Archives of Mechanics • Archiwum Mechaniki Stosowanej • 31, 6, pp. 861-874, 1979

On slip-line field solutions for steady-state and self-similar problems with stress-free boundaries

H. PETRYK (WARSZAWA)

IN THIS PAPER it is shown that the matrix technique for constructing slip-line field solutions proposed by COLLINS [6] can be used for solving steady-state and self-similar problems with free boundaries. Boundary operators are proposed which generate the velocity field, automatically satisfying the condition of invariable position or of maintenance of the geometric similarity of a free boundary. The presented examples of slip-line field solutions illustrate a practical applicability of the proposed free boundary operators.

Pokazano, że macierzowa metoda konstrukcji siatek linii poślizgów zaproponowana przez COLLINSA [6] może być zastosowana do rozwiązywania stacjonarnych i samopodobnych zagadnień ze swobodnym brzegiem. Wyprowadzono postać operatorów macierzowych generujących pole prędkości spełniające automatycznie warunek stałego położenia linii swobodnego brzegu, względnie zachowania jej geometrycznego podobieństwa. Podano przykłady rozwiązań ilustrujące możliwości praktycznego zastosowania proponowanych operatorów.

Показано, что матричный метод построения сеток линий скольжений, предложенный Коллинсом [6], может быть применен для решения стационарных и автомодельных задач со свободной границей. Выведен вид матричных операторов, генерирующих поле скорости, удовлетворяющее автоматически условию постоянного положения линии свободной границы или сохронения ее геометрического подобия. Приведены примеры решений иллюстрирующие возможности практического применения предложенных операторов.

1. Introduction

THE THEORY of slip-line fields has been successfully used to analyse a great number of plane strain metal deformation problems [1, 2, 3, 4, 5]. The results obtained are surprisingly close to experimental observations in spite of the use of a strongly idealized, incompressible rigid-perfectly plastic model of the material.

The class of solutions available has been remarkably widened by employing the matrix technique for constructing slip-line field solutions developed by COLLINS [6] and DEWHURST and COLLINS [7]. The matrix technique makes it possible to derive in a relatively simple way the solutions of the so-called indirect type, in which the shape of none of the slip-lines, or their hodograph images (at least in some region), can be deduced in advance. In such cases the base slip-lines must first be found by solving an integral equation in the case of the analytic formulation. In the matrix formulation the problem of finding the base slip-lines reduces to solving an algebraic matrix equation with vector representations of these slip-lines as unknowns. The use of the matrix procedure is, however, limited to problems with boundary conditions leading to a linear integral equation. This will be the case when a plastically deforming region is bounded by: 1) slip-lines constituting rigid-plastic boundaries, or 2) rectilinear contours of rigid tools (rotating or not) with constant shear stress along them [7, 8].

In the present paper it is shown that the matrix technique can also be used when a plastic region is bounded by 3) a curvi-linear stress-free surface in a steady-state or self-similar(¹) problem. In that case not only the shape of slip-lines, but also the position of the free surface are to be found. The shape of the body must be chosen such that the solution of the problem satisfies the condition of invariability or of maintenance of geometric similarity of the body shape during deformation for steady-state or self-similar problems, respectively. The boundary conditions to be satisfied by the solutions of steady state or self-similar problems will be discussed later in detail. Since these conditions are imposed along the free boundary of initially unknown position, it would seem that it is necessary to employ some iterative procedure involving successive changes of geometry of the free surface until these conditions are satisfied. Such iterative procedure has been in fact used in some papers [9, 10, 11], not only for the plane strain problems.

However, it is found that these free boundary conditions can be incorporated in the matrix procedure by introducing respective so-called free boundary operators. One such operator, which generates the slip-line field between a given slip-line and a stress-free surface of initially unknown shape, has been introduced by DEWHURST and COLLINS [7]. The other free boundary operators, which transform the hodograph characteristics according to the steady-state or self-similarity requirements for velocities at the free surface, are proposed in the present paper. Using these free boundary operators within the framework of the matrix technique, we obtain a solution automatically satisfying all the stress and velocity conditions at the stress-free boundary, without the need of employing any iterative procedure(²).

The important advantage of this approach, in addition to saving computing time due to the elimination of an iterative procedure, lies in the fact that it enables us to derive in a natural way not only one but a class of different possible solutions for the same boundary value problem. This is also important for any other solution method since numerous examples show [12, 13, 14, 15, 16] that non-unique solutions do exist. There is no contradiction with the uniqueness theorem due to HILL [17] since this theorem does not apply to problems with undefined a priori boundaries.

Three slip-line field solutions with a stress-free boundary illustrate the practical applicability of the proposed free boundary operators.

2. Stress-free boundary in steady-state problems

In steady-state problems the stress and velocity do not vary at any fixed point. In order to satisfy this condition, the position of a free surface must remain unchanged during the

(¹) By self-similar problems we understand problems of non-steady motion where geometric similarity of the entire configuration is maintained during the deformation.

(²) However, in most cases it is necessary to use iterative procedure to satisfy other conditions such as geometric or total force requirements, appearing in the problem but not covered by the matrix equation (see the examples given in Sect. 4).

deformation. This implies that the free boundary must coincide with a stream-line, i.e. along this line we have

$$\mathbf{v}\cdot\mathbf{n} = 0, \tag{2.1}$$

where $\mathbf{v}$ is the velocity vector and $\mathbf{n}$ denotes the vector normal to the free boundary.

