}_2 = \tfrac{1}{\sqrt{3}}\mathbf{h}_1 - \tfrac{1}{\sqrt{6}}\mathbf{h}_2 + \tfrac{1}{\sqrt{2}}\mathbf{h}_3, \quad \mathbf{a}_3 \equiv \mathbf{e}_3 \otimes \mathbf{e}_3 \tfrac{1}{\sqrt{3}}\mathbf{h}_1 - \tfrac{1}{\sqrt{6}}\mathbf{h}_2 - \tfrac{1}{\sqrt{2}}\mathbf{h}_3,$$

$$\mathbf{a}_4 \equiv \tfrac{1}{\sqrt{2}}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2), \quad \mathbf{a}_5 \equiv \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1), \quad \mathbf{a}_6 \equiv \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1).$$

$$\mathbf{a}_K \cdot \mathbf{a}_L = \delta_{KL}, \quad \mathbf{a}_K \in \mathcal{T}^s_{2(n=3)}, K, L = 1,..,6; \qquad \mathbf{1} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3.$$

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{\boldsymbol{a}_1\equiv}, \quad \underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{\boldsymbol{a}_2\equiv}, \quad \underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{\boldsymbol{a}_3\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{\boldsymbol{a}_4\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}}_{\boldsymbol{a}_5\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{\boldsymbol{a}_6\equiv} \{\mathbf{e}_i \otimes \mathbf{e}_j\}$$

$$\underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{\boldsymbol{h}_1\equiv}, \underbrace{\frac{1}{\sqrt{6}}\begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}}_{\boldsymbol{h}_2\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}}_{\boldsymbol{h}_3\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{\boldsymbol{h}_4\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}}_{\boldsymbol{h}_5\equiv}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{\boldsymbol{h}_6\equiv} \{\mathbf{e}_i \otimes \mathbf{e}_j\}$$

$$\boldsymbol{\sigma} = \tfrac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3)\mathbf{h}_1 + \tfrac{1}{\sqrt{6}}(2\sigma_1 - \sigma_2 - \sigma_3)\mathbf{h}_2 + \tfrac{1}{\sqrt{2}}(\sigma_2 - \sigma_3)\mathbf{h}_3 + \sigma_4 \mathbf{h}_4 + \sigma_5 \mathbf{h}_5 + \sigma_6 \mathbf{h}_6$$

$$\mathbf{h}_1 \equiv \tfrac{1}{\sqrt{3}}\mathbf{1} = \tfrac{1}{\sqrt{3}}[\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3], \qquad \mathbf{h}_2 \equiv \tfrac{1}{\sqrt{6}}[2\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3] = \tfrac{1}{\sqrt{6}}[2\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3],$$

$$\mathbf{h}_3 \equiv \tfrac{1}{\sqrt{2}}[\mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3] = \tfrac{1}{\sqrt{2}}[\mathbf{a}_2 - \mathbf{a}_3], \quad \mathbf{h}_4 \equiv \tfrac{1}{\sqrt{2}}[\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2] = \mathbf{a}_4,$$

$$\mathbf{h}_5 \equiv \tfrac{1}{\sqrt{2}}[\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1] = \mathbf{a}_5, \qquad \mathbf{h}_6 \equiv \tfrac{1}{\sqrt{2}}[\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1] = \mathbf{a}_6,$$

$$\mathbf{h}_K \cdot \mathbf{h}_L = \delta_{KL}, \quad \mathbf{h}_K \in \mathcal{T}^s_{2(n=3)}, K, L = 1,..,6;$$

Fourth order symmetric tensors, different representations of tensors in non-isometric bases.

Representations of the fourth order *isotropic tensors* in two *non-isometric orthonormal bases* $\mathbf{a}_K$ and $\mathbf{h}_K$ *may be different*

$$\mathbf{I}^{(4s)} = I^{(4s)}{}_{KL}\mathbf{t}_K \otimes \mathbf{t}_L, \qquad \mathbf{I}_{\mathcal{P}} = \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1} = I_{\mathcal{P}\,KL}\mathbf{t}_K \otimes \mathbf{t}_L, \qquad \mathbf{I}_{\mathcal{D}} \equiv \mathbf{I}^{(4s)} - \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1} = I_{\mathcal{D}\,KL}\mathbf{t}_K \otimes \mathbf{t}_L$$

$$I^{(4s)}_{KL} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_{\mathcal{P}\,KL} \sim \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad I_{\mathcal{D}\,KL} \sim \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{I}^{(4s)} = I^{(4s)}{}_{KL}\mathbf{h}_K \otimes \mathbf{h}_L, \qquad \mathbf{I}_{\mathcal{P}} = \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1} = I_{\mathcal{P}\,KL}\mathbf{h}_K \otimes \mathbf{h}_L, \qquad \mathbf{I}_{\mathcal{D}} \equiv \mathbf{I}^{(4s)} - \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1} = I_{\mathcal{D}\,KL}\mathbf{h}_K \otimes \mathbf{h}_L$$

$$I^{(4s)}_{KL} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_{\mathcal{P}\,KL} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad I_{\mathcal{D}\,KL} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Second order symmetric tensors, interaction with external system (environment).

A very interesting loop has been encircled. Ricci-Curbastro motivated by the idea of *quadratic forms invariance* devised objects − and the whole mathematical apparatus, which predicts that in the case of second order symmetric tensor its *six components transform in linear manner with change of coordinates system*.

Next, it was identified that from these six components there can always be formed a set of *three linearly independent invariants* independent from the change of coordinates system, and a *set of another three parameters changing with change of coordinates system.* This, when *tensor* is considered an *autonomous object* – an analogy with a free vector comes to ones mind.

When the tensor is considered in some environment, *in interaction with other tensors then* it turned out that *six invariants can be formed* out of its components – and analogy with hooked vector comes to ones mind.

Second order symmetric tensors, interaction with external system (environment).

Let us consider the following situation in order to better understand in what sense the hooking of the tensor takes place.

Let us take two autonomous (free) tensors, e.g. stress tensor and not coaxial with it strain tensor, a typical situation for non-isotropic materials. Each of these tensors is fully described by *three invariants* and *three Euler angles*. Respective Euler angles characterize orientation of each tensor with respect to any conceivable coordinates system (laboratory reference frame). While these angles change with change of coordinate system (reference frame) the *relative orientation* of specific stress tensor with respect to specific non-coaxial strain tensor does not change.

Upon the tensors interaction, e.g. taking their scalar product, *only their relative orientation is important*, what manifests itself in reported possibility of *generating six invariants*.

So, *hooking of the tensor* means that orientation of the principal axes of the first or the second tensor take over the role of reference frame and no other reference frame is needed, does not play any role.

Second order symmetric tensors, energy orthogonal decomposition

Rychlewski's *energy orthogonal decomposition* of Cauchy stress resulting from spectral decomposition of Hooke's tensor delivers yet another very inspiring and prolific hint for running research works.

It demonstrates that when with some physical phenomenon (start of plastic yield flow, cracking, damage, phase transition, etc) there can be associated some fourth order tensor $\mathbf{H}$ − possessing symmetries such as Hooke's tensor, then loadings inducing the occurrence of the phenomenon can be divided/decomposed, in the most general case, into maximum 6 *classes of loadings*, depending on the external symmetry of the tensor $\mathbf{H}$.

When, for example, safety of the structure is analyzed in view of the phenomenon then different coefficients of safety will in general be applicable in order to assure secure operation of the structure submitted to specific class of loadings. This in turn gives grounds for introduction of *weighted effective stress* notion. For example, in the form of quadratic involving tensor $\mathbf{H}$ and stress tensor $\boldsymbol{\sigma}$ ($\sim \boldsymbol{\sigma}\,\mathbf{H}\boldsymbol{\sigma}$). Such quantity will enable more precise evaluation of effort of materials (also other utility features) then classical *effective stress* notion σ_{ef} . Actually, this is return to the original idea of von Mises from 1928, but now better justified.

Second order symmetric tensors, energy orthogonal decomposition.

Importance of distinguishing classes of loadings

video: Railroad tank car vacuum implosion, Tom Brattain, https://www.youtube.com/watch?v=Zz95_VvTxZM&t=2s

While in majority of situations taking into account prevailing type/class of loadings (design loads) lead to safe exploitation of engineering structure. In some situations neglected (in design) specific class of loadings can lead to catastrophic situations

Second order symmetric tensors, Christoffel anisotropic waves equation.

Christoffel E.S. in 1877 published work in which he formulated the problem of propagation of plane waves in anisotropic elastic media.

$$\left(\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}, \quad \boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} \;\sim \sigma_{ij} = C_{ijkl}\varepsilon_{kl} \right) \;\Rightarrow\; \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathbf{C}\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} \;\sim \rho \frac{\partial^2 u_i}{\partial t^2} = C_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l}$$

Christoffel showed that the problem of *solution of the anisotropic waves propagation* can be reduced to the *eigenvalues problem* of second order symmetric matrix

$$\mathbf{u}(\boldsymbol{x},t) = \mathbf{p}e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \quad \boldsymbol{k} = (\omega/V)\boldsymbol{n} \quad \Rightarrow \quad (\boldsymbol{\chi} - \rho V_M^2 \mathbf{1})\boldsymbol{p}_M = \mathbf{0}$$

The matrix has been later identified to be representation the so so-called *acoustic (Christoffel) tensor* $\boldsymbol{\chi}(\boldsymbol{n}) \equiv \boldsymbol{n}\mathbf{C}\boldsymbol{n}$ having the components $\chi_{ik} = \mathbf{C}_{ijkl}\, n_j\, n_l$, where $\boldsymbol{n}$ denotes a wave propagation direction (phase) vector.

Christoffel wave equation reveals that generally in *solid anisotropic elastic medium three plane waves* can propagate in any direction $\mathbf{n}$. They have mutually perpendicular polarizations.

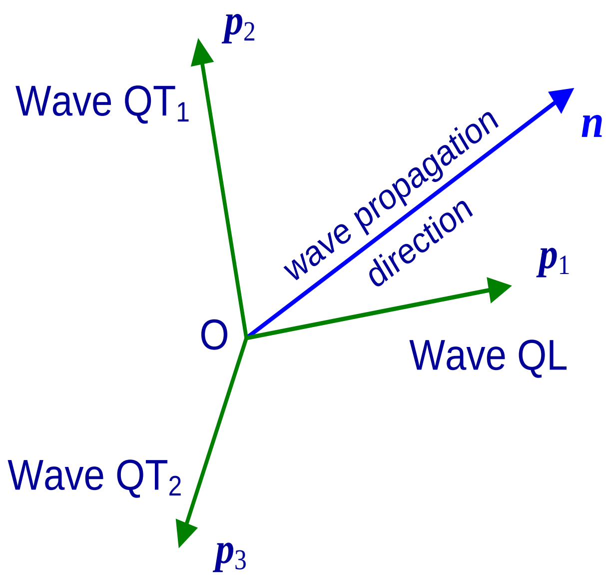
Christoffel E B. Ueber die Fortpflanzung von Stossen durch elastische feste Korper, Annali di Matematica, VIII, 1877.

Second order symmetric tensors, Christoffel anisotropic waves equation.

The wave with polarization **p** closest to the propagation direction vector **n** is called *quasi-longitudinal* (QL) wave.

