

Thermodynamic conditions for stability in materials with rate-independent dissipation

BY HENRYK PETRYK

*Institute of Fundamental Technological Research Polish Academy
of Sciences, 00-049 Warsaw, Poland
(hpetryk@ippt.gov.pl)*

A distinctive feature of the examined class of solids is that a part of the entropy production is due to rate-independent dissipation, as in models of plasticity, damage or martensitic transformations. The standard condition for thermodynamic stability is shown to be too restrictive for such solids and, therefore, an extended condition for stability of equilibrium is developed. The classical thermodynamic theory of irreversible processes is used along with the internal variable approach, with the emphasis on the macroscopic effects of micro-scale instabilities in the presence of two different scales of time. Specific conditions for material stability against internal structural rearrangements under deformation-sensitive loading are derived within the incremental constitutive framework of multi-mode inelasticity. Application to spontaneous formation of deformation bands in a continuum is presented. Conditions for stability or instability of a quasi-static process induced by varying loading are given under additional constitutive postulates of normality and symmetry. As illustration of the theory, the stability of equilibrium or a deformation path under uniaxial tension is analysed for a class of inelastic constitutive laws for a metal crystal deformed plastically by multi-slip or undergoing stress-induced martensitic transformation.

Keywords: thermodynamic stability; rate-independent solid; energy criterion;
multi-mode inelasticity; plasticity; martensitic transformation

1. Introduction

The classical (Gibbs) condition for thermodynamic stability of equilibrium (cf. Callen 1960) is a fundamental tool for examining stability of equilibrium states in various branches of science. In the form that requires maximum entropy for constant energy of an isolated system, it is directly based on the second law of thermodynamics. Nevertheless, for the particular but widely used class of material models examined in this paper, the maximum entropy condition turns out to be too restrictive and thus of limited applicability. A part of this paper is devoted to a discussion of this important point.

Concepts of the classical thermodynamics of irreversible processes (cf. Meixner & Reik 1959; de Groot & Mazur 1962) are accepted here, including the internal variable approach. The background formalism to be used is thus well

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known. A distinctive property of the examined class of solids is that a part of the entropy production in transition between two neighbouring equilibrium states does not depend on the duration of the transition. This property, referred to as *rate-independent dissipation*, is typical for number of macroscopic models of inelastic phenomena, including rate-independent plasticity of metals or geomaterials, propagation of microcracks or damage growth, mechanical twinning or diffusionless phase transformations, etc.

The distinctive feature of rate-independent dissipation may be attributed to the scale of size and time on which the phenomena are considered, in particular to how the neighbouring equilibrium states are understood. In this paper, the material description involves macro-scales of size and time much larger than the typical activation volume and relaxation time, respectively, of an individual micro-event in the material, such as a displacement of a single dislocation, microcrack or interface. The internal microstructure of a material sample may suffer rapid local rearrangements due to micro-scale instabilities, and the speed and duration of local non-equilibrium jumps are not controlled by an external loading rate; cf. e.g. Neuhäuser (1986) for experimental evidence in the context of metal plasticity. When smoothed out after transition to the macro-level, such local jumps may be treated as the physical source of macroscopic dissipation of rate-independent type. Consequences for the interpretation of certain macroscopic thermodynamic quantities are discussed in the introductory §2.

That discussion is only auxiliary for this paper which concentrates on macroscopic conditions for stability. However, it helps in explaining why the classical condition of thermodynamic stability does not comply with the concept of rate-independent dissipation. In short, rigorous fulfillment of this condition would exclude any local instabilities at a micro-level, eliminating thus the postulated physical source of the macroscopic rate-independent dissipation itself.

The difficulty can be overcome (Petryk 1995) by splitting the total dissipation rate into the intrinsic (rate-independent) and complementary parts assumed to be separately non-negative, and using only the *complementary* part to construct a modified Lyapunov functional whose value is non-increasing in an unperturbed process. This leads to *the extended condition* for thermodynamic stability of equilibrium which in general is *less* restrictive than the classical one, and reduces to it in the absence of rate-independent dissipation. Instabilities appearing only at a micro-scale are no longer treated as genuine macroscopic instabilities but just as a physical source of the rate-independent dissipation at the macro-level. If the temperature is assumed constant then the extended condition for stability of equilibrium can be reduced to those derived earlier for rate-independent solids on the basis of the energy balance. In those derivations, Hill (1958) used the work functional determined by integration of the stress power, and Nguyen (1993, 2000) split the required energy supply into an increment in the free energy and a time integral of the dissipation function. Both approaches receive a thermodynamic support via the extended condition applied under different constitutive assumptions.¹

¹ However, the reduced, second-order stability conditions, derived in this paper without postulating a normality law, are interpreted merely as sufficiency conditions for ‘directional’ (and not unconditional) stability of equilibrium.

In this paper, the extended condition for stability is further developed for a material sample with generalized displacements and forces as external variables and with a finite number of internal mechanisms of rate-independent inelastic deformation. Specific conditions for stability of equilibrium against spontaneous internal structural rearrangements within the sample are derived. These conditions, while being of thermodynamic origin, are applicable within a purely phenomenological, isothermal constitutive description of multi-mode inelasticity, without any need to specify internal state variables or associated thermodynamic driving forces.

A related but different topic addressed below is the stability of a quasi-static deformation *process* (or *path*, in alternative terminology). A quasi-static process, dissipative, rate-independent and induced by a loading program, may itself be unstable (in the rough sense of sensitivity to small disturbances) even if each of the traversed equilibrium states separately is stable. This possibility is closely related to the well-known phenomenon of path *bifurcation* under *varying* loading (cf. Hill 1978; Petryk 1993a) in the absence of any disturbances. While the bifurcation theory can predict in many cases the critical instant and mode of the loss of uniqueness of a theoretical solution, it does not provide a criterion of choice among the non-unique post-critical paths. Since, the condition for stability of equilibrium is also insufficient for that purpose, and a study of initial imperfections may be not effective, there is a need for a plausible criterion of path stability or instability.

A fully general theory of path stability in inelastic systems with rate-independent dissipation has not been developed yet. Specific results concerning a contractivity property of plastic flow were obtained so far under rather restrictive assumptions, e.g. Nguyen (1984), Simo (1991). Also the energy criterion of path stability proposed by the author (Petryk 1982, 1985), formulated later in an alternative thermodynamic form (Petryk 1993b; Fedelich & Ehrlacher 1997), becomes plausible only under additional assumptions. The energy criterion is examined in §5 within the isothermal constitutive framework of multi-mode inelasticity under the normality and symmetry hypotheses. Extensions of the previous results obtained in (Petryk 1991b, 1993b, 1999; Petryk & Thermann 1992) are presented.

The theory of stability of equilibrium or path is illustrated by the examples of spontaneous formation of shear bands in a homogeneous stressed continuum (§4) and of stability under uniaxial tension in single crystals deformed plastically by multi-slip or undergoing stress-induced phase transformation (§6).

2. Constitutive framework

(a) Local thermodynamic processes

Consider a material sample undergoing non-uniform inelastic deformation on a fine size scale such that the continuum approach is acceptable locally; phenomena at a still lower scale where discrete nature of matter becomes important are not investigated in this paper. A material point within the sample is regarded here as a closed system: chemical reactions and diffusion are disregarded. The axiom of local state is adopted as in the classical formalism of non-equilibrium thermodynamics (cf. de Groot & Mazur 1962). It is assumed

that a local state of a material point can be characterized by a triple $(\mathbf{e}, T, \boldsymbol{\beta})$, where \mathbf{e} is a measure of the local finite strain relative to a fixed reference configuration, $T > 0$ is the local absolute temperature, and $\boldsymbol{\beta}$ stands for a collection of local internal state variables considered in the *reference* configuration. A local state at fixed $(\mathbf{e}, T, \boldsymbol{\beta})$ is regarded as that of constrained equilibrium (cf. Kestin & Rice 1970; Maugin & Muschik 1994), and the specific entropy and other thermodynamic variables are defined locally as in thermostatics.

It will be convenient to work with thermodynamic quantities defined relative to an arbitrary, fixed reference configuration. To simplify the formulae by eliminating a mass density ρ which in the reference configuration does not vary, the densities s and ϕ of entropy and Helmholtz free energy, respectively, are taken *per unit reference volume*.

Variables conjugate to $(\mathbf{e}, T, \boldsymbol{\beta})$ are obtained in the usual way from the postulated Gibbs relation expressed in terms of the thermodynamic potential $\phi(\mathbf{e}, T, \boldsymbol{\beta})$,

$$d\phi = \mathbf{t} \cdot d\mathbf{e} - s dT - \mathbf{B} \cdot d\boldsymbol{\beta}, \quad \mathbf{t} = \frac{\partial \phi}{\partial \mathbf{e}}, \quad s = -\frac{\partial \phi}{\partial T}, \quad \mathbf{B} = -\frac{\partial \phi}{\partial \boldsymbol{\beta}}, \quad (2.1)$$

where a prefix d denotes a differential. Viscous effects are neglected, so that \mathbf{t} is the total stress, work-conjugate to the total strain \mathbf{e} . A central dot denotes a product with full contraction. For instance, if $\boldsymbol{\beta}$ is identified with a column matrix of components β_K (which may be scalars or arbitrarily ordered tensor components) then the set of thermodynamic forces (affinities) \mathbf{B} conjugate to $\boldsymbol{\beta}$ is identified with a row matrix of components $B_K = -\partial \phi / \partial \beta_K$, and $\mathbf{B} \cdot d\boldsymbol{\beta} = \sum_K B_K d\beta_K$. Similarly, $\mathbf{t} \cdot d\mathbf{e} = t_{ij} de_{ij}$ with the summation convention over repeated indices that denote tensor components on a fixed orthonormal triad. To simplify the notation, we will use the same symbol for a function and its value and omit the set of independent variables whenever it is evident or has been defined beforehand.

The Gibbs equation in the form (2.1) encompasses an alternative, frequently used description with an appropriately oriented stress-free configuration as a moving reference frame (Mandel 1971) provided the deformation gradient after unloading is regarded as a function of $\boldsymbol{\beta}$, cf. Kleiber & Raniecki (1985). For the purposes of the present paper, it is more appropriate to take as an independent variable of ϕ the total and not elastic strain.

Consider a transition from some state to a neighbouring state within an infinitesimal increment $d\tau$ of time τ . The local entropy production inequality, on using equation (2.1), takes the form

$$\mathbf{B} \cdot d\boldsymbol{\beta} + T \mathbf{h} \cdot \nabla(T^{-1}) d\tau \geq 0, \quad (2.2)$$

where ∇ denotes the spatial gradient and \mathbf{h} is the heat flux, both considered in the reference configuration.

We adopt the common assumption that the two terms in equation (2.2) are *separately* non-negative, which results in the thermal dissipation inequality

$$T \mathbf{h} \cdot \nabla(T^{-1}) \geq 0, \quad (2.3)$$

and the intrinsic dissipation inequality

$$\mathbf{B} \cdot d\boldsymbol{\beta} \geq 0. \quad (2.4)$$

This classical description of irreversible processes at the level of a material point is applied below to a macroscopic material sample, generally heterogeneous, which occupied in the reference configuration a given finite volume \bar{V} . Straightforward averaging procedures are employed for that purpose, but extended to averaging over a *time* interval as well.

The intrinsic dissipation increment in a time interval $(t, t + \Delta t)$ in the material sample reads

$$\Delta \mathcal{D} = \int_{\bar{V}} \left(\int_t^{t+\Delta t} \mathbf{B}(\tau) \cdot \frac{d\boldsymbol{\beta}}{d\tau}(\tau) d\tau \right) d\bar{V} \geq 0, \quad (2.5)$$

with an implicitly understood dependence of variables $\mathbf{B}, \boldsymbol{\beta}$ also on place.

(b) *Macroscopic thermodynamic process*

Henceforth, attention is focused on thermodynamic description of a generally heterogeneous material sample (or macroscopic cell) \mathcal{M} in terms of macroscopic variables. The assumption is introduced that the temperature T , while generally varying in time, remains *uniform* within the sample of sufficiently small size. Fields of strain and internal variables over the volume \bar{V} , say $\tilde{\boldsymbol{\epsilon}}$ and $\tilde{\boldsymbol{\beta}}$, respectively, constitute conceptually useful state variables for \mathcal{M} which, however, may be less convenient in analysis. To simplify the description, suppose that, to a suitable approximation, in place of a strain field $\tilde{\boldsymbol{\epsilon}}$ we may use a vector of *generalized displacements* (or strains) \mathbf{q} , and in place of a field $\tilde{\boldsymbol{\beta}}$ of local internal state variables we may use *averaging variables* (Rice 1971) denoted collectively by $\boldsymbol{\alpha}$. At the level of generality attempted here, the (scalar or tensorial) nature of elements of $\boldsymbol{\alpha}$ is not essential. Dimensionality of the spaces of \mathbf{q} and $\boldsymbol{\alpha}$ is assumed to be finite but is otherwise arbitrary; we may wish to keep it reasonably low in applications. However, the deformation of the material sample need not be macroscopically uniform in any sense. If it is then \mathbf{q} may have the meaning of an overall strain, but in general \mathbf{q} may have more degrees of freedom to accommodate, for instance, a mode of strain localization. In short, the material sample \mathcal{M} is regarded as an arbitrarily discretized system of uniform temperature.

