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ACCOUNTING FOR LODE ANGLE IN THE FAILURE CRITERION OF ROCKS

UWZGLĘDNIENIE KĄTA LODEGO W WARUNKU ZNISZCZENIA SKAŁ

Paper presents general hypothesis of the relationship between the strength of rocks and the third invariant of the stress tensor deviator at failure. It has been assumed that the normal and tangential stresses acting on the strictly determined planes defined by the directional cosines depending on the three invariants of the stress state should be taken into account in the calculation of the rock strength. The hypothesis is the extension of the former Burzyński's work. Geometrical interpretation of the hypothesis has been given and its verification for Sandstone and Dunham Dolomite, on the basis of experimental data known from the relevant literature, has been presented.

Keywords: rock strength, failure hypothesis, characteristic planes

Artykuł przedstawia ogólną hipotezę związku pomiędzy wytrzymałością skał, a trzecim niezmiennikiem dewiatora tensora naprężenia. Założono, że przy obliczeniach wytrzymałości skał należy wziąć pod uwagę naprężenia normalne i styczne działające na ściśle określonych płaszczyznach zdefiniowanych przez cosinusy kierunkowe zależne od trzech niezmienników stanu naprężenia. Hipoteza stanowi rozwinięcie wcześniejszych prac Burzyńskiego. Podano interpretację geometryczną hipotezy oraz jej weryfikację dla Piaskowca i Dunham Dolomite na podstawie danych eksperymentalnych z literatury.

Słowa kluczowe: wytrzymałość skał, hipoteza zniszczenia, płaszczyzny charakterystyczne

1. Introduction

The existing failure criteria take into account various stresses which are assumed to characterize the state of stress essential for rock failure. Among them are Mohr envelope with its linear Coulomb criterion form (the most commonly used one) and parabolic form of Griffith criterion. Although they are based on quite different physical concepts they both take into account exclusively the greatest and the least principal stresses (Jaeger & Cook, 1976). It is well known

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from the literature (Jaeger & Cook, 1976; Paterson, 1978) that laboratory investigations which show that failure of rocks is influenced also by the intermediate principal stress do not confirm these hypotheses. Hence another group of failure hypotheses has been developed which take into account volumetric and distortional strain energy at failure (Schleicher, 1926; Burzyński, 1928, 1929, 1929a) and they are sometimes referred to as the extended Mises criteria (Mogi, 1967), or the extended Griffith criterion (Jaeger & Cook, 1976; Murrell, 1963).

Another attempt has also been made to explain the differences between the distortional strain energy at failure for various states of stress at the same mean pressure (Burzyński, 1928; Akai & Mori, 1970; Bishop, 1966). It has been shown that, in order to obtain a better agreement with experimental results, it is usually necessary to introduce some additional correction in mathematical forms of the above mentioned failure criteria. This has been done either by introducing into mathematical forms of failure criterion an additional parameter which modifies the value of the internal friction coefficient of rock according to the relationship between the intermediate- and maximum-, minimum-principal stresses (in (Akai & Mori, 1970; Bishop, 1966) this is referred to as the extended Coulomb criterion), or by introducing into this expressions an additional function, of the third invariant of the stress tensor deviator, which multiplies octahedral shear stress at failure or by changing "the calculation plane"[1] (Burzyński, 1928, 1929, 1929a) to suitably modify both, normal and shear, octahedral stresses.

In these types of criteria the stresses taken into account are from one "calculation plane" (e.g. octahedral one), independently of the state of stress at failure and independently of the real physical failure plane and the correction is made because stresses on the primarily chosen "calculation plane" do not characterize adequately the strength of rocks. Burzyński criterion (Burzyński, 1928, 1929) is the only one which goes beyond these limits. In this criterion there is also used a single calculation plane but its choice depends on material properties. Questioning the idea of using the same calculation plane to describe material strength in compression, tension or shear (i.e. the idea that always the same combination of principal stresses should be responsible for material disintegration) the attempt was made to extend Burzyński's idea by introducing calculation planes which depend both on the material properties and on the state of stress.

The aim of this paper is to present a hypothesis of failure based on the assumption that the stresses acting on the single, a priori chosen "calculation plane" are not adequate for defining the whole failure surface. As a consequence, in the presented failure hypothesis the set of "calculation planes" is defined. The sections II, III and the Appendix deal with the presented theory and section IV with its verification on the basis of multiaxial strength tests of Sandstone (Akai & Mori, 1970) and Dunham Dolomite (Mogi, 1967).

2. Characteristic planes

For the state of stress given by principal stresses σ_1, σ_2, σ_3 one can calculate the normal σ and tangential τ components of the stress vector $\boldsymbol{v}$ acting on any plane in a stressed material, according to the following formulae:

$$\sigma = \sigma_2 j^2 + \sigma_2 k^2 + \sigma_3 l^2 \tag{1}$$

[1] We note that the idea of a „calculation plane“ depending on the material properties and independent of the state of stress has been suggested by Włodzimierz BURZYŃSKI in (Burzyński, 1928).

$$\tau^2 = (\sigma_1 - \sigma_2)^2 j^2 k^2 + (\sigma_2 - \sigma_3)^2 k^2 l^2 + (\sigma_3 - \sigma_1)^2 l^2 j^2 \tag{2}$$

if only the directional cosines j, k, l of the normal n to this plane, in the space of the principal stresses are known.