Thus, in steady-state problems we have a special type of boundary conditions at the free boundaries. Not only should the normal and tangential components of stress vanish there, but the additional condition (2.1) should also be satisfied. Instead, the shape of the stress-free boundary is to be found. Such conditions are particularly difficult to satisfy when a free surface bounds a plastically deforming region.

Let us consider a plastic region ABC bounded by a concave segment AB of a stress-free boundary(³) and by slip-lines AC and BC of angular range θ (Fig. 1a). Assume that the

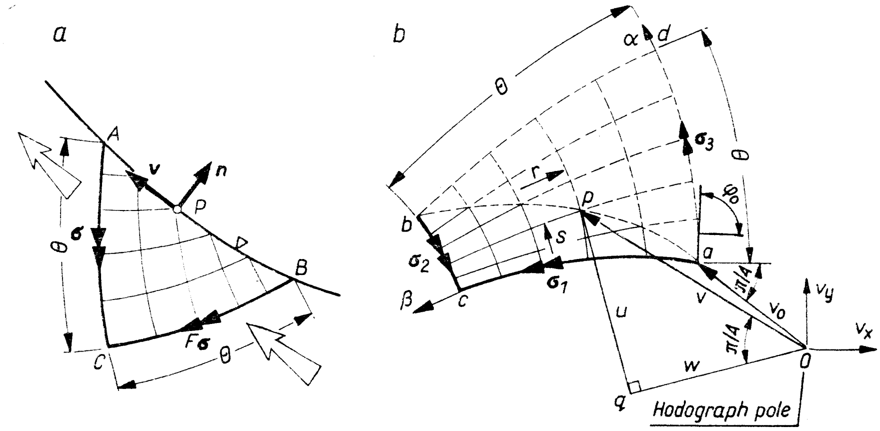

FIG. 1.

configuration of slip-lines images in the hodograph diagram abc for this region is as shown in Fig. 1b (the other possibilities will be discussed later). We will assume that both the stress and velocity fields in ABC are described by analytic functions. There is no loss of generality in so doing since in the opposite case the ABC region could be decomposed into analytic subregions. The ABC region is sought as a part of a slip-line solution for a steadystate problem; the shape of AB is a priori not known.

Let us extend the hodograph net beyond the image ab of AB up to obtaining the curvilinear quadrangle $abcd$. The hodograph characteristics form, as slip-lines do, the Hencky-Prandtl net [18]. Thus it is convenient to introduce [1] three pairs of parameters (α, β), (r, s), (u, w) in terms of which the relations describing the geometry of the hodograph net take a particularly simple form. The parameters (α, β) constitute the pair of curvilinear coordinates such that ad and ac are the reference α- and β-lines. α and β at a typical point p are the positive angles turned through to reach that point from the base point at a along either pair of α- and β-characteristics. Thus $\alpha = 0$ along ac and $\beta = 0$ along ad. (r,s) denote the radii of curvature of α- and β-lines taken with a positive sign, and (u, w) are

(³) Note that superimposing a uniformly distributed hydrostatic pressure has no influence on the following considerations.

so-called "moving coordinates" related to the Cartesian coordinates (v_x, v_y) in the hodograph plane by

$$u = v_x \cos\varphi + v_y \sin\varphi, \quad w = -v_x \sin\varphi + v_y \cos\varphi, \tag{2.2}$$

where $\varphi = \alpha+\beta+\varphi_0$, φ_0 is the angle of inclination of the tangent to the α-line at a to the positive v_x-axis (Fig. 1b).

The geometry of the hodograph net is governed by the equations (compare with Hencky's second theorem)

$$\partial s/\partial\alpha = r, \quad \partial r/\partial\beta = -s, \tag{2.3}$$

or (compare with Geiringer's equations)

$$\partial w/\partial\alpha = -u, \quad \partial u/\partial\beta = w. \tag{2.4}$$

The quantities (r, s) and (u, w) are related by

$$r = \partial u/\partial\alpha - w, \quad s = \partial w/\partial\beta + u. \tag{2.5}$$

Now we are ready to examine the influence of the boundary conditions at the free boundary AB on the stress and velocity fields in the plastic region ABC. Since the shape of AB is not known, it is impossible to determine in this region the shape of any slip-line. Neither can we determine the shape of any hodograph characteristic from the velocity conditions at AB alone. However, it is possible to determine the form of the boundary operators transforming slip-lines or hodograph characteristics in such a way that the boundary conditions at AB are automatically satisfied. The form of the operator F: $AC \to BC$ which generates the slip-line field in ABC such that AB is stress-free, has been derived in [7]. Below we will seek for the form of the operator H: $ac \to bc$, acting on the hodograph plane which generates in ABC the velocity field satisfying automatically the condition (2.1) at AB.

The hydrostatic pressure does not vary along AB. Thus from Hencky's relations we obtain

$$\alpha = \beta \quad \text{along } ab. \tag{2.6}$$

Since AB is stress-free, all slip-lines meet it at 45°. Thus the condition (2.1) may be written in the form

$$u(\alpha, \alpha) = w(\alpha, \alpha), \quad 0 \leqslant \alpha \leqslant \theta \tag{2.7}$$

or simply: $u = w$ along ab.