Macroscopic inelastic deformation of \mathcal{M} can be considered as a consequence of microstructural rearrangements (Rice 1971, 1975), described at the micro-level by changes in $\tilde{\boldsymbol{\beta}}$, e.g. due to dislocation movement, microcracking, phase transformation, etc. As indicated in the introduction, kinetics of those microstructural rearrangements can be highly irregular due to a large number of local instabilities. There are illustrative models which show associated macroscopic effects, e.g. Ericksen (1975), Müller & Villaggio (1977), Pontier *et al.* (1979). In view of such local instabilities, the concept of a macroscopic strain-rate or stress-rate for inelastic solids should involve averaging not only over a representative volume of the material, but also over a *representative time increment* as well. Accordingly, two *different time scales* are to be used, where the differentials at the micro-level in equation (2.1) are of a smaller order of magnitude than physically meaningful infinitesimal increments in averaging variables, to be denoted by a prefix ‘ δ ’ in the macroscopic description.

Consider a non-equilibrium process in the material sample \mathcal{M} between two states close to each other, of small duration $\delta t > 0$ which is regarded as infinitesimal at the macroscopic scale of time t . While processes at the

micro-level need not be continuous with respect to t , all *macroscopic* variables in view of their averaging sense are assumed to vary continuously in time t . The (forward) rate of any macroscopic variable with respect to time t is interpreted at the micro-scale in the averaged sense as the quotient of a small increment in that quantity divided by δt , i.e.

$$\dot{\Psi}(t) = \frac{\delta\Psi}{\delta t}, \quad \delta\Psi = \Psi(t + \delta t) - \Psi(t). \tag{2.6}$$

Following the well-known approach, at each instant of a non-equilibrium process we associate with \mathcal{M} an *accompanying equilibrium state* characterized by $\mathcal{G} = (\mathbf{q}, T, \boldsymbol{\alpha})$; see Maugin & Muschik (1994) and the references therein (e.g. Germain *et al.* 1983) for detailed exposition of this concept. If $(\mathbf{q}, T, \boldsymbol{\alpha})$ are imagined to be fixed then \mathcal{M} is regarded to be in a state of constrained equilibrium. In analogy to equation (2.1), variables conjugate to $(\mathbf{q}, T, \boldsymbol{\alpha})$ are obtained from the macroscopic Helmholtz free energy function $\Phi(\mathbf{q}, T, \boldsymbol{\alpha})$, viz.

$$\dot{\Phi} = \mathbf{Q} \cdot \dot{\mathbf{q}} - S \dot{T} - \mathbf{A} \cdot \dot{\boldsymbol{\alpha}}, \quad \mathbf{Q} = \partial\Phi/\partial\mathbf{q}, \quad S = -\partial\Phi/\partial T, \quad \mathbf{A} = -\partial\Phi/\partial\boldsymbol{\alpha}. \tag{2.7}$$

Here, \mathbf{Q} stands for the vector of generalized forces (or stresses) conjugate to \mathbf{q} , S is the macroscopic entropy, and \mathbf{A} denotes collectively macroscopic thermodynamic forces (or affinities) conjugate to $\boldsymbol{\alpha}$. Equations (2.7) provide, of course, only an approximation to a more accurate description at the micro-level in terms of fields $\tilde{\mathbf{e}}$ and $\tilde{\boldsymbol{\beta}}$. Macroscopic state variables S, Φ are identified with volume integrals of respective local quantities, $S = \int_V s \, d\bar{V}$, $\Phi = \int_V \phi \, d\bar{V}$ (recall that T is assumed uniform). The link between other variables appearing in equations (2.1) and (2.7) is more complex, especially due to different time scales involved in the description at the micro- and macro-levels.

The rate of intrinsic dissipation within the material sample \mathcal{M} is defined on account of equations (2.5) and (2.6) by

$$\dot{D} = \int_{\bar{V}} \left(\frac{1}{\delta t} \int_t^{t+\delta t} \mathbf{B}(\tau) \cdot \frac{d\boldsymbol{\beta}}{d\tau}(\tau) d\tau \right) d\bar{V} \geq 0. \tag{2.8}$$

This expression in general cannot be reduced to a single volume integral of the local dissipation rate, since a local \mathbf{B} may vary strongly with respect to a ‘fast’ local time variable τ for physically reasonable values of δt . A small change in an overall state of \mathcal{M} may correspond to a large jump in a local $\boldsymbol{\beta}$.

Similarly, the rate of deformation work within \mathcal{M} is defined as

$$\dot{W} = \int_{\bar{V}} \left(\frac{1}{\delta t} \int_t^{t+\delta t} \mathbf{t}(\tau) \cdot \frac{d\mathbf{e}}{d\tau}(\tau) d\tau \right) d\bar{V}, \tag{2.9}$$

with analogous comments. The deformation work and intrinsic dissipation are *functionals* dependent on the entire process within the material sample.

Calculation of $\dot{\Phi}$ by integrating equation (2.1) and neglecting an infinitesimal term of order $T\delta S$ now yields

$$\dot{\Phi} = \dot{W} - S\dot{T} - \dot{D}. \tag{2.10}$$

From the first law for the material sample,

$$\dot{U} = \dot{W} + \dot{Q}, \tag{2.11}$$

where $U = \Phi + TS$ is the internal energy of \mathcal{M} and Q is the heat absorbed by \mathcal{M} , it follows that equation (2.10) is in agreement with the usual definition $\dot{\mathcal{D}} = T\dot{S} - \dot{Q}$ of the rate of intrinsic dissipation. In turn, on eliminating $\dot{\Phi}$ between equations (2.10) and (2.7), we obtain

$$\dot{W} - \mathbf{Q} \cdot \dot{\mathbf{q}} = \dot{\mathcal{D}} - \mathbf{A} \cdot \dot{\boldsymbol{\alpha}}. \tag{2.12}$$

In the absence of external disturbances (see below), this is consistent with the following straightforward identification

$$\dot{W} = \mathbf{Q} \cdot \dot{\mathbf{q}}, \quad \dot{\mathcal{D}} = \mathbf{A} \cdot \dot{\boldsymbol{\alpha}} \geq 0 \quad \text{in an unperturbed process.} \tag{2.13}$$

By comparing the expressions for $\dot{\mathcal{D}}$ in equations (2.8) and (2.13)₂, a remarkable conclusion is obtained (Petryk 1995) that the averaging thermodynamic forces \mathbf{A} need not vanish in *unconstrained* equilibrium, even if the local thermodynamic forces \mathbf{B} do vanish in equilibrium states assumed at instants t and $t + \delta t$ and deviate from zero only in non-equilibrium processes between those states. This feature of \mathbf{A} is crucial in the thermodynamic description of plasticity and other inelastic phenomena with the *macroscopically rate-independent dissipation*, i.e. when a dissipation part in the transition between two neighbouring states at the macro-level does not depend on the macroscopic deformation rate. Local instabilities at the micro-level related to local jumps in $\boldsymbol{\beta}$ and associated peaks in \mathbf{B} , when smoothed out with the help of averaging variables $\boldsymbol{\alpha}$, are identified as the basic source of macroscopic dissipation of rate-independent type.

In stability investigations we wish to analyse perturbed processes. Denote by \dot{W}^{dist} the rate of work supplied directly to the material by a disturbing agency, independently of the rate of work done by generalized forces \mathbf{Q} . Accordingly, equation (2.13)₁ is replaced by

$$\dot{W} = \mathbf{Q} \cdot \dot{\mathbf{q}} + \dot{W}^{\text{dist}}, \tag{2.14}$$

while equation (2.9) is still valid. Note that the disturbing work-rate \dot{W}^{dist} need not vanish even if the respective differential quotient, cf. equation (2.6), concerns two states of equilibrium at instants t and $t + \delta t$. Physically, this might correspond to a disturbing wave travelling through the material sample in a short time interval interior to $(t, t + \delta t)$. The presence of two time scales manifests here again. The meaning of equation (2.14) can be seen better after the transformation of the formula (2.9) by expressing $d\mathbf{e}/d\boldsymbol{\tau}$ through the velocity gradient, using the divergence theorem and substituting the *perturbed* equations of motion; details of that standard transformation are omitted.

On substituting equation (2.14) into equation (2.12), we obtain that in a perturbed process

$$\dot{\mathcal{D}} = \mathbf{A} \cdot \dot{\boldsymbol{\alpha}} + \dot{W}^{\text{dist}} \geq 0, \tag{2.15}$$

which differs from equation (2.13)₂.

We shall assume that the knowledge of a forward rate $\dot{\boldsymbol{\alpha}}$ in a given state \mathcal{G} determines, with sufficient accuracy, the intrinsic dissipation rate $\dot{\mathcal{D}}$ in the material sample \mathcal{M} in *any* process, either perturbed or not. Moreover, the intrinsic dissipation rate *at the macro-level* is assumed to be entirely rate-independent. It means that *the dissipation function* D of $(\mathcal{G}, \dot{\boldsymbol{\alpha}})$ is given such that,

to a suitable approximation,

$$\dot{D} = D(\mathcal{G}, \dot{\boldsymbol{\alpha}}), \quad D(\mathcal{G}, r\dot{\boldsymbol{\alpha}}) = rD(\mathcal{G}, \dot{\boldsymbol{\alpha}}) \quad \text{for all } r > 0. \quad (2.16)$$

Since, $\dot{\boldsymbol{\alpha}}$ represents here only *virtual* structural rearrangements which generally can only be realized under suitable perturbing forces within the material sample, the function D defines a *virtual* dissipation rate. The difference between D as the virtual rate of intrinsic dissipation defined by equation (2.8) and the expression $\mathbf{A} \cdot \dot{\boldsymbol{\alpha}}$ as the actual dissipation rate in an unperturbed process is equal to the rate of work supplied to the material by a disturbing agency, cf. equation (2.15). Although the concept of a dissipation function has been widely used, the above distinction between the dissipation function and the actual dissipation rate was usually not emphasized.

Of course, if a particular $\dot{\boldsymbol{\alpha}}$ corresponds to an unperturbed (actual) process then D should yield the actual value of the dissipation rate. Denote by $\mathcal{L}(\mathcal{G})$ a cone in $\dot{\boldsymbol{\alpha}}$ -space that contains all $\dot{\boldsymbol{\alpha}}$ compatible with the current thermodynamic forces $\mathbf{A}(\mathcal{G})$. Consequently,

$$D(\mathcal{G}, \dot{\boldsymbol{\alpha}}) = \mathbf{A}(\mathcal{G}) \cdot \dot{\boldsymbol{\alpha}} \quad \text{for all } \dot{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{G}), \quad (2.17)$$

so that D is then linear in $\dot{\boldsymbol{\alpha}}$ within that cone.

(c) Multi-mode inelasticity

To complete the constitutive framework, we must specify the form of evolution rules for internal state variables (for which no balance equation is assumed). It is not required that the local behaviour of a material point within \mathcal{M} be rate-independent on the time scale adopted at the micro-level, especially in view of possible high strain-rates induced locally by micro-scale instabilities. However, the behaviour of the material sample \mathcal{M} *on the macro-scale* is investigated below only in a rate-independent limit (which need not be uniquely defined).

By adopting the well-known concept of time-independent multi-mode plasticity (e.g. for metal crystals) extended to other rate-independent inelastic phenomena, smooth loading functions $f_K(\mathcal{G})$, $K=1, \dots, N$, are defined such that in an admissible *unperturbed* process

$$f_K \leq 0 \quad \text{and} \quad (f_K < 0 \quad \text{for all } K \text{ implies } \dot{\boldsymbol{\alpha}} = 0). \quad (2.18)$$

Each loading function f_K is associated with an internal mechanism of inelastic deformation, which *may* be activated if f_K reaches the zero threshold value. The respective K -th direction of non-zero $\dot{\boldsymbol{\alpha}}$ is specified for each mechanism in the current state, while the magnitude of $\dot{\boldsymbol{\alpha}}$ is set by a scalar multiplier denoted by $\dot{\gamma}_K$. Accordingly, for possibly more than one mechanism activated simultaneously, we assume that

$$\dot{\boldsymbol{\alpha}} = \sum_K \boldsymbol{\eta}_K(\mathcal{G}) \dot{\gamma}_K. \quad (2.19)$$

A circle rather than a dot over γ_K is used to emphasize that $\dot{\gamma}_K$, unlike the rate of a state variable, is an independent variable not defined by a formula analogous to equation (2.6). The orientation of each $\boldsymbol{\eta}_K$ is assumed such that $\dot{\gamma}_K \geq 0$ by convention (a mechanism which may operate in two opposite directions is

regarded as split into two separate mechanisms). In result, the basic rule for activity of K -th mechanism takes the familiar form

$$\dot{\gamma}_K \geq 0, \quad f_K \leq 0, \quad f_K \dot{\gamma}_K = 0, \quad (2.20)$$

with *no* implicit summation assumed for the mechanism index. In the special case $N=1$, the framework of classical elastoplasticity is obtained with a single (averaged) mechanism of inelastic deformation. A case $N=2$ may correspond, for instance, to the classical elastoplasticity accompanied by damage or phase transformation, etc.

Values of inelastic multipliers $\dot{\gamma}_K$ in an unperturbed process are to be determined from *the consistency conditions*

$$\dot{f}_K \leq 0, \quad \dot{f}_K \dot{\gamma}_K = 0 \quad \text{for every } K \in \mathcal{P} \equiv \{K : f_K = 0\}, \quad (2.21)$$

which follow from equation (2.20) by differentiation, while $\dot{\gamma}_K = 0$ for $K \notin \mathcal{P}$. The set $\mathcal{P}(\mathcal{G})$ of indices of *potentially active* mechanisms in a state \mathcal{G} will play an essential role in the stability conditions.