The two remaining waves are called *quasi-transversal* (QT).

Experiments show that in metals the QL wave propagates approximately two times more quickly than QT waves.



According to the spectral decomposition formula of Hooke's tensor, the *Christoffel (acoustic) tensor* can be expressed as follows

$$\boldsymbol{\chi}(\mathbf{n}) \equiv \mathbf{nCn} = \lambda_I (\boldsymbol{\omega}_I \mathbf{n}) \otimes (\boldsymbol{\omega}_I \mathbf{n}) + \ldots + \lambda_{VI} (\boldsymbol{\omega}_{VI} \mathbf{n}) \otimes (\boldsymbol{\omega}_{VI} \mathbf{n})$$

Hint. The above formula suggests that *acoustic (elastic) waves* in *anisotropic media* can be divided into finite set of *classes of waves*. Similarly like all symmetries of *crystallographic materials* are structured into *32 classes of symmetry*. Structuralization of elastic waves in anisotropic materials makes an interesting open scientific problem.

Royer D., Valier-Brassier T., Elastic waves in Solids 1, Propagation, Iste, Wiley, 2022.
Rychlewski J., Elastic waves under unusual anisotropy, J. Mechanics and Physics of Solids, 49, 2001, pp.2651–2666

Second order symmetric tensors, Christoffel anisotropic waves equation.

Example

Cubic symmetry material has three distinct Kelvin moduli of single, double and triple multiplicity. The eigenstates of cubic materials Hooke's tensor presented in parametric form are as follows (where θ,φ,ψ, are parameters which can take values from $\langle 0, \pi/2 \rangle$)

$$\boldsymbol{\omega}_I = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\omega}_{II,III}(e.g.\ \theta = 0; \theta = \tfrac{\pi}{2}) = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos(\theta) & 0 & 0 \\ 0 & \cos(\theta + \tfrac{2}{3}\pi) & 0 \\ 0 & 0 & \cos(\theta - \tfrac{2}{3}\pi) \end{bmatrix},$$

$$\boldsymbol{\omega}_{IV,V,VI}(e.g.\ \varphi = 0; \varphi = \tfrac{\pi}{2} \wedge \psi = 0; \varphi = \tfrac{\pi}{2} \wedge \psi = \tfrac{\pi}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \sin\varphi\cos\psi & \sin\varphi\sin\psi \\ \sin\varphi\cos\psi & 0 & \cos\varphi \\ \sin\varphi\sin\psi & \cos\varphi & 0 \end{bmatrix}$$

Spectral decomposition of Hooke's tensor for cubic symmetry materials and acoustic tensor can be presented in the form

$$\mathbf{C}^{cub} = \lambda_I \boldsymbol{\omega}_I \otimes \boldsymbol{\omega}_I + \lambda_{II}[\boldsymbol{\omega}_{II} \otimes \boldsymbol{\omega}_{II} + \boldsymbol{\omega}_{III} \otimes \boldsymbol{\omega}_{III}] +$$
$$+ \lambda_{IV}[\boldsymbol{\omega}_{IV} \otimes \boldsymbol{\omega}_{IV} + \boldsymbol{\omega}_V \otimes \boldsymbol{\omega}_V + \boldsymbol{\omega}_{VI} \otimes \boldsymbol{\omega}_{VI}]; \quad \lambda_{II} = \lambda_{III}, \quad \lambda_{IV} = \lambda_V = \lambda_{VI}$$

$$\boldsymbol{\chi}^{cub}(\mathbf{n}) \equiv \mathbf{n}\mathbf{C}^{cub}\mathbf{n} = \lambda_I \boldsymbol{\omega}_I \mathbf{n} \otimes \boldsymbol{\omega}_I \mathbf{n} + \lambda_{II}(\boldsymbol{\omega}_{II}\mathbf{n} \otimes \boldsymbol{\omega}_{II}\mathbf{n} + \boldsymbol{\omega}_{III}\mathbf{n} \otimes \boldsymbol{\omega}_{III}\mathbf{n}) +$$
$$+ \lambda_{IV}(\boldsymbol{\omega}_{IV}\mathbf{n} \otimes \boldsymbol{\omega}_{IV}\mathbf{n} + \boldsymbol{\omega}_V\mathbf{n} \otimes \boldsymbol{\omega}_V\mathbf{n} + \boldsymbol{\omega}_{VI}\mathbf{n} \otimes \boldsymbol{\omega}_{VI}\mathbf{n})$$

Session 11 - synopsis

Hooke's law for isotropic materials. How and why it happens that elastic energy of linear elastic materials does not depend on skewness angle (third stress invariant). The mathematical reason for that can be identified in collinearity of stress and strain tensors. Pierre Curie symmetry principle of Causes and Effects delivers lucid explanation for otherwise mysterious fact that plastic flow condition of some elastically isotropic materials is non-isotropic (deviates from Huber-Mises condition). This reason is non-isotropy of stress tensor, i.e. the cause inducing plastic flow. Open scientific problem is identified that scalar measure of energy is somehow insufficient because it is devoid of information about microscopic internal ordering (entropy) of stress tensor, which causes that some stress loadings nominally leading to the same value of scalar values of elastic energy are more destructive than the other. This calls for development of the concept of energy containing information about microscopic ordering of loadings (depend also on skewness angle), i.e. the concept of „ordered energy" („directed energy"). For example, it is known that among shear loadings with the same deviator modulus (the same value of scalar elastic energy) the pure shear loading is the most damaging (dangerous). Simple shear and planar shear as experimental testing layouts implementing pure shear. Interesting observation of Jan Rychlewski who proved that experimental results of only five linearly independent pure shear loadings tests enable unique determination whether the material is elastically isotropic.

Second order symmetric tensors, interaction with external system (environment).

The classical Hooke's law describing elastic properties of isotropic linear elastic material leads to the following constitutive relations between stress and strain tensors,

$$\boldsymbol{\sigma} = \mathbf{S}\cdot\boldsymbol{\varepsilon} = \lambda(tr\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \mathbf{C}\cdot\boldsymbol{\sigma} = (1/9K)(tr\boldsymbol{\sigma})\mathbf{1} + (1/2\mu)\mathbf{s}, \quad \mathbf{S}\cdot\mathbf{C} = \mathbf{I}^{(4s)},$$

$$\sigma_m = K\varepsilon_v, \quad \mathbf{s} = 2\mu\boldsymbol{\varepsilon}^d, \quad \varepsilon_v \equiv tr(\boldsymbol{\varepsilon}), \quad \boldsymbol{\varepsilon}^d \equiv \boldsymbol{\varepsilon} - \tfrac{1}{3}\varepsilon_v, \ \mathbf{1}\,(\delta_{ij}), \ \mathbf{1}\otimes\mathbf{1}\,(\delta_{ij}\delta_{kl}),$$

$$K = \lambda + \tfrac{2}{3}\mu, \quad E = \mu(3\lambda + 2\mu)/(\lambda+\mu), \quad \nu = \tfrac{1}{2}\lambda/(\lambda+\mu), \quad \mu = \tfrac{1}{3}E/(1+\nu),$$

$$\mathbf{S} = \lambda\,\mathbf{1}\otimes\mathbf{1} + 2\mu\,\mathbf{I}^{(4s)} - S_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

$$\mathbf{1}\otimes\mathbf{1} - (\mathbf{1}\otimes\mathbf{1})_{ijkl} = \delta_{ij}\delta_{kl}, \quad \mathbf{I}^{(4s)} - I^{(4s)}_{ijkl} = \delta_{KL} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad KL = 1,6, \ ijkl = 1,3.$$

where $\boldsymbol{S}$, $\boldsymbol{C}$ denote linear elastic, isotropic stiffness and compliance tensors, λ, μ denote Lame constants, $\mu=G$ is shear modulus, E, K are Young and Bulk modules, ν denotes Poisson's ratio, $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^d$ are strain and strain deviator, respectively, $\boldsymbol{I}^{(4s)}$ is fourth order, symmetric unit tensor.

Second order symmetric tensors, interaction with external system (environment).

In the case of linear elastic isotropic materials elastic energy stored in the material fully decouples into part connected with pressure and part connected with distortion, what corresponds with decomposition of stress tensor (elastic strain tensor) into volumetric and shearing parts,

$$\Phi(\boldsymbol{\sigma}) \equiv \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{C}\cdot\boldsymbol{\sigma} = \frac{1}{2K}\sigma_m^{\ 2} + \frac{1}{4\mu}\boldsymbol{s}\cdot\boldsymbol{s} \quad (*); \quad \Phi(\boldsymbol{\sigma}) = \Phi(\sigma_m\mathbf{1}) + \Phi(\boldsymbol{s})$$

in view of property $\sigma_m\mathbf{1}\cdot\boldsymbol{s} = 0, \quad \|\boldsymbol{s}\|^2 = \boldsymbol{s}\cdot\boldsymbol{s} = s_{ij}s_{ij} = 2J_2 = \tfrac{2}{3}\sigma_{ef}^{\ 2}.$

The above formula delivers immediate proof of the following,

Theorem 1:

Elastic energy of linear elastic, isotropic material does not and cannot depend on skewness angle θ_{sk}, Lode angle θ_L.

While the formula (*) is very common knowledge, the present author has not encountered in the literature explicitly formulated in Theorem 1 its direct consequence.
Probably because being so obvious, it very often escapes attention, or it is somehow forgotten.

Second order symmetric tensors, interaction with external system (environment).

Why the formula for *elastic energy of linear elastic, isotropic materials does not depend on the skewness angle* can be fully grasped when one computes expression for elastic energy, in a little more elaborate manner, using decomposition of stress and strain tensors in principal axes coordinates system. The following sequence of relations is valid,

$$2\Phi(\boldsymbol{\sigma}) = \boldsymbol{\sigma}\cdot\mathbf{C}\cdot\boldsymbol{\sigma} = (tr(\boldsymbol{\sigma})\mathbf{1}+\mathbf{s})\cdot(\tfrac{1}{9K}tr(\boldsymbol{\sigma})\mathbf{1}+\tfrac{1}{2\mu}\mathbf{s}) = \frac{1}{2K}\sigma_m{}^2 + \frac{1}{4\mu}\|\,\mathbf{s}\,\|^2$$

$$\sigma_I = \sigma_m + \tfrac{2}{3}\sigma_{ef}\cos(\theta_L),\;\; \sigma_{II} = \sigma_m + \tfrac{2}{3}\sigma_{ef}\cos(\theta_L - 120^0),\;\; \sigma_{III} = \sigma_m + \tfrac{2}{3}\sigma_{ef}\cos(\theta_L + 120^0),$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\sigma_m,\sigma_{ef},\theta_L) = \sigma_I\,\mathbf{n}_I + \sigma_{II}\,\mathbf{n}_{II} + \sigma_{III}\,\mathbf{n}_{III} = \sigma_m\mathbf{1}+\mathbf{s}$$

$$\|\,\boldsymbol{s}\,\|^2 = \boldsymbol{s}\cdot\boldsymbol{s} = \tfrac{2}{3}\sigma_{ef}[\cos(\theta_L)\mathbf{n}_I + \cos(\theta_L - 120^0)\mathbf{n}_{II} + \cos(\theta_L + 120^0)\,\mathbf{n}_{III}]\cdot$$

$$\tfrac{2}{3}\sigma_{ef}[\cos(\theta_L)\mathbf{n}_I + \cos(\theta_L - 120^0)\mathbf{n}_{II} + \cos(\theta_L + 120^0)\,\mathbf{n}_{III}] =$$

$$= (\tfrac{2}{3}\sigma_{ef})^2[\cos^2(\theta_L) + \cos^2(\theta_L - 120^0) + \cos^2(\theta_L + 120^0)] = (\tfrac{2}{3}\sigma_{ef})^2\,\tfrac{3}{2} = \tfrac{2}{3}\sigma_{ef}{}^2 = 2J_2$$

Note: This specific case delivers a *hint* that in the concept of *macroscopic scalar energy* it is lost some important information on the interactions on micro level. It suggests that development of more comprehensive, precise, concept of *ordered (directed) energy* might be very useful.

