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A COMPLETENESS PROBLEM FOR STRESS EQUATIONS OF MOTION IN THE LINEAR ELASTICITY THEORY*

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Introduction

The present paper deals with the problem as to what system of stress equations of motion alone one can use to be sure that all fundamental field equations of linear theory elasticity are satisfied. It is shown that the dynamic stress problem can be treated in an entirely different way from that in the static case. A system of six stress equations of motion for six components of the stress tensor in non-homogeneous and anisotropic elastic solids is established. Some theorems that show the relations between these stress equations of motion and the fundamental system of field equations are proved. The uniqueness theorem for the dynamic stress equations is established. Stress equations of motion, the positive definiteness of the strain energy density and the accompanying initial and boundary conditions in terms of stress only, determine the problem completely.

It is also shown that another system of six equations of motion, called the generalization of the Beltrami–Michell stress equations for homogeneous and isotropic, elastic solid is necessary but not sufficient for all of the dynamic field equations to be satisfied. Comparisons are established between the pure stress treatment of the dynamical linear elasticity and mixed one, where the generalization of the Beltrami–Michell stress equations should be considered together with the displacement vector.

Some previous papers which have considered the problem of the stress equations of motion in linear elasticity are also discussed.

Further, the plane–strain and the generalized plane–stress equations of motion are taken into account. The theory is illustrated by the problem of Rayleigh waves in a non-homogeneous, isotropic, elastic semi–space (see [10]).

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1. Fundamental Stress Equations of Motion in Linear Elasticity

An anisotropic, non-homogeneous and elastic solid V is considered. We recall here from [1], the dynamic system of field equations in the isothermal three-dimensional linear theory of elasticity.

The linearized displacement-strain relations in the usual, indicial notation [1] appear as

$$2\varepsilon_{ij}(x,t) = u_{i,j}(x,t) + u_{j,i}(x,t), \tag{1.1}$$

where u_i and ε_{ij} designate respectively the cartesian components of the displacement vector and of the infinitesimal strain tensor, while t denotes time.

The equations of motion reduce to

$$\sigma_{ij,j}(x,t) + F_i(x,t) = \varrho(x)\ddot{u}_i(x,t), \tag{1.2}$$

where σ_{ij} are the components of the stress tensor, and F_i stands for the components of the body-force density. The dot denotes the partial differentiation with respect to t, and $\varrho(x)$ density of the medium.

The stress-strain relations take the form:

$$\varepsilon_{ij}(x,t) = \varkappa_{ijkl}(x)\sigma_{kl}(x,t), \tag{1.3}$$

where $\varkappa_{ijkl}$ denotes tensor of the elastic moduli.

Theorem 1.1. *If* $u_i = u_i(x,t)$, $\varepsilon_{ij} = \varepsilon_{ij}(x,t)$, $\sigma_{ij} = \sigma_{ij}(x,t)$ *meet* (1.1), (1.2), (1.3), *then* σ_{ij} *meets the equations*

$$2\varkappa_{ijkl}(x)\ddot{\sigma}_{kl}(x,t) = [\varrho^{-1}(x)\sigma_{ik,k}(x,t)]_{,j} + [\varrho^{-1}(x)\sigma_{jk,k}(x,t)]_{,i} + [\varrho^{-1}(x)F_i(x,t)]_{,j} + [\varrho^{-1}(x)F_j(x,t)]_{,i}. \tag{1.4}$$

Proof: We combine Eqs. (1.1), (1.2), (1.3), and arrive at (1.4).

The system (1.4) for an isotropic and homogeneous elastic solid was given in [2] and [3]. See also C. A. Truesdell, [4].

