Saturation of Estimates for the Maximum Enstrophy Growth in a Hydrodynamic System as an Optimal Control Problem

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Agenda

- Navier–Stokes equation ($\Omega = [0, L]^d$, $d = 2, 3$)

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} + \boldsymbol{\nabla} p - \nu \Delta \mathbf{v} = \mathbf{0}, & \text{in } \Omega \times (0, T] \\ \boldsymbol{\nabla} \cdot \mathbf{v} = 0, & \text{in } \Omega \times (0, T] \\ \text{Initial Condition} & \text{on } \Gamma \times (0, T] \\ \text{Boundary Condition (periodic)} & \text{in } \Omega \text{ at } t = 0 \end{cases}$$

- 2D Case
  - Existence Theory Complete — smooth and unique solutions exist for arbitrary times and arbitrarily large data
- 3D Case
  - Weak solutions (possibly nonsmooth) exist for arbitrary times
  - Classical (smooth) solutions (possibly nonsmooth) exist for *finite* times only
  - Possibility of “blow–up” (finite–time singularity formation)
  - One of the Clay Institute “Millennium Problems” ($ 1M!)

    http://www.claymath.org/millennium/Navier-Stokes_Equations

What is known? — Available Estimates

- A Key Quantity — Enstrophy

$$\mathcal{E}(t) \triangleq \int_\Omega |\boldsymbol{\nabla} \times \mathbf{v}|^2 \, d\Omega \qquad (= \|\boldsymbol{\nabla}\mathbf{v}\|_2^2)$$

- Smoothness of Solutions $\iff$ Bounded Enstrophy (Foias & Temam, 1989)

$$\max_{t\in[0,T]} \mathcal{E}(t) < \infty \quad ???$$

- Can estimate $\frac{d\mathcal{E}(t)}{dt}$ using the momentum equation, Sobolev's embeddings, Young and Cauchy–Schwartz inequalities, ...
  - REMARK: incompressibility not used in these estimates ....

- 2D Case:

$$\frac{d\mathcal{E}(t)}{dt} \leq \frac{C^2}{\nu}\mathcal{E}(t)^2$$

  - Gronwall's lemma and energy equation yield $\forall_t \ \mathcal{E}(t) < \infty$
  - smooth solutions exist for all times

- 3D Case:

$$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3}\mathcal{E}(t)^3$$

  - corresponding estimate not available ....
  - upper bound on $\mathcal{E}(t)$ blows up in finite time

$$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$$

  - singularity in finite time cannot be ruled out!

Problem of Lu & Doering (2008), I

- Can we actually find solutions which "saturate" a given estimate?
- Estimate $\frac{d\mathcal{E}(t)}{dt} \leq c\mathcal{E}(t)^3$ at a *fixed* instant of time t

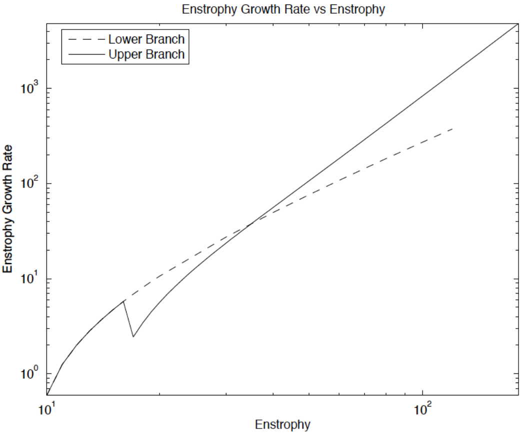
$$\max_{\mathbf{v}\in H^1(\Omega),\ \boldsymbol{\nabla}\cdot\mathbf{v}=0} \frac{d\mathcal{E}(t)}{dt}$$
$$\text{subject to} \quad \mathcal{E}(t) = \mathcal{E}_0$$

where

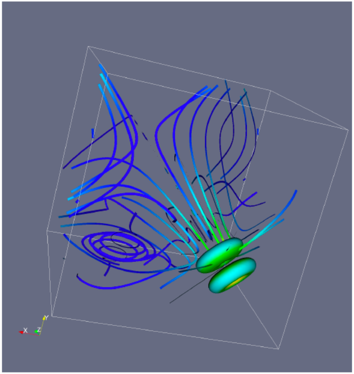
- $$\frac{d\mathcal{E}(t)}{dt} = -\nu\|\boldsymbol{\Delta}\mathbf{v}\|_2^2 + \int_\Omega \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\cdot\boldsymbol{\Delta}\mathbf{v}\, d\Omega$$
- $\mathcal{E}_0$ is a parameter