Second order symmetric tensors, interaction with external system (environment).

In numerous works devoted to more advanced materials research, *linear elastic, isotropic constitutive relation is assumed* for the investigated material behavior, and next frequently it is searched e.g. "elastic energy" criterion of material effort depending on Lode angle.

In view of Theorem 1 such proceedings lead at the best to methodological inconsistencies. Their removal requires in each specific case clearly formulated and well justified additional assumptions, usually missing.

What factors can be identified to be responsible for very often encountered in experimental works dependence of e.g. critical stress of plastic yielding on skewness (Lode) angle, besides at the same time material apparently exhibiting with good approximation linear elastic and isotropic behavior.

There can be identified at least there such factors (causes):

i) material is actually *not linear elastic*,
ii) material is *not isotropic*,
iii) so called, *internal constraints* actually operate in the material (of force or kinematic character, of known or unknown physical origins).

Second order symmetric tensors, interaction with external system (environment).

The *first factor (i)* can be identified to be the primary reason, why it is a standard that elastic energy functions proposed for rubberlike and/or polymeric materials as potentials for derivation of their constitutive relations are proposed as functions of all three principal invariants of strain tensor. Due to that, correctly, it is assumed for polymeric materials that their elastic energy depends on skewness (Lode) angle of strain tensor.

In the case of polymers also the *second factor (ii)* plays the role.

Elasticity in polymeric materials is physically generated by change of internal entropy of these materials and not internal energy.

Due to that polymeric material, even when isotropic at zero loading, changes its internal symmetry, usually into transversely isotropic one, when loaded to moderate strains. It returns to original symmetry (isotropy) upon removal of loading.

Typical situation when the *third factor (iii)* becomes important is in the case of for example composite materials, in which there are present, some kind of *reinforcement elements*.

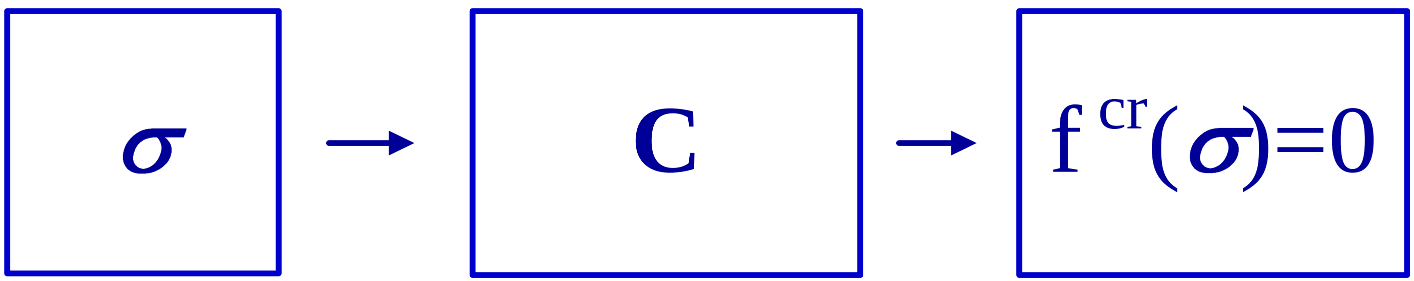
Müller I., History of Thermodynamics, Springer, Berlin, 2007. pp. 111-112 Ziółkowski A., Methodology and micromechanical estimation of macroscopic elastic energy, coherence energy, and phase transition strains for SMA materials, Researchgate 10.13140/RG.2.2.30420.50561, 2018 pp. 31-33.

Second order symmetric tensors, interaction with external system (environment).

There exists one more factor finding roots in Pierre Curie's *Principle of Superposition of Dissymmetries*, which says:

Since certain causes produce certain effects, the elements of symmetry of the causes must find reflection in the elements of symmetry of the caused effects.

When certain effects exhibit a certain dissymmetry, this dissymmetry must manifest itself in the causes that generated these effects.



Input *anisotropic*

System *isotropic*

Output *can be isotropic or anisotropic*

The input (stress σ) is *anisotropic*, in general.
The system (material described with Hooke's tensor) is *isotropic*.
The result (critical condition, e.g., for plastic flow) can be isotropic or anisotropic depending on the "symmetrization strength" of the system (here material).

Curie P., Ziółkowski A. (author of Polish translation and Commentary) O symetrii zjawisk fizycznych, symetrii pola elektrycznego i pola magnetycznego (in Polish). Studia Historiae Scientiarum 22, pp. 23–67, 2023 (original work in French, 1894). DOI: 10.4467/2543702XSHS.23.002.17693. https://www.researchgate.net/publication/374524761_O_symetrii_zjawisk_fizycznych_symetrii_pola_elektrycznego_i_pola_magnetycznego

Second order symmetric tensors, experimental testing layouts of simple shear versus planar shear.

Considerable attention has been devoted in the present work to the theoretical issues connected with pure shear mode/state.

Let us discuss at present so called *"planar shear"* and *"simple shear"*, i.e., two major *experimental testing layouts* leading to *actual physical realization* of the *pure shear*.

A lot of misunderstandings exist in the literature regarding difference between *simple shear* versus *planar shear* testing.

In mechanics the "pure shear mode" is considered in terms of not only stress but also in terms of strain. These later interpretation, i.e. *pure shear strain*, is more convenient for the present discussion.

No conceptual difference between stress and strain interpretation of pure shear mode exists when tested material is isotropic because in such a case straightforward equivalence exists between stress and strain tensors due to their coaxiality.

Second order symmetric tensors, experimental testing layouts of simple shear versus planar shear.

The *simple shear* and *planar shear* testing layouts belong to the class of *biaxial tests*. In order to clarify the issues, *simple shear* and *planar shear are both pure shear modes* of deformation because in both cases trace and determinant of strain tensor during testing are equal to zero,

$$tr(\boldsymbol{\varepsilon}) = 0, \det(\boldsymbol{\varepsilon}) = 0 \Rightarrow tr(\boldsymbol{\varepsilon}^3) = 0$$

The difference in kinematics, i.e. physical motion of material points, *exists between these two testing layouts*.

While different kinematics means different deformation gradients, the principal values of stretch tensor $\mathbf{U}$ ($\mathbf{F}=\mathbf{RU}$) are exactly the same in both layouts, though differently situated in laboratory frame.

Here, only the most important characteristics of simple and planar shear are succinctly and explicitly recalled in order to possibly facilitate taking decision on the selection of one or the other experimental layout for attaining specific experimental research tasks.

More detailed discussion of pure shear and simple shear interested reader can find for example in Ogden and/or Ziółkowski.

Ogden R.W., Non-linear elastic deformations. Dover Publications, Inc. 1997.
Ziółkowski A., Simple shear test in identification of constitutive behavior of materials submitted to large deformations – hyperelastic materials case, Engng. Trans., 54, 4, 2006, pp. 251-269.

Second order symmetric tensors, experimental testing layouts of simple shear versus planar shear.

The *major matching feature* of *simple shear* and *planar shear* testing layouts is *identical strain pattern* shared by both layouts.

This finds reflection in *identical values of principal stretches*.

In the case of *simple shear* principal stretches take the form,

$$\lambda_I^{ss} = \lambda^{ss},\ \lambda_{II}^{ss} = 1/\lambda^{ss},\ \lambda_{III}^{ss} = 1, \quad \lambda^{ss} = \tfrac{1}{2}\gamma + \sqrt{1+(\tfrac{1}{2}\gamma)^2}$$

where γ denotes so called *shear parameter*.

In the case of *planar shear* principal stretches take the form,

$$\lambda_I^{ps} = \lambda^{ps},\ \lambda_{II}^{ps} = 1,\ \lambda_{III}^{ps} = 1/\lambda^{ps}, \quad \lambda^{ps} = \Delta L / L_0$$

When λ^{ss} is equated to λ^{ps} one to one correspondence can be immediately found between γ and ΔL.

In both experimental testing layouts volume is preserved

$$\det(\mathbf{F}) = \lambda_I^{ss}\lambda_{II}^{ss}\lambda_{III}^{ss} = \lambda_I^{ps}\lambda_{II}^{ps}\lambda_{III}^{ps} = dv/dV = 1$$

where dv denotes elementary volume in actual configuration and dV is elementary volume in initial configuration.

Second order symmetric tensors, experimental testing layouts of simple shear versus planar shear.

The *major distinctive feature* differing *simple shear* from *planar shear* is that

- in *simple shear* layout *principal axes constantly rotate*, while
- in *planar shear* layout *principal axes remain fixed* relative to laboratory frame at all times during advancement of shear loading.

Pure Shear experimental layouts

$$\mathbf{E}^{(0)} \equiv ln(\mathbf{U}),\ \ tr(\mathbf{E}^{(0)}) = 0$$

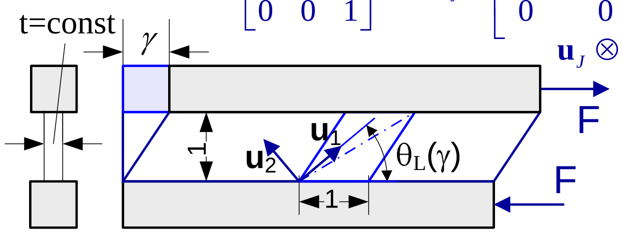
$$\det(\mathbf{E}^{(0)}) = tr((\mathbf{E}^{(0)})^3) = 0$$

$$\det(\mathbf{F}) = V / V_0 = \lambda_I \lambda_{II} \lambda_{III} = 1$$

Simple Shear

(= Pure Shear with constantly rotating principal axes)

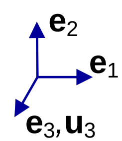
$$\mathbf{F}^{ss} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_j = \begin{bmatrix} \lambda^{ss} & 0 & 0 \\ 0 & 1/\lambda^{ss} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{u}_J \otimes \mathbf{u}_J$$



$$\lambda_I^{ss} = \lambda^{ss},\ \lambda_{II}^{ss} = 1/\lambda^{ss},\ \lambda_{III}^{ss} = 1$$

$$\lambda^{ss} = \tfrac{1}{2}\gamma + \sqrt{1 + (\tfrac{1}{2}\gamma)^2},\ \gamma = \lambda^{ss} - 1/\lambda^{ss}$$

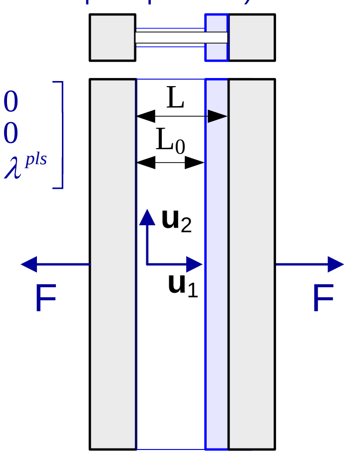
$$\tan(2\theta_L) = -2/\gamma,\ \ \theta_L \in <\pi/4 \nearrow \pi/2>$$



Planar Shear

(= Pure Shear with fixed principal axes)

$$\mathbf{F}^{pls} = \begin{bmatrix} \lambda^{pls} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda^{pls} \end{bmatrix} \mathbf{e}_i \otimes \mathbf{e}_j$$



$$\lambda_I^{pls} = \lambda^{pls},\ \ \lambda_{II}^{pls} = 1,\ \ \lambda_{III}^{pls} = 1/\lambda^{pls}$$

$$\lambda^{pls} = (L - L_0)/L_0$$

The $\mathbf{u}_i$ denote Lagrangian principal axes, θ_L denotes orientation angle of Lagrangian principal axes with respect to fixed laboratory frame, $\mathbf{E}^{(0)}$ denotes logarithmic Lagrangian strain measure, λ_J are principal stretches

Second order symmetric tensors, experimental testing layouts of simple shear versus planar shear.

