# ON THE THEORY OF STEADY PLASTIC CYCLES IN STRUCTURES

Z. MRÓZ,

*Polish Academy of Sciences,*
*Institute of Fundamental Problems of Technology, Warsaw, Poland*

ABSTRACT

Steady plastic cycles are shown to constitute limit states of transitory processes of plastic deformation under cyclically varying loads and external temperature. Uniqueness of steady cycles is discussed and it is shown that superposed fixed displacements on the structure cannot affect the form of the steady cycle. Some extensions of Melan's shake-down theorem are shown for hardening materials and for the case of partially elastic shake-down. Starting from the notion of statically and kinematically admissible cycles, some inequalities are proved; these may be useful in finding an approximate solution to the steady cycle.

## 1. INTRODUCTION

When metals are subjected to monotonically increasing loading beyond the elastic range, they usually harden; however, during cyclic stressing or straining the metals may harden or soften depending on their initial state. When cyclic loading is applied to the annealed state, the material hardens during cyclic straining; on the other hand, metals initially hardened by previous plastic deformation may soften. In general, there exists a steady limiting state to which cycles are tending during initial transitory period. For a large class metals and other structural materials this state does not seem to depend upon initial state or previous deformation history and for a given material it is uniquely defined by prescribed cyclic loads or displacements. Such steady plastic cycles have been investigated experimentally, mostly for uniaxial or biaxial stress states, cf., for instance, ref. [1-6].

Becouse the stady cyclic state sets in befor failure phenomenon occurs, its properties are fundamental in establishing proper low-cycle fatigue criteria. Furthermore, the possibility of finding a solution for steady cyclic state without following the whole previous deformation history can greatly facilitate the analysis of structures subjected to cyclic loads. In the next section, we shall discuss the uniqueness of steady cycles and in Section 4 some inequalities will be derived which may be helpful in finding the approximate solution. In Section 3, two extensions of Melan's shakedown theorem are presented. The considered material will be regarded as a composition of elastic, viscous, and perfectly plastic or hardening elements.

The existence of steady cycles independent of the initial state, exhibits the phenomenon of fading memory /or asymptotic stability / in dissipative media. However, this is not a general property of all workhardening materials. It is known, for instance, that some alloys are sensitive to previous deformation history during cyclic loading. Therefore the phenomenological approach based upon the assunption that the material is a composition of elastic-plastic and viscous elements may not simulate fundamental properties of some workhardening materials during cyclic loading.

## 2. UNIQUENESS THEOREM FOR STEADY CYCLES

### i/ Convergence of transitory processes to steady cycles

Let the body be composed of linearly elastic and viscoplastic, linearly hardening elements according to the rule of kinematic hardening. In this case, the yield surface translates in the stress space like a rigid body; for the stress point lying on the yield surface, the instantaneous translation is assumed to occur in the direction of the outward normal at this point. For the associated flow law, we have

$$f(\underset{\sim}{\sigma}-\underset{\sim}{\alpha})-\sigma_0=0\;, \qquad \overset{\circ}{\underset{\sim}{\alpha}}=c\,\dot{\underset{\sim}{\varepsilon}}^{p}, \qquad /1/$$

where $c$ and $\sigma_0$ are material constants of the element. This model of hardening, first introduced my Melan /10/, was later discussed by numerous authors. It was also recently applied in numerical calculations of plastic cycles in structures by Armen, Isakson and Pifko /13/. Our analysis will later be generalized by considering an arbitrary form of the stress-strain curve and assuming that the element is a composition of kinematically hardenin sub-elements.

Viscous strain rates $\dot{\underset{\sim}{\varepsilon}}^{v}$ and the temperature $\theta$ define a potential function $D_v(\dot{\underset{\sim}{\varepsilon}}^{v},\theta)$ , such that

$$\underset{\sim}{\sigma} = \frac{\partial D_v(\dot{\underset{\sim}{\varepsilon}}^v, \theta)}{\partial \dot{\underset{\sim}{\varepsilon}}^v} , \qquad /2/$$

and satisfying the convexity condition

$$D_v(\dot{\underset{\sim}{\varepsilon}}_2^v, \theta) - D_v(\dot{\underset{\sim}{\varepsilon}}_1^v, \theta) - (\dot{\underset{\sim}{\varepsilon}}_2^v - \dot{\underset{\sim}{\varepsilon}}_1^v) \cdot \frac{\partial D_v(\dot{\underset{\sim}{\varepsilon}}_1^o, \theta)}{\partial \dot{\underset{\sim}{\varepsilon}}_1^v} > 0 \qquad /3/$$

for any pair of strain rates $\dot{\underset{\sim}{\varepsilon}}_2^v$ and $\dot{\underset{\sim}{\varepsilon}}_1^v$ and the fixed temperature $\theta$ . This condition can alternatively be written in the form

$$(\underset{\sim}{\sigma}_2 - \underset{\sim}{\sigma}_1) \cdot (\dot{\underset{\sim}{\varepsilon}}_2^v - \dot{\underset{\sim}{\varepsilon}}_1^v) > 0 , \qquad /4/$$

where $\underset{\sim}{\sigma}_2$ and $\underset{\sim}{\sigma}_1$ are related to $\dot{\underset{\sim}{\varepsilon}}_2^v$ and $\dot{\underset{\sim}{\varepsilon}}_1^v$ by /2/. Similarly, for a convex yield surface and the associated flow law, by the principle of maximum plastic work, we have