Now we assume, that for any state of stress it is possible to find such a plane that the normal (1) and tangential (2) stresses acting on it can be effectively used for the calculations of rock strength. We shall define this plane in terms of the invariants of the state of stress at failure, i.e. the octahedral stresses σ_{oct}, τ_{oct} and the Lode angle ω

$$\omega = \frac{1}{3} \arccos\left(\frac{\sqrt{2}J_3}{\tau_{oct}^3}\right) \tag{3}$$

where J_3 is the third invariant of the stress tensor deviator.[2]

The plane with normal n_p (σ_1, σ_2, σ_3) the directional cosines of which are equal to

$$j_P^2 = \frac{\left(|\sigma_{oct}| - \sqrt{2}P\tau_{oct}\cos\omega_n\right)^2}{3\left(\sigma_{oct}^2 + P^2\tau_{oct}^2\right)} \tag{4}$$

$$k_P^2 = \frac{\left(|\sigma_{oct}| + \sqrt{2}P\tau_{oct}\cos\left(\omega_n + \frac{\pi}{3}\right)\right)^2}{3\left(\sigma_{oct}^2 + P^2\tau_{oct}^2\right)} \tag{5}$$

$$l_P^2 = \frac{\left(|\sigma_{oct}| + \sqrt{2}P\tau_{oct}\cos\left(\omega_n - \frac{\pi}{3}\right)\right)^2}{3\left(\sigma_{oct}^2 + P^2\tau_{oct}^2\right)} \tag{6}$$

is called a **characteristic plane** and the number P is assumed to be a constant for a given rock.

It can be easily seen that for $P \neq 0$ equations (4)-(6) characterize the various planes for different stress states at failure. Hence for further calculations the set of characteristic planes will be taken into account.

The failure hypothesis based on the idea of characteristic planes is presented in the following section.

3. Failure hypothesis

Failure hypotheses can be expressed in terms of normal and tangential stresses acting on the plane that is predetermined on the basis of some assumptions concerning a model of a failure mechanism, e.g. shear mechanism in the Coulomb hypothesis, Griffith's minute cracks or the

[2] Further comments, calculations and geometrical interpretation are given in the Appendix.

dependence of the critical distortional strain energy on the mean pressure, e.g. Burzyński criterion (Burzyński, 1929, 1929a).

Thus these hypotheses may be written in a general form as

$$\tau = f(\sigma) \tag{7}$$

which should be understood as the relations

(e.g. Coulomb and Griffith criteria) $$|\sigma_1 - \sigma_3| = f(\sigma_1 + \sigma_3) \tag{8}$$

or

(e.g. Mises-Schleicher) $$\tau = f(\sigma_{oct}) \tag{9}$$

according to the model of failure mechanism (σ_1, σ_3 are the greatest and the least principal stresses respectively), or

(Burzyński criterion) $$\tau^* = f(\sigma^*) \tag{10}$$

where

$\boldsymbol{\sigma}^* = [\lambda\, \boldsymbol{\sigma_1} + (1 - \lambda)\, \boldsymbol{\sigma_2} + \lambda\, \boldsymbol{\sigma_3}]/(1 + \lambda)$;

$\boldsymbol{\tau}^{*2} = \lambda/(1 + \lambda)^2[(1 - \lambda)(\boldsymbol{\sigma_1} - \boldsymbol{\sigma_2})^2 + (1 - \lambda)(\boldsymbol{\sigma_2} - \boldsymbol{\sigma_3})^{\mathbf{2}} + \lambda(\boldsymbol{\sigma_3} - \boldsymbol{\sigma_1})^2]$

and $1/2 \le \lambda \le 1$ is a number depending on material properties.

It can be seen from equations (8, 9, 10) that directional cosines of the "calculation plane" in the case of Mohr, Coulomb and Griffith criteria are equal to $j^2 = l^2 = 1/2$; $k^2 = 0$, for the extended Mises criteria they are equal to $j^2 = k^2 = l^2 = 1/3$ and for Burzyński criterion $j^2 = l^2 = \lambda/(1 + \lambda)$; $k^2 = (1 - \lambda)/(1 + \lambda)$. So, the directional cosines of "calculation planes" are predetermined and independent of the state of stress at failure.

The proposed idea of characteristic planes enables to formulate a new criterion of rock failure. The great advantage of such a criterion is that we are not predetermining the stress components which should be accounted for in the criterion. The more general rule allowing to calculate the directional cosines of the characteristic planes should be found out instead.

The suggested failure hypothesis can be expressed in the way similar to equation (7), namely

$$\tau_P = f(\sigma_P) \tag{11}$$

where

$$\sigma_P = \sigma_2 j_P^2 + \sigma_2 k_P^2 + \sigma_3 l_P^2 \tag{12}$$

$$\tau_P^2 = (\sigma_1 - \sigma_2)^2 j_P^2 k_P^2 + (\sigma_2 - \sigma_3)^2 k_P^2 l_P^2 + (\sigma_3 - \sigma_1)^2 l_P^2 j_P^2 \tag{13}$$

and j_P^2, k_P^2, l_P^2 are defined by equations (4)-(6) ($_P$ index is used to emphasize that they are dependent not only on the state of stress but also on the constant P) The hypothesis (11) may be rewritten in the following form

$$\tau_P(\sigma_{oct}, \tau_{oct}, \omega) = f[\sigma_P(\sigma_{oct}, \tau_{oct}, \omega) \tag{14}$$

and in the case of $\boldsymbol{\sigma_{oct}}$ = const and P = const it becomes a relation between $\boldsymbol{\tau_{oct}}$ and ω:

$$\tau_P(\tau_{oct}, \omega) = f[\sigma_P(\tau_{oct}, \omega) \tag{15}$$

which, having in mind (3) gives the relation between $\boldsymbol{\tau_{oct}}$ and the third invariant of the stress tensor deviator

$$g(\tau_{oct}, J_3) = 0; \quad \sigma_{oct} = \text{const}, \quad P = \text{const} \tag{16}$$

It can be solved numerically if only a mathematical form of the function *f* is known, but this is not the case under consideration.