The *simple shear* testing layout is very popular (standard) in experimental testing of behavior and/or properties of *metallic materials*.

The *planar shear* testing layout is very often used (standard) in examination of *polymeric materials*.

Many additional factors, besides strain pattern, may have influence on choosing one layout or the other. For example stiffness of metallic samples prevents early warping of the sample during simple shear testing. On the other hand testing metallic sheets in pure shear scheme might require considerably larger forces in comparison to simple shear scheme of testing.

It is worth to indicate that *loadings* used in testing *of metallic samples* as a standard *does not involve change of symmetry of the material*.

In the case of testing *polymeric materials* used in their testing *loadings* as a standard *do induce change of their symmetry* – due to entropic origin of polymeric elasticity, e.g. initially isotropic polymeric material changes its symmetry to transversely isotropic under testing load. From the above discussion it can be concluded that execution of simple shear and pure shear tests on the same material allows to evaluate the influence of principal axes rotation on the behavior of the material.

Second order symmetric tensors, tests for determination of isotropy of elastic materials.

A very interesting experimental application of pure shear modes is that *results of 5 tests* in which linear elastic material is submitted to a set of 5 linearly independent *pure shear loadings* enable to *uniquely determine experimentally whether the material is elastically isotropic*. This finding was published for the first time in Spanish language by Jan Rychlewski in 1984. It has been recalled by Blinowski and Rychlewski in a very concise form in 1998 (see proof of Theorem 4.1).

The result gives at the same time information what is the minimum number of tests necessary and sufficient for finding out whether the elastic material is isotropic.

Indeed, when linear elastic material is submitted to five tests with pure shear loadings, for example the ones with the following representations in the fixed laboratory frame, $(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \neq 0,\ \boldsymbol{\tau}_2 \cdot \boldsymbol{\tau}_3 \neq 0,\ \boldsymbol{\tau}_3 \cdot \boldsymbol{\tau}_4 \neq 0,\ \boldsymbol{\tau}_4 \cdot \boldsymbol{\tau}_5 \neq 0)$

$$\overbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}^{\boldsymbol{\tau}_1 =}, \overbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}^{\boldsymbol{\tau}_2 =}, \overbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}^{\boldsymbol{\tau}_3 =}, \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}}^{\boldsymbol{\tau}_4 =}, \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}^{\boldsymbol{\tau}_5 =} \mathbf{e}_i \otimes \mathbf{e}_j$$

and in response to these loadings, the shear moduli determined from charts $\boldsymbol{\varepsilon}_i = (1/2\mu_i)\boldsymbol{\tau}_i$ - elaborated on the base of experimental data, will show to have the same value $\mu_i = \mu, i = 1,\dots,5$. Then, this will prove that the tested material is isotropic, linear elastic. The set $\boldsymbol{\tau}_i, i = 1, 5$ must be linearly independent.

Rychlewski J. Una propriedad notable de los materiales elasticos isotropos (On some remarkable property of isotropic elastic materials), *Ciencias Tecnicas. Fisicas y Matematicas*, Vol 5, pp. 99-104, 1984.
Blinowski A., Rychlewski J. Pure shears in the mechanics of materials, Math. Mech. Solids, 3,4, pp. 471-503, 1998.

Session 12 - synopsis

The concept of triaxiality factor or a measure characterizing magnitude of pressure forces in relation to shear forces. Wierzbicki and Xue constraint equation for triaxiality factor valid in biaxial stress states. Explicit relations linking triaxiality factor and skewness (Lode) angle in biaxial tests. Observation that radial lines (rays) coming out from the origin of rectangular coordinates system of biaxial tests domain (two dimensional) are simultaneously lines of constant values of triaxiality factor and lines of constant values of skewness angle. Parametrization of biaxial tests domain. Existence of three one to one relations between skewness angle and triaxiality factor in biaxial tests domain. Limitations of biaxial tests in examination of material properties. Convenient numerically formulas for determination of skewness (Lode) angle from the value of triaxiality factor in biaxial tests. Observation that the value of triaxiality factor exceeding two thirds ($\frac{2}{3}$) value in biaxial tests makes an excellent indicator that conditions of biaxiality were lost and general triaxial stress state starts to exist.

Second order symmetric tensors, triaxiality factor.

In 1959 Davies and Connelly introduced so called *triaxiality factor*, defined as quotient of stress first principal invariant divided by effective stress

$$\eta_{DC} \equiv I_1 / \sqrt{3J_2} = 3\sigma_m / \sigma_{ef}, \quad \sigma_{ef} \neq 0$$

They were motivated in this proposal by supposition, correct in view of their own and later research, that spherical tension (σ_{m}) called by them rather exotically *triaxial tension* has strong influence on the *loss of ductility of metals*, and the need to have some parameter to describe this effect.

The name *triaxiality factor* for the parameter is rather *unfortunate* because it gives false impression that *not pressure but general 3D multiaxial stress states* are subject of description with this parameter.

The triaxiality factor gained considerable attention and use when Wierzbicki and his collaborators pointed out that not only *spherical tension* (*negative pressure*) but also Lode angle can considerably *influence ductility and other properties of metals*.

Davies E.A., Connelly F.M., Stress Distribution and Plastic Deformation in Rotating Cylinders of Strain-Hardening Material, J. Applied Mech., March 1959, pp. 25-30.

Second order symmetric tensors, constraint relation for Triaxiality factor in biaxial (planar - 2D) tests.

The class of *biaxial tests* is defined by the condition that *always one of the principal values of stress tensor is equal to zero* ($\sigma_{III}=0$).

According to the ordering convention of principal values this could be smallest, middle or the largest principal value ($\sigma_I \geq \sigma_{II} \geq \sigma_{III}$) but usually, conventionally it is written that the third principal value is zero, regardless of the standard ordering convention.

Wierzbicki and Xue in 2005 found that in the case of *biaxial tests* unique relation exists between *Lode angle* (normalized principal third invariant of deviator) and *triaxiality factor*, formula (8) in Bai and Wierzbicki

$$\bar{J}_3 \equiv \cos(3\theta_L) = -\tfrac{27}{2}\eta(\eta^2 - \tfrac{1}{3}).$$

Wierzbicki et. al. adopted slightly modified definition of the triaxiality factor than the original one

$$\eta \equiv \sigma_m / \sigma_{ef} = \tfrac{1}{3}\eta_{DC}.$$

Since that time *triaxiality factor* started to be very frequently used in charts as *governing parameter* to present experimental results obtained in *biaxial tests* in order to present *influence the value of Lode angle* on various properties of metals and other materials.

Bai, Y., Wierzbicki, T., A new model of metal plasticity and fracture with pressure and Lode dependence, Int. J. Plast. 24, 6, 2008, pp. 1071-1096.

Second order symmetric tensors,
property of planar (2D) tensors arising in biaxial tests.

During whatever kind of biaxial tests, in view of $\sigma_{\mathrm{III}} \equiv 0$, two control *parameters* only, e.g., *two principal values* $\{\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}\}$, *uniquely determine* any set of *three principal stress invariants* fully characterizing stress tensor treated as sovereign object, e.g. $\{\sigma_m, J_2, J_3\}$. Some other convenient pair of control parameters can be selected, for example $\{\sigma_{\mathrm{m}}, \Delta\sigma\}$.

Tthe following relations are valid in the case of *biaxial tests*,

$$\sigma_{III} = 0 \;\Rightarrow\; \sigma_m = \tfrac{1}{3}(\sigma_I + \sigma_{II}),\;\; \Delta\sigma = (\sigma_I - \sigma_{II}),$$

$$s_I = \sigma_I - \sigma_m,\; s_{II} = \sigma_{II} - \sigma_m,\; s_{III} = -\sigma_m,$$

$$J_2 = s_{III}{}^2 - (s_I s_{II}) = \tfrac{1}{3}[\sigma_I{}^2 + \sigma_{II}{}^2 - \sigma_I\sigma_{II}] = \tfrac{1}{4}[3\sigma_m{}^2 + \Delta\sigma^2],$$

$$J_3 = s_{III}(s_I s_{II}) = \tfrac{1}{27}(\sigma_I + \sigma_{II})(\sigma_I - 2\sigma_{II})(2\sigma_I - \sigma_{II}) = \tfrac{1}{4}\sigma_m[\Delta\sigma^2 - \sigma_m{}^2] = \sigma_m[J_2 - \sigma_m{}^2]$$

Thus, for biaxial tests ($\sigma_{\mathrm{III}}=0$) the following inequality is valid

$$\sigma_{ef} = \sqrt{3J_2} = \sqrt{\tfrac{9}{4}[\sigma_m{}^2 + \tfrac{1}{3}\Delta\sigma^2]} \geq \tfrac{3}{2}|\sigma_m| \geq 0$$

The above reveals interesting information that *effective stress* − measure of shearing stresses intensity, *is always greater from absolute value of mean stress* for any planar (2D) stress state appearing in biaxial tests.

Second order symmetric tensors, property of planar (2D) tensors arising in biaxial tests.

The observation can be reformulated as follows

Property

The *modulus of deviatoric (shearing) part* of any non-zero (non-trivial) *planar tensor* is *always greater* than the *modulus of pressure part*.

$$\| \mathbf{s} \| = \sqrt{2J_2} \geq \sqrt{\tfrac{3}{2}} \, | \sigma_m | \geq 0$$

Direct consequence of the above property is the conclusion that

- *no purely spherical (isotropic) planar tensors exist*, or equivalently
- *the only purely spherical (isotropic) planar tensor is zero tensor*.

Giving the above physical interpretation it can be stated that
in the case of any not trivial planar tensor its shearing part dominates over its spherical part.

Second order symmetric tensors, explicit relations linking Triaxiality factor and Skewness angle in biaxial tests.