The differentials $du = (\partial u/\partial\alpha)d\alpha + (\partial u/\partial\beta)d\beta$ and $dw = (\partial w/\partial\alpha)d\alpha + (\partial w/\partial\beta)d\beta$ after substituting Eqs. (2.4) and (2.5) take the form

$$\begin{aligned} du &= (r+w)d\alpha + w d\beta, \\ dw &= \quad -u d\alpha + (s-u)d\beta. \end{aligned}$$

In virtue of Eq. (2.7) the differentials du and dw taken in the direction $d\alpha = d\beta$ must be equal along ab. Thus we have

$$r = s - 2(u+w) \quad \text{along } ab. \tag{2.8}$$

Equating in turn the differentials of both sides of Eq. (2.8) taken in the direction $d\alpha = d\beta$ and using the relations (2.3) to (2.7) in a way similar to the one presented above, we obtain

$$\partial r/\partial\alpha + r = \partial s/\partial\beta - s \quad \text{along } ab. \tag{2.9}$$

It can be proved by mathematical induction that the steady-state condition (2.1) leads to a more general, recurrence relationship on higher-order derivatives of the radii of curvature of hodograph characteristics, namely

$$\partial^{n+1}r/\partial\alpha^{n+1} + \partial^n r/\partial\alpha^n = \partial^{n+1}s/\partial\beta^{n+1} - \partial^n s/\partial\beta^n \quad \text{along } ab \quad (n = 0, 1, 2, ...). \tag{2.10}$$

Now let us introduce the vector representations [6, 7] $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_3$ of the basic hodograph characteristics ac and ad

$$ac \to \boldsymbol{\sigma}_1 = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \end{bmatrix}, \quad ad \to \boldsymbol{\sigma}_3 = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix},$$

where $s_0, s_1, \ldots$ and $r_0, r_1, \ldots$ denote the coefficients of the power series expansions of the radii of curvature of the lines ac and bc:

$$s(0, \beta) = \sum_{n=0}^{\infty} s_n \beta^n/n!, \quad r(\alpha, 0) = \sum_{n=0}^{\infty} r_n \alpha^n/n!.$$

From Eq. (2.10) written at the point a it immediately follows that

$$r_{n+1} + r_n = s_{n+1} - s_n, \quad n = 0, 1, 2, \ldots. \tag{2.11}$$

From Eq. (2.8) we obtain

$$r_0 = s_0 \cdot \xi, \tag{2.12}$$

where ξ is a parameter introduced for convenience according to the formulae

$$\begin{aligned} \xi &= 1 - 2\sqrt{2} v_0/s_0, \\ v_0 &= |\overline{oa}| = \sqrt{2} \cdot u(0, 0) = \sqrt{2} \cdot w(0, 0). \end{aligned} \tag{2.13}$$

The parameter ξ must be taken from the interval (0, 1] in order to satisfy the conditions $r_0 > 0$, $s_0 > 0$ and $v_0 \geqslant 0$, but is otherwise at this moment arbitrary. If the shape of ac is determined, the position of ac referred to the hodograph pole o will be defined by the value of ξ according to the formulae (2.13).

The relationship (2.11) together with Eq. (2.12) can be written in the matrix form

$$\boldsymbol{\sigma}_3 = A_\xi \boldsymbol{\sigma}_1, \tag{2.14}$$

where

$$A_\xi = \begin{bmatrix} \xi & 0 & 0 & 0 & \ldots \\ -1-\xi & 1 & 0 & 0 & \ldots \\ 1+\xi & -2 & 1 & 0 & \ldots \\ -1-\xi & 2 & -2 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{2.15}$$

The required operator H: $ac \to bc$ can be obtained by using the relation

$$\boldsymbol{\sigma}_2 = R_\theta (P_\theta^*)^{-1}(\boldsymbol{\sigma}_3 - Q_\theta^* R_\theta \boldsymbol{\sigma}_1) \tag{2.16}$$

between the vector representation $\boldsymbol{\sigma}_2$ of bc and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_3$. P, Q, R in the above relation are the basic matrix operators introduced by COLLINS [6] and discussed by DEWHURST and COLLINS [7]. The equality [2.16] results from the relation $(15)_1$ of the paper [7].

Combining Eqs. (2.14) and (2.16) we finally obtain

$$\boldsymbol{\sigma}_2 = H_{\theta\xi}\boldsymbol{\sigma}_1, \tag{2.17}$$

where

$$H_{\theta\xi} = R_\theta (P_\theta^*)^{-1}(A_\xi - Q_\theta^* R_\theta). \tag{2.18}$$

From Eq. (2.18) it is evident that the operator H: $ac \to bc$ has the form of an infinite matrix with elements depending only on θ and ξ as the corresponding subscripts indicate in the operator symbol. For the fixed values θ and ξ the operator H is then linear and can be determined independently of the shape of a free boundary AB. So the operator H may be used in the framework of the matrix technique for solving steady-state problems with a stress-free surface. As it has been proved above, the relation (2.17) must be satisfied for the position of the free boundary AB to be fixed during the deformation. On the other hand, if the velocity field in ABC is such that Eq. (2.17) is satisfied and the hodograph pole has a position according to Eq. (2.13), then the steady-state condition (2.1) is automatically satisfied along AB, no matter what the actual shape of the free boundary is. This can be shown by inverting the course of argumentation presented above and using the assumption that the velocity field in ABC is described by analytic functions.