The postulated expressions for the rates of \mathbf{Q} and f_K in terms of $\dot{\mathbf{q}}$, \dot{T} and $\dot{\gamma}_K$ are as follows

$$\dot{\mathbf{Q}} = \mathbf{K}^e \cdot \dot{\mathbf{q}} - \mathbf{k} \dot{T} - \sum_K N_K^p \dot{\gamma}_K, \quad \dot{f}_K = N_K \cdot \dot{\mathbf{q}} + k_K \dot{T} - \sum_L g_{KL} \dot{\gamma}_L. \quad (2.22)$$

The state-dependent coefficients, which can be expressed from equations (2.7) and (2.19) under appropriate smoothness assumptions as

$$\begin{aligned} \mathbf{K}^e &= \Phi_{,qq}, & \mathbf{k} &= -\Phi_{,qT}, & N_K^p &= -\Phi_{,q\alpha} \cdot \boldsymbol{\eta}_K, \\ N_K &= f_{K,q}, & k_K &= f_{K,T}, & g_{KL} &= -f_{K,\alpha} \cdot \boldsymbol{\eta}_L, \end{aligned} \quad (2.23)$$

are assumed to vary in any process continuously and with a bounded rate if the $\dot{\gamma}_K$'s, $\dot{\mathbf{q}}$ and \dot{T} are bounded. Henceforth, the usual short-hand notation is used that a comma as a lower index denotes partial derivative with respect to quantities that follow the comma. Note that \mathbf{K}^e is symmetric but the matrix (g_{KL}) need not be symmetric in general.

Although the expression (2.22) is linear in the rate-variables, the resulting incremental constitutive law defined by equations (2.22) and (2.21) as a whole is *not* linear, since the set of *active* mechanisms (for which $\dot{\gamma}_K > 0$) is not known in advance. It should be noted that the constitutive rate equations (2.22) may be postulated as phenomenological relationships without explicit reference to thermodynamics.

The last term in the expression for $\dot{\mathbf{Q}}$ in equation (2.22), on using $\mathbf{Q} = \mathbf{Q}(q, T, \alpha)$ equal to

$$\dot{\mathbf{Q}}^p \equiv \mathbf{Q}_{,\alpha} \cdot \dot{\alpha} = -\Phi_{,q\alpha} \cdot \dot{\alpha} = -\sum_K N_K^p \dot{\gamma}_K, \quad (2.24)$$

defines the inelastic (plastic) part of the rate of generalized forces, cf. Hill & Rice (1973). If \mathbf{K}^e is invertible then an equivalent dual formulation is available where the roles of $\dot{\mathbf{Q}}$ and $\dot{\mathbf{q}}$ are interchanged. The aspects of duality and invariance have been discussed in detail elsewhere (cf. Hill & Rice 1973; Hill 1978; Kleiber & Raniecki 1985; Petryk 2000a). Therefore, we only mention here that in the dual

formulation, where $\mathbf{q} = \mathbf{q}(\mathbf{Q}, T, \boldsymbol{\alpha})$, the inelastic (plastic) part of the rate of generalized displacements is defined as follows

$$\dot{\mathbf{q}}^p \equiv \mathbf{q}_{,\alpha} \cdot \dot{\boldsymbol{\alpha}} = (\mathbf{K}^e)^{-1} \cdot \sum_K N_K^p \dot{\gamma}_K = -(\mathbf{K}^e)^{-1} \cdot \dot{\mathbf{Q}}^p. \tag{2.25}$$

The rate of free energy defined by formula (2.7) can be expressed as

$$\dot{\Phi} = \mathbf{Q} \cdot \dot{\mathbf{q}} - S \dot{T} - \sum_K \pi_K \dot{\gamma}_K, \quad \pi_K = \pi_K(\mathcal{G}), \tag{2.26}$$

where

$$\pi_K = \mathbf{A} \cdot \boldsymbol{\eta}_K, \quad \sum_K \pi_K \dot{\gamma}_K = \mathbf{A} \cdot \dot{\boldsymbol{\alpha}}. \tag{2.27}$$

It is emphasized that Φ itself is *not* regarded here as a function of the γ_K 's obtained formally by time integration of $\dot{\gamma}_K$'s since usually such γ_K 's are not *state* variables.

In analogy to Rice (1971), the loading functions f_K are now given the following thermodynamic interpretation

$$f_K = \pi_K - \pi_K^c, \quad \pi_K^c \geq 0, \quad \pi_K^c = \pi_K^c(\mathcal{G}), \tag{2.28}$$

where π_K^c is a state-dependent threshold value for a thermodynamic driving force π_K . However, an essential distinction from Rice's theory is that π_K^c may depend here not only on internal variables $\boldsymbol{\alpha}$ but also on *elastic* changes in strain or stress. In this way, the restriction to a normality rule is removed and non-associative plasticity is included in equation (2.28). This is related to stress-dependence of the dissipation function, cf. Collins & Houlsby (1997).

The dissipation function equations (2.16) is specified as follows

$$\dot{\mathcal{D}} = D(\mathcal{G}, \dot{\gamma}_K) = \sum_K \pi_K^c(\mathcal{G}) \dot{\gamma}_K \geq 0, \tag{2.29}$$

in accord to the interpretation of π_K^c as a non-negative value of π_K at which the K -th mechanism actually operates. A virtual $\dot{\gamma}_K \geq 0$ not compatible with the current $f_K(\mathcal{G}) \neq 0$ may be induced after compensating the difference between $\pi_K(\mathcal{G})$ and $\pi_K^c(\mathcal{G})$ by applying to \mathcal{M} an extra set of internal forces (e.g. a perturbing wave) at fixed \mathbf{q} and T . From equations (2.15), (2.27)–(2.29) it follows that the associated rate of work supplied by a disturbing agency reads

$$\dot{W}^{\text{dist}} = - \sum_K f_K \dot{\gamma}_K. \tag{2.30}$$

Hence, we arrive at the following version of equation (2.14)

$$\dot{W} = \mathbf{Q} \cdot \dot{\mathbf{q}} - \sum_K f_K \dot{\gamma}_K. \tag{2.31}$$

Note that the dissipation function \mathcal{D} in the multi-mode inelasticity framework is linear on the whole set of the inelastic multipliers $\dot{\gamma}_K$ in a given state \mathcal{G} . This is a simplification in comparison with a dissipation function (2.16) which, according to equation (2.17), must be linear only within some cone in $\boldsymbol{\alpha}$ -space. In related approaches, the dissipation function D is usually assumed to be convex in $\boldsymbol{\alpha}$, cf. Maugin (1992), Nguyen (2000).

(d) Normality and symmetry

The additional assumptions to be formulated in this subsection will not be needed to formulate conditions for stability of *equilibrium*; however, they will be used later in the analysis of stability of a quasi-static deformation *process*.

The normality flow rule in time-independent inelasticity (plasticity), correctly extended to finite deformation and generalized variables (cf. Hill & Rice 1973; Hill 1978), can be written down for each mechanism *separately* as follows

$$\dot{\mathbf{q}}^p = \dot{\Lambda}_K \frac{\partial \hat{f}_K(\mathbf{Q}, T, \boldsymbol{\alpha})}{\partial \mathbf{Q}} \Leftrightarrow \dot{\mathbf{Q}}^p = -\dot{\Lambda}_K \frac{\partial f_K(\mathbf{q}, T, \boldsymbol{\alpha})}{\partial \mathbf{q}}, \quad \dot{\Lambda}_K \geq 0, \quad (2.32)$$

where $\hat{f}_K(\mathbf{Q}, T, \boldsymbol{\alpha}) \equiv f_K(\mathbf{q}, T, \boldsymbol{\alpha})$ under the assumed invertibility of \mathbf{K}^e . It is essential that the inelastic (plastic) parts of the rates of generalized displacements and forces are defined as in equations (2.25) and (2.24), respectively. The equivalence in equation (2.32) follows from the chain rule of differentiation of f at fixed T and $\boldsymbol{\alpha}$ along with the symmetry of \mathbf{K}^e . The normality rule (2.32) can be deduced from the classical Ilyushin postulate (Hill 1968; Hill & Rice 1973). Equation (2.32) can be rewritten as

$$\dot{\gamma}_K \mathbf{N}_K^p = \dot{\Lambda}_K \mathbf{N}_K. \quad (2.33)$$

Let each loading function f_K be normalized such that $|\mathbf{N}_K| = |\mathbf{N}_K^p|$. Then, the normality rule (2.32) reduces to

$$\mathbf{N}_K^p = \mathbf{N}_K. \quad (2.34)$$

If the normality rule in the form (2.34) is assumed then the stability analysis is more conclusive when the following symmetry restriction is also imposed

$$g_{KL} = g_{LK} \quad \text{for } K, L \in \mathcal{P}. \quad (2.35)$$

It is worth mentioning that the above normality and symmetry conditions can be deduced from the thermodynamic postulate that requires the dissipation evaluated to the second-order terms along any path connecting two adjacent states to be always minimized on a straight (or direct) path (Petryk 2000b). Both equations (2.34) and (2.35), although not universally valid, may thus be regarded as physically plausible restrictions.

3. Thermodynamic conditions for stability of equilibrium

(a) Compound thermodynamic system

We shall consider a compound system \mathcal{A} that consists of the material sample \mathcal{M} , a loading device as a work source, and a surrounding heat reservoir of constant absolute temperature $\bar{T} > 0$. According to the approximations adopted, the Helmholtz free energy Φ and entropy S of \mathcal{M} are treated as functions of $\mathcal{G} = (\mathbf{q}, T, \boldsymbol{\alpha})$, i.e. of generalized displacements \mathbf{q} , internal state variables $\boldsymbol{\alpha}$ and a *uniform* temperature T which may differ from \bar{T} .

The loading device is defined as a mechanical conservative system, of a potential energy \mathcal{Q} being a twice differentiable function of (\mathbf{q}, λ) , which supplies

work W^{ext} (but no heat) to \mathcal{M} , such that

$$\dot{W}^{\text{ext}} = \mathbf{P} \cdot \dot{\mathbf{q}}, \quad \mathbf{P} = \mathbf{P}(\mathbf{q}, \lambda) \equiv -\mathcal{Q}_{,q}(\mathbf{q}, \lambda), \tag{3.1}$$

where λ is a loading parameter. The considerations are restricted to cases where \mathbf{q} is not constrained kinematically. Entropy of the mechanical loading device is neglected for simplicity, so that \mathcal{Q} is interpreted as the internal energy of the device. As a counterpart to equation (2.22), we have the following constitutive rate equation for the vector of generalized loads \mathbf{P}

$$\dot{\mathbf{P}} = \dot{\mathbf{P}}^* - \mathbf{K}^* \cdot \dot{\mathbf{q}}, \quad \dot{\mathbf{P}}^* = -\mathcal{Q}_{,q\lambda} \dot{\lambda}, \quad \mathbf{K}^* = \mathcal{Q}_{,qq}. \tag{3.2}$$

The matrix \mathbf{K}^* describes the effect of *deformation-sensitive loading* which should not be excluded from general investigations of material stability (cf. Hill 1978; Petryk 2000a,b).

Kinetic energy K_A of the system \mathcal{A} is merely assumed to be non-negative and to vanish in equilibrium.

Internal energy and entropy of the heat reservoir are denoted by E_h and S_h , respectively. We shall assume that the system \mathcal{A} is thermally isolated, so that heat Q supplied to the material sample \mathcal{M} from the reservoir satisfies $\delta Q = -\delta E_h = -\bar{T} \delta S_h$. Any extra heat supplied to \mathcal{M} by a disturbing agency is also included in δQ .

The total (internal + kinetic) energy E_A and total entropy S_A of the compound system \mathcal{A} are

$$E_A = U + \mathcal{Q} + E_h + K_A, \quad S_A = S + S_h, \tag{3.3}$$

with U as the internal energy of \mathcal{M} .

At a fixed value of the loading parameter λ and in the absence of external disturbances, the system \mathcal{A} is regarded as fully isolated. This fundamental property motivates the selection of \mathcal{A} for thermodynamic stability investigations. An equilibrium state of \mathcal{A} is characterized by (and for convenience identified with) $\mathcal{G} = \mathcal{G}^0$ constant in time at fixed λ .

The use of the first law (2.11) yields

$$\dot{E}_A = \dot{W} + \dot{\mathcal{Q}} + \dot{K}_A. \tag{3.4}$$

From the second law we obtain

$$\dot{S}_A = \dot{S} - \dot{Q}/\bar{T} \equiv \dot{S}^i \geq 0. \tag{3.5}$$

Under the assumptions introduced, the entropy production rate \dot{S}^i consists of two parts, thermal \dot{S}^{th} and intrinsic rate-independent \dot{S}^{in} ,

$$\dot{S}^i = \dot{S}^{\text{th}} + \dot{S}^{\text{in}}, \quad \dot{S}^{\text{th}} = \left(\frac{1}{T} - \frac{1}{\bar{T}} \right) \dot{Q}, \quad \dot{S}^{\text{in}} = \frac{1}{T} \dot{D}. \tag{3.6}$$

More generally, other contributions to $(\dot{S}^i - \dot{S}^{\text{in}})$ might exist, e.g. due to viscous dissipation, which are neglected here. This would not influence the extended condition for stability, to be based on the second law assumed in the following strengthened form: *both \dot{S}^{in} and the complementary part of \dot{S}^i are separately*

non-negative

$$\dot{S}^{\text{in}} \geq 0, \quad \dot{S}^{\text{i}} - \dot{S}^{\text{in}} \geq 0. \tag{3.7}$$

Before formulating the extended condition, the classical condition for stability is briefly discussed first.

(b) *Classical condition for stability of equilibrium*

Let the prefix Δ denote an increment in some quantity in a process, perturbed in general, leading from an equilibrium state \mathcal{G}^0 to a state \mathcal{G} ,

$$\Delta\Psi = \Psi(\mathcal{G}) - \Psi(\mathcal{G}^0) \quad \text{or} \quad \Delta\Psi = \Psi|_{\mathcal{G}} - \Psi|_{\mathcal{G}^0}, \tag{3.8}$$

for a state variable or a quantity which depends also on the transition path between the two states, respectively. Henceforth up to the end of §4, all increments are considered at a *fixed loading parameter* λ .