Wierzbicki and Xue constraint relation valid for *biaxial tests* can be expressed in the equivalent form of classical third power polynomial equation

$$\eta^3 - \tfrac{1}{3}\eta + \tfrac{2}{27}\sin(3\theta_{sk}) = 0; \qquad \eta \equiv \sigma_m / \sigma_{ef}\,, \quad \bar{J}_3 = \sin(3\theta_{sk}) = \tfrac{27}{2}[\tfrac{1}{3}\eta - \eta^3]$$

where instead of Lode angle the Skewness angle was used as governing parameter. This equation can be solved with the same method as the one used for finding stress principal values from characteristic equation.

The solution can be written in the following form

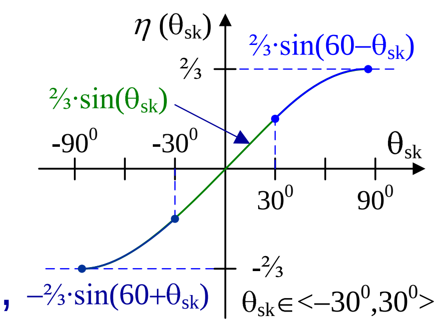
$$\sin(60^0 - \theta_{sk}) = \tfrac{3}{2}\eta \quad when \quad \sigma_{III} = 0 \le \sigma_{II} < \sigma_I, \quad \eta \in < \tfrac{2}{3}, \ \tfrac{1}{3} >, \quad \theta_{sk} \in < -30^0, 30^0 >,$$

$$\sin(\theta_{sk}) = \tfrac{3}{2}\eta \quad when \quad \sigma_{III} \le \sigma_{II} = 0 \le \sigma_I, \quad \eta \in < -\tfrac{1}{3}, \ \tfrac{1}{3} >, \quad \theta_{sk} \in < -30^0, 30^0 >,$$

$$\sin(60^0 + \theta_{sk}) = -\tfrac{3}{2}\eta \quad when \quad \sigma_{III} < \sigma_{II} \le \sigma_I = 0, \quad \eta \in < -\tfrac{1}{3}, -\tfrac{2}{3} > \quad \theta_{sk} \in < -30^0, 30^0 >.$$

$$sign(\eta) = sign(\sigma_m), \quad sign(\theta_{sk}) = sign(\bar{J}_3), \quad \eta \in < -\tfrac{2}{3}, \tfrac{2}{3} >,$$

$$\sigma_{ij} \Rightarrow \sigma_m = \tfrac{1}{3}\sigma_{ii}, \sigma_{ef} = \sqrt{3J_2} = \sqrt{\tfrac{3}{2} s_{ij} s_{ij}} \Rightarrow \eta \equiv \sigma_m / \sigma_{ef}\,.$$



In the above formulas standard denotation convention of principal stresses ($\sigma_{I} \ge \sigma_{II} \ge \sigma_{III}$) and the following identities were employed,

$$4\sin^3(\theta_{sk}) - 3\sin(\theta_{sk}) + \sin(3\theta_{sk}) = 0, \ \sin(\theta - 120^0) = -\sin(\theta + 60^0), \ \sin(\theta + 120^0) = \sin(60^0 - \theta)$$

Second order symmetric tensors, in biaxial tests paths of constant value of triaxiality factor η are paths of constant value of θ_{sk} angle.

Valid for biaxial tests explicit relations linking triaxiality factor and skewness angle $\eta \leftrightarrow \theta_{sk}$ are three bijections (one to one relations) in three separate areas (sharing edges) making together entire *two parameters* domain (half-plane) of biaxial tests states ($\sigma_I \geq \sigma_{II}; \sigma_{III}=0$),

$$\eta = \tfrac{2}{3}\sin(60^0 - \theta_{sk}) \;\; when \;\; 0 \leq \sigma_{II}, \sigma_I, \quad \eta = \tfrac{2}{3}\sin(\theta_{sk}) \;\; when \;\; \sigma_{II} \leq 0 \leq \sigma_I, \quad (*)$$

$$\eta = -\tfrac{2}{3}\sin(60^0 + \theta_{sk}) \;\; when \;\; \sigma_{II}, \sigma_I \leq 0$$

Theorem I

The *radial lines (rays) coming out from the origin* ($\sigma_I=0, \sigma_{II}=0$) of coordinates frame of biaxial tests domain, i.e., half plane $\sigma_I \geq \sigma_{II}$, are lines of constant values of triaxiality factor $\eta=const$ and at the same time lines of constant values of skewness (Lode) angle $\theta_{sk}=const$.

Proof

The radial lines running from the origin can be described as follows,

$$\sigma_{II} = a\,\sigma_I \;(a = const) \quad \Rightarrow \quad \sigma_m = \tfrac{1}{3}(\sigma_I + \sigma_{II}) = \tfrac{1}{3}(1+a)\sigma_I,$$

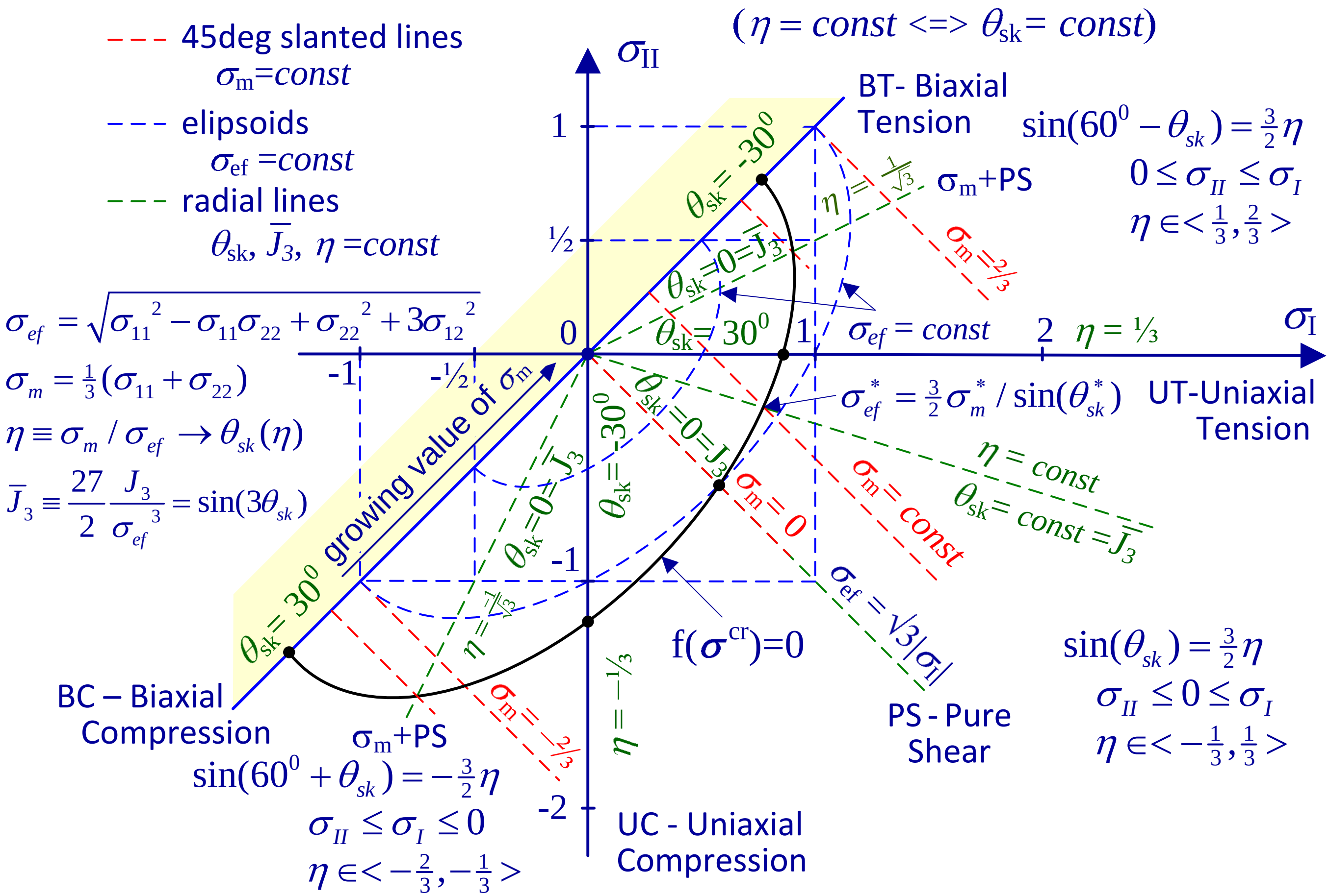
$$\sigma_{ef} = [\sigma_I{}^2 + \sigma_{II}{}^2 - \sigma_I\sigma_{II}]^{1/2} = [1-a+a^2]^{1/2}\,|\sigma_I| \Rightarrow \eta = \frac{\sigma_m}{\sigma_{ef}} = \frac{1}{3}\frac{(1+a)}{[1-a+a^2]^{1/2}}\,sign(\sigma_I),$$

$$a = const \Leftrightarrow \eta = const \Leftrightarrow \theta_{sk} = const \quad \text{this last upon relations (*)} \quad \text{q.e.d.}$$

In the case $\sigma_I=0$, σ_{II} can take any value, and it is $\eta=-⅓=const$, $\theta_{sk}=-30^0= const$

Second order symmetric tensors, parametrization of biaxial tests domain.

Biaxial tests domain $(\sigma_I \geq \sigma_{II};\ \sigma_{III} \equiv 0)$



Graphical illustration of biaxial tests domain parametrization in terms of parameters $(\sigma_m,\ \sigma_{ef},\ \theta_{sk})$.

The $f(\sigma^{cr})=0$ illustrates some hypothetical convex, critical surface (e.g. plastic yield), for which critical values of effective stress depend on skewness angle.

Second order symmetric tensors, property of relations linking $(\sigma_m, \sigma_{ef}, \theta_{sk})$ in biaxial tests.

Theorem II

The relations $\sigma_m(\sigma_{ef}, \theta_{sk})$, $\sigma_{ef}(\sigma_m, \theta_{sk})$, $\theta_{sk}(\sigma_m, \sigma_{ef})$, are bijections (one to one relations) in three complementary areas of the whole domain of biaxial tests domain parameterized with $(\sigma_I \geq \sigma_{II};\ \sigma_{III} = 0)$, except on the line $\sigma_m = \sigma_I + \sigma_{II} = 0$, on which $\theta_{sk} = \eta = 0$ for any value of $\sigma_{ef} = \sqrt{3}\sigma_I$.

Proof

It is straightforward to show that:

- *skewness angle* θ_{sk} maintains *constant* value on *radial lines* running from the origin $(\sigma_I = 0, \sigma_{II} = 0)$ of biaxial tests domain coordinates frame,
- *mean value of stress* σ_m maintains *constant* value on *45 degrees slanted lines* in the biaxial test domain,
- *effective stress* σ_{ef} maintains *constant* value on *ellipsoids with centers in the origin* $(\sigma_I = 0, \sigma_{II} = 0)$ of biaxial tests domain.

In view of the above, at any specific point of three subdomains of biaxial tests domain (mutually separate), the value of any variable chosen from triple set $\{\sigma_m, \sigma_{ef}, \theta_{sk}\}$ can be uniquely determined by the values of two remaining ones.