Further the presented hypothesis will be called the hypothesis of characteristic planes or, simply, the *P*-hypothesis.

The hypothesis of characteristic planes may now be given in an explicit form as follows: "the various states of stress at failure and the material constant *P* define the set of characteristic planes and for each of the planes the relation (11) between the normal and tangential stresses at failure is the same for a given material".

Therefore, in order to verify the *P*-hypothesis it is necessary to show that the experimental data of the rock strength, represented in the $\boldsymbol{\sigma_P}$, $\boldsymbol{\tau_P}$ coordinates, form one line (not necessarily the straight one) for the adequate constant *P* value. Two examples of such a verification are given in the next section.

4. Verification of the hypothesis of characteristic planes

It is necessary for the experimental verification of the hypothesis of characteristic planes to measure the strength of rock under multiaxial stress conditions. Especially, the angle ω related to the third invariant of the stress tensor deviator should be tested in a wide range.

Although the comparison of the experimental results from triaxial compression (ω = 0) and triaxial extension ($\omega = \pi/3$) tests is sufficient for this verification it is preferable to use also the results of other laboratory tests to make sure that the number *P* is really independent of the state of stress.

The presented verification of the *P*-hypothesis is based on the results of the polyaxial compression tests of Sandstone (Akai & Mori, 1970) and Dunham Dolomite (Mogi, 1967). In both experiments the strength of rock was measured for several different values of the angle ω. The experimental points are represented in various coordinates, namely, in Mohr, in octahedral and in $\boldsymbol{\sigma_P}$, $\boldsymbol{\tau_P}$ coordinates in order to allow the comparison of the hypothesis of characteristic planes with the usually used criteria. Figures 1, 2 and 3 relate to the results of the strength tests of Sandstone and Figs. 4, 5, 6 and 7 are graphical representations of Dunham Dolomite strength tests.

The strength of Sandstone (Akai & Mori, 1970) was measured by the polyaxial compressing of the cubic specimens with edge length 5.5 cm. It is shown in Fig. 1 that there does not exists a single Mohr envelope characterizing the strength of Sandstone. Also, as it is seen in Fig. 2, there is no single relation between the normal and tangential octahedral stresses and, hence, no one form of Mises-Schleicher criterion can describe the strength of Sandstone. An additional

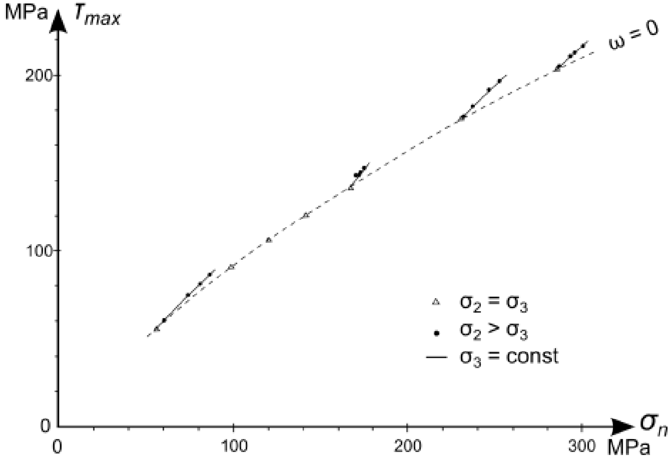

Fig. 1. The relation between shear stress τ_{max} = ″($\sigma_1 - \sigma_3$) and normal stress σ_n = ″($\sigma_1 + \sigma_3$) in Sandstone – results of tests from (Akai & Mori, 1970)

conclusion can be drawn if the P-hypothesis is considered. Namely, an assumption of $P = 0$ in formulae (12, 13) gives the result $\sigma_P = \sigma_{oct}$ and $\tau_P = \tau_{oct}$ but Fig. 2 shows that the octahedral plane cannot be the characteristic one. Therefore number $P \neq 0$ should be used in calculation of the directional cosines of the characteristic planes.

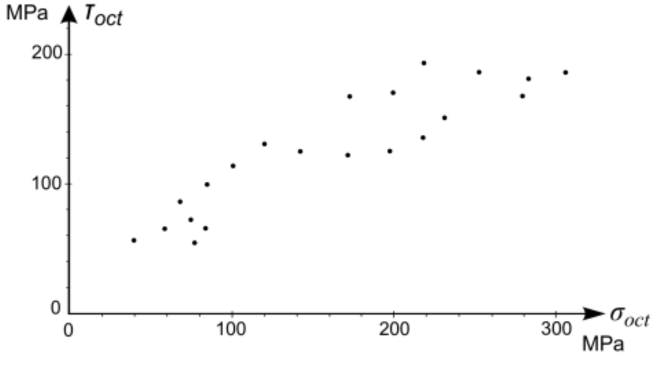

Fig. 2. Failure stress τ_{oct} as a function of the mean stress σ_{oct} for Sandstone – results of tests from (Akai & Mori, 1970)

If the function $\tau_P = f(\sigma_P)$ is not given in an explicit form, as it is in the case of our consideration, it is necessary to make a few calculation steps before the right value of the number P will be find.