The classical condition for stability of an equilibrium state \mathcal{G}^0 of an isolated system \mathcal{A} can be written down in the form

$$\Delta S_{\mathcal{A}} < 0 \quad \text{whenever} \quad \Delta E_{\mathcal{A}} = 0. \tag{3.9}$$

Importance of inequality (3.9) as a fundamental condition for thermodynamic stability is unquestionable. However, for the specific class of solids with *rate-independent* dissipation, the condition (3.9) turns out to be too restrictive. To show this, is it convenient to reformulate inequality (3.9), for the compound system \mathcal{A} defined above, equivalently as

$$\Delta E_{\mathcal{A}} - \bar{T}\Delta S_{\mathcal{A}} > 0 \quad \text{for every } \mathcal{G} \neq \mathcal{G}^0. \tag{3.10}$$

The equivalence follows from the possibility of manipulating the value of $\Delta E_{\mathcal{A}}$ by exchanging extra heat between external sources and the heat reservoir, without affecting the value of the left-hand expression in inequality (3.10). It is a standard result that the conditions

$$T = \bar{T} \quad \text{and} \quad \mathbf{Q} = \mathbf{P} \quad \text{in } \mathcal{G}^0, \tag{3.11}$$

of thermal and mechanical equilibrium in \mathcal{A} , respectively, follow as consequences of inequality (3.10) by equating to zero the first quasi-static variation of the left-hand expression in inequality (3.10) *at fixed* α . However, if α is allowed to vary then inequality (3.10) in an equilibrium state implies also (cf. §3c) that

$$\delta E_{\mathcal{A}} - \bar{T}\delta S_{\mathcal{A}} = -\mathbf{A}(\mathcal{G}^0) \cdot \delta\alpha \geq 0 \quad \text{in } \mathcal{G}^0. \tag{3.12}$$

As discussed in §2, the thermodynamic forces \mathbf{A} generally do not vanish in an equilibrium state, moreover, $\dot{\alpha}$ compatible with \mathbf{A} and corresponding to a positive intrinsic dissipation rate must satisfy equation (2.13)₂ with strict inequality. This contradicts inequality (3.12) unless there is no rate-independent intrinsic dissipation.

It may be concluded that the classical thermodynamic condition (3.9) does not provide a satisfactory basis for investigations of stability in materials or systems with rate-independent dissipation. The existing attempts (cf. Bažant & Cedolin 1991) refer, in effect, only to equivalent (essentially elastic) systems without an explicit definition of intrinsic dissipation of rate-independent type.

(c) *Extended condition for stability of equilibrium*

The above conclusion that the maximum entropy inequality (3.9) is too restrictive for materials or systems with rate-independent dissipation is not surprising. In view of its interpretation as a *sufficiency* condition for stability of equilibrium, which follows from the second law, its rigorous fulfillment would also exclude local material instabilities at the micro-level. Adopting the point of view that just local instabilities appearing at a lower scale represent the physical source of the rate-independent inelastic behaviour of the macroscopic material sample, the reason for the over-sufficiency of the classical condition becomes clear.

The classical condition for thermodynamic stability must thus be appropriately extended to comply with the concept of rate-independent dissipation. This can be done (Petryk 1995) by taking, as a basis, the second law of thermodynamics not in its fundamental form (3.5) but rather in its *strengthened* version (3.7). More specifically, it is the *second* inequality in equation (3.7) which replaces that in equation (3.5) in the usual argument.

Define a *functional* V by

$$\Delta V \equiv \Delta E_{\mathcal{A}} - T\Delta S_{\mathcal{A}} + \bar{T}\Delta S^{\text{in}}, \tag{3.13}$$

where

$$\Delta S^{\text{in}} = \int_{t^0}^t \dot{S}^{\text{in}} dt = \int_{t^0}^t \frac{1}{T} \dot{D} dt \tag{3.14}$$

is the *path-dependent* intrinsic production of entropy in any *perturbed* process leading from an equilibrium state $\mathcal{G}^0 = \mathcal{G}(t^0)$ to a current state $\mathcal{G}(t)$. The value of V varies continuously with respect to time t in any process, since the energy and entropy do. Moreover, in a process free of actual (persistent) disturbances, which follows any earlier temporary disturbance, we have

$$\dot{V} = \dot{E}_{\mathcal{A}} - \bar{T}(\dot{S}^{\text{i}} - \dot{S}^{\text{in}}) \leq 0, \tag{3.15}$$

from equation (3.7)₂ and the energy conservation law $\dot{E}_{\mathcal{A}} = 0$ at fixed λ . This shows that ΔV is a natural candidate for the Lyapunov functional.

We arrive thus at the following conclusion (Petryk 1993b, 1995): *If*

$$\Delta V > 0 \quad \text{for every } \mathcal{G} \neq \mathcal{G}^0, \tag{3.16}$$

and for every path connecting \mathcal{G}^0 with \mathcal{G} , then the equilibrium state \mathcal{G}^0 of an isolated system \mathcal{A} is stable with respect to the distance $|\Delta V|$.

The condition (3.16) may be regarded as an extension of the classical condition (3.10) to systems with rate-independent dissipation. Further discussion of condition (3.16) and its consequences for stability in continuous systems can be found in the references quoted above.

On substituting equation (3.3) into (3.13), we obtain

$$\Delta V = \Delta U - \bar{T}\Delta S + \bar{T}\Delta S^{\text{in}} + \Delta\Omega + \Delta K_{\mathcal{A}}. \tag{3.17}$$

The theory of thermodynamic stability can be developed by using the functional E defined by

$$\Delta E = \Delta U - \bar{T}\Delta S + \bar{T}\Delta S^{\text{in}} + \Delta\Omega. \tag{3.18}$$

Since, the kinetic energy $K_{\mathcal{A}}$ is non-negative and vanishes in equilibrium, from condition (3.16) we have the basic corollary: *Stability of an equilibrium state \mathcal{G}^0*

(with respect to the distance $|\Delta V|$) is ensured if

$$\Delta E > 0 \quad \text{for every } \mathcal{G} \neq \mathcal{G}^0, \tag{3.19}$$

and for every path connecting \mathcal{G}^0 with \mathcal{G} .

It is worth mentioning that condition (3.16) implies $K_{\mathcal{A}} < \Delta V$, so that the kinetic energy in a process satisfying inequality (3.15) is bounded by the initial positive value of ΔV adopted as a measure of an initial disturbance strength.

For isothermal processes, equation (3.18) simplifies to

$$\Delta E = \Delta \Phi + \Delta \mathcal{D} + \Delta \mathcal{Q} \quad (T \equiv \bar{T}), \tag{3.20}$$

or equivalently, on using equation (2.10), to

$$\Delta E = \Delta W + \Delta \mathcal{Q} \quad (T \equiv \bar{T}). \tag{3.21}$$

Under this restriction to constant temperature, the quantity E can be identified (cf. equation (3.4)) with the internal energy of the compound system \mathcal{A} . Isothermal stability conditions for inelastic solids, which can be expressed in terms of the quantities in equation (3.20) or (3.21), have been previously based on the energy balance, cf. Hill (1958), Nguyen & Radenkovic (1976), Petryk (1982) and Nguyen (1993). Their thermodynamic justification follows from the extended condition (3.19) by the limiting passage from a non-isothermal case (Petryk 1995).

Our present aim is to examine consequences of the condition (3.19) for the discretized material sample \mathcal{M} . The primary task is to examine the sign of ΔE after infinitesimal deviations from \mathcal{G}^0 . In view of the non-existence of the standard variation of ΔS^{in} , an infinitesimal increment denoted by a prefix δ or the associated rate (2.6) denoted by a superimposed dot have the meaning of a *one-sided* directional variation or derivative, respectively.

A virtual one-sided variation of equation (3.18), on substituting equations (2.7), (2.16), (3.1) and (3.6)₃ can be written down as follows

$$\delta E = (\mathbf{Q} - \mathbf{P}) \cdot \delta \mathbf{q} + (T - \bar{T}) \delta S + \frac{\bar{T}}{T} D(\mathcal{G}, \delta \boldsymbol{\alpha}) - \mathbf{A} \cdot \delta \boldsymbol{\alpha}. \tag{3.22}$$

As a condition necessary for fulfillment of equation (3.19), in place of the previous unacceptable condition (3.12), in an equilibrium state we now obtain

$$\delta E = D(\mathcal{G}^0, \delta \boldsymbol{\alpha}) - \mathbf{A}(\mathcal{G}^0) \cdot \delta \boldsymbol{\alpha} \geq 0 \quad \text{in } \mathcal{G}^0. \tag{3.23}$$

Of course, the infinitesimal increments can be replaced by *forward* rates defined by equation (2.6).

It should be remarked that while the condition (3.23) is necessary for the fulfillment of inequality (3.19), its status as a necessity condition for stability itself is less clear. All $\dot{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{G}^0)$ compatible with the actual thermodynamic forces $\mathbf{A}(\mathcal{G}^0)$ satisfy equation (3.23) with the equality sign, cf. equation (2.17). Any other $\dot{\boldsymbol{\alpha}}$ requires an instantaneous *finite* perturbation of averaging thermodynamic forces \mathbf{A} . It is an open question in which circumstances such perturbations may be regarded as physically realizable. We will consider below only those perturbations which satisfy equation (3.23) with the equality sign and thus do not require instantaneous finite perturbation of actual thermodynamic forces. Consequently, second-order terms are to be analysed.

(d) *Second-order stability conditions*

The stability analysis will now be continued within the framework of multi-mode inelasticity, where $\dot{\mathbf{a}}$ is expressible through equation (2.19) in terms of $\overset{\circ}{\gamma}_K$'s. The results can be compared with differently formulated second-order stability conditions given elsewhere (cf. Hill 1958, 1978; Nguyen 1993, 2000; Petryk 1993*b*, 1995).

Infinitesimal increments are henceforth replaced by forward rates (2.6). Equation (3.22) is transformed to

$$\dot{E} = (\mathbf{Q} - \mathbf{P}) \cdot \dot{\mathbf{q}} + (T - \bar{T})\dot{S} + \sum_K \left(\frac{\bar{T}}{T} \pi_K^c - \pi_K \right) \overset{\circ}{\gamma}_K, \tag{3.24}$$

which is applicable in any state. In particular, inequality (3.23) is satisfied automatically for all $\overset{\circ}{\gamma}_K \geq 0$ if $f_K(\mathcal{G}^0) \leq 0$.

Under no instantaneous finite perturbations of driving forces π_K , all the conditions in equation (2.20) are satisfied in the examined equilibrium state \mathcal{G}^0 . Denote by $\mathcal{P}^0 = \mathcal{P}(\mathcal{G}^0)$ the set of indices of *potentially active* mechanisms in the state \mathcal{G}^0 , i.e. $K \in \mathcal{P}^0 \Leftrightarrow f_K(\mathcal{G}^0) = 0$, and by $[\psi]$ a column matrix of components ψ_K with $K \in \mathcal{P}^0$. Relations like $[\psi] \geq [0]$ are to be read componentwise. Then, the set of all $[\overset{\circ}{\gamma}] \geq [0]$ can be identified with the set of virtual modes of inelastic deformation realizable under *arbitrary small* perturbations of driving forces π_K . The required perturbations of π_K are initially zero in \mathcal{G}^0 and their magnitude grows in time with a finite rate if the consistency conditions (2.21) are violated in the state \mathcal{G}^0 .

From equation (3.24) it follows that $\dot{E} = 0$ for every set $(\dot{\mathbf{q}}, \dot{T}, [\overset{\circ}{\gamma}])$ in an equilibrium state \mathcal{G}^0 . A natural further step is to examine the sign of the forward second-order rate of ΔE (with respect to an appropriate time scale) calculated as the rate of equation (3.24). On substituting next the equilibrium conditions and rearranging, this yields²

$$\ddot{E} = (\dot{\mathbf{Q}} - \dot{\mathbf{P}}) \cdot \dot{\mathbf{q}} + \dot{T}(\dot{S} - \dot{S}^{\text{in}}) - \sum_{K \in \mathcal{P}^0} \dot{f}_K \overset{\circ}{\gamma}_K \quad \text{in } \mathcal{G}^0. \tag{3.25}$$

The second-order condition for stability of equilibrium, corresponding to the basic condition (3.19), is expressed as follows³

$$\ddot{E} > 0 \quad \text{for every non-zero } \left(\dot{\mathbf{q}}, \dot{T}, [\overset{\circ}{\gamma}] \geq [0] \right) \quad \text{in } \mathcal{G}^0. \tag{3.26}$$

It is pointed out that condition (3.26) is, in general, not sufficient for stability but only for directional stability of an equilibrium state \mathcal{G}^0 . It excludes the most straightforward form of instability, namely, a spontaneous departure from equilibrium on a *direct* path along which all macroscopic state variables vary proportionally with respect to some distance measure, with accuracy to the first order.

It has also not been shown that violation of inequality (3.26) implies a physically meaningful instability. Accordingly, inequality (3.26) or its particular

² A non-zero increment $\delta \overset{\circ}{\gamma}_K$ for $K \notin \mathcal{P}^0$ has been excluded as requiring an instantaneous finite perturbation of π_K . It may be noted that such $\delta \overset{\circ}{\gamma}_K$ would give a positive contribution to \ddot{E} , in view of the implication $\overset{\circ}{\gamma}_K = 0 \Rightarrow \delta \overset{\circ}{\gamma}_K \geq 0$, not influencing thus the stability condition (3.26).