By using GREEN'S method [19] it can be shown that for the configuration of slip-lines and their hodograph images as illustrated in Fig. 1 the rate of plastic work is positive everywhere in ABC if, and only if, AB is a trajectory of the algebraically smaller principal stress. However, if the boundary AB is subjected to tension, then the hodograph must take another form. In such a case, as well as when AB is convex, the analysis can be carried out in a way analogous to that presented above. In any case the operator H depends on the relative distance of the hodograph net from the hodograph pole, expressed by the parameter ξ. The particular case when one of the hodograph characteristics represents a given circular arc or is reduced to a singular point, has been examined by EWING [20] and COLLINS [25] by using a somewhat different approach.

3. Stress-free boundary in self-similar problems

In the general problem of non-steady motion the stress and velocity depend on the element position, defined by the position vector $\mathbf{r}$ referred to some fixed point O, as well as on the stage of the deformation defined by a characteristic length c. In self-similar problems the stress and velocity are functions only of the single variable $\mathbf{r}/c$. The analysis of self-similar problems becomes much easier when using the concept of a unit diagram [21]. An element whose position vector is $\mathbf{r}$ in physical space is represented in the unit diagram by a point whose position vector is $\mathbf{r}^* = \mathbf{r}/c$. When geometric similarity is preserved the unit diagram is exactly the same for all stages of the deformation. Thus, in the unit diagram the curve corresponding to the free boundary must coincide with a stream-line. Here a close resemblance between self-similar and steady-state problems is apparent [1].

Thus, in self-similar problems along a free boundary we have the additional condition

$$\mathbf{v}^* \cdot \mathbf{n} = 0 \tag{3.1}$$

analogous to Eq. (2.1); $\mathbf{v}^* = d\mathbf{r}^*/dc$ is the velocity vector of an element image in the unit diagram. Since $\mathbf{v}^*$ is directed towards the point whose position vector is the non-dimensionalized velocity vector $\mathbf{v} = d\mathbf{r}/dc$ [1] (see Fig. 2), the condition (3.1) can be written down in the equivalent form

$$(\mathbf{v}-\mathbf{r}^*) \cdot \mathbf{n} = 0. \tag{3.2}$$

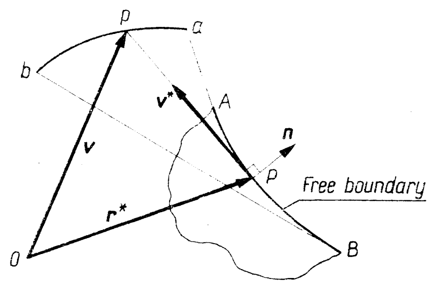

FIG. 2.

It can be shown [18] that when a segment of a free boundary bounds a rigid and rotating region, then Eq. (3.2) is satisfied if and only if this segment has the form of a logarithmic spiral. It must obviously be straight when such a rigid region does not rotate.

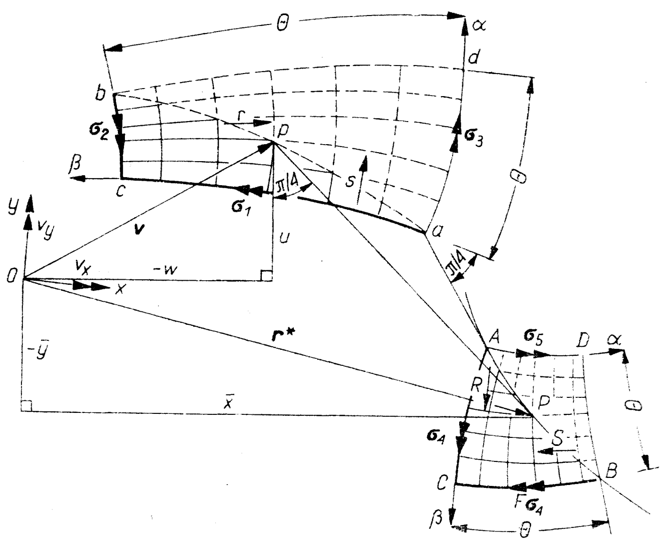

FIG. 3.

Now we will analyse the most difficult case when a stress-free surface bounds a plastically deforming region. This will be done in a way similar to that presented in the previous section. However, it can already be seen from Eq. (3.2) that in self-similar problems the geometry of the hodograph net is directly affected by the shape of the free boundary, contrary to the case of steady-state problems.

Let us consider the problem in terms of the unit diagram (Fig. 3). Let the plastic region ABC be bounded by a concave segment AB of a stress-free boundary and by the slip-lines AC and BC of the angular range θ (for simplicity we use the same terminology as in the physical plane). The corresponding hodograph diagram abc for non-dimensionalized velocities $\mathbf{v} = d\mathbf{r}/dc$ is superimposed on the unit diagram in such a way that the hodograph pole and the fixed point O coincide. We assume that the configuration of slip-lines and their hodograph images is as shown in (Fig. 3); other possibilities can be examined in an analogical way. As previously we make the assumption that the functions describing the stress and velocity fields in ABC are analytic.