Theorem 1.2. *We shall assume the body to be initially undisturbed in the sense that*

$$\left.\begin{aligned} u_i(x,t) = \sigma_{ij}(x,t) = 0 \\ \dot{u}_i(x,t) = \dot{\sigma}_{ij}(x,t) = 0 \end{aligned}\right\} \text{ in } V, \quad -\infty < t \leqslant 0. \tag{1.5}$$

Let σ_{ij} *meet* (1.4) *and* ε_{ij} *satisfy* (1.3). *We define*

$$u_i(x,t) = \varrho^{-1}(x)\int_0^t (t-\tau)[\sigma_{ij,j}(x,\tau) + F_i(x,\tau)]\,d\tau. \tag{1.6}$$

Then

(a) u_i *meet* (1.2),

(b) u_i, ε_{ij} *meet the strain-displacement relations.*

[1] All Latin subscripts take on the value 1, 2, 3, the single argument x stands for (x_1, x_2, x_3), and summation over repeated subscripts is implied. Moreover: $u_{i,j} = \partial u_i/\partial x_j$.

P r o o f : Part (a) of this theorem is readily inferred from the definition (1.6) and the homogeneous initial conditions (1.5). To justify (b) we differentiate (1.6) twice with respect to t and combine with (1.4); we find

$$2\varkappa_{ijkl}(x)\ddot{\sigma}_{kl}(x,t)=\ddot{u}_{i,j}(x,t)+\ddot{u}_{j,i}(x,t)\,, \tag{1.7}$$

from which homogeneous initial conditions (1.5) and stress-strain relations (1.3) lead us to (1.1).

According to Theorems 1.1 and 1.2, the system (1.4) is a fundamental, dynamic stress system. Having found σ_{ij} which meets (1.4), the displacement vector can be found from (1.6)[2].

Theorem 1.3. Uniqueness in terms of stresses only[3]. *Let σ_{ij} meet the stress equations of motion in the form* (1.4). *The equation* (1.4), *which must hold throughout the region of space V occupied by the medium, is subject to the initial conditions*

$$\sigma_{ij}(x,0)=\sigma^{\circ}_{ij}(x)\,,\quad \dot{\sigma}_{ij}(x,0)=\dot{\sigma}^{\circ}_{ij}(x)\ \text{in}\ V, \tag{1.8}$$

where $\sigma^{\circ}_{ij}(x)$ and $\dot{\sigma}^{\circ}_{ij}(x)$ are the prescribed initial distributions of stress tensor and velocity of stress tensor. The accompanying boundary conditions are characterized by

$$\sigma_{ij}(x,t)n_j(x)=p_i(x,t)\ \text{on}\ B,\quad 0<t<\infty\,, \tag{1.9}$$

if B is the boundary of V, while $p_i(x,t)$ are the given surface tractions. Let V be a bounded regular region of space with the boundary B, and moreover suppose that

$$\left.\begin{array}{ll}\varkappa_{ijkl}(x)f_{ij}f_{kl}>0, & \text{if}\quad f_{ij}\neq 0\\ \varkappa_{ijkl}=\varkappa_{jikl}=\varkappa_{klij}, & \varrho(x)>0\end{array}\right\}\ \text{in}\ V. \tag{1.10}$$
[4]

Then there exists at most one tensor $\sigma_{ij}(x,t)$, which satisfies (1.4) *in $V(0<t<\infty)$, and meets* (1.8), (1.9).

P r o o f : In view of the linearity of (1.4), (1.8), (1.9), it is sufficient to show that

$$2\varkappa_{ijkl}(x)\ddot{\sigma}_{kl}(x,t)=[\varrho^{-1}(x)\sigma_{ik,k}(x,t)]_{,j}+[\varrho^{-1}(x)\sigma_{jk,k}(x,t)]_{,i}\quad \text{in}\ V\quad 0<t<\infty, \tag{1.11}$$

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[2] Among many pecularities of linear elastodynamics which do not belong to elastostatics we draw attention to the fact that in dynamic case displacement vector has been defined twice by strain tensor: first definition is given by (1.6) and (1.3) and the second one follows from (1.1). Compatibility of these two definitions leads us to (1.4). In the static case only strain — displacement relations define displacement vector provided strain tensor is prescribed.

[3] The uniqueness theorem is a well known fact from classical linear elasticity and here follows from Theorem 1.2 provided the strain-energy density is a positive definite function of the components of strain and suitable initial and boundary conditions are to be met. We prove once more uniqueness of the dynamic stress solution without making use of the kinetic energy notion which has been introduced in the conventional proof of Kirchhoff-Neuman's theorem.