³ Non-zero $(\dot{\mathbf{q}}, \dot{T}, [\overset{\circ}{\gamma}] \geq [0])$ means that *some* of $\dot{\mathbf{q}}$, \dot{T} or $\overset{\circ}{\gamma}_K$ differs from zero, for $K \in \mathcal{P}^0$ and all $\overset{\circ}{\gamma}_K \geq 0$.

versions below are interpreted merely as sufficiency conditions for directional stability of a general or specified type, respectively.

From inequality (3.26) we easily deduce several more specialized conditions (all formulated in \mathcal{G}^0):

For thermal stability ($\dot{\mathbf{q}} = \mathbf{0}, [\dot{\gamma}] = [0]$)

$$-\Phi_{,TT} > 0, \tag{3.27}$$

i.e. the classical condition of positiveness of the specific heat at constant deformation.

For elastic stability ($\dot{T} = 0, [\dot{\gamma}] = [0]$)

$$\dot{\mathbf{q}} \cdot (\mathbf{K}^e + \mathbf{K}^*) \cdot \dot{\mathbf{q}} > 0 \text{ for every } \dot{\mathbf{q}} \neq \mathbf{0}, \tag{3.28}$$

which coincides with the classical condition of positive definiteness of the second variation of the total potential energy of the *system*.

For thermoelastic stability ($[\dot{\gamma}] = [0]$): the conditions (3.27) and (3.28) jointly, and no other condition is generated.

For intrinsic stability ($\dot{\mathbf{q}} = \mathbf{0}, \dot{T} = 0$)

$$\sum_{K,L \in \mathcal{P}^0} g_{KL} \dot{\gamma}_K \dot{\gamma}_L > 0 \text{ for every non-zero } [\dot{\gamma}] \geq [0], \tag{3.29}$$

i.e. stability against spontaneous internal structural rearrangements in the material sample at fixed macroscopic deformation and temperature. As a thermodynamic condition for material stability, equation (3.29) was apparently derived first in (Petryk 1993b), although that inequality appeared earlier in the literature in a different context, cf. Havner (1992).

Henceforth, we restrict attention to isothermal stability, assuming that $T \equiv \bar{T}$.

On substituting equations (2.22) and (3.2), the condition (3.26) reduces to the following second-order condition for isothermal stability of equilibrium of the material sample under deformation-sensitive loading

$$\dot{\mathbf{q}} \cdot (\mathbf{K}^e + \mathbf{K}^*) \cdot \dot{\mathbf{q}} - \dot{\mathbf{q}} \cdot \sum_{K \in \mathcal{P}^0} (\mathbf{N}_K^p + \mathbf{N}_K) \dot{\gamma}_K + \sum_{K,L \in \mathcal{P}^0} g_{KL} \dot{\gamma}_K \dot{\gamma}_L > 0 \tag{3.30}$$

for every non-zero $\left(\dot{\mathbf{q}}, [\dot{\gamma}] \geq [0] \right)$.

This condition obviously includes conditions (3.28) and (3.29) as immediate consequences.

It may be important to note that the stability condition (3.30) and its consequences that follow, although derived from thermodynamic considerations, eventually involve only those constitutive parameters that appear in the *phenomenological* constitutive rate equations, (2.22) for the material and (3.2) for the loads. In effect, applications of condition (3.30) do not require specification of internal state variables or thermodynamic forces. The particular form (3.30) of the stability condition appears to be new.

We shall assume that the condition (3.28) of elastic stability is satisfied, which allows the rate of generalized displacements to be eliminated from the stability condition (cf. Nguyen 1984). Then the left-hand expression in condition (3.30) for

given $[\overset{\circ}{\gamma}]$ reaches a minimum with respect to $\dot{\mathbf{q}}$ at

$$\dot{\mathbf{q}}^m = \frac{1}{2} \mathbf{G}^* \cdot \sum_{K \in \mathcal{P}^0} (\mathbf{N}_K^p + \mathbf{N}_K) \overset{\circ}{\gamma}_K, \quad \mathbf{G}^* \equiv (\mathbf{K}^e + \mathbf{K}^*)^{-1}. \quad (3.31)$$

On substituting $\dot{\mathbf{q}} = \dot{\mathbf{q}}^m$ into equation (3.30), the following condition is obtained

$$\sum_{K,L \in \mathcal{P}^0} \left(g_{KL} - \frac{1}{4} (\mathbf{N}_K^p + \mathbf{N}_K) \cdot \mathbf{G}^* \cdot (\mathbf{N}_L^p + \mathbf{N}_L) \right) \overset{\circ}{\gamma}_K \overset{\circ}{\gamma}_L > 0 \quad (3.32)$$

for every non-zero $[\overset{\circ}{\gamma}] \geq [0]$.

Clearly, the conditions (3.28) and (3.32) considered jointly are equivalent to condition (3.30). The condition (3.32) is more restrictive than condition (3.29) on account of the positive definiteness of \mathbf{G}^* implied by inequality (3.28).

The condition (3.32) ensures directional stability of equilibrium for the most unfavourable combination of disturbances both at the macro-level (associated with $\dot{\mathbf{Q}} \neq \dot{\mathbf{P}}$) and at the micro-level (associated with $f_K \overset{\circ}{\gamma}_K \neq 0$), and may be thus over-sufficient for directional stability. Two less restrictive versions of inequality (3.32) are as follows.

First, if external quasi-static equilibrium remains unperturbed then, from equations (2.22) and (3.2),

$$\dot{\mathbf{Q}} = \dot{\mathbf{P}} \quad \Leftrightarrow \quad \dot{\mathbf{q}} = \mathbf{G}^* \cdot \sum_K \mathbf{N}_K^p \overset{\circ}{\gamma}_K. \quad (3.33)$$

On substituting equation (3.33), the condition (3.30) reduces to the following condition for internal stability under deformation-sensitive loading

$$- \sum_{K \in \mathcal{P}^0} f_K \overset{\circ}{\gamma}_K = \sum_{K,L \in \mathcal{P}^0} g_{KL}^* \overset{\circ}{\gamma}_K \overset{\circ}{\gamma}_L > 0 \quad \text{for every non-zero } [\overset{\circ}{\gamma}] \geq [0], \quad (3.34)$$

where

$$g_{KL}^* = g_{KL} - \mathbf{N}_K \cdot (\mathbf{K}^e + \mathbf{K}^*)^{-1} \cdot \mathbf{N}_L^p. \quad (3.35)$$

This is another condition for stability against spontaneous internal structural rearrangements in the material sample. In contrast to condition (3.29) applicable under rigid boundary constraints for \mathcal{M} , the condition (3.34) is explicitly dependent on the constraints stiffness \mathbf{K}^* .

It should be noted that if the additional assumption of normality ($\mathbf{N}_K^p = \mathbf{N}_K$) is introduced then the conditions (3.32) and (3.34) coincide. This means that a macroscopically quasi-static deformation mode in equation (3.33) is then most critical for internal stability.

Second, assume alternatively that equation (3.33) is perturbed while there are no internal disturbances that would perturb directly the values of driving forces π_K . Then the consistency condition (2.21) must be satisfied, which for each given $\dot{\mathbf{q}}$ leads to the linear complementarity problem for unknown $[\overset{\circ}{\gamma}]$. Suppose that inequality (3.29) holds, which ensures existence of a solution $[\overset{\circ}{\gamma}]$ to that problem for every $\dot{\mathbf{q}}$ (cf. Cottle *et al.* 1992). Then, $\dot{\mathbf{Q}}$ in a given state is expressible in terms of $\dot{\mathbf{q}}$, although not necessarily unique. For those $\dot{\mathbf{Q}}$, the condition (3.30) reduces

now to

$$\dot{\mathbf{Q}} \cdot \dot{\mathbf{q}} > -\dot{\mathbf{q}} \cdot \mathbf{K}^* \cdot \dot{\mathbf{q}} \quad \text{for every } \dot{\mathbf{q}} \neq 0. \tag{3.36}$$

For dead loading ($\mathbf{K}^* = 0$), the familiar condition $\dot{\mathbf{Q}} \cdot \dot{\mathbf{q}} > 0$ is recovered, which corresponds to Hill's (1958) *integral* condition for stability in a continuum. Conditions in the form (3.36) and (3.37) for material stability at locally uniform strain were given by Hill (1978).

If all principal minors of the matrix (g_{KL}) with $K, L \in \mathcal{P}^0$ are positive then the solution $[\dot{\gamma}]$ to the above mentioned linear complementarity problem is unique; cf. (Cottle *et al.* 1992) for the mathematical theory and (Maier 1969; Petryk 2000a) for its applications to inelastic materials. Then $\dot{\mathbf{Q}}$ in a given state becomes a continuous and piecewise linear function of $\dot{\mathbf{q}}$, and inequality (3.36) can be written down as follows

$$\dot{\mathbf{q}} \cdot (\mathbf{K}(\dot{\mathbf{q}}) + \mathbf{K}^*) \cdot \dot{\mathbf{q}} > 0 \quad \text{for every } \dot{\mathbf{q}} \neq 0, \quad \mathbf{K} = \frac{\partial \dot{\mathbf{Q}}}{\partial \dot{\mathbf{q}}}. \tag{3.37}$$

If $\dot{\mathbf{Q}}(\dot{\mathbf{q}})$ is not differentiable at some $\dot{\mathbf{q}}$ then inequality (3.37) is to be understood in the limit sense as this $\dot{\mathbf{q}}$ is approached from any side. \mathbf{K} is identified as *the tangent stiffness matrix* for the material sample, dependent on the *direction* of $\dot{\mathbf{q}}$ in a piecewise-constant manner. Of course, inequality (3.37) is different from (and less restrictive than) the requirement of positive definiteness of every well-defined matrix $(\mathbf{K}(\dot{\mathbf{q}}) + \mathbf{K}^*)$.

4. Stability against spontaneous formation of deformation bands

The specific kind of material instability related to the localization of deformation into planar bands has attracted much attention in the last decades. It is beyond the scope of this paper to make even a brief review of the relevant vast literature; cf. (Rice 1977) as a basic reference and a survey, e.g. (Petryk 1997b). The present aim is only to show how the second-order condition (3.26) for stability of equilibrium, and its versions discussed above, can be directly applied to the problem of *spontaneous* formation of shear bands under *fixed* remote loads *or* displacements. For this purpose, the isothermal stability condition (3.30) will be specified for a material element placed *within* a possible deformation band in an infinite homogeneous continuum. The material is supposed to be stressed to a level such that multiple internal mechanisms of inelastic deformation may be activated.

Constitutive rate equations which accompany equations (2.20) and (2.21) for a material element at finite deformation are assumed in the following general form

$$\dot{\mathbf{S}} = \mathbf{C}^e \cdot \dot{\mathbf{F}} - \sum_K \mathbf{A}_K^p \dot{\gamma}_K, \quad \dot{f}_K = \mathbf{A}_K \cdot \dot{\mathbf{F}} - \sum_L g_{KL} \dot{\gamma}_L, \tag{4.1}$$

analogous to equation (2.22) at fixed T , where \mathbf{F} , \mathbf{S}^T and \mathbf{C}^e are the deformation gradient, nominal stress and related elastic stiffness moduli tensor, respectively. The reader is referred to Hill (1978) and Petryk (2000a) for the relations between equations (4.1) and constitutive equations expressed in other (objective) rate-variables.

It is assumed that the deformation gradient outside a planar band remains unaffected by the band formation, and is thus kept *fixed* in the present study of

stability of *equilibrium*. Then, the rate of \mathbf{F} within a deformation band of a unit normal \mathbf{n} in the reference configuration must satisfy the standard kinematic compatibility condition in the form

$$\dot{\mathbf{F}} = \dot{\mathbf{g}} \otimes \mathbf{n}, \tag{4.2}$$

where $\dot{\mathbf{g}}$ is an arbitrary spatial vector and \otimes denotes a dyadic product. On substituting equation (4.2) into equation (4.1) and multiplying the former equation (4.1) by \mathbf{n} , we obtain

$$\dot{\mathbf{S}}\mathbf{n} = \mathbf{A}^e \dot{\mathbf{g}} - \sum_K \mathbf{A}_K^p \mathbf{n} \dot{\gamma}_K, \quad \dot{f}_K = (\mathbf{A}_K \mathbf{n}) \cdot \dot{\mathbf{g}} - \sum_L g_{KL} \dot{\gamma}_L. \tag{4.3}$$

Here, the notation $(\mathbf{S}\mathbf{n})_i = S_{ij}n_j$ is used, and \mathbf{A}^e is the acoustic tensor for the *purely elastic* constitutive branch, of components

$$A_{ik}^e = A_{ik}^e(\mathbf{C}^e, \mathbf{n}) = C_{ijkl}^e n_j n_l. \tag{4.4}$$

The condition for purely elastic stability is that \mathbf{A}^e is positive definite, i.e. \mathbf{C}^e is strongly elliptic, which is henceforth assumed.

From the assumption of constant deformation gradient outside the band it follows that the nominal surface tractions on the band boundaries are also constant. The *second* variation of the potential energy of loads thus vanishes, and the condition (3.26) reduces to $\ddot{W} > 0$ as can be seen from equation (3.21). Generally, $\dot{\mathbf{g}}$ may vary across the band, and inelastic deformation in some layer within the band may be compensated by elastic unloading in other layers. However, the stability condition $\ddot{W} > 0$ is readily satisfied for all possible distributions of $\dot{\mathbf{g}}$ across the band if and only if it is satisfied for every constant $\dot{\mathbf{g}}$. Hence, on using equations (4.2) and (4.3), it follows that the stability conditions from §3d become directly applicable with the following correspondence

$$(\dot{\mathbf{g}}, \dot{\mathbf{Q}}, \mathbf{K}^e) \leftrightarrow (\dot{\mathbf{g}}, \dot{\mathbf{S}}\mathbf{n}, \mathbf{A}^e) \quad \text{and} \quad (\mathbf{N}_K^p, \mathbf{N}_K) \leftrightarrow (\mathbf{A}_K^p \mathbf{n}, \mathbf{A}_K \mathbf{n}), \tag{4.5}$$

along with $\mathbf{K}^* = \mathbf{0}$.