Let us extend the slip-line field from the ABC-region beyond the free boundary AB up to obtaining the curvilinear quadrangle $ACBD$, and repeat such operation for the hodograph net. The geometry of the hodograph net can be described in terms of parameters (α, β), (r, s) and (u, w) defined in Sect. 2; the corresponding equations (2.3), (2.4) and (2.5) remain unchanged. The same pair of variables (α, β) parametrize the slip-line field in $ACBD$ as well, and any point P in $ACBD$ can have the same (α, β) coordinates as its hodograph image p. Denote by R and S the radii of curvature of α- and β- slip-lines respectively, taken with a positive sign. Introduce also the moving coordinates $(\bar{x}, \bar{y})$ for points of the slip-line net, related to the Cartesian coordinates (x, y) in the unit diagram plane by formulaes analogous to Eq. (2.2). Then,

$$\partial S/\partial\alpha = -R, \quad \partial R/\partial\beta = S, \tag{3.3}$$

$$\partial\bar{y}/\partial\alpha = -\bar{x}, \quad \partial\bar{x}/\partial\beta = \bar{y} \tag{3.4}$$

and

$$R = \partial\bar{x}/\partial\alpha - \bar{y}, \quad S = -\partial\bar{y}/\partial\beta + \bar{x}. \tag{3.5}$$

Let us now consider the boundary conditions imposed on the free boundary AB. We remind that the shape of AB is not known; if the slip-line AC were known it could be determined uniquely with the help of the operator F: $AC \to BC$ mentioned above. Now we will seek for the operators, which relate the shape of the hodograph characteristics ac and bc and the slip-line AC in such a way that Eq. (3.2) is automatically satisfied.

Since the pressure along AB is constant, then by Hencky's relations

$$\alpha = \beta \quad \text{along } AB. \tag{3.6}$$

Moreover, all slip-lines meet the free boundary at 45°. Thus $Rd\alpha = Sd\beta$ along AB and

$$R(\alpha, \alpha) = S(\alpha, \alpha), \quad 0 \leqslant \alpha \leqslant \theta. \tag{3.7}$$

The condition (3.2) expressed in terms of moving coordinates $(\bar{x}, \bar{y})$ and (u, w) takes the form

$$u(\alpha, \alpha) - \bar{y}(\alpha, \alpha) = w(\alpha, \alpha) + \bar{x}(\alpha, \alpha), \quad 0 \leqslant \alpha \leqslant \theta \tag{3.8}$$

or simply: $u - \bar{y} = w + \bar{x}$ along AB.

The differentials $d(u-\bar{y})$ and $d(w+\bar{x})$ after substituting Eqs. (2.4), (2.5) (3.4) and (3.5) take the form

$$d(u-\bar{y}) = (r+w+\bar{x})d\alpha+(w+\bar{x}-S)d\beta,$$
$$d(w+\bar{x}) = (-u+R+\bar{y})d\alpha+(s-u+\bar{y})d\beta.$$

Equating, in virtue of Eq. (3.8), these differentials taken in the direction $d\alpha = d\beta$ and making use of Eqs. (3.7) and (3.8), we obtain

(3.9) $$r = s-4(w+\bar{x}) \quad \text{along } AB$$

or, since

$$\sqrt{2}\,[w(\alpha, \alpha)+\bar{x}(\alpha, \alpha)] = \sqrt{2}\,[u(\alpha, \alpha)-\bar{y}(\alpha, \alpha)] = |\overrightarrow{Pp}|,$$

(3.10) $$r = s-2\sqrt{2}\,|\overrightarrow{Pp}| \quad \text{along } AB.$$

Equating in turn the differentials of both sides of Eq. (3.9) taken in the direction $d\alpha = d\beta$, we get after some transformations

(3.11) $$\partial r/\partial\alpha+r = \partial s/\partial\beta-s-2(R+S) \quad \text{along } AB.$$

It can be proved by mathematical induction that the condition (3.2) leads to a more general, recurrence relationship

(3.12) $$\partial^{n+1}r/\partial\alpha^{n+1}+\partial^n r/\partial\alpha^n = \partial^{n+1}s/\partial\beta^{n+1}-\partial^n s/\partial\beta^n-2(\partial^n R/\partial\alpha^n+\partial^n S/\partial\beta^n) \quad \text{along } AB$$
$$(n = 0, 1, 2, \ldots).$$

Now let us introduce the vector representations of the base slip-lines and hodograph char-acteristics, namely

$$ac \mapsto \boldsymbol{\sigma}_1 = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \end{bmatrix}, \quad AC \mapsto \boldsymbol{\sigma}_4 = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \end{bmatrix}, \quad ad \mapsto \boldsymbol{\sigma}_3 = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix}, \quad AD \mapsto \boldsymbol{\sigma}_5 = \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ \vdots \end{bmatrix}, \quad bc \mapsto \boldsymbol{\sigma}_2.$$

From Eq. (3.12) it follows that

(3.13) $$r_{n+1}+r_n = s_{n+1}-s_n-2(R_n+S_n), \quad n = 0, 1, 2, \ldots$$

and from Eq. (3.10) we obtain

(3.14) $$r_0 = s_0 \cdot \zeta,$$

where

(3.15) $$\zeta = 1-2\sqrt{2}\,|\overrightarrow{Aa}|/s_0.$$

The parameter ζ plays a similar role as ξ in steady-state problems and must also be taken from the interval $(0, 1]$.

Another recurrence relationship follows from Eq. (3.7), namely [22]

(3.16) $$R_{n+1}+R_n = S_{n+1}-S_n, \quad n = 0, 1, 2, \ldots$$

(note the formal analogy with Eq. (2.11)) with $R_0 = S_0$.