[4] We admit $f_{ij}=f_{ij}(x,t),\ x\in V,\ t\in\langle 0,\infty)$.

together with

(1.12) $\sigma_{ij}(x, 0) = \dot{\sigma}_{ij}(x, 0) = 0 \quad \text{in } V,$

(1.13) $\sigma_{ij}(x, t) n_j(x) = 0 \quad \text{on } B, \quad 0 < t < \infty,$

imply $\sigma_{ij}(x, t) \equiv 0 \quad \text{in } V, \quad 0 < t < \infty.$

To this end we multiply (1.11) by $\dot{\sigma}_{ij}(x, t)$, and integrate both sides of the equation over the region V and $0 < \tau < t < \infty$. We obtain:

(1.14) $$\int_V dV \int_0^t d\tau [2\varkappa_{ijkl} \ddot{\sigma}_{kl} \dot{\sigma}_{ij} - (\varrho^{-1}\sigma_{ik,k})_{,j} \dot{\sigma}_{ij} - (\varrho^{-1}\sigma_{jk,k})_{,i} \dot{\sigma}_{ij}] = 0 .$$

Since

$$2\varkappa_{ijkl} \dot{\sigma}_{ij} \ddot{\sigma}_{kl} = \varkappa_{ijkl}(\ddot{\sigma}_{ij}\dot{\sigma}_{kl} + \dot{\sigma}_{ij}\ddot{\sigma}_{kl}) = \frac{d}{d\tau} \varkappa_{ijkl} \dot{\sigma}_{ij} \dot{\sigma}_{kl}, \quad \varkappa_{ijkl} \ddot{\sigma}_{ij} \dot{\sigma}_{kl} = \varkappa_{klij} \dot{\sigma}_{kl} \ddot{\sigma}_{ij},$$

by virtue of the last Eqs. (1.10), and

$$-(\varrho^{-1}\sigma_{ik,k})_{,j} \dot{\sigma}_{ij} = -(\varrho^{-1}\sigma_{ik,k} \dot{\sigma}_{ij})_{,j} + \varrho^{-1}\sigma_{ik,k} \dot{\sigma}_{ij,j},$$

$$-(\varrho^{-1}\sigma_{jk,k})_{,i} \dot{\sigma}_{ij} = -(\varrho^{-1}\sigma_{jk,k} \dot{\sigma}_{ji})_{,i} + \varrho^{-1}\sigma_{jk,k} \dot{\sigma}_{ji,i},$$

$$\int_V [\varrho^{-1}\sigma_{ik,k} \dot{\sigma}_{ij}]_{,j} dV = \int_B \varrho^{-1}\sigma_{ik,k} \dot{\sigma}_{ij} n_j dB = 0 ;$$

whence, from (1.14)

$$\int_V dV \int_0^t d\tau \left[\frac{d}{d\tau} (\varkappa_{ijkl} \dot{\sigma}_{ij} \dot{\sigma}_{kl}) + \frac{d}{d\tau} (\varrho^{-1}\sigma_{ik,k} \sigma_{is,s}) \right] = 0$$

and

$$\int_V dV (\varkappa_{ijkl} \dot{\sigma}_{ij} \dot{\sigma}_{kl} + \varrho^{-1}\sigma_{ik,k} \sigma_{is,s}) = 0$$

from which, because of (1.10), $\dot{\sigma}_{ij} \equiv 0$, $\sigma_{is,s} \equiv 0$, whence according to (1.12) $\sigma_{ij}(x, t) \equiv 0$ in $V(0 \leqslant t < \infty)$. This completes the proof.

2. Stress Equations of Motion for an Isotropic and Non-homogeneous Elastic Solid ($\varrho = \text{const}$)

We particularize (1.4) to the isotropic case and obtain:

(2.1) $$\frac{1}{c_2^2(x)} \left[\ddot{\sigma}_{ij}(x, t) - \frac{\lambda(x)\delta_{ij}}{3\lambda(x) + 2\mu(x)} \ddot{\sigma}_{kk}(x, t) \right]$$

$$= \sigma_{ij,kj}(x, t) + \sigma_{jk,ki}(x, t) + F_{i,j}(x, t) + F_{j,i}(x, t),$$

where $\lambda = \lambda(x)$, $\mu = \mu(x)$ elastic moduli for isotropic media, δ_{ij} is the Kronecker delta and $1/c_2^2(x) = \varrho/\mu(x)$.