- Solution using a gradient–based descent method

Problem of Lu & Doering (2008), II



$$\left[\frac{d\mathcal{E}(t)}{dt}\right]_{max} = 8.97 \times 10^{-4}\ \mathcal{E}_0^{2.997}$$



vorticity field (top branch)

- How about solutions which saturate $\frac{d\mathcal{E}(t)}{dt} \leq c\mathcal{E}(t)^3$ over a *finite* time window $[0, T]$?

$$\max_{\mathbf{v}\in H^1(\Omega),\ \boldsymbol{\nabla}\cdot\mathbf{v}=0} \left[\max_{t\in[0,T]} \mathcal{E}(t) \right]$$
$$\text{subject to} \ \ \mathcal{E}(t) = \mathcal{E}_0$$

  where

  - $$\mathcal{E}(t) = \int_0^t \frac{d\mathcal{E}(\tau)}{d\tau}\, d\tau + \mathcal{E}_0$$
  - $\mathcal{E}_0$ is a parameter
  - $\max_{t\in[0,T]} \mathcal{E}(t)$ nondifferentiable w.r.t initial condition $\Longrightarrow$ non–smooth optimization problem

- In principle doable, but will try something simpler first ...

- Burgers equation ($\Omega = [0,1]$, $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}$)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{in } \Omega$$

$$u(x) = \phi(x) \qquad \text{at } t = 0$$

$$\text{Periodic B.C.}$$

Enstrophy : $\mathcal{E}(t) = \frac{1}{2}\int_0^1 |u_x(x,t)|^2\, dx$

- Solutions smooth for all times
- Questions of sharpness of enstrophy estimates still relevant

$$\frac{d\mathcal{E}(t)}{dt} \le \frac{3}{2}\left(\frac{1}{\pi^2\nu}\right)^{1/3}\mathcal{E}(t)^{5/3}$$

- Best available finite-time estimate

$$\max_{t\in[0,T]} \mathcal{E}(t) \le \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2\left(\frac{1}{\pi^2\nu}\right)^{4/3}\mathcal{E}_0\right]^3 \xrightarrow[\mathcal{E}_0\to\infty]{} C_2\mathcal{E}_0^3$$

"Small" Problem of Lu & Doering (2008), I

- Estimate $\frac{d\mathcal{E}(t)}{dt} \le c\mathcal{E}(t)^{5/3}$ at a *fixed* instant of time t

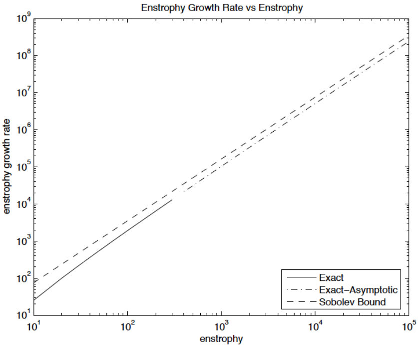
$$\max_{u \in H^1(\Omega)} \frac{d\mathcal{E}(t)}{dt}$$
$$\text{subject to } \ \mathcal{E}(t) = \mathcal{E}_0$$