These calculations give the value $P = 0.35$ and the corresponding stresses in characteristic planes are shown in Fig. 3. The calculated points are not positioned exactly on one line $\tau_P = f(\sigma_P)$

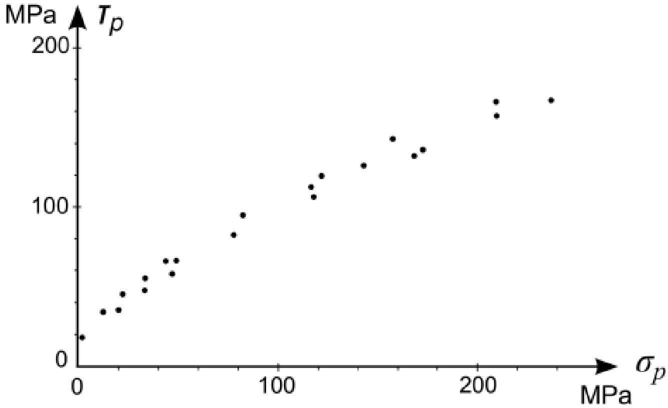

Fig. 3. The relation between shear τ_p and normal σ_p stresses acting on characteristic planes in Sandstone for P = 0,35 – results of tests from (Akai & Mori, 1970)

due to the scattering of the experimental results. The important result is P = const for the whole range of the ω values, and thus P may be assumed to be the material constant.

The strength of Dunham Dolomite (Mogi, 1967) was measured on cylindrical specimens in triaxial compression ($\omega = 0$) and triaxial extension ($\omega = \pi/3$) tests and in tests of biaxial compression of rectangular prisms ($0 \leq \omega \leq \pi/3$). The results shown in Fig. 4 in Mohr coordinates suggest that one Mohr envelope can be used as an acceptable approximation of the failure criterion but it is only an accidental coincidence of the results. Namely Mogi (1967) has shown that the intermediate principal stress σ_2 in biaxial compression tests ($\sigma_3 = 0$) has the great influence on the strength of Dunham Dolomite – Fig. 5 – so the Mohr theory is not valid.

In Fig. 6 stresses τ_{oct} are plotted versus σ_{oct} and the influence of the third invariant of the stress tensor deviator on the strength of Dunham Dolomite is easily seen. Much better results for

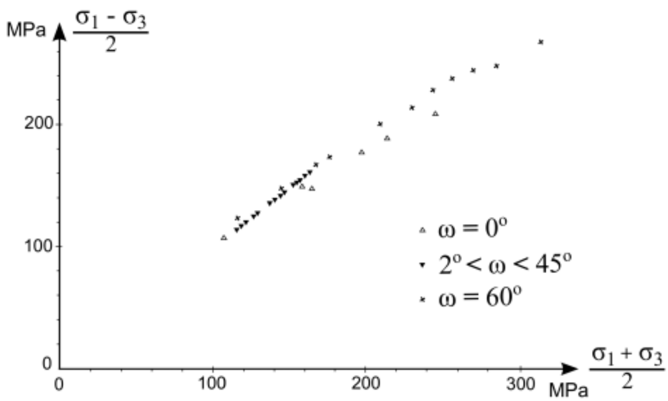

Fig. 4. The relation between shear stress τ_{max} = ″($\sigma_1 - \sigma_3$) and normal stress σ_n = ″($\sigma_1 + \sigma_3$) for Dunham Dolomite – results of tests from (Mogi, 1967)

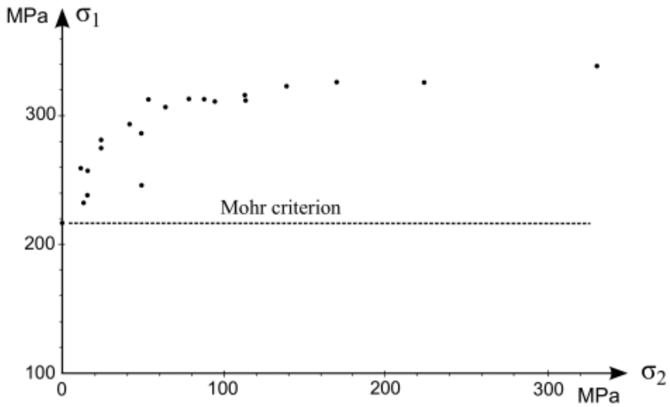

Fig. 5. Effect of the intermediate principal stress $\boldsymbol{\sigma_2}$ on the strength of Dunham Dolomite (Mogi, 1967)

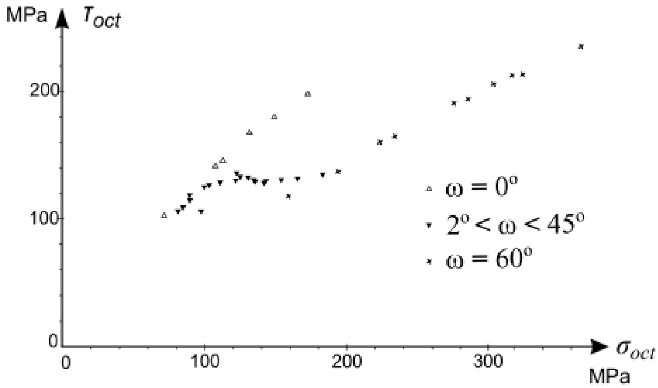

Fig. 6. Failure stress $\boldsymbol{\tau_{oct}}$ as a function of the mean stress $\boldsymbol{\sigma_{oct}}$ for Dunham Dolomite – results of tests from (Mogi, 1967)

various ω one can get if the P-hypothesis is used. For $P = 0.43$ the normal and tangential stress components on the characteristic planes follow the relation $\boldsymbol{\tau_P} = f(\boldsymbol{\sigma_P})$ with a better accuracy as it is shown in Fig. 7.