On line $\sigma_m = \sigma_I + \sigma_{II} = 0$ it is

$(\sigma_m = \sigma_I + \sigma_{II} = 0) \Rightarrow (J_3 = 0,\ \sigma_{ef} = \sqrt{3}\sigma_I) \Rightarrow \theta_{sk} = \eta = 0.$ q.e.d.

Insufficiency of biaxial (planar) tests for experimental examination of materials behavior sensitivity to skewness (Lode) angle.

An important open problem of *experimental mechanics of materials* is determination of *critical stress states surfaces* conditioning initiation of some physical processes in materials, for example plastic yield flow, damage, cracking or start of phase transition. In that respect the following observation can be formulated.

Corollary I

In the case of *convex critical surface* with the aid of *whatever type of biaxial test,* for any *fixed value of mean stress (pressure)* $\sigma_m=\sigma_m^*$, *critical effective stress* σ_{ef}^* can be determined for *only single value of skewness (Lode) angle* θ_{sk}^*.

In the case of *convex critical surface* with the aid of *whatever type of biaxial test,* for any *fixed value of skewness (Lode) angle* θ_{sk}^*, *critical effective stresses* σ_{ef}^* can be determined for *only three values of mean stress (pressure)* $\sigma_m=\sigma_m^*$

The above Corollary I is direct consequence of Theorems I and II.

Insufficiency of biaxial (planar) tests for experimental examination of materials behavior sensitivity to skewness (Lode) angle.

The direct conclusion from the **Corollary I** is that planar (biaxial) tests, among them very common tension (compression)-torsion tests on tubular samples (also the ones with internal pressure), are *not suitable for executing methodologically correct experimental examination of the influence of skewness (Lode) angle on materials behavior*.

This is so because using biaxial tests only, no sufficient experimental data can be collected to reliably separate the influence of mean stress and/or skewness angle on the possible variations of critical effective stresses. One value for any fixed pressure and/or three values for any fixed skewness angle are rather insufficient for such purpose.

This observation delivers a clear incentive for development and use of experimental techniques in which all three parameters characterizing stress state can be independently controlled to induce in the specimen not only 2D planar stress state but fully 3D complex stress state loadings.

They should make possible determination of critical effective stresses, or other parameters, e.g. example effective fracture strains in the whole domain of skewness angle values at freely prescribed, fixed mean stress.

Second order symmetric tensors, triaxiality factor as convenient indicator for loss of biaxiality.

Hint I

Formulas for triaxiality factor valid for *biaxial tests* show that in such a case the values of triaxiality factor must always remain in the range $\eta \in < -\frac{2}{3}, \frac{2}{3} >$, while in general case of *unconditioned 3D multiaxial tests* the triaxiality factor can take any value from the range $\eta \in < -\infty, \infty >$.

In many *experimental mechanics* publications, in which results from *biaxial tests* are presented, there can be noticed values of triaxiality factor exceeding the two third value $\frac{2}{3} \leq \eta$, which may seem to be *incorrect*.

However, experimental observation of *triaxiality factor greater than* $\frac{2}{3}$ rather indicates that *conditions of biaxiality were lost*, and in the sample true general triaxial stress state started to exist.

This delivers a hint to develop experimental methodologies, in which *triaxiality factor* is used as an effective *indicator of passing from plane (2D) state of stress to three dimensional (3D) state of stress*.

Second order symmetric tensors, convenient numerically formulas for determination of skewness (Lode) angle in biaxial tests.

Hint II

The uncoupled relations linking triaxiality factor and skewness angle $\eta \leftrightarrow \theta_{sk}$ are *very convenient for numerical computations*, because they enable determination of the value of skewness (Lode) angle θ_{sk} from the value of triaxiality factor η *more efficiently* than Wierzbicki, Xue formula,

$$\sigma_{ij} \Rightarrow \sigma_m = \tfrac{1}{3}\sigma_{ii}, \sigma_{ef} = \sqrt{3J_2} = \sqrt{\tfrac{3}{2}s_{ij}s_{ij}} \Rightarrow \eta \equiv \sigma_m / \sigma_{ef} \Rightarrow$$

$$\eta \in < \tfrac{2}{3}, \tfrac{1}{3} > \quad (0 \le \sigma_{II} \le \sigma_I) \Leftrightarrow \quad \theta_{sk} = -\arcsin(\tfrac{3}{2}\eta) + 60^0, \quad \theta_{sk} \in < -30^0, 30^0 >$$

$$\eta \in < -\tfrac{1}{3}, \tfrac{1}{3} > \quad (\sigma_{II} \le 0 \le \sigma_I) \Leftrightarrow \quad \theta_{sk} = \arcsin(\tfrac{3}{2}\eta), \quad \theta_{sk} \in < -30^0, 30^0 >$$

$$\eta \in < -\tfrac{1}{3}, -\tfrac{2}{3} > (\sigma_{II} \le \sigma_I \le 0) \Leftrightarrow \quad \theta_{sk} = -\arcsin(\tfrac{3}{2}\eta) - 60^0, \quad \theta_{sk} \in < -30^0, 30^0 >$$

Selection of the *proper formula* for calculation of skweness angle does not require computation of principal values of stress tensor because it can be decided upon *the value of* η falling into specific range of values of η.

For example when $\eta^* = 0.51 \in < ⅔, ⅓ >$ then $\theta_{sk}^* = -\sin^{-1}(1.5 \cdot \eta^*) + 60^0$.
The above formula is more efficient numerically then the formula

$$\theta_{sk} = \tfrac{1}{3}\sin^{-1}(\tfrac{27}{2}(\tfrac{1}{3}\eta - \eta^3)).$$

Second order symmetric tensors, disadvantage of presenting biaxial tests results using triaxiality factor as a parameter.

Hint III

Triaxiality factor η is not convenient operand, in general, to be used *for presentation of experimental biaxial tests results*.

This is so because when taken at its face value it contains tangled together information on two in principle linearly independent parameters characterizing (loading) stress tensor, i.e. σ_{m} and σ_{ef}., so this entanglement projects to the presented results making them somehow blurred.

In the case of *biaxial tests* in view of the existence of one to one relation between triaxiality factor and skewness (Lode) angle, actually *constant value of triaxiality factor corresponds to constant value of skewness (Lode) angle*. It is advisable to directly use *skewness angle as governing parameter in charts* presenting experimental biaxial tests results. Possibly, with information indicating the mode of loading: tensioning ($0<\sigma_{\mathrm{I}},\sigma_{\mathrm{II}}$), mixed ($\sigma_{\mathrm{II}}<0<\sigma_{\mathrm{I}}$) or compressive ($\sigma_{\mathrm{II}},\sigma_{\mathrm{I}} < 0$).

In this manner specific information presented in the chart will be delivered in transparent, methodologically unambiguous manner.

Session 13 - synopsis

Summary and concluding remarks. Other physical interpretations of second order tensors. Two Novozhilov (material) tensors very useful in materials characterization. Christoffel (acoustic) tensor very useful in characterization of waves propagation. Observation that Hooke's tensor for two dimensional (2D) materials can be presented in the form of second order symmetric tensor. Physical interpretations of invariants valid for Cauchy stress are invalid/meaningless for 2D Hooke's tensor. List of some interesting open scientific problems identified as a result of this study.

Summary and Concluding Remarks

1. *Historical survey* is delivered, necessarily quite compact, on how, why, when and by whom the key steps in development of *tensor notion* and *tensor calculus* were achieved, looking primarily from the perspective of mechanical community. The survey gives grounds to the view that the prime *philosophical and practical reasons* why tensors became the language of all advanced engineering and the other sciences is *invariance (when changing coordinates system)* and *linearity*.

2. The analysis is focused on identification and finding possibly best manner of description of *eigenproperties of Euclidean, second order symmetric tensors*. The *Cauchy stress tensor*, generic instant of symmetric second order tensor, is taken as primary subject of the discussion due to obvious for mechanical community reasons.

All the presented observations, conclusions, remarks and proposals are mutatis mutandis applicable to any second order symmetric tensor, which may possibly find application in modeling description of state and/or properties of real physical phenomena and/or objects.

3. It is pointed out that *tensors* make very rich and comprehensive idea and can be viewed and/or understood from many different perspectives for example as: *algebraic objects, matrices, linear transformations or geometrical objects*.

It is very succinctly recalled mathematically precise *algebraic definition* of tensors together with a set of underlying it algebraic structures. This to show what is the complete building structure *necessary* and *sufficient* to reach the notion of the tensor on technical side of being able to execute precise mathematically quantitative analyses.

This section excellently illustrates that mere craftsmanship of technical algebraic calculations does not allow fully grasping all the *richness, flavors and beauty* of tensor notion.

It is observed that some difficult aspects of tensors and tensor calculus can be much easier overcome when they are treated as *geometrical objects*. Advantage is taken of *algebraic-geometric duality* of tensors. In the case of second order symmetric tensor, this means an object characterized by *three features* (set of three invariants) and *specific orientation in space* (set of three *Euler angles*).

4. There are explained several, as it would seem at first sight paradoxes, and it is delivered information not widely known or at least perceived in mechanical community.

For example, how comes that second order symmetric *tensor,* an object *invariant with respect to change of coordinates system* - the very philosophical idea underlying tensor concept, *characterized by six linearly independent components* has only *three invariants of its components*?
It is shown that actually *six invariants* of the second order symmetric tensor *can be uniquely constructed* when the tensor is considered in interaction with other tensorial objects (some environment).

Another example is as follows. It is not a common knowledge or at least it is not commonly perceived, the existence of so called *non-isometric orthonormal tensorial bases*, i.e., bases which *are generated in different manner than tensorial product of rotated basis (triple orthogonal unit vectors)* of 3D Euclidean space generating higher order tensor spaces.
Isotropic tensors have *identical* representation components in *mutually isometric orthonormal* bases, but *isotropic tensors* in general have *different* representation components in *orthonormal* bases, which are not mutually isometric.

Summary and Concluding Remarks

5. Continuum mechanics literature is plenty of discussions on different *sets of invariants* of second symmetric order tensors, e.g., interpreted as stress or strain tensor. However, it is not adequately underlined that actually *infinite number of triplets of invariants* can be distinguished for second order symmetric tensors.

There has been explicitly specified the most popular sets of triplets of invariants of second order tensor, e.g. *basic (main) invariants*, *set of principal values*, *principal invariants*, with their unique naming and mutual mathematical relations between them.

While the relevant formulas are known, they are scattered among many different publications (books and papers) with many different denotations. Gathering them in one place and their specification in consistent notation makes very convenient and *handy reference resource*.

Summary and Concluding Remarks

6. *Historical survey* is delivered and *state of the art review* of the parametric description of Cauchy stress tensor, generic instant of second order symmetric tensors, to reach the so called *isomorphic orthogonal cylindrical coordinates* ($\sqrt{3}\cdot\sigma_m$, $r=(2J_2)^{1/2}$, θ_L) where σ_m is *mean value of stress*, r is *modulus of deviator* and θ_L is *Lode angle*.

It is put right here *erroneous information*, spread widely since several years in mechanical literature, about so called *Haigh-Westergaard (H-W) space*.

Haigh and independently Westergaard, nearly at the same time in 1920, introduced the concept of 3D space in which principal values of stress (σ_I,σ_{II},σ_{III}) were proposed as independent coordinates (parameters), and this is the *actual definition of Haigh-Westergaard space*.