$$(\underset{\sim}{\sigma}_2' - \underset{\sim}{\sigma}_1') \cdot (\dot{\underset{\sim}{\varepsilon}}_2^p - \dot{\underset{\sim}{\varepsilon}}_1^p) > 0 , \qquad /5/$$

where $\underset{\sim}{\sigma}_1' = \underset{\sim}{\sigma}_1 - \underset{\sim}{\alpha}_1$ , $\underset{\sim}{\sigma}_2' = \underset{\sim}{\sigma}_2 - \underset{\sim}{\alpha}_2$ are stress vectors issuing from the centers of the yield surfaces corresponding to the arbitrary states $\underset{\sim}{\sigma}_1$, $\dot{\underset{\sim}{\varepsilon}}_1^p$ and $\underset{\sim}{\sigma}_2$, $\dot{\underset{\sim}{\varepsilon}}_2^p$ /see Fig. 1a/. The dot between two symbols denotes the scalar product and vector notation is used for the second order tensors. The inequality /5/ implies that there exists a plastic dissipation function $D_p(\dot{\underset{\sim}{\varepsilon}}^p, \theta)$ which defines the effective stresses $\underset{\sim}{\sigma}'$ by the potential law

$$\underset{\sim}{\sigma}' = \frac{\partial D_p(\dot{\underset{\sim}{\varepsilon}}^p, \theta)}{\partial \dot{\underset{\sim}{\varepsilon}}^p} .$$

We assume that at any point the total strain is a sum of elastic, plastic, viscous and thermal components

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^e + \underset{\sim}{\varepsilon}^p + \underset{\sim}{\varepsilon}^v + \underset{\sim}{\varepsilon}^\theta \qquad /6/$$

where elastic strains are linearly related to stresses and the specific elastic energy is a quadratic, positive definite function of strains. It is assumed that thermal strain is defined by the prescribed temperature field and temperature variation due to internal dissipation and thermoelastic coupling is neglected. The body can be inhomogeneous; thus plastic, elastic and viscous properties can vary from element to element. The deformations are assumed to be small and geometry changes do not affect the equilibrium equations.

One of general properties of such model is convergence of any two deformation processes starting from different initial states, but induced by the same boundary tractions and displacements. This question was previously discussed for an elastic, perfectly plastic material by Mróz /11/. Now, we shall generalize somewhat the analysis by considering dynamical processes and assuming more general material model.

Consider any two deformation processes A and B corresponding to the same loading and deformation programmes defined by specifying the surface tractions and displacements on the portions $S_T$ and $S_u$ of the boundary as functions of time. The initial states from which these two processes start for $t=t_0$ are different since one of the bodies was previously subjected to another plastic deformation process or heat treatment, so that residual stress distributions before initiating the processes A and B are different. As a measure of distance apart of these two processes at any instant $t>t_0$, the following positive-definite quantity is assumed

$$E = \int \left[\tfrac{1}{2}(\underset{\sim}{\sigma}_2-\underset{\sim}{\sigma}_1)\cdot(\underset{\sim}{\varepsilon}_2^e-\underset{\sim}{\varepsilon}_1^e) + \tfrac{1}{2}c(\underset{\sim}{\varepsilon}_2^p-\underset{\sim}{\varepsilon}_1^p)^2 + \tfrac{1}{2}\rho(\dot{\underset{\sim}{u}}_2-\dot{\underset{\sim}{u}}_1)^2\right] \qquad /7/$$

where $\underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2$ and $\underset{\sim}{\varepsilon}_1^e, \underset{\sim}{\varepsilon}_2^e$ are stresses and elastic strains for the processes A and B, interrelated by the Hookes law; $\underset{\sim}{\varepsilon}_1^p, \underset{\sim}{\varepsilon}_2^p$ are the corresponding plastic strains and $\dot{\underset{\sim}{u}}_1, \dot{\underset{\sim}{u}}_2$ are the rates of displacements. The first term of /2.7/ represents the elastic energy $U(\underset{\sim}{\varepsilon}_2^e-\underset{\sim}{\varepsilon}_1^e)$ of the strain difference and the second term can be interpreted as the elastic energy of residual microstresses $U_r(\underset{\sim}{\varepsilon}_2^p-\underset{\sim}{\varepsilon}_1^p)$ within the element; the last term in /2.7/ represents the kinetic energy $K(\dot{\underset{\sim}{u}}_2-\dot{\underset{\sim}{u}}_1)$ corresponding to the difference $\dot{\underset{\sim}{u}}_2-\dot{\underset{\sim}{u}}_1$ and $\rho$ is the specific material density. The time derivative of E equals

$$\dot{E} = \int\left[(\underset{\sim}{\sigma}_2-\underset{\sim}{\sigma}_1)\cdot(\dot{\underset{\sim}{\varepsilon}}_2^e-\dot{\underset{\sim}{\varepsilon}}_1^e) + c(\underset{\sim}{\varepsilon}_2^p-\underset{\sim}{\varepsilon}_1^p)\cdot(\dot{\underset{\sim}{\varepsilon}}_2^p-\dot{\underset{\sim}{\varepsilon}}_1^p) + \rho(\dot{\underset{\sim}{u}}_2-\dot{\underset{\sim}{u}}_1)(\ddot{\underset{\sim}{u}}_2-\ddot{\underset{\sim}{u}}_1)\right]dV \qquad /8/$$