Using Eq. (3.16) we can write Eq. (3.13) together with Eq. (3.14) in the matrix form

(3.17) $$\boldsymbol{\sigma}_3 = A_\zeta\boldsymbol{\sigma}_1-B\boldsymbol{\sigma}_4,$$

where A_ζ is defined by Eq. (2.15), and

$$B = 4\begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 1 & -1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{3.18}$$

Substituting Eq. (3.17) into Eq. (2.16) and denoting

$$E_\theta = R_\theta(P_\theta^*)^{-1}B, \tag{3.19}$$

we finally obtain

$$\boldsymbol{\sigma}_2 = H_{\theta\zeta}\boldsymbol{\sigma}_1 - E_\theta\boldsymbol{\sigma}_4. \tag{3.20}$$

The relation (3.20) is equivalent to the condition (3.2) of maintenance of geometric similarity of the AB-line, provided Eq. (3.15) is satisfied (compare the similar equivalency of Eq. (2.17) with Eq. (2.1)). The operator H occurring in Eq. (3.20) is identical with the one for steady-state problems, i.e. is defined by Eq. (2.18). Thus, Eq. (3.20) differs from Eq. (2.16) by the second term which expresses the influence of the shape of the free boundary AB on the velocity field in ABC. However, since the operator E is linear and depends on the value of θ only, as it is seen from Eq. (3.19), it is possible to use the matrix technique for solving self-similar problems with a stress-free boundary as well.

4. Examples

In order to illustrate the possibilities of using the free boundary operators, three new slip-line field solutions are discussed. All solutions involve a plastically deforming region bounded by a stress-free surface of initially unknown shape, all can be determined by using the matrix technique and all are non-unique. Only the main features of the solutions will be discussed here; for a more detailed analysis see [13, 16, 23].

4.1. A solution for the steady-state problem of rolling of a rigid cylinder on a plastic half-space

The slip-line field and corresponding hodograph in Fig. 4 represent the steady stage of rolling of a rigid, perfectly rough cylinder on a rigid-perfectly plastic half-space. Accord-

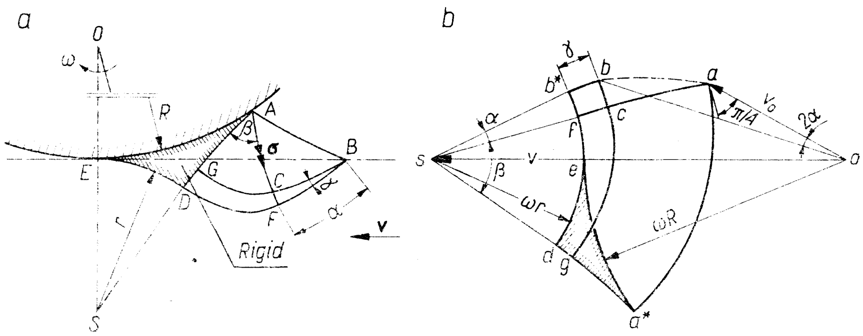

FIG. 4.

ing to the incompressibility condition, free surface elements before and after deformation are on the same level. The additional portion of the material which forms a "stationary wave" in front of the cylinder has been bulged out from the half-space during the initial, non-steady stage of the deformation which is not considered here.

The solution shown in Fig. 4 differs from those proposed by other authors [10, 24, 25] mainly on the basis of the stress singularity at B. The region EAD is rigid and rotates together with the cylinder with the angular speed ω. The velocity discontinuity of the magnitude ωr occurs across the $BFDE$-line, r being the radius of an isolated slip-line arc DE. The plastically deforming region ABC is exactly of the same type as that discussed in Sect. 2. Thus the results obtained there may now be directly applied.

Let us denote the vector representation of the slip-line AC by $\boldsymbol{\sigma}$. We can express the vector representations of subsequent slip-lines and hodograph characteristics up to the ac- and bc-lines by $\boldsymbol{\sigma}$, using the matrix operators P, Q, R, F discussed in [7]. Moreover, in order to satisfy the steady state requirements, the ac-line transformed by the operator H defined by Eq. (2.18) must give the line bc as proved in Sect. 2. From this we obtain the matrix equation on $\boldsymbol{\sigma}$, viz.

$$K\boldsymbol{\sigma} = rL\mathbf{c}, \tag{4.1}$$

where $\mathbf{c}$ is the unit circle vector and

$$\begin{aligned} K &= MQ_\gamma^*(P_{\gamma\beta}Q_\alpha^*F_\alpha + Q_{\beta\gamma}Q_\alpha^*) - H_{\alpha\xi}P_{\alpha\beta}P_{\alpha\beta}, \\ L &= -MP_\gamma^*, \\ M &= R_{\alpha+\beta} - H_{\alpha\xi}Q_{\beta\alpha}. \end{aligned}$$

α, β and γ are the field angles as shown in Fig. 4 and the parameter ξ has a geometrical interpretation according to Eq. (2.13). Three of these parameters are independent since one geometrical condition $oe = oa^*$ must be satisfied. When $\boldsymbol{\sigma}$ is found form Eq. (4.1), all geometrical parameters of the solution as well as the moment and forces acting on the cylinder can be conveniently determined by the series method due to EWING [22].

We may assume arbitrarily the magnitudes of two components of loading (e.g. the moment and the vertical component of the pushing force). Since the solution has three degrees of freedom, then an infinite number of solutions can be constructed each for a different value of the third component of loading. It should be added that for some range of parameters the solution is proved to be complete since the statically admissible extension of the stress field into rigid regions can be constructed [23].