Disseminated in many continuum mechanics papers information that cylindrical orthogonal coordinates ($\sqrt{3}\cdot\sigma_m$, $r=(2J_2)^{1/2}$, θ_L) make Haigh-Westergaard space is *incorrect information*.

The set of specific orthogonal cylindrical coordinates ($\sqrt{3}\cdot\sigma_m$, $r=(2J_2)^{1/2}$,θ_L) based on invariants of stress tensor, according to the present author best knowledge were for the first time introduced in 1958 by J. Murzewski.

Summary and Concluding Remarks

7. It is proposed *completely new generic parametrization* of the Cauchy stress tensor employing *new notions* of so called *isotropy angle* and *skewness angle* (shear stress mode angle), this last to replace *Lode angle*. The definition of skewness angle is based on the notion of *reference comparison state.* It is proposed to accept *pure shear* as *reference comparison state* for defining the skewness angle.

8. The physical interpretation is delivered of *pure shears* to be *elementary (atomic) elements* of any *deviator of second order symmetric tensor*. The *pure shears* can be identified to be generators of deviatoric space of second order symmetric tensors.

9. It is derived and explicitly specified completely new formula for index of *anisotropy degree* of second order symmetric tensors based on the notion of a *tensor orbit,* this being further advancement of the original Jan Rychlewski ideas. The Rychlewski's anisotropy degree index (factor) based on tensor orbit is much more precise and subtle measure than the index based on the size of *modulus of tensor deviator* only.

The formula for *anisotropy factor* becomes extremely simple when it is expressed in terms of *isotropy angle* and *skewness angle*.

10. It is delivered *statistical interpretation of the invariants* of *the Cauchy stress* tensor.

Firstly, results of this precursory approach delivered motivation and justification for giving the deviatoric (shear) part of Cauchy stress (second order symmetric) tensor *mode angle*, which defining *reference comparison state* is *pure shear*, the name *"skewness" angle*.

Secondly, the statistical interpretation allowed for identifying the reason and deliver explanation why the *degree of anisotropy* of Cauchy stress *decreases with its departure* from *pure shear* stress.

Thirdly, the statistical interpretation of the Cauchy stress invariants delivered grounds for introduction of the concept of *internal entropy of the Cauchy stress*.

In this manner a very interesting *link* has been discovered between *continuum mechanics* and *thermodynamics*.

Summary and Concluding Remarks

11. It is expounded that *elastic energy of linear elastic isotropic material does not and cannot depend on skewness (Lode) angle*. This is in contrast with the conjectures made in many papers that the material is assumed to be linear elastic and at the same time strength criteria for the same material, based on stored in it elastic energy, is assumed to be dependent on Lode angle. Such an action is a methodological error.

Lack of presence of skewess angle (carrying information on force interactions on micro level) in the formula for (macroscopic) *energy* delivers hint that development of new more comprehensive concept (and definition) of *ordered (directed) energy* might be very useful.

12. It is shortly outlined that *interaction* of second order symmetric tensor with other tensorial objects, e.g. fourth order (Hooke's) tensor representing elastic properties of a material, enables construction of *not only three* but *six quantities* remaining *invariant* upon change of coordinates system.

It is indicated that this gives premises for introduction and development of a *notion of weighted effective stress*, for example in the shape of stress quadratic form, which takes into account interaction of stress tensor with other tensorial object characterizing material in order to improve the classical *effective stress notion*.

Summary and Concluding Remarks

13. It is clearly explained essential difference between very popular in experimental mechanics, testing layouts of so called *simple shear* and *planar shear* tests, being practical implementations of *pure shear*. In the first case *principal axes rotate constantly* with increasing loading while in the second the orientation of *principal axes remain all the time stationary* with respect to laboratory reference frame. The kinematics in simple shear and kinematics in planar shear testing layouts, described by deformation gradient ***F***, are different but strains are exactly the same in the both layouts.

14. History of introduction of *triaxiality factor* into mechanical literature is presented very concisely. It is derived new, by solving third degree polynomial introduced by Wierzbicki and Xiao, original *explicit formulas linking triaxiality factor and skewness angle* (Lode angle) valid in the case *biaxial tests*.

It is indicated that with the use of *biaxial tests only* it is *impossible in correct methodologically manner* to precisely separate out the influence of mean stress and skewness angle on strength of material, a factor very important when formulating. e.g. criteria of plastic yielding, phase transition start or initiation of fracture. The finding delivers strong argument for experimental mechanics researchers to develop 3D multiaxial tests adequate for the purpose.

15. A very strong need and demand can be contemporary noticed for development of efficient methods of *computer visualization of second order (and higher order) tensorial fields*.

The classical visualization approaches e.g. in the form of *principal axes ellipsoid* can be evaluated as not insufficient but rather a completely unsatisfactory.

The present study indicates that it is practically impossible to deliver efficient and lucid graphical representation of second order symmetric tensor fields without its prior *structuralization* (construction of adequate set of invariants).

The set of invariants must be constructed in such a way to relevantly describe the features of interest. The present study showed that even second order symmetric tensors structure is too reach to show graphically all of its properties simultaneously.

Without structuring the visualization results usually prove to be very obscure, incomprehensible and intricate. Proposed here new parametrization of the Cauchy stress eigenproperties delivers good example of such structuralization for visulization purposes.

Summary and Concluding Remarks, V. Novozhilov tensors, E.B. Christoffel (acoustic) tensor.

V.V. Novozhilov drew attention to two very interesting tensors

$$\boldsymbol{\mu} \equiv \mathbf{C}\cdot\mathbf{1} \qquad \sim \mu_{ij} \equiv C_{ijkk} \qquad \Rightarrow \quad tr(\boldsymbol{\mu}) = \mathbf{1}\cdot\mathbf{C}\cdot\mathbf{1},$$

$$\boldsymbol{\nu} \equiv \mathbf{C}^{<3,2>}\cdot\mathbf{1} \quad \sim \nu_{ij} \equiv C_{ikkj} = C_{ikjk} \quad \Rightarrow \quad tr(\boldsymbol{\nu}) = \boldsymbol{\nu}\cdot\mathbf{1} = tr(\mathbf{C})$$

These are *second order symmetric tensors* constituting linear isotropic functions of Hooke's tensor, i.e. *material tensors* of the body.

Tensor $\boldsymbol{\mu}$ describes the body's reaction to spherical deformation. If $\boldsymbol{\varepsilon} = \mathbf{1}$ then $\boldsymbol{\sigma} = \mathbf{C}\cdot\mathbf{1} = \boldsymbol{\mu}$. Taking advantage of *spectral decomposition* of Hooke's tensor, it can be expressed as follows

$$\boldsymbol{\mu} = \mathbf{C}\cdot\mathbf{1} = \lambda_I\, tr(\boldsymbol{\omega}_I)\,\boldsymbol{\omega}_I + \ldots + \lambda_{VI}\, tr(\boldsymbol{\omega}_{VI})\,\boldsymbol{\omega}_{VI}$$

Tensor $\boldsymbol{\nu}$ plays a role in the dynamics of elastic waves. Taking advantage of *spectral decomposition* of Hooke's tensor, acoustic tensor can be expressed as follows

$$\boldsymbol{\nu} = \mathbf{C}^{<3,2>}\cdot\mathbf{1} = \lambda_I\,\boldsymbol{\omega}_I^{\,2} + \ldots + \lambda_{VI}\,\boldsymbol{\omega}_{VI}^{\,2}$$

E.B. Christoffel *acoustic tensor* is also *second order symmetric tensor*

$$\boldsymbol{\chi}(\mathbf{n}) \equiv \mathbf{n}\,\mathbf{C}\,\mathbf{n} = \mathbf{C}\cdot\mathbf{n}\otimes\mathbf{n} = \lambda_I\,(\boldsymbol{\omega}_I\mathbf{n})\otimes(\boldsymbol{\omega}_I\mathbf{n}) + \ldots + \lambda_{VI}\,(\boldsymbol{\omega}_{VI}\mathbf{n})\otimes(\boldsymbol{\omega}_{VI}\mathbf{n})$$

Novozhilov V.V. Theory of elasticity, Sudpromgiz, Leningrad, 1958.
Rychlewski J. Mathematical structure of elastic bodies, IPPT PAN Reports, Warszawa 2023; (English translation of original in Russian 1984).

Summary and Concluding Remarks, 2D materials Hooke's tensor.

A very interesting physical interpretation of *second order symmetric tensors* makes *two dimensional materials Hooke's tensor*

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\varepsilon} \sim \sigma_K^{Ke} = C_{KL}^{Ke} \varepsilon_L^{Ke}, \quad (K, L = 1, 3)$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & \sqrt{2}C_{1112} \\ C_{2211} & C_{2222} & \sqrt{2}C_{2212} \\ \sqrt{2}C_{1211} & \sqrt{2}C_{1222} & 2C_{1212} \end{bmatrix}^{Ke} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix}$$

$$H_1 = \tfrac{1}{2}[C_{11} + C_{22} + 2C_{12}], \qquad H_3 = \tfrac{1}{\sqrt{2}}[C_{11} - C_{22}],$$

$$H_2 = \tfrac{1}{2\sqrt{2}}[C_{11} + C_{22} - 2C_{12} + 2C_{33}], \quad H_4 = C_{13} + C_{23},$$

$$H_5 = \frac{C_{11} + C_{22} - 2C_{12} - 2C_{33}}{2\sqrt{2}}, \qquad H_6 = C_{13} - C_{23}$$

$$J_1 \equiv H_1 = \frac{1}{2}(C_{11} + C_{22} + 2C_{12}) = \lambda_{\mathcal{P}}, \quad J_2 \equiv H_2 = \frac{1}{2\sqrt{2}}(C_{11} + C_{22} - 2C_{12} + 2C_{33}) = \sqrt{2}\,\lambda_{\mathcal{D}},$$

$$J_3 \equiv R_1{}^2 = H_3{}^2 + H_4{}^2 = \frac{1}{2}(C_{11} - C_{22})^2 + (C_{13} + C_{23})^2 = |\boldsymbol{\tau}|^2,$$

$$J_4 \equiv R_2{}^2 = H_5{}^2 + H_6{}^2 = \frac{1}{8}(C_{11} + C_{22} - 2C_{12} - 2C_{13})^2 + (C_{13} - C_{23})^2 = |\mathbf{D}|^2$$

$$J_5 = (H_3{}^2 - H_4{}^2)H_6 - 2H_3H_4H_5 = 0 \text{ condition for orthotropic symmetry}$$

The Novozhilov's tensors, Christoffel tensor, 2D-Hooke's tensor, being second order symmetric tensors, require and deserve separate works, analogous to this presented here for Cauchy stress tensor, for identification of meaningful physically and useful sets of their invariants.

Blinowski A. , Ostrowska-Maciejewska J., Rychlewski J., Two-dimensional Hooke's tensors − isotropic decomposition, effective symmetry criteria, Arch. Mech., 48, 2, pp. 325-345, Warszawa 1996.