Since the two processes proceed for the same surface tractions and displacements, we have

$$\int(\underset{\sim}{\sigma}_2-\underset{\sim}{\sigma}_1)(\dot{\underset{\sim}{\varepsilon}}_2-\dot{\underset{\sim}{\varepsilon}}_1)\,dV + \int\rho(\ddot{\underset{\sim}{u}}_2-\ddot{\underset{\sim}{u}}_1)\cdot(\dot{\underset{\sim}{u}}_2-\dot{\underset{\sim}{u}}_1)\,dV = 0 \qquad /9/$$

Substituting /7/ into /6/ and accounting for /.1/, and /4/, we have

$$\dot{E} = -\int\left[(\underset{\sim}{\sigma}_2-\underset{\sim}{\alpha}_2)-(\underset{\sim}{\sigma}_1-\underset{\sim}{\alpha}_1)\right]\cdot(\dot{\underset{\sim}{\varepsilon}}_2^p-\dot{\underset{\sim}{\varepsilon}}_1^p)\,dV_p - \int(\underset{\sim}{\sigma}_2-\underset{\sim}{\sigma}_1)\cdot(\dot{\underset{\sim}{\varepsilon}}_2^v-\dot{\underset{\sim}{\varepsilon}}_1^v)\,dV_v \qquad /10/$$

where $V_p$ and $V_v$ denote the volumes of regions where plastic and viscous flow occurs, respectively. In view of /4/ and /5/, it is easy to see that both integrands in /8/ are positive-definite for all $\dot{\underset{\sim}{\varepsilon}}_2^p \neq \dot{\underset{\sim}{\varepsilon}}_1^p$ and $\dot{\underset{\sim}{\varepsilon}}_2^v \neq \dot{\underset{\sim}{\varepsilon}}_1^v$. Thus $\dot{E}<0$ and the processes A and B starting from their initial states will approach each other so that the assumed measure of distance $E$ will diminuish. For a final state which can be reached asymptotically, there is $\dot{E}=0$ and $\dot{\underset{\sim}{\varepsilon}}_2^p=\dot{\underset{\sim}{\varepsilon}}_1^p$, $\dot{\underset{\sim}{\varepsilon}}_2^v=\dot{\underset{\sim}{\varepsilon}}_1^v$. For regular functions of viscous and plastic dissipation this implies that $\underset{\sim}{\sigma}_2=\underset{\sim}{\sigma}_1$ within the viscous and plastic regions. On the other hand, the plastic deformations and hence the total translations $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2$ of the yield surfaces and the stress states within the elastic regions can differ. Thus in the space of external loads or displacements the two processes in the final state need not coincide but can proceed parallelly /fig. 1b/.

Suppose now that the surface loads are prescribed in the form of repeating cycles. We can assume that the process A corresponds to the instant t of loading and a set of processes B to the instant $t+nT$ , where $T$ denotes the period of the loading and temperature cycle, and n is an arbitrary integer. Since for A and B the external loads vary identically in time, the transitory process will tend to a steady cycle; this cycle is characterized by the property that for any $t$ and $t+nT$ the fields of stress and irreversible strain rate are the same within both viscous and plastic regions.

ii/ Uniqueness of limiting cycles

From the previous analysis it follows that steady cycles exist as limiting states for bodies subjected to cyclically varying loads. The uniqueness of steady cycles can also be deduced from this analysis. In fact, $\dot{E}=0$ only when the fields $\dot{\underset{\sim}{\varepsilon}}^p, \dot{\underset{\sim}{\varepsilon}}^v$ and hence the stress field $\underset{\sim}{\sigma}$ are uniquely defined within the plastic and viscous regions and do not depend on previous deformation history; similarly, the displacement rates $\dot{\underset{\sim}{u}}$ are also unique throughout the whole body. However, the stress distribution within the elastic region and the position of the centre of the yield surface is not uniquely defined in the steady state. For $\dot{E}=0$ , let us integrate /8/ from $t_1$ to $t_1+T$ and suppose that labels 1 and 2 refer to any two supposed solutions for the steady cycle. Since the stress distribution at the instants $t_1$ and $t_1+T$ should be the same at each point of the body, we have

$$\int_{t_1}^{t_1+T} (\underset{\sim}{\sigma}_2-\underset{\sim}{\sigma}_1)\cdot(\dot{\underset{\sim}{\varepsilon}}_2^e-\dot{\underset{\sim}{\varepsilon}}_1^e)\, dt = 0, \qquad /11/$$

and

$$\frac{1}{2}c\int_{t_1}^{t_1+T} \frac{d}{dt}\left(\dot{\underset{\sim}{\varepsilon}}_2^p-\dot{\underset{\sim}{\varepsilon}}_1^p\right)^2 dt + \frac{1}{2}\rho\int_{t_1}^{t_1+T} \frac{d}{dt}(\dot{\underset{\sim}{u}}_2-\dot{\underset{\sim}{u}}_1)(\dot{\underset{\sim}{u}}_2-\dot{\underset{\sim}{u}}_1)\, dt = \qquad /12/$$
$$= \frac{1}{2c}\left[(\underset{\sim}{\alpha}_2-\underset{\sim}{\alpha}_1)^2_{t=t_1+T} - (\underset{\sim}{\alpha}_2-\underset{\sim}{\alpha}_1)^2_{t=t_1}\right] = 0$$

since $\dot{\underset{\sim}{u}}_2=\dot{\underset{\sim}{u}}_1$ . From /12/ it follows that $\underset{\sim}{\alpha}_2-\underset{\sim}{\alpha}_1$ must be the same for any $t_1$ and $t_1+T$ . In other words, the centres of the yield surfaces trace the same curve in the stress space but these surfaces need not coincide.