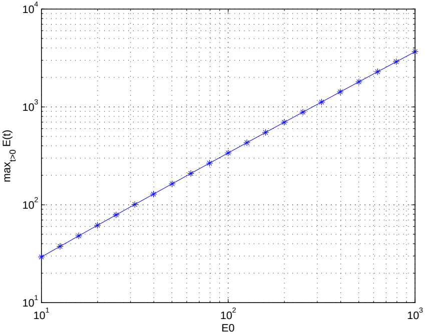
  where

  - $$\frac{d\mathcal{E}(t)}{dt} = -\nu \left\| \frac{\partial^2 u}{\partial x^2} \right\|_2^2 + \frac{1}{2} \int_0^1 \left(\frac{\partial u}{\partial x} \right)^3 \, d\Omega$$
  - $\mathcal{E}_0$ is a parameter

- Solution (maximizing field) found analytically!
  (in terms of elliptic integrals and Jacobi elliptic functions)

"Small" Problem of Lu & Doering (2008), II





$$\left[\frac{d\mathcal{E}(t)}{dt}\right]_{max} = 0.2476\,\frac{\mathcal{E}_0^{5/3}}{\nu^{1/3}}$$

$$\max_{t\in[0,T]} \mathcal{E}(t) \le C\mathcal{E}_0^{1.048}$$

instantaneous estimate is sharp

finite–time estimate far from saturated

Finite–Time Optimization Problem (I)

- Statement

$$\max_{u \in H^1(\Omega)} \mathcal{E}(T)$$
$$\text{subject to } \ \mathcal{E}(t) = \mathcal{E}_0$$

T, $\mathcal{E}_0$ — parameters

- Optimality Condition

$$\forall_{\phi' \in H^1} \qquad \mathcal{J}'_\lambda(\phi;\phi') = -\int_0^1 \frac{\partial^2 u}{\partial x^2}\Big|_{t=T} u'\Big|_{t=T}\, dx - \lambda \int_0^1 \frac{\partial^2 \phi}{\partial x^2}\Big|_{t=0} u'\Big|_{t=0}\, dx$$

Finite–Time Optimization Problem (II)

- Gradient Descent

$$\begin{aligned} \phi^{(n+1)} &= \phi^{(n)} - \tau^{(n)} \nabla \mathcal{J}(\phi^{(n)}), \qquad n = 1, \dots, \\ \phi^{(0)} &= \phi_0, \end{aligned}$$

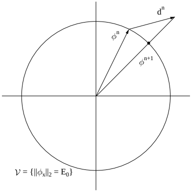
where $\nabla \mathcal{J}$ determined from *adjoint system* via H^1 Sobolev preconditioning

$$-\frac{\partial u^*}{\partial t} - u\frac{\partial u^*}{\partial x} - \nu\frac{\partial^2 u^*}{\partial x^2} = 0 \qquad \text{in } \Omega$$

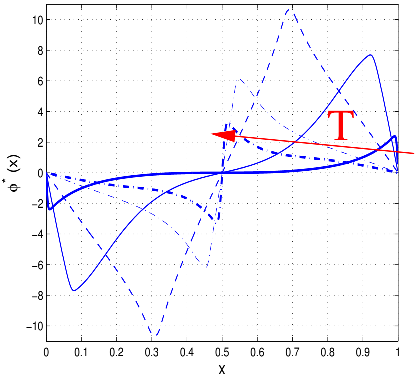
$$u^*(x) = -\frac{\partial^2 u}{\partial x^2}(x) \ \text{ at } t = T$$

Periodic B.C.

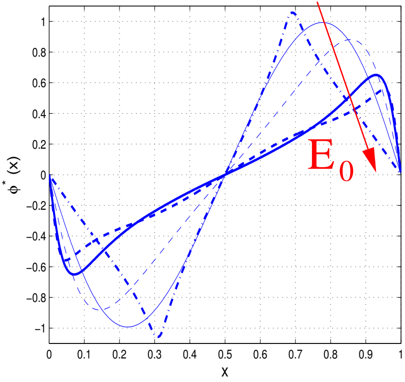
- Step size $\tau^{(n)}$ found via *arc minimization*



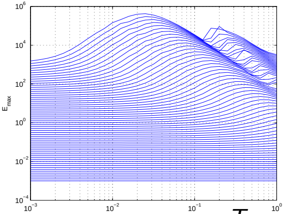
- Two parameters: T, $\mathcal{E}_0$ $(\nu = 10^{-3})$
- Optimal initial conditions corresponding to initial guess with wavenumber $m = 1$ (local maximizers)