Again, as in the case of the strength of Sandstone, the number P may be accepted as the material constant.

Analogous verification can be done for other rock materials reported in the relevant literature (e.g. Carrara Marble, Westerly Granite) but the rock strength is usually measured in triaxial extension and compression tests only e.g. in the case of LGOM Sandstone and Dolomite (Cieślik, 2007) or Flysh Sanstone (Łukaszewski, 2007). Thus the experimental data represent only two ω values and can be recognized as incomplete for the verification of the P-hypothesis so they are not presented in the paper although the preliminary results of verification are also promising. Nowadays research in this field is much wider than previous works thanks to availability of the

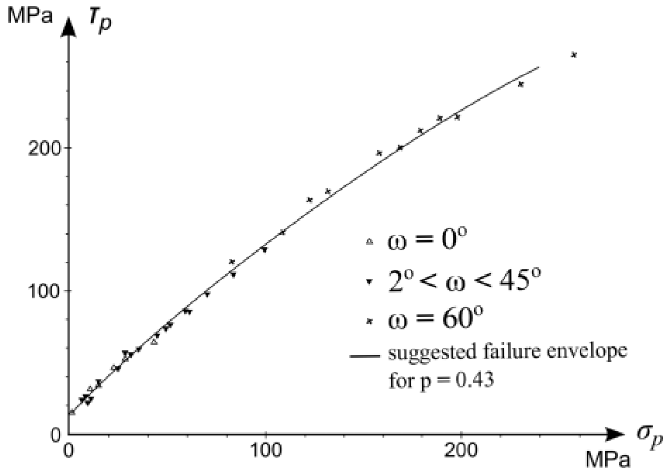

Fig. 7. The relation between shear $\boldsymbol{\tau_p}$ and normal $\boldsymbol{\sigma_p}$ stresses acting on characteristic planes in Dunham Dolomite for P = 0,43 – results of tests from (Mogi, 1967)

better testing equipment. However the results are usually presented in the form of elaborated graphs and general conclusions with lack of information about the raw measured data of $\boldsymbol{\sigma_1}$, $\boldsymbol{\sigma_2}$, $\boldsymbol{\sigma_3}$ (Cieślik, 2007; Kwaśniewski, 2007; Lis & Kijewski, 2007). Without such a raw data further verification of the P-hypothesis cannot be done.

5. Conclusions

Presented failure hypothesis follows Burzyński's idea of variable volumetric-distortional strain energy at failure where calculation planes depend on material properties. It also takes into account experimental facts that relation between volumetric and distortional strain energy depends both on the material properties and on the stress state including Lode angle. The suggested rock failure hypothesis $\boldsymbol{\tau_P} = f(\boldsymbol{\sigma_P})$, with an aid of the additional material constant P, quantitatively evaluates the influence of the third invariant of the stress tensor deviator on the strength of Sandstone and Dunham Dolomite as measured in the laboratory tests.

The main idea of the hypothesis is that of "characteristic planes" i.e. the planes characterized by the state of stress at failure and the material properties.

Although no assumption has been made concerning the phenomenological mechanism underlying the failure of rocks e.g. cleavage, slip, the phenomenological model is also given. Namely, it is assumed that it is always the same relation between the components of the stress vector acting on any characteristic plane. As a consequence, for two different states of stress and two adequate characteristic planes with normal stress $\boldsymbol{\sigma_P}^*$ acting on them, the failure should take place under the same tangential stress $\boldsymbol{\tau_P}^*$ on the both characteristic planes. For a convenience it can be understood as an analogy to the Mohr hypothesis – with the great difference that in this case one has a set of planes uniquely determined by the stress tensor invariants instead of the only one plane (the plane of the extreme principal stresses).

The characteristic planes have also a simple geometrical interpretation in the space of principal stresses. For the stress vector $\boldsymbol{v}(\sigma_1, \sigma_2, \sigma_3)$ the characteristic plane has its normal vector $\boldsymbol{n}_p(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ always coplanar with the stress vector **v** and the mean pressure axis.

The general formula of the *P*-hypothesis used in this paper has been chosen for discussion instead of more explicit mathematical forms because it has been intended only to show that the system of characteristic planes can be used to describe the strength of rock material. For the calculation of the rock strength for engineering applications it would be necessary to specify the mathematical form of the criterion (linear or non-linear one) based on the idea of the characteristic planes.

The *P*-hypothesis should also be regarded as another approach to the failure of rocks rather than the solution of the problem of rock failure prediction, although the given examples of its experimental verification for more than 50 experimental points for various states of stress yields much better results than those for others hypothesis.