iii/ Superposition of constant displacements

Consider two deformation programmes differing by prescribed displacements on the portion $S_u$ of the boundary, the surface tractions being the same on the portion $S_T$. For one deformation programme, let the displacements be prescribed in the form $\underset{\sim}{u}_1 = \underset{\sim}{u}_0 + \underset{\sim}{u}'(t)$ whereas for the other $\underset{\sim}{u}_2 = \underset{\sim}{u}'(t)$. Here $\underset{\sim}{u}_0$ denotes the fixed displacement field applied on $S_u$ before imposing the time-dependent displacement $\underset{\sim}{u}'(t)$. Let $\underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2, \dot{\underset{\sim}{\varepsilon}}_1, \dot{\underset{\sim}{\varepsilon}}_2$ be the stresses and strain rates corresponding to these two deformation programmes. It is seen that the equality /9/ is still valid since $\dot{\underset{\sim}{\varepsilon}}_1, \dot{\underset{\sim}{\varepsilon}}_2$ are derived from the velocity fields satisfying the same boundary conditions $\dot{\underset{\sim}{u}}_1 = \dot{\underset{\sim}{u}}_2 = \dot{\underset{\sim}{u}}'(t)$ on $S_u$ and $\underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2$ are in equilibrium with the same surface tractions on $S_T$. In view of /10/, we have $\dot{E} < 0$ and in the limiting state there should be $\dot{\underset{\sim}{\varepsilon}}_1^v = \dot{\underset{\sim}{\varepsilon}}_2^v$, $\dot{\underset{\sim}{\varepsilon}}_1^p = \dot{\underset{\sim}{\varepsilon}}_2^p$ in both viscous and plastic regions. Thus the following conclusion can be stated: a fixed initial displacement of a part of boundary <u>cannot affect</u> the form of the steady cycle for any superposed loading programme. Obviously, this property does not depend on existence of viscous terms. For instance, if a beam is subjected to initial extension which is maintained constant during subsequent cyclic flexural deformation of prescribed magnitude, a form of the steady cycle in bending should not depend on this extension. Experimental observations by Ross and Dean Morrow /12/ for the case of uniaxial stress cycle superposed on initial extension seem to confirm this general property.

<u>iv/ Generalizations to more complex workhardening models</u>

All general conclusions concerning existence, uniqueness of steady cycles and their independence of fixed displacements remain valid for a material element regarded as an arbitrary composition of elastic and kinematically hardening elements. In fact, any hardening curve can can be modelled by a properly arranged set of subelements satisfying /1/, Fig.1c. For instance, for series or parallel connections of such subelements, we have respectively

$$a)\quad d\underset{\sim}{\varepsilon}^p = \frac{1}{n}\sum_{i=1}^{n} d\underset{\sim}{\varepsilon}_i^p, \qquad \underset{\sim}{\sigma} = \underset{\sim}{\sigma}_i, \qquad i = 1, 2, \dots n$$
$$b)\quad d\underset{\sim}{\varepsilon} = d\underset{\sim}{\varepsilon}_i, \qquad \underset{\sim}{\sigma} = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{\sigma}_i, \qquad /13/$$

where all plastic elements obey /1/. Such hardening models have been recently applied by Iwan /16/ and Wells and Pasley/17/ to description of hysteretic phenomena and hardening for complex loading paths. The function $E$ now takes the form

$$E = \int \Big[\tfrac{1}{2}(\underset{\sim}{\sigma}_2 - \underset{\sim}{\sigma}_1)\cdot(\underset{\sim}{\varepsilon}_2^e - \underset{\sim}{\varepsilon}_1^e) + \sum \tfrac{1}{2} C_i \big(\underset{\sim}{\varepsilon}_{2i}^p - \underset{\sim}{\varepsilon}_{1i}^p\big)^2\Big] dV, \qquad /14/$$

where $C_i$ are constants corresponding to $C$ in /1/.

One characteristic feature of such complex models is inability to describe progressive flow during plastic cycling. This fact is schematically illustrated for series connection of kinematically hardening elements satisfying Mises yield condition and for the case of biaxial stress state. Let the stress $\tau_{xy}$, alternating between the values at M and $M'$, Fig 1d, be superposed upon fixed initial stress $\sigma_x$. Asymptotic state will always be represented by particular yield surfaces with centres on the line $MM'$ and with no strain accumulation, $\varepsilon_x^p = 0$. This prediction is contrary to numerous experimental data /1-6/. It seems therefore that such models cannot be generally applied to predict structural behaviour under cyclic loads. This problem is discussed in detail in refs. /14, 15/ where alternative models are proposed and their properties are discussed.