The displacement vector is given by formula (1.6), provided $u_i = u_i(x, t)$ and $\sigma_{ij} = \sigma_{ij}(x, t)$ meet homogeneous initial conditions (1.5).

3. Connection of Stress Equations of Motion with the Generalization of the Beltrami-Michell Stress Equations (Isotropic and Homogeneous Elastic Solid)

Theorem 3.1. *If u_i, ε_{ij}, σ_{ij} meet the fundamental field equations of linear, isotropic and homogeneous elasticity*

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i}, \tag{3.1}$$

$$\sigma_{ij,j} + F_i = \varrho \ddot{u}_i \tag{3.2}$$

$$\sigma_{ij} = 2\mu\, \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij}, \tag{3.3}$$

then

$$\Box_2^2 \sigma_{ij} + \frac{2\lambda+2\mu}{3\lambda+2\mu}\sigma_{kk,ij} + \left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right)\frac{\lambda\delta_{ij}}{3\lambda+2\mu}\ddot{\sigma}_{kk} + \frac{\lambda\delta_{ij}}{\lambda+2\mu}F_{k,k} + F_{i,j} + F_{j,i} = 0 \tag{3.4}$$

and

$$\Box_1^2\Box_2^2\sigma_{ij} + \Box_1^2(F_{i,j} + F_{j,i}) + \frac{\lambda\delta_{ij}}{\lambda+2\mu}\Box_2^2 F_{k,k} - \frac{2\lambda+2\mu}{\lambda+2\mu}F_{k,kij} = 0, \tag{3.5}$$

where

$$\Box_\alpha^2 = \nabla^2 - \frac{1}{c_\alpha^2}\frac{\partial^2}{\partial t^2}, \quad \frac{1}{c_1^2} = \frac{\varrho}{\lambda+2\mu}, \quad \frac{1}{c_2^2} = \frac{\varrho}{\mu}. \tag{3.6}$$

P r o o f : To prove this we combine (3.1), (3.2), (3.3) and arrive at (3.4) and (3.5). (see [5], and [6]).

Theorem 3.2. *Let σ_{ij} meet the generalization of the Beltrami-Michell stress equations with $F_i = 0$*

$$\Box_2^2\sigma_{ij} + \frac{2\lambda+2\mu}{3\lambda+2\mu}\sigma_{kk,ij} + \left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right)\frac{\lambda\delta_{ij}}{3\lambda+2\mu}\ddot{\sigma}_{kk} = 0, \tag{3.7}$$

and moreover

(3.8)[5]
$$\frac{1}{c_3^2}\ddot{\sigma}_{kk} = \sigma_{ij,ij}, \quad \frac{1}{c_3^2} = \frac{\varrho}{3\lambda+2\mu}.$$

then the associated vector field $U_i = U_i(x, t)$ given by the formula

$$\varrho U_i(x,t) = \int_0^t (t-\tau)\sigma_{ij,j}(x,\tau)\,d\tau \tag{3.9}$$

meets the displacement equations of motion (Lamé equations)

$$\mu U_{i,kk} + (\lambda+\mu)U_{k,ki} = \varrho \ddot{U}_i. \tag{3.10}$$

P r o o f : If follows from (3.7) that

$$\Box_1^2\sigma_{kk} = 0. \tag{3.11}$$

[5] (3.8) is the contraction of (2.1) with $F_i = 0$.

We differentiate (3.7) with respect to x_j, and use (3.11) and (3.7). We find:

$$\square_2^2 \sigma_{ij,j} + \frac{\lambda+\mu}{\mu} \cdot \frac{\varrho}{3\lambda+2\mu} \ddot{\sigma}_{kk,i} = 0 , \tag{3.12}$$

from which, because of (3.8), (3.9)

$$\square_2^2 U_i + \frac{\lambda+\mu}{\mu} U_{k,ki} = 0 . \tag{3.13}$$

We have just integrated (3.12) twice with respect to t, and used homogeneous initial conditions for U_i and σ_{ij}.