Presented verification of the *P*-hypothesis is based on the experimental data for rock samples having, for a given rock, the same geometrical orientation. Thus the measured data cannot reveal possible rock anisotropy and hence *P*-hypothesis was verified under assumption of rock isotropy which is not adequate in general case. Recent theoretical studies (Ostrowska-Maciejewska et al., 2011) on material effort and limit condition for anisotropic materials with asymmetric elastic range show that it is possible to use the third invariants of the stress tensor deviator projected onto one of energetically orthogonal subspaces of the stress space to make the qualitative distinction between various deviators belonging to the same subspace. It is connected with the notion of an “abstract angle”, which corresponds to Lode angle in case of isotropic materials. However no attempt has been made to generalize the *P*-hypothesis to anisotropic materials yet. Some guidelines how to approach this problem can be found in the papers of Pęcherski et al. (2011) and Nowak et al. (2011).

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APPENDIX
Geometrical interpretation of the characteristic planes

In order to introduce the idea of characteristic planes let us consider the geometrical interpretation of the state of stress in the space of the principal stresses. In Fig. A1 the state of stress is represented by the stress vector ***v*** from the origin *O* to the point *A* and its projections onto the principal stress axes are equal to:

$$\sigma_1 = |v| \cos \alpha_A \tag{A1}$$

$$\sigma_2 = |v| \cos \beta_A \tag{A2}$$

$$\sigma_3 = |v| \cos \gamma_A \tag{A3}$$

where $\boldsymbol{\sigma_1}$, $\boldsymbol{\sigma_2}$, $\boldsymbol{\sigma_3}$ denote the values of the greatest, intermediate and the least principal stresses respectively and α_A, β_A, γ_A are the angles as shown in Fig. A1 (compressive stresses are taken as positive).

The stresses $\boldsymbol{\sigma_1}$, $\boldsymbol{\sigma_2}$, $\boldsymbol{\sigma_3}$ can also be expressed in terms of invariants of the stress tensor:

$$\sigma_1 = \frac{1}{3} I_1 + \frac{2}{\sqrt{3}} J_2^{1/2} \cos\left[\frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right)\right] \tag{A4}$$

$$\sigma_2 = \frac{1}{3} I_1 - \frac{2}{\sqrt{3}} J_2^{1/2} \cos\left[\frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right) + \frac{\pi}{3}\right] \tag{A5}$$

$$\sigma_3 = \frac{1}{3} I_1 - \frac{2}{\sqrt{3}} J_2^{1/2} \cos\left[\frac{1}{3} \arccos\left(\frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}\right) - \frac{\pi}{3}\right] \tag{A6}$$

where:

$$I_1 = {}^3\sigma_{oct} = \sigma_1 + \sigma_2 + \sigma_3 \tag{A7}$$

is the first invariant of the stress tensor,

$$J_2 = \frac{3}{2} \tau_{oct}^2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{A8}$$

is the second invariant of the stress tensor deviator and

$$J_3 = \frac{1}{27}(2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_3 - \sigma_1)(2\sigma_3 - \sigma_1 - \sigma_2) \tag{A9}$$

is the third invariant of the stress tensor deviator.

In formulae (A4)-(A6) it is convenient to introduce the octahedral stresses $\boldsymbol{\sigma_{oct}}$, $\boldsymbol{\tau_{oct}}$ and the Lode angle ω shown in Fig. A1 and given by the following relation:

$$\cos 3\omega = \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}} = \sqrt{2} \frac{J_3}{\tau_{oct}^3} \tag{A10}$$

Thus the stresses $\boldsymbol{\sigma_1}$, $\boldsymbol{\sigma_2}$, $\boldsymbol{\sigma_3}$ are equal to:

$$\sigma_1 = \sigma_{oct} + \sqrt{2}\tau_{oct} \cos\omega \tag{A11}$$

$$\sigma_2 = \sigma_{oct} - \sqrt{2}\tau_{oct} \cos\left(\omega + \frac{\pi}{3}\right) \tag{A12}$$

$$\sigma_3 = \sigma_{oct} - \sqrt{2}\tau_{oct} \cos\left(\omega - \frac{\pi}{3}\right) \tag{A13}$$

and directional cosines of the vector $\boldsymbol{v}(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{\sigma_3})$ are:

$$\cos^2\alpha_A = \frac{(\sigma_{oct} + \sqrt{2}\tau_{oct} \cos\omega)^2}{3(\sigma_{oct}^2 + \tau_{oct}^2)} \tag{A14}$$

$$\cos^2\beta_A = \frac{\left(\sigma_{oct} - \sqrt{2}\tau_{oct} \cos\left(\omega + \frac{\pi}{3}\right)\right)^2}{3(\sigma_{oct}^2 + \tau_{oct}^2)} \tag{A15}$$

$$\cos^2\gamma_A = \frac{\left(\sigma_{oct} - \sqrt{2}\tau_{oct}\cos\left(\omega - \frac{\pi}{3}\right)\right)^2}{3\left(\sigma_{oct}^2 + \tau_{oct}^2\right)} \tag{A16}$$

We may find at the point *O* of the stressed material a plane Pc (Fig. A2) on which the normal and tangential stresses $\boldsymbol{\sigma_P}$, $\boldsymbol{\tau_P}$ can be found from the relations:

$$\sigma_P = \sigma_1\cos^2\alpha + \sigma_2\cos^2\beta + \sigma_3\cos^2\gamma \tag{A17}$$

$$\tau_P^2 = (\sigma_1 - \sigma_2)^2\cos^2\alpha\cos^2\beta + (\sigma_2 - \sigma_3)^2\cos^2\beta\cos^2\gamma + (\sigma_3 - \sigma_1)^2\cos^2\gamma\cos^2\alpha \tag{A18}$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the directional cosines of the normal $\boldsymbol{n_p}$ with respect to the directions of the principal stresses. Next we suppose that to characterize the strength of the material the stresses $\boldsymbol{\sigma_P}$, $\boldsymbol{\tau_P}$ should be used rather than the stresses $\boldsymbol{\sigma_{oct}}$, $\boldsymbol{\tau_{oct}}$ or the stresses $\boldsymbol{\sigma_n}$, $\boldsymbol{\tau_{oct}}$.