### 3. ON THE EXTENSION OF MELAN'S SHAKE-DOWN THEOREM

The preceding analysis can easily be adjusted to extend the Melan's shake-down theorem for the case of dynamic loads and occurence of partially elastic shake-down.

i/ Assume that viscous terms are not present and such residual stress state $\underset{\sim}{\sigma}_r^*$ and such position of the centre of yield surface $\underset{\sim}{\alpha}_1^*$ can be found that superposed elastic stress state does not violate the yield condition, that is

$$\underset{\sim}{\sigma}_1 = \underset{\sim}{\sigma}_e + \underset{\sim}{\sigma}_r^* \quad , \qquad f\left(\underset{\sim}{\sigma}_1 - \underset{\sim}{\alpha}_1^*\right) \leqslant \sigma_c \; . \qquad /15/$$

for any dynamic process satisfying the equations of motion and boundary conditions; here $\underset{\sim}{\sigma}_e$ denotes the stress state corresponding to purely elastic solution. Let the first process correspond to elastic behaviour whereas the second be the actual process occuring within the body. Defining E by /7/, we have

$$\dot{E} = -\int \left[\left(\underset{\sim}{\sigma}_2 - \underset{\sim}{\alpha}_2\right) - \left(\underset{\sim}{\sigma}_1 - \underset{\sim}{\alpha}_1^*\right)\right] \cdot \dot{\underset{\sim}{\varepsilon}}_2^p \, dV_p < 0 \qquad /16/$$

Thus any elastic-plastic deformation process will tend to an elastic state. The dynamic shake-down theorem for more particular assumptions was also proved by Ceradini /7/. Existence of viscous strains within the body invalidates this theorem since any residual stress state will relax to viscoelastic solution. Thus if we find such position of $\alpha_1^*$ that the viscoelastic stress nowhere exceeds the yield surface, no plastic flow will occur in the asymptotic state for slow cycles.

ii/ Consider now the case when under the prescribed cyclic loading system $\underset{\sim}{T}_1$, the steady static or dynamic plastic cycle is established with a variable plastic region $V_p$. Let the second loading system $\underset{\sim}{T}_2$ bes superposed on the portion of the boundary. If such a residual stress state exists that the superposed elastic stress does not violate the yield condition beyond $V_p$ and does not change the stress distribution within $V_p$ , the second loading system will not affect the plastic behaviour due to the loading system $\underset{\sim}{T_1}$ and the body beyond $V_p$ will shake-down to elastic state. This obvious theorem may prove very useful when studying the behaviour of structures under two-parameter variable load systems. Let us illustrate its application by a simple example.

Consider a rectangular beam of unit width and the thickness 2h, subjected to alternating bending moment $\pm M$, superposed upon constant axial force $N$ . Assume that the material is elastic, perfectly plastic with no viscous effects. Our aim is to determine on the /M-N/-plane the regions of variation of M when: i/ elastic shake-down occurs, ii/ flexural plastic cycles occur with no axial elongation, iii/ there is ... plastic interaction between M and N, and axial elongation occurs in the course of flexural cycling. Let the bending moment involve plastic flow during each cycle. Then the stress distribution for extreme values of M will be that presented in Fig. 2b. Since in the elastic region between $M$ and $M'$ there is no increase of stress during unloading and reverse loading, the admissible axial force will correspond to the stress distribution $OMM'$. Thus we have $M = \sigma_o (h^2 - \frac{1}{3}\xi^2)$, $N_o = \sigma_o \xi$ , and

$$m = M/M_o = 1 - \frac{4}{3}\left(\frac{N}{N_o}\right)^2 = 1 - \frac{4}{3}n^2, \qquad /17/$$

where $M_o = \sigma_o h^2$ and $N_o = \sigma_o 2h$ denote limit values of M and N. Equation /17/ defines the region where only flexural plastic cycles occur with no influence of the axial force. To obtain the region of elastic shake-down, we assume the stress distribution of Fig. 2c. The admissible axial force corresponds to the stress distribution $MNOM'N'$. This gives us the relation

$$n = \frac{N}{N_o} = 1 - \frac{3}{4}\frac{M}{M_o} = 1 - \frac{3}{4}n. \qquad /18/$$

In Fig. 2a the parallelogram ABCDA defines the initial elastic region whereas AKCLA is the limit yield locus. The region of elastic shake-down is denoted by AEFCHGA, whereas the region of partial elastic shake-down is bounded by parabolas EKF and GHL defined by Eq. /17/ and the two horizontal lines EF and GH. Thus the flexural plastic cycle is not affected by the axial force provided provided the minimum and maximum values of the bending moment are represented by points within EKF and HGL. Beyond AEKFC, the flexural cycles will be accompanied by accumulating axial elongation. It seems that regions of elastic or partial elastic shake-down can be found for any structure by applying this generalization of Melan's theorem.

## 4. TWO INEQUALITIES FOR STEADY PLASTIC CYCLES

Let us consider now sufficiently slow cycles, so that inertia terms can be neglected. The material is assumed to be elastic, perfectly-plastic with no viscous deformations, First, we shall introduce notions od statically and kinematically admissible velocity cycles.

The statically admissible stress cycle $\underset{\sim}{\sigma}_s(t,\underset{\sim}{x})$ is a continuous, cyclically varying stress field which satisfies the boundary conditions on $S_T$ and nowhere violates the yield condition. Thus

$$\sigma_{ij\,s}\,\nu_i = T_i \quad \text{on } S_T \quad , \quad f(\underset{\sim}{\sigma}_s) < \underset{\sim}{\sigma}_o$$
$$\underset{\sim}{\sigma}_s(t+T) = \underset{\sim}{\sigma}_s(t) \qquad \text{for any} \quad x \in V \qquad /19/$$

where $\nu_i$ is the unit normal vector to the boundary $S_T$ and $T$ denotes the cycle period.