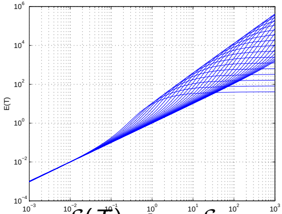
Fixed $\mathcal{E}_0 = 10^3$, different T



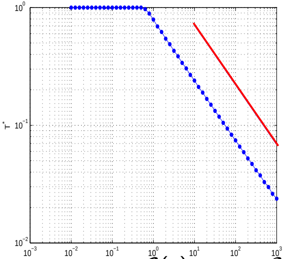
Fixed $T = 0.0316$, different $\mathcal{E}_0$



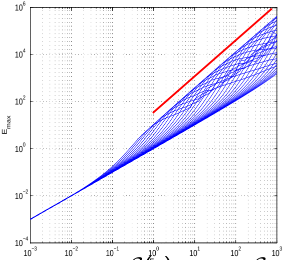
$\max_{t\in[0,T]}$ versus T



$\mathcal{E}(T)$ versus $\mathcal{E}_0$



$\operatorname{argmax}_{t\in[0,T]} \mathcal{E}(t)$ versus $\mathcal{E}_0$

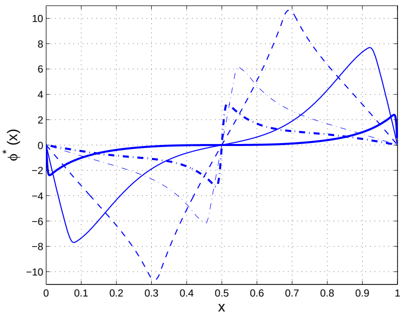


$\max_{t\in[0,T]} \mathcal{E}(t)$ versus $\mathcal{E}_0$

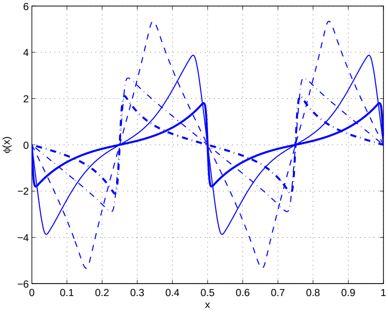
$$\boxed{\operatorname{argmax}_{t\in[0,T]} \mathcal{E}(t) \sim C\mathcal{E}_0^{-0.5}}$$

$$\boxed{\max_{t\in[0,T]} \mathcal{E}(t) \sim C\mathcal{E}_0^{1.5}}$$

- Sol'ns found with initial guesses $\phi^{(m)}(x) = \sin(2\pi m x)$, $m = 1, 2, \ldots$



$m = 1$, $\mathcal{E}_0 = 10^3$



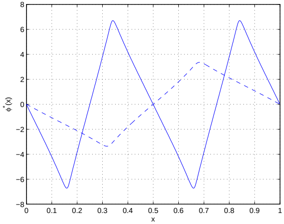
$m = 2$, $\mathcal{E}_0 = 10^3$

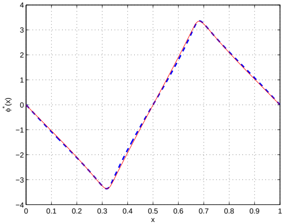
- Change of variables leaving Burgers equation invariant ($L \in \mathbb{Z}^+$):

$$x = L\xi, \ (x \in [0,1], \ \xi \in [0, 1/L]), \qquad \tau = t/L^2$$

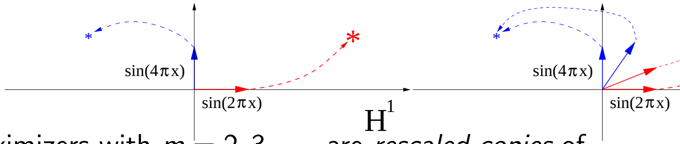
$$v(\tau, \xi) = Lu(x(\xi), t(\tau)), \qquad \mathcal{E}_v(\tau) = L^4 \mathcal{E}_u\left(\frac{t}{L^2}\right)$$