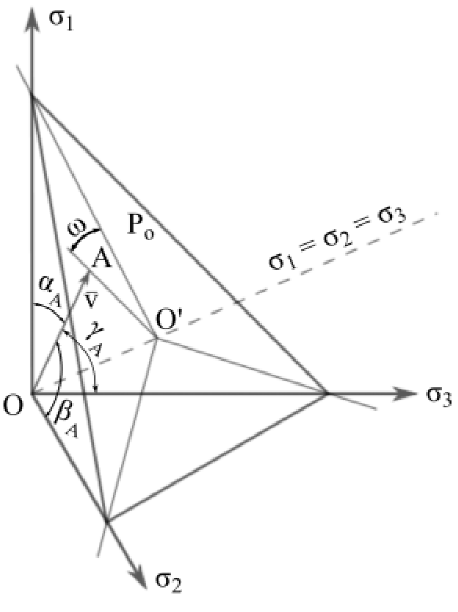

Fig. A1. Geometrical interpretation of the state of stress in the principal stress space – plane P_o is the octahedral one

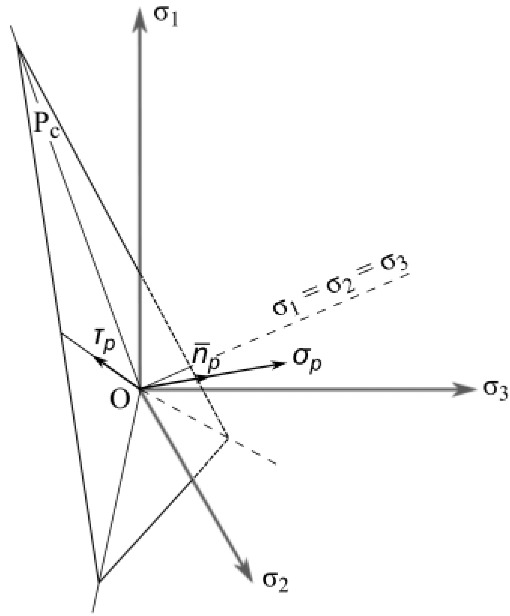

Fig. A2. Characteristic plane P_c with the stresses $\boldsymbol{\tau_p}$, $\boldsymbol{\sigma_p}$ and normal $\boldsymbol{n_p}$

In order to find the adequate stresses $\boldsymbol{\sigma_P}$, $\boldsymbol{\tau_P}$ we assume that the normal $\boldsymbol{n_p}$ is coplanar with the stress vector **v** and the mean pressure axis so that:

$$\omega_n = \omega + k\pi; \quad k = 0.1 \tag{A19}$$

where ω_n is the angular coordinate of the vector $\boldsymbol{n_p}$ measured on the octahedral plane in the same manner as the Lode angle ω defined by (A10) for the tress vector $\boldsymbol{v}$ (Fig. A3).

For further calculations it is convenient to take advantage of the following formulae:

$$\arccos\{[1,1,1], n_p\} \leq \frac{\pi}{2} \tag{A20}$$

$$|OO'| = \sqrt{3}\,|\sigma_{oct}| \tag{A21}$$

where the absolute value $|\boldsymbol{\sigma_{oct}}|$ follows from (A20), and to scale |O'B| in terms of $\boldsymbol{\tau_{oct}}$

$$|O'B| = P|O'A| = \sqrt{3}P\tau_{oct} \tag{A22}$$

where P is the number that is assumed to depend on the rock material (Fig. A4).

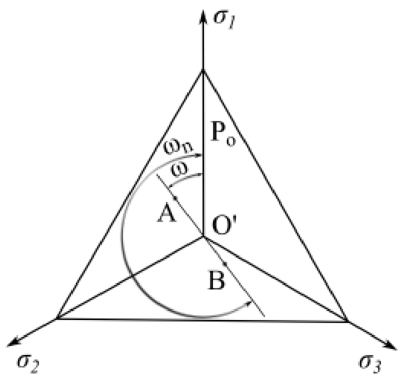

Fig. A3. Geometrical interpretation of equation (A19)

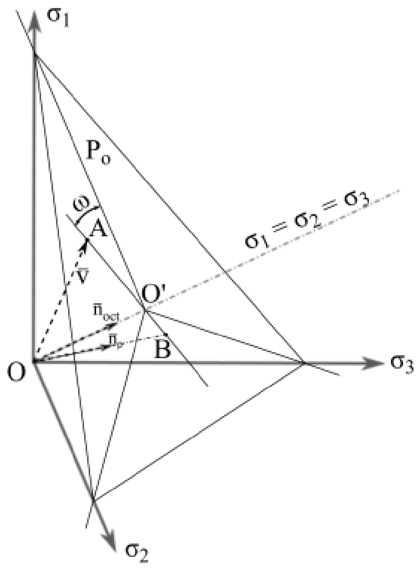

Fig. A4. Geometrical interpretation of the vector $\boldsymbol{n_p}$ normal to the characteristic plane and its relation to the stress vector $\boldsymbol{v}$ in the principal stress space