List of identified open scientific problems

In connection with elaboration new parametrization of Cauchy stress tensor several interesting open scientific problems could be identified:

1. Development of lucid graphical illustration of the tensor orbit concept.

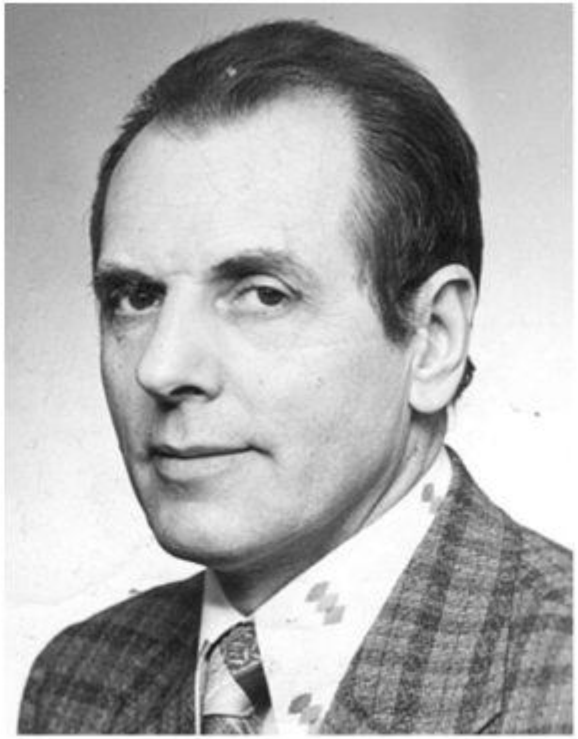
2. Development of classification (structuring) of elastic waves in anisotropic materials taking advantage of spectral decomposition of Hooke's tensor.

3. Development of strength of elastic material hypotheses taking advantage of the information that energy stored in the elastic material can always be decomposed into six mutually independent parts.

4. Development of extended definition of energy, that is elaboration of a concept of energy measure, which will contain information on entropy of microscopic force interactions, the "ordered (directed) energy".

Master of Polish School of Mechanics

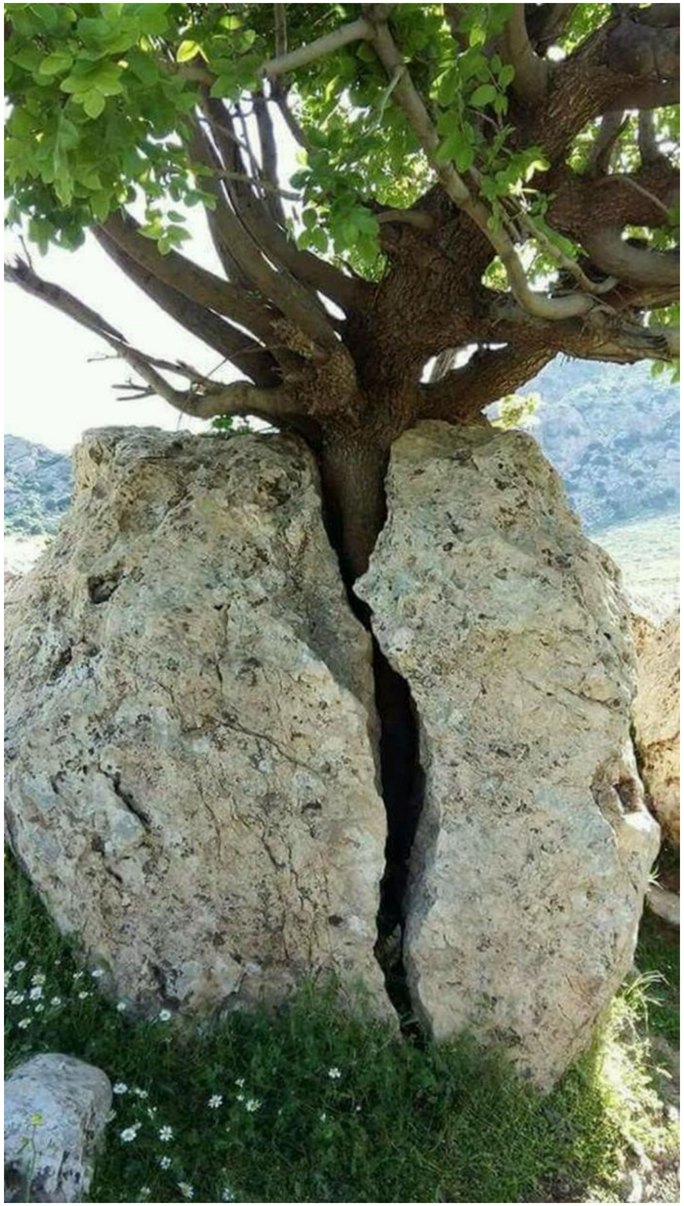
Jan Rychlewski



(1934-2011)

I dedicate this work to Professor Jan Rychlewski my teacher and tutor in tensorial calculus Andrzej Ziółkowski.

Summary and Concluding Remarks



vis vitalis

Some supplementary information, wisdom.

Jan Kochanowski z "Wykładu cnoty"; from the "Lecture on Virtues"

"Dwie tedy rzeczy człowieka szlachcią: obyczaje a rozum,
obyczaje z cnót pochodzą, a rozum z nauki,
obiedwie rzeczy w sobie mieć rzecz nieprzepłacona człowiekowi.
Ale jeśli tylko przy jednej masz zostać,
raczej przy cnocie niż przy nauce zostań,
bo NAUKA BEZ CNOTY, jako miecz u szalonego,
I SOBIE, I LUDZIOM SZKODZI,
cnota, choć dobrze sama będzie, chwalebna jest i pożyteczna".

"Two things then give a man nobility: morals and reason;
morals come from virtue, and reason from science;
to have both attributes, a thing priceless to man.
But if you only have to stay with one,
stay with virtue rather than with science,
because SCIENCE WITHOUT A VIRTUE, as a sword at the crazyman,
HARMS ITSELF AND PEOPLE;
virtue, even if it is alone, is glorious and profitable."

Some supplementary information, linear elasticity (Hooke's) constitutive law.

Robert Hooke initially (1676) announced his law of elastic materials behavior, linking force with deformation, in the form of Latin anagram

ceiiinosssttuv

He decoded his anagram two years later (1678) to read

ut tensio sic vis   (F=k·x)

or in the form that we know it today

$$\boldsymbol{\sigma} = \mathbf{L}\boldsymbol{\varepsilon}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ denote stress and strain tensors, $\mathbf{L}$ denotes Hooke's tensor.

J. Rychlewski (1983) *"CEIIINOSSSTTUV". Mathematical structure of elastic bodies [in Russian]*, Technical Report 217, Inst. Mech. Probl. USSR Acad. Sci., Moskva.

Some supplementary information, linear elasticity (Hooke's) constitutive law.

Spectral decomposition of elastic stiffness (Hooke's) tensor

A motivating question arises *execution of how many and what kind of experimental tests* is necessary and effective to uniquely determine elastic properties of the most general elastic, anisotropic material, or speaking otherwise all components of elastic stiffness (compliance) tensor?

Experimental answer to this question as a first delivered Woldemar Voigt in 1887 in favor of 21 constants. Enlightening structure of this constants was for the first time revealed by Jan Rychlewski in 1983, where he proved that any symmetric fourth order tensor can be *spectrally decomposed* into 6 mutually orthogonal subspaces. Each subspace is characterized by *stiffness (Kelvin) modulus* λ_K – scalar, and *elastic eigenstate* ω_K -symmetric second order tensor ($K=1,..,6$). Each elastic eigenstate is characterized by 2 so called *stiffness distributors* $\aleph_\alpha$ ($\alpha=1,..,12$).

Rychlewski (1984) J. *Elastic energy decompositions and limit criteria* [in Russian], Uspekhi Mekh., 7, 3.
Rychlewski J. (1985) Unconventional approach to linear elasticity, Arch. Mech.

Some supplementary information, spectral decomposition of elastic stiffness (Hooke's) tensor.

In summary set of 21 components/parameters determining any symmetric stiffness tensor can be divided into 3 classes

$$6 + 12 + 3 = 21$$

1. The first group consists of *6 Kelvin moduli* $\lambda_I, \ldots, \lambda_{VI}$

2. The second group consists of *12 stiffness distributors* $\aleph_1, \ldots, \aleph_{12}$, generators of 6 *elastic eigenstates* $\omega_I, \ldots, \omega_{VI}$

3. The third group consists of *3 Euler angles* ϕ_1, ϕ_2, ϕ_3

The *18 parameters* from the first and the second group are *invariants* of elastic stiffness tensor.

Question: When 21 components of elastic stiffness tensor determined in an experimental testing program, with fixed reference frame, for two otherwise unknown specimen shows to have the same values.
Does that mean the specimen were made of the same material?
(The answer is No. A.Z.)

See also Kowalczyk–Gajewska K., Ostrowska –Maciejewska J. (2009) Review on spectral decomposition of Hooke's tensor for all symmetry groups of linear elastic material, Arch. Mech.

Some supplementary information, qualitatively different types of (micro) ordering patterns in materials.

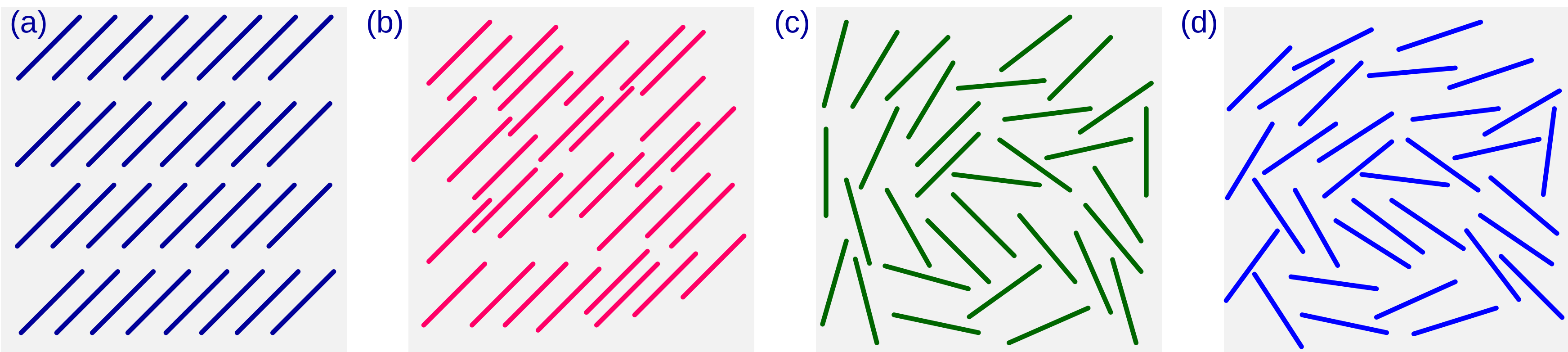
Abeyaratne presented nice schematic diagram illustrating different types of possible ordering patterns characterizing different materials, in Figure 13.1 on page 372 of his Lecture notes.

Both positional and orientational order

orientational order but no positional order

no positional order, some orientational order (Note AZ).

no positional order, more orientational order (Note AZ).



Abeyaratne , Continuum Mechanics, Vol. 2, Lecture Notes on the mechanics of elastic solids.
http://web.mit.edu/abeyaratne/lecture notes.html

Notes