Theorem 3. 3. *Let $a_i(x, t)$, $\sigma_{ij}(x, t)$ meet the equations:*

$$\sigma_{ij,j} + F_i = \varrho a_i , \tag{3.14}$$

$$\sigma_{ij,kk} + \frac{2\lambda+2\mu}{3\lambda+2\mu} \sigma_{kk,ij} + (F_i - \varrho a_i)_{,j} + (F_j - \varrho a_j)_{,i} + \frac{\lambda \delta_{ij}}{\lambda+2\mu} (F_s - \varrho a_s)_{,s} = 0 . \tag{3.15}$$

We define

$$\varepsilon_{ij} = \frac{1}{2\mu} \left(\sigma_{ij} - \frac{\lambda \delta_{ij}}{3\lambda+2\mu} \sigma_{kk} \right) , \tag{3.16}$$

then

$$\varepsilon_{ij,kk} + \varepsilon_{kk,ij} - \varepsilon_{ik,jk} - \varepsilon_{jk,ik} = 0 . \tag{3.17}$$

P r o o f : We combine (3.16), (3.15), (3.14) and check (3.17), (see also [11]).

4. The Plane–Strain and the Generalized Plane–Stress Equations of Motion in a Non–Homogeneous, Isotropic and Elastic Solid

4.1. Plane-strain solutions[6] **(ϱ constant, $F_\alpha = 0$).** ***Theorem 4.1.*** *If $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(x, t)$ meets equation*

$$\frac{\varrho}{\mu(x)} \left[\ddot{\sigma}_{\alpha\beta}(x, t) - \frac{\lambda(x)\delta_{\alpha\beta}}{2\lambda(x)+2\mu(x)} \ddot{\sigma}_{\gamma\gamma}(x, t) \right] = \sigma_{\alpha\gamma,\gamma\beta}(x, t) + \sigma_{\beta\gamma,\gamma\alpha}(x, t), \tag{4.1}$$

then the associate displacement vector

$$u_\alpha(x, t) = \varrho^{-1} \int_0^t (t-\tau) \sigma_{\alpha\beta,\beta}(x, \tau) \, d\tau \tag{4.2}$$

together with the strain tensor $\varepsilon_{\alpha\beta}$

$$\varepsilon_{\alpha\beta}(x, t) = \frac{1}{2\mu(x)} \left[\sigma_{\alpha\beta}(x, t) - \frac{\lambda(x)\delta_{\alpha\beta}}{2\lambda(x)+2\mu(x)} \sigma_{\gamma\gamma}(x, t) \right] \tag{4.3}$$

meets the strain-displacement relations and v_α satisfies the equations of motion in the elastic plane–strain state.

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[6] All Greek subscripts take on the values 1, 2, the single argument x stands for (x_1, x_2).

4.2. Plane–stress equations of motion. *Theorem 4.2. In the dynamic plane–stress problem, it is sufficient to find*

a solution $\bar{\sigma}_{\alpha\beta}$, *which meets equation*

$$(4.4)\qquad \frac{\varrho}{\mu(x)}\left[\ddot{\bar{\sigma}}_{\alpha\beta}(x,t)-\frac{\lambda(x)\,\delta_{\alpha\beta}}{3\lambda(x)+2\mu(x)}\ddot{\bar{\sigma}}_{\gamma\gamma}\right]=\bar{\sigma}_{\alpha\gamma,\,\gamma\beta}(x,t)+\bar{\sigma}_{\beta\gamma,\,\gamma\alpha}(x,t)\,;$$

then the associate displacement vector is given by

$$(4.5)\qquad \bar{u}_{\alpha}(x,t)=\varrho^{-1}\int_0^t (t-\tau)\bar{\sigma}_{\alpha\gamma,\,\gamma}(x,\tau)\,d\tau\,.$$