The kinematically admissible velocity cycle $\dot{u}_K(\underset{\sim}{x},t)$ is a cyclically varying rate of displacement field, satisfying the kinematic boundary conditions on $S_u$ and the equality

$$\int\int_{t_1}^{t_1+T} \underset{\sim}{T}\cdot\dot{\underset{\sim}{u}}_K \, dt\, dS_T = \int\int_{t_1}^{t_1+T} \underset{\sim}{\sigma}_K\cdot\dot{\underset{\sim}{\varepsilon}}_K^{\,p}\, dt\, dV \; ; \qquad /20/$$

moreover

$$\int_{t_1}^{t_1+T} \dot{\underset{\sim}{u}}_K\, dt = \Delta\underset{\sim}{u}_K^{\,p} \qquad /21/$$

where $\Delta\underset{\sim}{u}_p^{\,K}$ denotes the permanent displacement accumulating after each cycle. These two definitions are similar to those introduced by Koiter [9].

Assume that the body is rigidly supported on $S_u$ and the cyclically varying loads are applied on $S_T$. For any pair of kinematically and statically admissible cycles $\dot{\underset{\sim}{u}}_K$ and $\underset{\sim}{\sigma}_s$, we can write

$$\int\int_{t_1}^{t_1+T} \underset{\sim}{T}\cdot\dot{\underset{\sim}{u}}_K\, dt\, dS_T = \int\int_{t_1}^{t_1+T} \underset{\sim}{\sigma}_s\cdot\dot{\underset{\sim}{\varepsilon}}_K^{\,p}\, dt\, dV + \int\int_{t_1}^{t_1+T} \underset{\sim}{\sigma}_s\cdot\dot{\underset{\sim}{\varepsilon}}_K^{\,e}\, dt\, dV . \qquad /22/$$

Subtracting /19/ from /22/, we obtain

$$I = \int\int \underset{\sim}{\sigma}_s\cdot\dot{\underset{\sim}{\varepsilon}}_K^{\,e}\, dt\, dV = \int(\underset{\sim}{\sigma}_K - \underset{\sim}{\sigma}_s)\cdot\dot{\underset{\sim}{\varepsilon}}_K^{\,p}\, dt\, dV > 0 . \qquad /23/$$

Thus the work of any statically admissible stress cycle on the elastic strain of kinematically admissible velocity cycle within the period is positive-definite and vanishes when both cycles represent the actual solution. The inequality /23/ may prove helpful in constructing an approximate solution for the steady cycle.

Consider now the case when the alternating loads $\underset{\sim}{T}^{a}$ are superposed on fixed loads $\underset{\sim}{T}^{o}$. For the steady state, we have

$$\int\int_{t_1}^{t_1+T} \underset{\sim}{T}^{a} \cdot \dot{\underset{\sim}{u}}\, dt\, dS_2 + \int \underset{\sim}{T}^{o} \cdot \Delta \underset{\sim}{u}^{p} dS_1 = \int\int_{t_1}^{t_1+T} \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\varepsilon}}^{p} dt\, dV. \qquad /24/$$

Introduce now any statically admissible stress field $\underset{\sim}{\sigma}^{s}$ satisfying equilibrium equations and boundary conditions $\sigma_{ij}^{s} \nu_j = T_i^{s}$ on $S_1$ and $\sigma_{ij}^{s} \nu_j = 0$ on $S_2$. We can thus write

$$\int \underset{\sim}{T}^{s} \cdot \Delta \underset{\sim}{u}^{p} dS_1 = \int\int_{t_1}^{t_1+T} \underset{\sim}{\sigma}^{s} \cdot \dot{\underset{\sim}{\varepsilon}}^{p} dt\, dV. \qquad /2.5/$$

Subtracting /24/ from /25/, we obtain

$$\int \left(\underset{\sim}{T}^{s} - \underset{\sim}{T}^{o}\right) \cdot \Delta \underset{\sim}{u}^{p} dS_1 - \int\int_{t_1}^{t_1+T} \underset{\sim}{T}^{a} \cdot \dot{\underset{\sim}{u}}\, dt\, dS_2 = \int\int_{t_1}^{t_1+T} (\underset{\sim}{\sigma}^{s} - \underset{\sim}{\sigma}) \cdot \dot{\underset{\sim}{\varepsilon}}^{p} dt\, dV < 0. \qquad /26/$$

The inequality /26/ can be used to find an upper bound on plastic displacement accumulating after each cycle. Let, for instance, $\underset{\sim}{T}^{o}$ and $\underset{\sim}{T}^{s}$ be concentrated loads, and $\Delta u_T^{p}$ denotes the permanent displacement in the direction of $T^{o}$. From /26/, we have

$$\Delta u_T^{p} \leq \frac{\int\int_{t_1}^{t_1+T} \underset{\sim}{T}^{a} \cdot \dot{\underset{\sim}{u}}\, dt\, dS_2}{\left|\underset{\sim}{T}^{s} - \underset{\sim}{T}^{o}\right|}. \qquad /27/$$

The best bound is obtained when $T^{s} = T^{c}$, where $T^{c}$ denotes the static collapse load of the body under the fixed load $T^{o}$. In order to apply /27/, the value of the plastic work of alternating forces /or at least its upper bound/ should be known.