4.2. A solution for the steady-state problem of machining

The slip-line field and corresponding hodograph in Fig. 5 describe the steady plane flow of material cut by a rigid wedge-shaped tool. The shear stress τ along the tool rake face FH is assumed to be constant. Thus the slip-line field in FGH is generated by the straight rough boundary operator G discussed in [7]. The ABC-region is of the same type as that discussed in Sect. 2. It can be shown that for the slip-line field to statisfy the stress and velocity boundary conditions (including these at the free boundary AB), the vector

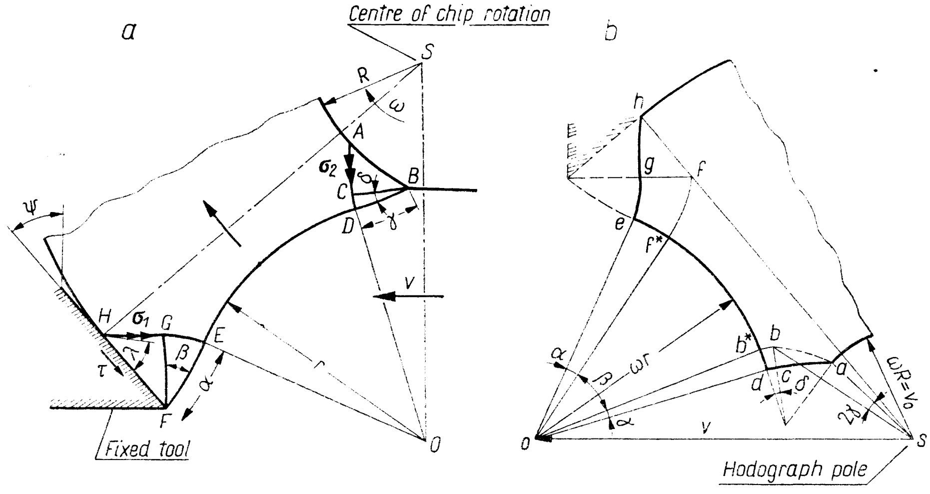

FIG. 5.

representations $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ defining the shape of the HG- and AC-lines must satisfy the matrix equations

$$(4.2) \qquad (G_{\lambda\alpha}Q_{\beta\alpha}Q_{\alpha\beta}G_{\lambda\alpha}-I)\boldsymbol{\sigma}_1 = rG_{\lambda\alpha}P_{\alpha\beta}\mathbf{c},$$

$$(4.3) \qquad (H_{\gamma\xi}-Q_{\delta\gamma}Q_{\gamma\delta}F_{\gamma})\boldsymbol{\sigma}_2 = rP_{\gamma\delta}\mathbf{c},$$

where I is the unit matrix and r is the radius of the isolated slip-line arc ED.

By solving Eqs. (4.2) and (4.3) we find $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, after which the geometry of the whole solution can be shown to be uniquely defined. The field has six degrees of freedom defined by five field angles α, β, γ, δ, λ as shown in Fig. 5 and the parameter ξ. Five conditions only exist to determine the values of these six parameters, namely three conditions of equilibrium of the chip and two conditions resulting from the assumed values of the rake angle ψ and of the shear stress τ along HF. This indicates that for given values of ψ and τ an infinite family of solutions may exist. However, numerical analysis suggests [13] that this might be the case only if a chip were not force-free, since the resultant moment and force acting across $ACDEGH$ did not simultaneously vanish for a remarkably wide range of the parameters examined(4).

In spite of this, the solution of the type shown in Fig. 5 presents a good example of the use of the operator H and confirms that such solutions may be non-unique.

4.3. A solution for the self-similar problem of cutting

The solution presented in Fig. 6 is of a similar type as that given in Fig. 5 but it represents now the process of indentation of a wedge-shaped tool into a plastic half-space at some small angle ϑ. The problem is self-similar; the unit diagram with a superimposed hodograph is shown in the figure. The plastic region ABC is of the same type as that considered in Sect. 3.

(4) Dewhurst [14] considered the solution which is a special case of that given in Fig. 5 when $\delta = 0$ and did not find a solution which satisfied all conditions of equilibrium of the chip, either.

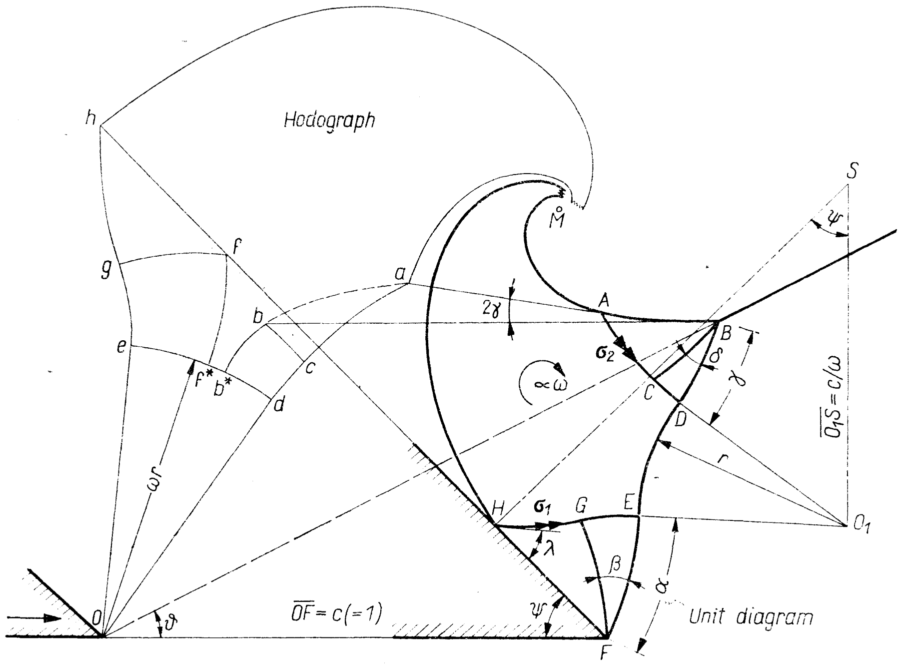

FIG. 6.