Strain–stress relations take the form

$$(4.6)\qquad \bar{\varepsilon}_{\alpha\beta}(x,t)=\frac{1}{2\mu(x)}\left[\bar{\sigma}_{\alpha\beta}(x,t)-\frac{\lambda(x)\delta_{\alpha\beta}}{3\lambda(x)+2\mu(x)}\,\bar{\sigma}_{\gamma\gamma}(x,t)\right].$$

Proof: Both the Theorems 4.1 and 4.2 are proved if the definitions of the plane–strain and the generalized plane–stress problem respectively and homogeneous initial conditions for $u_\alpha(x,t)$, $\varepsilon_{\alpha\beta}(x,t)$, $\sigma_{\alpha\beta}(x,t)$ are taken into account [1].

Some problems of dynamic plane elasticity have been studied by J. R. M. Radok [7], and P. P. Teodorescu, [8]. No pure stress methods of analysis have been used by them to solve the problems they formulated in [7] and [8]. Although three stress equations have been derived in [7][7] (plane–strain solution and homogeneous case),

$$(4.7)\qquad \begin{gathered}\Box_1^2\sigma_{\alpha\alpha}=0\,,\qquad \Box_2^2\sigma_{12}+\sigma_{\alpha\alpha,\,12}=0\,,\\ \left(\sigma_{11,\,11}-\frac{\varrho}{2\mu}\ddot{\sigma}_{11}\right)-\left(\sigma_{22,\,22}-\frac{\varrho}{2\mu}\ddot{\sigma}_{22}\right)=0.\end{gathered}$$

The author had to use the displacement vector to find σ_{12}. Teodorescu, [8], has also found σ_{12} by «mixed» method. The system (4.7), although it includes three components σ_{11}, σ_{22}, σ_{12} and accounts for the plane compatibility equation, is not sufficient to determine the dynamic plane problem . To prove this, it is sufficient to show that there exists a stress tensor $\sigma^*_{\alpha\beta}$ which meets (4.7) but not (4.1) (homogeneous case). $\sigma^*_{\alpha\beta}$ can be assumed in the form (see W. Nowacki, [9]):

$$(4.8)\qquad \sigma^*_{\alpha\beta}=-\Phi_{,\,\alpha\beta}+\delta_{\alpha\beta}\left(\Phi_{,\,\gamma\gamma}-\frac{\varrho}{2\mu}\ddot{\Phi}\right),$$

where

$$(4.9)\qquad \Box_1^2\Box_2^2\Phi=0.$$

[7] See also [12], p. 118, formulae (75.4)-(75.6).

It is easy to check, $\sigma^*_{\alpha\beta}$ meets (4.7), provided (4.9). However, if we substitute $\sigma^*_{\alpha\beta}$ into (4.1) we find after certain manipulations

$$\delta_{\alpha\beta}\frac{\lambda+2\mu}{2\lambda+2\mu}\Box_1^2\ddot{\Phi}=0. \tag{4.10}$$

Equation (4.10) cannot be met by any function Φ, such that

$$\Box_2^2\Phi=0\,. \tag{4.11}$$

It is also seen, that, although $\sigma^*_{\alpha\beta}$ for the static case ($\ddot{\Phi}=0$) reduces to the general solution in terms of Airy function, it does not represent a general solution of the dynamic problem.

Since

$$\sigma^*_{\alpha\beta,\,\beta}=-\frac{\varrho}{2\mu}\ddot{\Phi}_{,\alpha} \tag{4.12}$$

and because of

$$\sigma^*_{\alpha\beta,\,\beta}=\varrho\ddot{u}^*_\alpha \tag{4.13}$$

the displacement vector u^*_α has potential Φ only[8]. No shear waves are included in representation (4.8).

Some advantages of the method of pure stress–equations of motion can be seen if non–homogeneous, isotropic elastic solids are taken into consideration. The problem of Rayleigh waves in a non–homogeneous, isotropic elastic solid is chosen in [10] to expound the dynamic stress method.

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[8] We have assumed homogeneous initial conditions for u^*_α and Φ.