From condition (3.32) we obtain the following condition for stability of equilibrium against spontaneous formation of deformation bands of orientation \mathbf{n}

$$\sum_{K,L \in \mathcal{P}} g_{KL}^{\text{band}^\circ} \dot{\gamma}_K \dot{\gamma}_L > 0 \quad \text{for every non-zero } [\dot{\gamma}] \geq [0], \tag{4.6}$$

where

$$g_{KL}^{\text{band}} = g_{KL} - \frac{1}{4} (\mathbf{A}_K^p \mathbf{n} + \mathbf{A}_L \mathbf{n}) \cdot (\mathbf{A}^e)^{-1} (\mathbf{A}_L^p \mathbf{n} + \mathbf{A}_K \mathbf{n}). \tag{4.7}$$

If the less restrictive version (3.34) of the stability condition is used then we arrive at equation (4.6) with g_{KL}^{band} defined by

$$g_{KL}^{\text{band}} = g_{KL} - (\mathbf{A}_K \mathbf{n}) \cdot (\mathbf{A}^e)^{-1} (\mathbf{A}_L^p \mathbf{n}), \tag{4.8}$$

in place of equation (4.7). In either version, the condition (4.6) appears to be new.

The stability condition (4.6) can be compared with a *different* condition for uniqueness, also expressed in terms of the matrix (4.8), which excludes quasi-static *bifurcation* within a band at *varying* deformation gradient outside the band (Petryk 2000a).

5. Stability of a quasi-static process

(a) Assumptions

We proceed now to examining stability of a quasi-static *process* (or path) induced in the compound system \mathcal{A} by a loading program with $\dot{\lambda} \neq 0$. The author has not found in the literature a counterpart to the condition (3.19) which would be of similar generality and at the same time pertinent to stability of deformation processes with rate-independent dissipation. From the argument given in §3*b* it follows that the maximum entropy criterion, referred to in such context by Bažant & Cedolin (1991), does not provide a satisfactory basis for examining path stability of systems with rate-independent dissipation.

In comparison with the general circumstances in which the condition (3.19) for stability of equilibrium has been obtained, the path stability analysis will be carried out below under the following specific assumptions

- (i) constant temperature ($T \equiv \bar{T}$),
- (ii) multi-mode inelasticity (§2*c*),
- (iii) normality rule (2.34),
- (iv) symmetry property (2.35),
- (v) the first approximation, in which all state-dependent parameters in equations (2.22) and (3.2) are fixed.

It will be shown that under the assumptions (i)–(v) the criterion of path stability, although different from equation (3.19), can still be formulated in terms of ΔE expressed equivalently by equation (3.20) or (3.21). For isothermal processes, ΔE can be defined (Petryk 1982) as the amount of energy to be supplied from external sources to the examined *system* in order to produce quasi-statically a deformation increment, in general with the help of perturbing forces.

From equation (3.24) complemented by the term proportional to $\dot{\lambda}$, we now have

$$\dot{E} = (\mathbf{Q} - \mathbf{P}) \cdot \dot{\mathbf{q}} - \sum_K f_K \dot{\gamma}_K + \Omega_{,\lambda} \dot{\lambda}. \quad (5.1)$$

Consider a fundamental, unperturbed process \mathcal{F}^0 defined as a theoretical, isothermal and quasi-static solution $\mathcal{G}(t)$ that satisfies (2.20) and (3.11) at every instant t , obtained for the system \mathcal{A} for a given function $\lambda(t)$. The fundamental process is assumed to go on sufficiently slowly to allow inertia effects to be neglected. Quantities pertinent to the fundamental process will be distinguished by a superscript ‘0’. Evidently, $\dot{E}^0 = \Omega_{,\lambda} \dot{\lambda}$.

We will examine stability of the fundamental process \mathcal{F}^0 at a selected deformation stage, defined by a small interval of λ starting from a given equilibrium state, say \mathcal{G}_R^0 . According to the assumption (v), the process \mathcal{F}^0 is linearized in that interval, i.e. directions of $\dot{\mathbf{q}}^0$ and $[\dot{\gamma}]^0$ are treated as fixed while their magnitude is proportional to $\dot{\lambda}$ (which may vary in time). However, the stability problem itself is still non-linear. The state \mathcal{G}_R^0 is chosen such that no abrupt unloading takes place in \mathcal{G}_R^0 along the fundamental path; consequently, the set \mathcal{P}^0 of the indices of potentially active mechanisms does not change in the linearized process \mathcal{F}^0 , including the state \mathcal{G}_R^0 . In the notation $\mathcal{P}^0 = \mathcal{P}(\mathcal{G}_R^0)$ and $[\psi] = (\psi_K)$, $K \in \mathcal{P}^0$, analogous to that used previously, we thus select \mathcal{G}_R^0 such

that

$$[f]^0 = [0] \quad \text{and} \quad [\dot{f}]^0 = [0] \quad \text{in } \mathcal{G}_R^0. \tag{5.2}$$

Note that such \mathcal{G}_R^0 corresponds to a *regular* point on the fundamental path, in the sense that the set of other (singular) points is discrete on account of finite N . All increments denoted by a prefix Δ are to be taken now relative to \mathcal{G}_R^0 .

Suppose that \mathcal{G}_R^0 is a stable equilibrium state. A temporary disturbance is imagined to initiate in \mathcal{G}_R^0 a perturbed process which, after a transitory period, reaches an equilibrium state \mathcal{G} close to \mathcal{G}_R^0 . No further disturbances are considered, and *at slowly varying* λ the perturbed process is continued from \mathcal{G} as a quasi-static process \mathcal{F}^s in which $\mathbf{Q}=\mathbf{P}$ and $f_K \overset{\circ}{\gamma}_K = 0$ at each instant. The question of stability of the process \mathcal{F}^0 is examined below by checking whether the energy measure $(\Delta E - \Delta E^0)$ of the distance between \mathcal{F}^s and \mathcal{F}^0 possesses the usual properties of a Lyapunov function.

As in §3*d*, only such temporary disturbances are considered which are not associated with *finite* perturbations of thermodynamic driving forces. By the continuity argument, there exists some neighbourhood \mathcal{R} of the state \mathcal{G}_R^0 such that $\mathcal{G} \in \mathcal{R}$ implies $f_K(\mathcal{G}) < 0$ for $K \notin \mathcal{P}^0$. Accordingly, for the examined class of disturbances, we have

$$f_K(\mathcal{G}_R^0) \overset{\circ}{\gamma}_K = 0 \quad \text{in every } \mathcal{G} \in \mathcal{R}. \tag{5.3}$$

(b) *Energy condition for quasi-static stability*

For two quasi-static solution paths \mathcal{F}^s and \mathcal{F}^0 close to each other, the difference between respective values of \dot{E} at a given λ reads

$$\dot{E}^s - \dot{E}^0 = -(\mathbf{q} - \mathbf{q}^0) \cdot \dot{\mathbf{P}}^*, \tag{5.4}$$

in the first approximation (v), from (3.2) and (5.1). In the sequel, all quantities are also evaluated at a given value of λ . Equation (5.4) can be transformed as follows

$$\begin{aligned} \dot{E}^s - \dot{E}^0 &= -(\mathbf{q} - \mathbf{q}^0) \cdot (\mathbf{K}^* \cdot \dot{\mathbf{q}}^0 + \dot{\mathbf{P}}^0) = (\mathbf{P} - \mathbf{P}^0) \cdot \dot{\mathbf{q}}^0 - \dot{\mathbf{P}}^0 \cdot (\mathbf{q} - \mathbf{q}^0) \\ &= (\mathbf{Q} - \mathbf{Q}^0) \cdot \dot{\mathbf{q}}^0 - \dot{\mathbf{Q}}^0 \cdot (\mathbf{q} - \mathbf{q}^0) \\ &= (\Delta \mathbf{Q} - \Delta \mathbf{Q}^0) \cdot \dot{\mathbf{q}}^0 - \dot{\mathbf{Q}}^0 \cdot (\Delta \mathbf{q} - \Delta \mathbf{q}^0). \end{aligned} \tag{5.5}$$

The first equality follows from equation (3.2), the second from the first approximation of $(\mathbf{P} - \mathbf{P}^0)$, the third from the assumption of quasi-static processes, and the last is obvious. On substituting now the constitutive rate equations (2.22) in either direct or straightforwardly integrated form using (v), we obtain

$$\begin{aligned} \dot{E}^s - \dot{E}^0 &= - \sum_K (\Delta \gamma_k - \Delta \gamma_k^0) N_K^p \cdot \dot{\mathbf{q}}^0 + \sum_K \overset{\circ}{\gamma}_K^0 N_K^p \cdot (\Delta \mathbf{q} - \Delta \mathbf{q}^0) \\ &= \sum_K \left(f_K \overset{\circ}{\gamma}_k^0 - (\Delta \gamma_K - \Delta \gamma_K^0) \overset{\circ}{f}_K \right) - \sum_{K,L} (g_{KL} - g_{LK}) (\Delta \gamma_K - \Delta \gamma_K^0) \overset{\circ}{\gamma}_L^0, \end{aligned} \tag{5.6}$$

where the former equality follows from equation (2.22)₁ and symmetry of \mathbf{K}^e , and the latter from (2.20)₃, (2.22)₂ and the assumed normality rule (2.34). The values of $\Delta\gamma_K$ and $\Delta\gamma_K^0$ represent time integrals of $\dot{\gamma}_K$ in the perturbed and fundamental processes, respectively, starting from the common initial state \mathcal{G}_R^0 .

From equation (5.3) it follows that the summation in equation (5.6) need only be performed over the set \mathcal{P}^0 as long as both processes \mathcal{F}^s and \mathcal{F}^0 continue within \mathcal{R} . On substituting into equation (5.6) the assumed conditions of symmetry (2.35) and regularity (5.2), from $f_K \leq 0$ in \mathcal{F}^s we finally obtain

$$\dot{E}^s - \dot{E}^0 = \sum_{K \in \mathcal{P}^0} f_K \dot{\gamma}_K^0 \leq 0 \quad \text{in every } \mathcal{G} \in \mathcal{R}. \tag{5.7}$$

We arrive thus at the following conclusion (cf. [Petryk 1993b](#)). If

$$\Delta E > \Delta E^0 \quad \text{for every } \mathcal{G} \neq \mathcal{G}_R^0, \quad \mathcal{G} \in \mathcal{R}, \tag{5.8}$$

and for every path connecting \mathcal{G}_R^0 with \mathcal{G} within \mathcal{R} then the fundamental process \mathcal{F}^0 is quasi-statically stable in the first approximation at \mathcal{G}_R^0 with respect to the energy distance $|\Delta E - \Delta E^0|$.

The energy condition (5.8) may be regarded as an extension of the condition (3.19) for stability of an equilibrium state \mathcal{G}_R^0 to an isothermal quasi-static process going through \mathcal{G}_R^0 . The energy excess $(\Delta E - \Delta E^0)$ reduces to ΔE if the process \mathcal{F}^0 degenerates to a single equilibrium state. The character of stability implied by equation (5.8) is, however, different from and complementary to the dynamic stability of equilibrium implied by equation (3.19) at fixed λ . The perturbed process now takes place at *varying* λ and, except in an initial transitory period, is regarded as quasi-static; this is why the term quasi-statically stable has been used. The stability is established for the non-linear problem but only in the first approximation, cf. assumption (v), and as such has a physical meaning within a sufficiently small interval of the loading parameter λ . Note that this may correspond to an unbounded interval of time t . A further discussion on the concept of stability in the first approximation (which represents a generalization of the linear stability concept) can be found in ([Petryk 1993b](#)).

It should be pointed out that on account of the specific assumptions (i)–(v), the path stability condition (5.8) is much less universal than the condition (3.19) for stability of equilibrium.

(c) *Second-order conditions for path stability*

The rate \dot{W} of deformation work in a perturbed process is expressed by equation (2.31) which includes the work-rate \dot{W}^{dist} of internal disturbing forces. Along any deformation path starting from \mathcal{G}_R^0 , that expression can be rearranged under the assumptions (i)–(iii) and (v) as follows

$$\begin{aligned} \dot{W} = & \mathbf{Q} \cdot \dot{\mathbf{q}} - \sum_K f_K \dot{\gamma}_K = \mathbf{Q}(\mathcal{G}_R^0) \cdot \dot{\mathbf{q}} - \sum_K f_K(\mathcal{G}_R^0) \dot{\gamma}_K + \dot{\mathbf{q}} \cdot \mathbf{K}^e \cdot \Delta \mathbf{q} + \sum_{K,L} g_{KL} \Delta \gamma_L \dot{\gamma}_K \\ & - \sum_K \mathbf{N}_K \cdot (\dot{\mathbf{q}} \Delta \gamma_K + \dot{\gamma}_K \Delta \mathbf{q}). \end{aligned} \tag{5.9}$$

Advantage has been taken of the fact that, under the assumption (v), the rates in equation (2.22) can simply be replaced by the respective increments reached on

an arbitrary path. Denote by ψ_R^0 a quantity ψ evaluated in the state \mathcal{G}_R^0 . If equation (5.3) is satisfied then straightforward integration of equation (5.9) along any path within \mathcal{R} with the use of symmetries of \mathbf{K}^e and (g_{KL}) (from (iv)) yields

$$\begin{aligned} \Delta W &= \mathbf{Q}_R^0 \cdot \Delta \mathbf{q} + \frac{1}{2} \Delta \mathbf{q} \cdot \mathbf{K}^e \cdot \Delta \mathbf{q} + \frac{1}{2} \sum_{K,L \in \mathcal{P}^0} g_{KL} \Delta \gamma_K \Delta \gamma_L - \sum_{K \in \mathcal{P}^0} \mathbf{N}_K \cdot \Delta \mathbf{q} \Delta \gamma_K \\ &= \mathbf{Q}_R^0 \cdot \Delta \mathbf{q} + \frac{1}{2} \Delta \mathbf{q} \cdot \mathbf{K}^e \cdot \Delta \mathbf{q} - \frac{1}{2} \sum_{K,L \in \mathcal{P}^0} g_{KL} \Delta \gamma_K \Delta \gamma_L - \sum_{K \in \mathcal{P}^0} \Delta f_K \Delta \gamma_K \\ &= \mathbf{Q}_R^0 \cdot \Delta \mathbf{q} + \frac{1}{2} \Delta \mathbf{Q} \cdot \Delta \mathbf{q} - \frac{1}{2} \sum_{K \in \mathcal{P}^0} \Delta f_K \Delta \gamma_K. \end{aligned} \tag{5.10}$$

These expressions are exact due to the assumption (v); analogous expressions valid to the second-order terms were derived without that assumption in (Petryk 1991a) for a uniformly strained material element. From equation (5.10) it follows that a work increment ΔW is independent of the path in $(\Delta \mathbf{q}, \Delta \gamma_K)$ -space within \mathcal{R} , being only dependent on the final increments.