- Solutions for $m = 1$ and $m = 2$, after rescaling





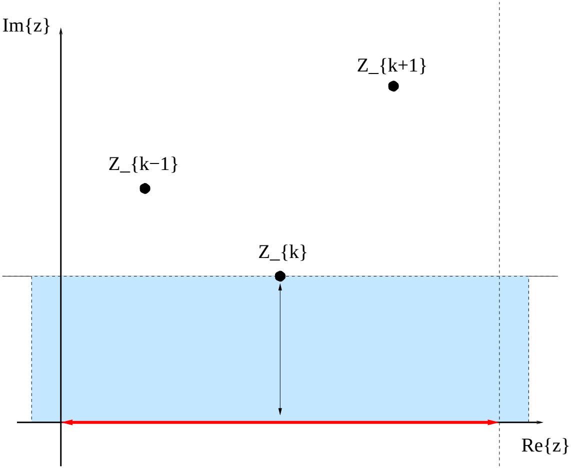
- Using initial guess: $\phi^{(0)}(x) = \sin(2\pi m x)$, $m = 1$, or $m = 2$
  $\phi^{(0)}(x) = \epsilon \sin(2\pi m x) + (1 - \epsilon) \sin(2\pi n x)$, $m \neq n$, $\epsilon > 0$



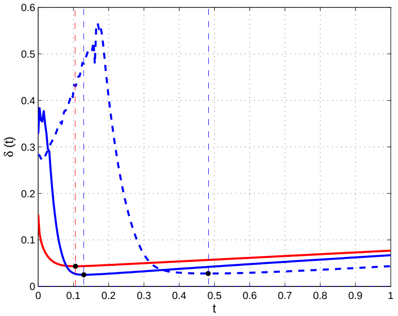
- All local maximizers with $m = 2, 3, \ldots$ are *rescaled copies* of the $m = 1$ maximizer

Location of Singularities in $\mathbb{C}$ from the Fourier spectrum

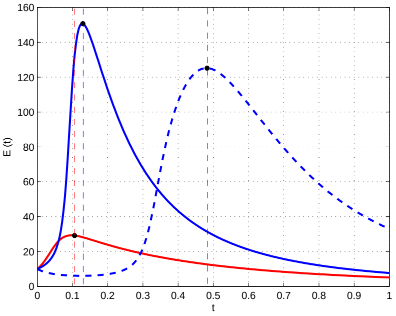
$$|\hat{u}_k| \sim C|k|^{-\alpha} e^{iz^*} \quad \text{as} \quad k \to \infty$$



Analyticity strip for a meromorphic function



$\Im\{z^*(t)\}$.



$\mathcal{E}(t)$

- RED — instantaneously optimal (Lu & Doering, 2008)
- BOLD BLUE — finite–time optimal ($T = 0.1$)
- DASHED BLUE — finite–time optimal ($T = 1$)

Summary & Conclusions

- Some evidence that optimizers found are in fact *global*
- Exponents in $\max_{t\in[0,T]} \mathcal{E}(t) = C\mathcal{E}_0^{\alpha}$ as $\mathcal{E}_0 \to \infty$

	theoretical estimate	optimal (instantaneous) [Lu & Doering, 2008]	optimal (finite–time) [present study]
α	3	1	$3/2$

  - more rapid enstrophy build–up in finite–time optimizers than in instantaneous optimizers
  - theoretical estimate *not sharp* $\Longrightarrow$ finite–time optimizers offer insights re: refinements required (work in progress)
- Finite–time maximizers saturate Poincaré's inequality (largest kinetic energy for a given enstrophy)
- Future work: Navier–Stokes 2D, 3D...

Reference

D. Ayala and B. Protas, "On Maximum Enstrophy Growth in a Hydrodynamic System", *Physica D* **240**, 1553–1563, (2011).