## REFERENCES

/1/ N. H. Polakowski, A. Polchdudhuri, Softening of certain cold-worked metals under the action of fatigue loads, Proc. Am. Soc. Test. Mat. vol. 54, 701-716, /1954/

/2/ L. F. Coffin, Jr. The stability of metals under cyclic plastic strain, J. Bas. Eng. 82D, 671-682, /1960/.

/3/ D. S. Dugdale, Stress-strain cycles of large amplitude, Journ. Mech. Phys. Sol. 7, 135-142, /1959/.

/4/ A. P. Gusenkov, Properties of curves of cyclic deformation at room temperatures, /in Russian/, in "Low-cycle failure and deformation", Moscow, 1967.

/5/ S.S. Mansen, Thermal stresses and low-cycle fatigue, Mc Graw-Hill Book Co.,/1966/.

/6/ V.V. Moskvitin, Plasticity under varying loads /in Russian/, Moscow University, /1965/.

/7/ G. Ceradini, Sull'adattamento dei corpi elastoplastici ad azioni dinamiche, Giorn. del Genio Civile, 4/5, 239-251, /1969/.

/8/ G. Augusti, Rigid-plastic structures subject to dynamic loads, Meccanica, 5, 1-12, /1970/.

9/ W. T. Koiter, A new general theorem on shakedown of elastic-plastic structures, Koninkl. Nederl. Ak. Wetensch. Ser. B, 1, /1956/.

/10/ E. Melan, Zur Plastizität des räumlichen Kontinuums, Ing. Arch., /1938/.

/11/ Z. Mróz, On the description of anisotropic hardening, Journ. Mech. Phys. Sol., 15, 163-175, /1967/.

/12/ A.S. Ross, Jo Dean Morrow, Cycle-dependent stress relaxation of A-286 Alloy, Journ. Bas. Eng. , 82D, 654-660, /1960/.

13/ H. Armen, G. Isakson, A. Pifko, Discrete element approach for the plastic analysis of structures subjected to cyclic loading, Intern Journ. Num. Meth. in Eng., 2, 189-206, /1970/.

/14/ Z. Mróz, An attempt to describe the behaviour of metals under cyclic loads using a more general workhardening model, Acta Mech. 7, 199-212, /1969/.

/15/ Z. Mróz, C. Goss, On composite models of plastic workhardening / in Polish/, Mech. Teoret. Stos. 1971 /in print/.

/16/ W. D. Iwan, On class of models for the yielding behaviour of continuous and composite systems, Journ. Appl. Mech., vol. 34, /1967/.

/17/ C. H. Wells, P. R. Paslay, A small-strain plasticity theory for planar slip materials, Journ. Appl. Mech., vol. 35,/1968/.