Two main differences arise when the solution from Fig. 5 is compared with the present one. Firstly, the free boundaries of the (rigid and rotating) chip have now the form of logarithmic spirals instead of circular arcs. Secondly, the vector representation of the AC-line according to Eq. (3.20) must now satisfy, instead of Eq. (4.3), the following equation:

$$\left(H_{\gamma\zeta}-Q_{\delta\gamma}Q_{\gamma\delta}F_{\gamma}-\frac{1}{\omega}E_{\gamma}\right)\boldsymbol{\sigma}_2 = rP_{\gamma\delta}\mathbf{c}, \tag{4.4}$$

where $\omega = \overline{OF}/\overline{O_1S}$ is the non-dimensional parameter proportional to the angular speed of a chip.

The parameter ω represents now an additional degree of freedom. Therefore, we have 7 degrees of freedom instead of 6 as in the previous case. Also we now have 6 conditions (instead of 5 in the previous case) since, additionally, ϑ must have the assumed value. Thus the comments concerning the non-uniqueness of the analogous solution of the steady-state problem remain valid when the problem becomes self-similar. Unfortunately, it is also doubtful whether the solution presented in Fig. 6 can satisfy the condition of total equilibrium of a chip.

References

1. R. HILL, *The mathematical theory of plasticity*, Oxford 1950.
2. W. SZCZEPIŃSKI, *Introduction to the mechanics of plastic forming of metals*, Sijthoff and Noordhoff 1979.
3. V. V. SOKOLOVSKII, *Theory of plasticity* [in Russian], Moscow 1969.

4. W. JOHNSON, R. SOWERBY and J. B. HADDOW, *Plane-strain slip-line fields: theory and bibliography*, Arnold London 1970.
5. L. M. KACHANOV, *Foundations of the theory of plasticity*, North Holland, Amsterdam 1971.
6. I. F. COLLINS, *The algebraic-geometry of slip line fields with applications to boundary value problems*, Proc. Roy. Soc., A **303**, 317, 1968.
7. P. DEWHURST, I. F. COLLINS, *A matrix technique for constructing slip-line field solutions to a class of plane strain plasticity problems*, Int. J. Num. Meth. Engng., **7**, 357, 1973.
8. I. F. COLLINS, *Compression of a rigid-perfectly plastic strip between parallel rotating smooth dies*, Q. J. Mech. Appl. Math., **23**, 329, 1970.
9. F. J. LOCKETT, *Indentation of a rigid/plastic material by a conical indenter*, J. Mech. Phys. Solids, **11**, 345, 1963.
10. E. A. MARSHALL, *Rolling contact with plastic deformation*, J. Mech. Phys. Solids, **16**, 243, 1968.
11. S. I. OH, S. KOBAYASHI, *An approximate method for a three-dimensional analysis of rolling*, Int. J. Mech. Sci., **17**, 293, 1975.
12. H. KUDO, *Some new slip line solutions for two-dimensional steady state machining*, Int. J. Mech. Sci., **7**, 43, 1965.
13. H. PETRYK, Ph. D. Thesis [in Polish], IFTR Reports 46/1977.
14. P. DEWHURST, *On the non-uniqueness of the machining process*, Proc. Roy. Soc., A **360**, 587, 1978.
15. H. PETRYK, *The plastic yielding due to cutting by wedge-shaped tools*, XX-th Polish Solid Mech. Conf., Kozubnik 1978.
16. H. PETRYK *Non-unique slip-line fieldso lutions for the wedgeinde ntation problem* [to be published].
17. R. HILL, *On the state of stress in a plastic-rigid body at the yield point*, Phil. Mag., **42**, 868, 1951.
18. A. P. GREEN, *On the use of hodographs in problems of plane plastic strain*, J. Mech. Phys. Solids, **2**, 73, 1954.
19. A. P. GREEN, *The plastic yielding of notched bars due to bending*, Quart J. Mech. Appl. Math., **6**, 223, 1953.
20. D. J. F. EWING, *A mass-flux method for deducing dimensions of plastic slipline fields*, J. Mech. Phys. Solids, **16**, 267, 1968.
21. R. HILL, E. H. LEE, S. J. TUPPER, *A theory of wedge indentation of ductile materials*, Proc. Roy. Soc., A **188**, 273, 1947.
22. D. J. F. EWING, *A series-method for constructing plastic slip-line fields*, J. Mech. Phys. Solids, **15**, 105, 1967.
23. H. PETRYK [to be published].
24. V. M. SEGAL, *Plastic contact in the case of the rolling of a rough cylinder on a perfectly-plastic half-space* [in Russian], Mekh. Tv. T., no 3, 184, 1971.
25. I. F. COLLINS, *On the rolling of a rigid cylinder on a rigid/perfectly plastic-space*, J. Mec. Appl., **2**, 431, 1978.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received October 10, 1978.