This allows to calculate the directional cosines of the vector $\boldsymbol{n_p}$, collinear with the vector **OB**, as functions of the number P and the invariants of the state of stress. Analogically to the expressions (A14)-(A16) the directional cosines of **OB** are equal to:

$$\cos^2\alpha = \frac{\left(|\sigma_{oct}| + \sqrt{2}P\tau_{oct}\cos\omega_n\right)^2}{3\left(\sigma_{oct}^2 + P^2\tau_{oct}^2\right)} \tag{A23}$$

$$\cos^2\beta = \frac{\left(|\sigma_{oct}| - \sqrt{2}P\tau_{oct}\cos\left(\omega_n + \frac{\pi}{3}\right)\right)^2}{3\left(\sigma_{oct}^2 + P^2\tau_{oct}^2\right)} \tag{A24}$$

$$\cos^2\gamma = \frac{\left(|\sigma_{oct}| - \sqrt{2}P\tau_{oct}\cos\left(\omega_n - \frac{\pi}{3}\right)\right)^2}{3\left(\sigma_{oct}^2 + P^2\tau_{oct}^2\right)} \tag{A25}$$

Moreover, taking into account general observations of fracture planes of rocks in various laboratory and field tests it has been decided to chose $\omega_n = \omega + \pi$. In this case the inclination of $\boldsymbol{n_p}$ with respect to the greatest principal stress direction is similar to the observed analogues inclination of the normal to the fracture plane.[3]

Hence, the directional cosines of the vector $\boldsymbol{n_p}$ become equal to:

$$\cos^2\alpha = \frac{\left(|\sigma_{oct}| - \sqrt{2}P\tau_{oct}\cos\omega\right)^2}{3\left(\sigma_{oct}^2 + P^2\tau_{oct}^2\right)} = j_P^2 \tag{A26}$$

$$\cos^2\beta = \frac{\left(|\sigma_{oct}| + \sqrt{2}P\tau_{oct}\cos\left(\omega + \frac{\pi}{3}\right)\right)^2}{3\left(\sigma_{oct}^2 + P^2\tau_{oct}^2\right)} = k_P^2 \tag{A27}$$

$$\cos^2\gamma = \frac{\left(|\sigma_{oct}| + \sqrt{2}P\tau_{oct}\cos\left(\omega - \frac{\pi}{3}\right)\right)^2}{3\left(\sigma_{oct}^2 + P^2\tau_{oct}^2\right)} = l_P^2 \tag{A28}$$

which is the same as formulae (4)-(6) defining the characteristic planes.

As a consequence of the formulae (A26)-(A28) two important features of the idea of characteristic planes should be pointed out:

1. For standard tests the following relations between the directional cosines of the normal to the characteristic plane are true for $P \neq 0$

$\cos\alpha < \cos\beta < \cos\gamma$ for polyaxial compression $(\sigma_1 > \sigma_2 > \sigma_3)$ (A29)

$\cos\alpha < \cos\beta + \cos\gamma$ for the triaxial (and uniaxial) compression $(\sigma_1 > \sigma_2 = \sigma_3)$ (A30)

$\cos\alpha = \cos\beta < \cos\gamma$ for the triaxial (and uniaxial extension) $(\sigma_1 = \sigma_2 > \sigma_3)$ (A31)

$-\cos\alpha = \cos\gamma \neq \cos\beta$ for pure shear $(\sigma_1 = -\sigma_3;\ \sigma_2 = 0)$ (A32)

[3] Usually fracture surface observed in a macro-scale is called "fracture plane" although it is not necessarily a plane one, so we will also follow this convention. Thus, it is not supposed that the vector $\boldsymbol{n_p}$ is identical with the normal to the fracture plane.

Hence, the phenomenological model implies that if the principal stresses at failure have the same value, the inclinations of the vector $\boldsymbol{n_p}$ with respect to the related principal stress axes are identical and that these principal stresses equally contribute to the normal and tangential stresses, $\boldsymbol{\sigma_P}$ and $\boldsymbol{\tau_P}$, acting on the characteristic plane. This is the important difference between the P-hypothesis and the other failure hypotheses used in rock mechanics, because for example in the case of triaxial compression the intermediate principal stress $\boldsymbol{\sigma_2}$ has exactly the same influence on the strength of a rock as the minimum principal stress $\boldsymbol{\sigma_3}$. This is a reasonable conclusion as for the experiment in which $\boldsymbol{\sigma_2}$ can be replaced by $\boldsymbol{\sigma_3}$, and vice versa, depending upon the decision of the experimentalist.

2. For $P = 0$ the directional cosines of $\boldsymbol{n_p}$ become equal

$$j_P^2 = k_P^2 = l_P^2 = 1/3 \tag{A33}$$

and hence the octahedral plane becomes the characteristic one, independently of the state of stress and the relation $\boldsymbol{\tau_P} = f(\boldsymbol{\sigma_P})$ becomes the relation $\boldsymbol{\tau_{oct}} = f(\boldsymbol{\sigma_{oct}})$ as in the case of Mises-Schleicher or Huber-Mises criteria.