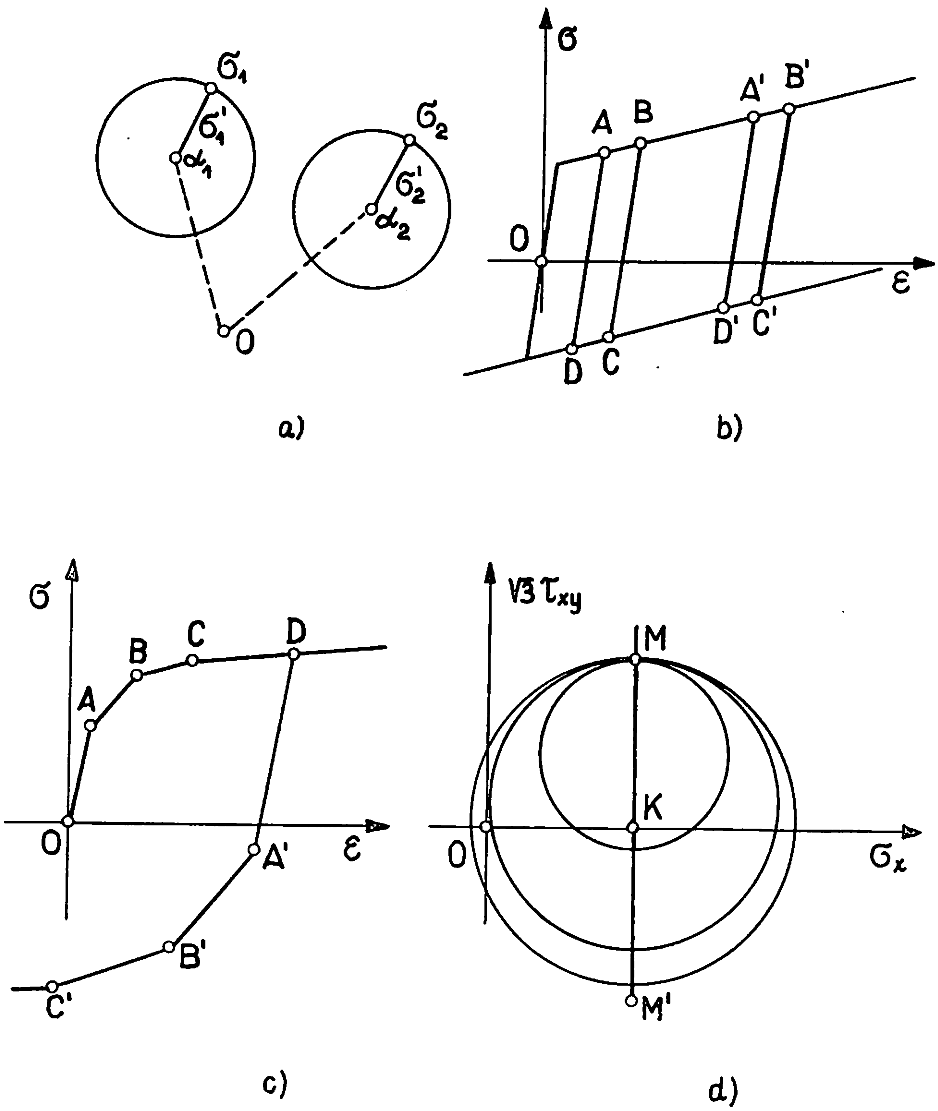


Figure 1. Simulation of workhardening by kinematically hardening elements: a/ possible positions of two yield surfaces in the steady state, b/ two equivalent steady cycles ABCD and A'B'C'D'; c/ hardening ouve for connection of kinematically hardening elements, d/ inability of describing plastic strain accumulation induced by cyclically varying stress $\tau_{xy}$ between M and M' superposed upon constant stress $\sigma_x$ .

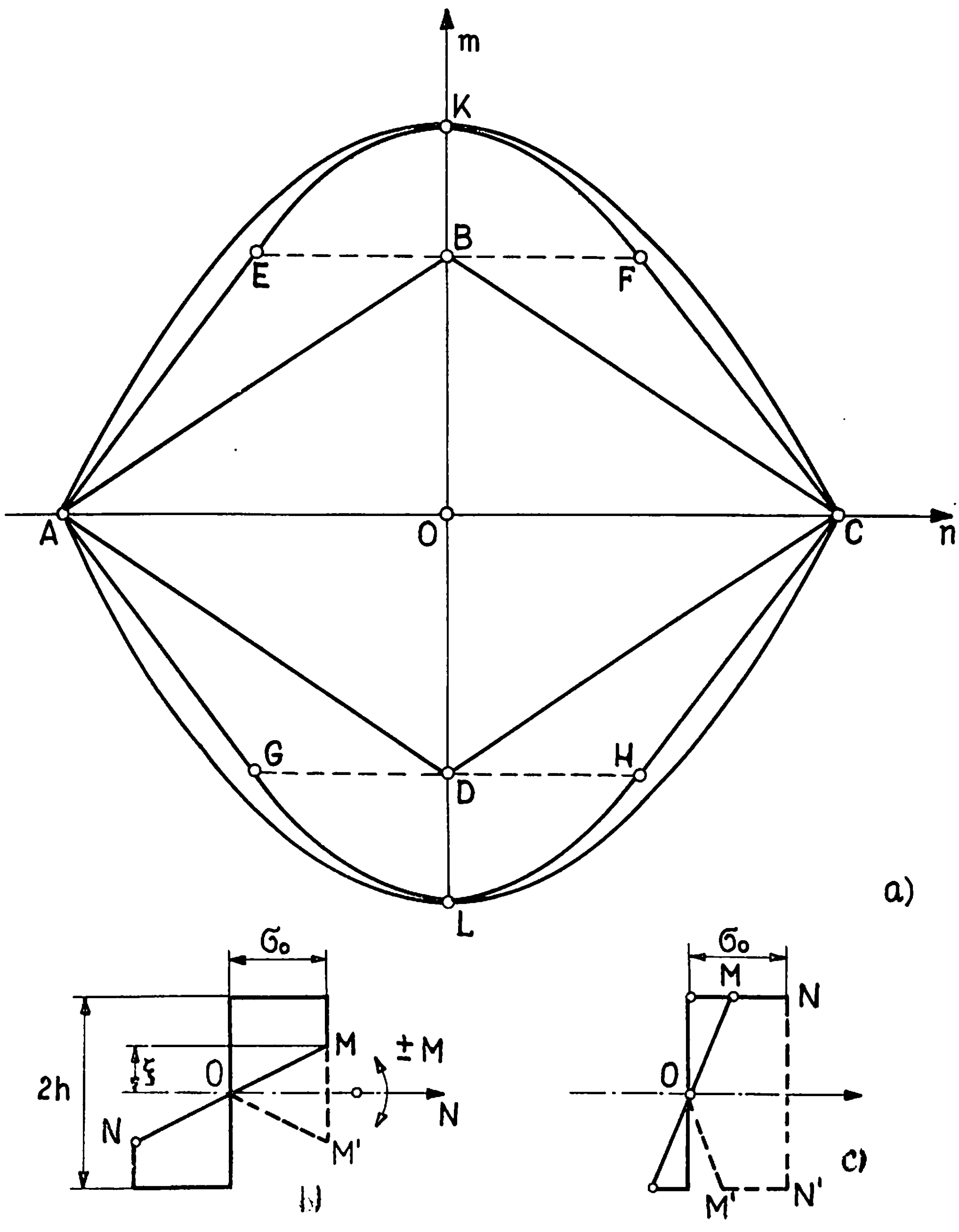


Figure 2 Regions of elastic and partially elastic shake-down for alternating bending momemnt M superposed upon constant axial force N, a/, and corresponding stress distributions, b/ and c/.