With the help of equation (3.2), the following expression for $\Delta \Omega$ is obtained

$$\Delta \Omega = \Omega(\mathbf{q}_R^0, \lambda) - \Omega(\mathbf{q}_R^0, \lambda_R^0) - (\mathbf{P}_R^0 - \Omega_{,q\lambda} \Delta \lambda) \cdot \Delta \mathbf{q} + \frac{1}{2} \Delta \mathbf{q} \cdot \mathbf{K}^* \cdot \Delta \mathbf{q}. \tag{5.11}$$

When combined with equation (5.10), it yields

$$\begin{aligned} \Delta E &= \frac{1}{2} \Delta \mathbf{q} \cdot (\mathbf{K}^e + \mathbf{K}^*) \cdot \Delta \mathbf{q} + \frac{1}{2} \sum_{K,L \in \mathcal{P}^0} g_{KL} \Delta \gamma_K \Delta \gamma_L + \left(\Omega_{,q\lambda} \Delta \lambda - \sum_{K \in \mathcal{P}^0} \mathbf{N}_K \Delta \gamma_K \right) \cdot \Delta \mathbf{q} \\ &\quad + \Omega(\mathbf{q}_R^0, \lambda) - \Omega(\mathbf{q}_R^0, \lambda_R^0) = \frac{1}{2} (\Delta \mathbf{Q} - \Delta \mathbf{P}) \cdot \Delta \mathbf{q} - \frac{1}{2} \sum_{K \in \mathcal{P}^0} \Delta f_K \Delta \gamma_K + \frac{1}{2} \Delta \mathbf{q} \cdot \Omega_{,q\lambda} \Delta \lambda \\ &\quad + \Omega(\mathbf{q}_R^0, \lambda) - \Omega(\mathbf{q}_R^0, \lambda_R^0). \end{aligned} \tag{5.12}$$

Clearly, ΔE is path-independent in $(\Delta \mathbf{q}, \Delta \gamma_K)$ -space within \mathcal{R} .

Alternatively, the forward rate of the expression (5.1) in the state \mathcal{G}_R^0 under the restriction (5.3) reads

$$\begin{aligned} \ddot{E}|_{\mathcal{G}_R^0} &= (\dot{\mathbf{Q}} - \dot{\mathbf{P}}) \cdot \dot{\mathbf{q}} - \sum_{K \in \mathcal{P}^0} \dot{f}_K \dot{\gamma}_K + \dot{\mathbf{q}} \cdot \Omega_{,q\lambda} \dot{\lambda} + \Omega_{,\lambda\lambda}(\mathcal{G}_R^0) (\dot{\lambda})^2 + \Omega_{,\lambda}(\mathcal{G}_R^0) \ddot{\lambda} \\ &= 2J(\dot{\mathbf{q}}, [\dot{\gamma}]) + \Omega_{,\lambda\lambda}(\mathcal{G}_R^0) (\dot{\lambda})^2 + \Omega_{,\lambda}(\mathcal{G}_R^0) \ddot{\lambda}, \end{aligned} \tag{5.13}$$

where the last two terms have prescribed values, and

$$J(\dot{\mathbf{q}}, [\dot{\gamma}]) = \frac{1}{2} \dot{\mathbf{q}} \cdot (\mathbf{K}^e + \mathbf{K}^*) \cdot \dot{\mathbf{q}} + \frac{1}{2} \sum_{K,L \in \mathcal{P}^0} g_{KL} \dot{\gamma}_K \dot{\gamma}_L - \left(\dot{\mathbf{P}}^* + \sum_{K \in \mathcal{P}^0} \mathbf{N}_K \dot{\gamma}_K \right) \cdot \dot{\mathbf{q}}. \tag{5.14}$$

Comparison between equations (5.12) and (5.13) shows that under the assumptions (i)–(v) and equation (5.3), the condition (5.8) is *equivalent* to the

following *second-order energy condition for path stability*

$$\ddot{E}(\dot{\mathbf{q}}, [\dot{\gamma}]) > \ddot{E}(\dot{\mathbf{q}}^0, [\dot{\gamma}]^0) \text{ for every } (\dot{\mathbf{q}}, [\dot{\gamma}]) \neq (\dot{\mathbf{q}}^0, [\dot{\gamma}]^0), \quad [\dot{\gamma}] \geq [0] \text{ in } \mathcal{G}_R^0, \tag{5.15}$$

which in turn is equivalent to

$$J(\dot{\mathbf{q}}, [\dot{\gamma}]) > J(\dot{\mathbf{q}}^0, [\dot{\gamma}]^0) \text{ for every } (\dot{\mathbf{q}}, [\dot{\gamma}]) \neq (\dot{\mathbf{q}}^0, [\dot{\gamma}]^0), \quad [\dot{\gamma}] \geq [0] \text{ in } \mathcal{G}_R^0. \tag{5.16}$$

For comparison, the condition (3.30) for stability of *equilibrium* under the normality assumption (iii) takes the form

$$J(\dot{\mathbf{q}}, [\dot{\gamma}]) > -\dot{\mathbf{P}}^* \cdot \dot{\mathbf{q}} \text{ for every non-zero } (\dot{\mathbf{q}}, [\dot{\gamma}] \geq [0]). \tag{5.17}$$

Moreover, for any quasi-static solution $(\dot{\mathbf{q}}^s, [\dot{\gamma}]^s)$, we readily have

$$J(\dot{\mathbf{q}}^s, [\dot{\gamma}]^s) = -\frac{1}{2} \dot{\mathbf{P}}^* \cdot \dot{\mathbf{q}}^s. \tag{5.18}$$

Some immediate implications follow, e.g. that inequality (3.30) implies $\dot{\mathbf{P}}^* \cdot \dot{\mathbf{q}}^0 > 0$ or that inequality (5.15) is necessarily violated if there exists $\dot{\mathbf{q}}^s \neq \dot{\mathbf{q}}^0$ such that $\dot{\mathbf{P}}^* \cdot \dot{\mathbf{q}}^s \geq \dot{\mathbf{P}}^* \cdot \dot{\mathbf{q}}^0$.

Suppose that the condition (3.28) for elastic stability of equilibrium is satisfied. Then, the minimum of equation (5.14) with respect to $\dot{\mathbf{q}}$ for given $[\dot{\gamma}]$ is attained at

$$\dot{\mathbf{q}}^m = \dot{\mathbf{q}}^m([\dot{\gamma}]) = \mathbf{G}^* \cdot \left(\dot{\mathbf{P}}^* + \sum_{K \in \mathcal{P}^0} N_K \dot{\gamma}_K \right) \Leftrightarrow \dot{\mathbf{Q}}^m = \dot{\mathbf{P}}, \tag{5.19}$$

where \mathbf{G}^* is defined in equation (3.31) and $\dot{\mathbf{Q}}^m$ is constitutively related to $\dot{\mathbf{q}}^m$. Thus, the ‘most critical’ mode associated with path stability condition (5.15) is a quasi-static one. From (2.22)₂ and (5.19)₁ we obtain that f_K^m related constitutively to $(\dot{\mathbf{q}}^m, [\dot{\gamma}])$ reads

$$f_K^m = \dot{f}_K^* - \sum_{L \in \mathcal{P}^0} g_{KL}^* \dot{\gamma}_L, \quad \dot{f}_K^* = \mathbf{N}_K \cdot \mathbf{G}^* \cdot \dot{\mathbf{P}}^*, \tag{5.20}$$

with g_{KL}^* defined by equation (3.35). It is essential that the matrix (g_{KL}^*) is now symmetric on account of the assumptions (iii) and (iv). On substituting equation (5.19) into equation (5.14) and rearranging, the path stability condition (5.15) is reduced to

$$J^m([\dot{\gamma}]) > J^m([\dot{\gamma}]^0) \text{ for every } [\dot{\gamma}] \neq [\dot{\gamma}]^0, \quad [\dot{\gamma}] \geq [0] \text{ in } \mathcal{G}_R^0, \tag{5.21}$$

where

$$J^m([\dot{\gamma}]) = J(\dot{\mathbf{q}}^m([\dot{\gamma}]), [\dot{\gamma}]) + \frac{1}{2} \dot{\mathbf{P}}^* \cdot \mathbf{G}^* \cdot \dot{\mathbf{P}}^* = \frac{1}{2} \sum_{K,L \in \mathcal{P}^0} g_{KL}^* \dot{\gamma}_K \dot{\gamma}_L - \sum_{K \in \mathcal{P}^0} \dot{f}_K^* \dot{\gamma}_K. \tag{5.22}$$

A minimizer $[\overset{\circ}{\gamma}]^m$ of J^m on the set of $[\overset{\circ}{\gamma}] \geq [0]$ must satisfy the Kuhn–Tucker conditions

$$\sum_{L \in \mathcal{P}^0} g_{KL}^* \overset{\circ}{\gamma}_L^m - j_K^* \geq 0 \quad \text{and} \quad \left(\sum_{L \in \mathcal{P}^0} g_{KL}^* \overset{\circ}{\gamma}_L^m - j_K^* \right) \overset{\circ}{\gamma}_K^m = 0 \quad \text{for every } K \in \mathcal{P}^0. \tag{5.23}$$

From (5.20) it follows that conditions (5.23) coincide with the consistency conditions (2.21) imposed on $[\overset{\circ}{\gamma}]^m$ in the state \mathcal{G}_R^0 , while equation (5.19)₂ is the condition of continuing equilibrium. Hence, a pair $(\overset{\circ}{q}^m([\overset{\circ}{\gamma}]^m), [\overset{\circ}{\gamma}]^m)$ which minimizes the function $J(\overset{\circ}{q}, [\overset{\circ}{\gamma}])$ defined by equation (5.14) satisfies automatically all the requirements for a quasi-static solution to the rate-problem defined by equations (2.21), (2.22) and (3.2) and the rate form of equation (3.11).

From the theory of quadratic programming it follows that the minimizer of J^m on the set of $[\overset{\circ}{\gamma}] \geq [0]$ exists if equation (3.34) holds and is unique if

$$\sum_{K, L \in \mathcal{P}^0} g_{KL}^* z_K z_L > 0 \quad \text{for every } [z] \neq [0]. \tag{5.24}$$

Moreover, if equation (5.24) holds then the unique minimizer $[\overset{\circ}{\gamma}]^m$ is the only solution to the problem (5.23) and must thus coincide with $[\overset{\circ}{\gamma}]^0$, implying inequality (5.21). Hence, we arrive at the conclusion that *simultaneous fulfillment of the conditions (3.28) and (5.24) implies fulfillment of the path stability condition (5.15)* (and also ensures uniqueness of a quasi-static solution to the rate-problem).

If equation (5.2) is replaced by a somewhat stronger condition of fully active loading in the fundamental process, that is, if

$$[\overset{\circ}{\gamma}]^0 > [0] \quad \text{in } \mathcal{G}_R^0, \tag{5.25}$$

then, in turn, inequality (5.24) is implied by inequality (5.15) by the usual second-variation argument. In that typical case, conditions (3.28) and (5.24) considered jointly become *equivalent* to the energy condition (5.15) for path stability.

In conclusion, equation (5.24) *represents the final condition for path stability*. Its fulfillment under the assumptions (i)–(v) and (5.3) implies that the energy condition (5.8) for path stability is satisfied whenever elastic stability is ensured by inequality (3.28).

The condition (5.24) has been derived from inequality (5.16) on eliminating $\overset{\circ}{q}$ under the assumed conditions (3.28) of elastic stability. It is possible to proceed in an alternative way by eliminating $[\overset{\circ}{\gamma}]$ with the help of the consistency condition (2.21), which means that the driving forces π_K are assumed to be not perturbed directly. In analogy to the implications of problem (5.24), a unique solution $[\overset{\circ}{\gamma}] \geq [0]$ to equation (2.21) exists for any given $\overset{\circ}{q}$ if

$$\sum_{K, L \in \mathcal{P}^0} g_{KL} z_K z_L > 0 \quad \text{for every } [z] \neq [0]. \tag{5.26}$$

Consequently, if inequality (5.26) holds then $\overset{\circ}{Q}$ in a given state becomes a function of $\overset{\circ}{q}$ which under the assumptions (iii) and (iv) admits a potential. Further analysis can follow known lines, cf. [Petryk & Thermann \(1992\)](#),

Petryk (1997*b*). Under the additional assumptions (5.25) and (5.26), the path stability condition (5.15) can be reduced to

$$\dot{\mathbf{q}} \cdot (\mathbf{K}(\dot{\mathbf{q}}^0) + \mathbf{K}^*) \cdot \dot{\mathbf{q}} > 0 \quad \text{for every } \dot{\mathbf{q}} \neq 0. \quad (5.27)$$

Note the difference between this requirement of positive definiteness of the *fundamental* tangent stiffness matrix $(\mathbf{K}(\dot{\mathbf{q}}^0) + \mathbf{K}^*)$ for the system and the previous condition (3.37) for stability of equilibrium. The condition (5.27) has also the interpretation as the *uniqueness* condition.