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Streszczenie

ZAGADNIENIE ZUPEŁNOŚCI DLA NAPRĘŻENIOWYCH RÓWNAŃ RUCHU W LINIOWEJ TEORII SPRĘŻYSTOŚCI

Praca rozważa problem, jaki układ naprężeniowych równań ruchu jedynie powinien być użyty, aby zapewnić spełnienie wszystkich podstawowych równań pola liniowej teorii sprężystości.

Wykazano, że dynamiczne naprężeniowe zagadnienie może być traktowane w pewien całkowicie odmienny sposób od stosowanego w przypadku statycznym. Wyróżniono pewien układ sześciu naprężeniowych równań ruchu w ciałach niejednorodnych anizotropowych i sprężystych. Udowodniono twierdzenia przedstawiające relacje między naprężeniowymi równaniami i podstawowym układem równań dynamicznych pola. Dowiedziono również twierdzenia o jednoznaczności dla dynamicznych równań naprężeniowych. Tylko równania naprężeniowe ruchu, dodatnio określona gęstość energii odkształcenia sprężystego i towarzyszące początkowe i brzegowe warunki wyrażone przez naprężenia określają zagadnienie zupełnie. Wykazano także, że pewien inny układ sześciu naprężeniowych równań ruchu, tzw. uogólnione naprężeniowe równania Beltramiego-Michella są warunkiem koniecznym, lecz nie dostatecznym, na to, aby były spełnione wszystkie dynamiczne równania pola. Dokonano porównania między czysto naprężeniowym traktowaniem dynamicznej teorii sprężystości i jej traktowaniem mieszanym, gdy uogólnione równania Beltramiego-Michella powinny być rozważane wraz z wektorem przemieszczenia. Przedyskutowano także pewne poprzednie prace dotyczące naprężeniowych równań ruchu w liniowej teorii sprężystości.

Rozważono także naprężeniowe równania ruchu dla płaskiego odkształcenia i uogólnionego płaskiego stanu naprężenia w niejednorodnym i izotropowym ciele sprężystym. Teorię zilustrowano w pracy [10] zagadnieniem rozchodzenia się fal Rayleigha w niejednorodnej, izotropowej półprzestrzeni.

Резюме

ПОЛНОТА ЗАДАЧИ ДЛЯ УРАВНЕНИЙ ДВИЖЕНИЯ В НАПРЯЖЕНИЯХ В ЛИНЕЙНОЙ ТЕОРИИ УПРУГОСТИ

Рассматривается вопрос, какую именно систему уравнений движения в напряжениях можно единственно использовать для того, чтобы обеспечить удовлетворение основным уравнениям поля линейной теории упругости.

Показано, что динамическую задачу в напряжениях можно рассматривать на некоторым, совершенно различным по сравнению с применяемым в статическом случае, способом. Постулируется некоторая система шести уравнений движения в напряжениях в неоднородных анизотропных и упругих телах. Доказываются теоремы, показывающие зависимости между уравнениями в напряжениях и основной динамической системе поля. Доказывается также теорема однозначности для динамических уравнений в напряжениях. Только уравнение движения в напряжениях, положительно определенная плотность энергии упругой деформаций и соответствующие краевые и начальные условия, выраженные с помощью напряжений полностью определяют задачу. Показано также, что некоторая другая система уравнений движения в напряжениях так наз. обобщенные уравнения в напряжениях Бельтрами-Мичелла, является необходимым, но недостаточным условием для того, чтобы удовлетворить всем динамическим уравнениям поля. Проводится сравнение между трактовкой динамической теории упругости исключительно с точки зрения напряжений и смешанной трактовкой, в случае когда обобщенные уравнения Бельтрами-Мичелла следует рассматривать одновременно с вектором перемещения. Проводится также дискуссия некоторых предыдущих работ, касающихся уравнений движе ния в напряжениях в линейной теории упругости.

Рассматриваются также уравнения движения в напряжениях для плоской деформации и обобщенного плоского напряженного состояния в неоднородном упругом теле. Теория иллюстрируется (в работе [10]) задачей о распространений волн Рэлея в неоднородном, изотропном полупространстве.

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