(d) *Energy criterion of path instability*

Suppose that inequality (5.15) fails due to

$$\ddot{E}(\dot{\mathbf{q}}, [\dot{\gamma}]) < \ddot{E}(\dot{\mathbf{q}}^0, [\dot{\gamma}]^0) \quad \text{for some } (\dot{\mathbf{q}}, [\dot{\gamma}] \geq [0]) \quad \text{in } \mathcal{G}_R^0, \quad (5.28)$$

while the condition (3.26) for stability of equilibrium is still satisfied.

We conclude first from inequality (3.26) that the condition (3.28) of elastic stability must hold, which by using the argument employed to derive condition (5.21) allows inequality (5.28) to be transformed to

$$J^m([\dot{\gamma}]) < J^m([\dot{\gamma}]^0) \quad \text{for some } [\dot{\gamma}] \geq [0] \quad \text{in } \mathcal{G}_R^0. \quad (5.29)$$

Moreover, the condition (3.34) for internal stability of equilibrium under deformation-sensitive loading must also hold as a consequence of inequality (3.26), here for $\mathbf{N}_K^p = \mathbf{N}_K$. From formula (5.22) and inequality (3.34) it follows that $J^m([\dot{\gamma}])$ is bounded from below on the set of $[\dot{\gamma}] \geq [0]$ since J^m is a continuous function and $J^m([\dot{\gamma}]) \rightarrow +\infty$ as $\dot{\gamma}_K \rightarrow +\infty$ for some $K \in \mathcal{P}$. From the Frank-Wolf theorem we obtain that $J^m([\dot{\gamma}])$ attains at some $[\dot{\gamma}]^m$ its absolute minimum on the set $[\dot{\gamma}] \geq [0]$.

Further, from inequality (5.29) it follows that the absolute minimizer $[\dot{\gamma}]^m \geq 0$ of J^m must differ from $[\dot{\gamma}]^0$. As stated earlier after the formula (5.23), each minimizer generates a quasi-static solution to the rate-problem. In the first approximation (v), the solution $\dot{\mathbf{q}}([\dot{\gamma}]^m, [\dot{\gamma}]^m)$ can be extended to a straight path, simply by replacing the rates by finite increments. We have thus proven the following theorem:

If the inequality (5.28) is satisfied simultaneously with the condition (3.26) for stability of equilibrium then, under the assumptions (i)–(v), there exists another quasi-static solution path emanating from \mathcal{G}_R^0 , which is energetically preferable to the fundamental path in the sense of assigning a smaller value to ΔE .

This conclusion is similar in spirit to the previous theorems proven under different specific assumptions in (Petryk 1991*b*; Petryk & Thermann 1992; Petryk 1999). It shows that a fundamental path along which conditions (5.28) and (3.26) hold simultaneously at every instant represents a *continuous range of quasi-static bifurcations* and as such may be regarded as unstable, the more that the branching solutions are energetically preferable. The inequality (5.28) may thus be regarded as the condition *sufficient for path instability*. Taken jointly with the condition (5.15) for path stability, it provides *the energy criterion of instability of quasi-static processes* under consideration.

Along a sufficiently regular fundamental path, the critical stage between the ranges (5.24) of path stability and (5.29) of path instability is met when

$$\det(g_{KL}^*) = 0 \quad \text{with } K, L \in \mathcal{P}^0, \tag{5.30}$$

i.e. when the *symmetric* matrix (g_{KL}^*) for potentially active mechanisms ceases to be positive definite. We shall prove that if $[\dot{\gamma}]^0 > [0]$ then this is the instant of *primary bifurcation*, i.e. the first instant of non-uniqueness of a quasi-static solution to the rate-problem along the path.

Since earlier bifurcations are excluded by inequality (5.24), we must only show that equation (5.30) *corresponds to a bifurcation point* if $[\dot{\gamma}]^0 > [0]$. To prove this, notice that an eigenvector $[w] \neq [0]$ of the singular matrix (g_{KL}^*) , of a sufficiently small magnitude $|w|$, can be added to $[\dot{\gamma}]^0$ to give $[\dot{\gamma}] = [\dot{\gamma}]^0 + [w] > 0$. For $\dot{\mathbf{q}} = \dot{\mathbf{q}}^m$ defined as in equation (5.19), from equation (5.20) we obtain that the vector $[\dot{\gamma}]$ corresponds to $[\dot{f}]^m = [\dot{f}]^0 = [0]$ as required by equation (2.21). It follows that the pair $(\dot{\mathbf{q}}^m([\dot{\gamma}]), [\dot{\gamma}])$ is a rate-solution different from the fundamental one, which completes the proof.

The condition (5.28) can be applied just *after* the critical instant (5.30) to reject post-bifurcation paths which are expected to have no physical meaning, cf. Petryk & Thermann (1992), Petryk (1997a).

6. Illustration: stability under uniaxial tension

Consider a material sample which obeys a constitutive law of multi-mode inelasticity (§2c) at constant temperature, and examine material stability within the sample with respect to internal structural rearrangements. Attention is now confined to *macroscopically* uniform deformations under uniaxial tension; the necking problem is not addressed, and stability against spontaneous formation of shear bands is examined in §4.

Stability analyses in solids should generally include geometric effects, so that a finite deformation theory is used here. The material sample of a uniform macroscopic deformation gradient \mathbf{F} is thought of to represent a gauge section of a tensile specimen. In a general case when the tensile axis does not coincide with an axis of material anisotropy, shearing parallel to the tensile axis must be allowed for (e.g. by using rotation-free grips). Since testing machines have always some flexibility, tensile component of \mathbf{F} is also not constrained explicitly. On the other hand, we disregard here any rotations of the tensile axis with respect to the material and fix this axis in space by aligning it along the first axis of a fixed orthogonal reference frame $(x_1, x_2, x_3; \text{figure 1})$. Trivial rotations around the tensile axis are eliminated by placing the second axis of the reference frame in a chosen material plane containing the tensile axis. In effect, the system has six external degrees of freedom. The restrictions imposed can be written in a concise form as follows

$$F_{ij} = \frac{\partial x_i}{\partial \xi_j} = 0 \quad \text{for } i > j, \quad \sigma_{ij} = \sigma_{ji} = 0 \quad \text{for } i > j, \tag{6.1}$$

where the indices denote vector or tensor components on the fixed orthonormal reference triad, x_i and ξ_j are coordinates of a material point in the current and reference configuration, respectively, and σ_{ij} are components of the macroscopic Cauchy stress.

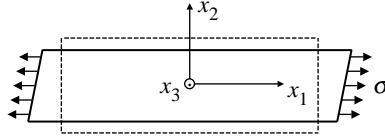


Figure 1. Macroscopically uniform deformation under uniaxial tension.

A potential energy of the loading device is defined as

$$\Omega = \frac{1}{2} C^* \cdot (F_{11} - F_{11}^*)^2 = -\frac{1}{2} S_{11} \cdot (F_{11} - F_{11}^*), \tag{6.2}$$

where S_{11} denotes the tensile *nominal* stress, $C^* \geq 0$ defines the stiffness of the loading device,⁴ and F_{11}^* is determined from a controlled external elongation and may be identified with the loading parameter λ .

Convenient identification of the quantities that define the basic matrix (g_{KL}^*) (cf. equation (3.35)) used to formulate the stability conditions may depend on how the constitutive law for the material is originally specified. We consider below two classes of material models; the first, well known, for single metal crystals deformed plastically by multi-slip, and the second for a two-phase elastic material undergoing stress-induced martensitic phase transformation with coherent parallel interfaces. The main purpose is to illustrate possible applications of more general stability conditions given in this paper, however, some novel conclusions may be of interest in themselves.

(a) *Plasticity of single crystals*

We adopt the constitutive framework of multi-mode plasticity with the normality structure (cf. Hill & Rice 1972; Asaro 1983; Havner 1992). For simplicity, the reference configuration it has taken to coincide momentarily with the current configuration. Accordingly, the basic formulae (2.20) and (2.21) are complemented with the following constitutive rate equations

$$\overset{\nabla}{\boldsymbol{\tau}} = \mathbf{L}^e \cdot \mathbf{D} - \sum_K \lambda_K \overset{\circ}{\boldsymbol{\gamma}}_K, \quad \dot{f}_K = \lambda_K \cdot \mathbf{D} - \sum_L g_{KL} \overset{\circ}{\boldsymbol{\gamma}}_L. \tag{6.3}$$

Here, $\overset{\nabla}{\boldsymbol{\tau}}$ is the corotational (Zaremba–Jaumann) flux of the Kirchhoff stress, \mathbf{D} is the Eulerian strain-rate, \mathbf{L}^e is the elastic moduli tensor for *these* rate-variables under the current Cauchy stress, and λ_K is an outward normal to the yield surface for the K -th slip system ($f_K=0$) in the strain space.

Assuming that \mathbf{L}^e as an operator acting in the space of symmetric second-order tensor possesses an inverse \mathbf{M}^e (the current elastic compliance tensor), these equations can be equivalently written down in a dual form, viz.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p, \quad \mathbf{D}^e = \mathbf{M}^e \cdot \overset{\nabla}{\boldsymbol{\tau}}, \quad \mathbf{D}^p = \sum_K \mu_K \overset{\circ}{\boldsymbol{\gamma}}_K, \quad \dot{f}_K = \mu_K \cdot \overset{\nabla}{\boldsymbol{\tau}} - \sum_L h_{KL} \overset{\circ}{\boldsymbol{\gamma}}_L. \tag{6.4}$$

⁴It is assumed for simplicity that C^* is constant, but the incremental analysis below can easily be extended to non-quadratic Ω .

A yield-surface normal $\boldsymbol{\mu}_K$ in the stress space and *effective* hardening moduli h_{KL} satisfy

$$\boldsymbol{\mu}_K = \mathbf{M}^e \cdot \boldsymbol{\lambda}_K, \quad h_{KL} = g_{KL} - \boldsymbol{\lambda}_K \cdot \mathbf{M}^e \cdot \boldsymbol{\lambda}_L. \tag{6.5}$$

For this familiar class of constitutive equations, it is convenient to identify the vector \mathbf{q} of generalized strains with the *logarithmic* strain measure \mathbf{e} based on the current configuration as reference, both of six independent components. Then, we have the correspondence $(\dot{\mathbf{q}}, \dot{\mathbf{Q}}, \mathbf{K}^e) \leftrightarrow (\mathbf{D}, \overset{\vee}{\boldsymbol{\tau}}, \mathbf{L}^e)$, which follows from the known fact (Hill 1968) that $\overset{\vee}{\boldsymbol{\tau}}$ coincides with the material time derivative of the stress work-conjugate to *this* \mathbf{e} . More specifically, we define the correspondence between a single index (say ζ) of a vector and a pair of indices (say (ij)) of a *symmetric* tensor as $(1, 2, 3, 4, 5, 6) \leftrightarrow (11, 22, 33, 23, 13, 12)$, and put

$$\begin{aligned} (\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4, \dot{q}_5, \dot{q}_6) &= (D_{11}, D_{22}, D_{33}, \sqrt{2}D_{23}, \sqrt{2}D_{13}, \sqrt{2}D_{12}), \\ (\dot{Q}_1, \dot{Q}_2, \dot{Q}_3, \dot{Q}_4, \dot{Q}_5, \dot{Q}_6) &= (\overset{\vee}{\tau}_{11}, \overset{\vee}{\tau}_{22}, \overset{\vee}{\tau}_{33}, \sqrt{2}\overset{\vee}{\tau}_{23}, \sqrt{2}\overset{\vee}{\tau}_{13}, \sqrt{2}\overset{\vee}{\tau}_{12}), \end{aligned} \tag{6.6}$$

and in consequence,

$$\begin{aligned} K_{\zeta\eta}^e &\leftrightarrow r_\zeta r_\eta L_{ijkl}^e, \quad \zeta, \eta = 1 \leftrightarrow (11), \dots, 6 \leftrightarrow (12), \\ r_\zeta &= \begin{cases} 1 & \text{if } \zeta = 1, 2, 3, \\ \sqrt{2} & \text{if } \zeta = 4, 5, 6 \end{cases}. \end{aligned} \tag{6.7}$$

For instance, $K_{11}^e = L_{1111}^e$, $K_{16}^e = \sqrt{2}L_{1112}^e$, $K_{66}^e = 2L_{1212}^e$. Assuming that $\overset{\circ}{\gamma}_K$ and f_K are invariant, the correspondence between \mathbf{N}_K and $\boldsymbol{\lambda}_K$ must reproduce that between $\dot{\mathbf{Q}}$ and $\overset{\vee}{\boldsymbol{\tau}}$, and the matrix (g_{KL}) is also invariant.

It remains to identify \mathbf{K}^* defined in equation (3.2) from the invariance of the potential energy \mathcal{Q} of the loading device, calculated to the *second-order* terms. In the current configuration taken as reference, under uniaxial tensile stress $\sigma > 0$ we have $P_1 = S_{11} = \sigma$, and all other components of \mathbf{P} are zero. Suppose that F_{11}^* is fixed; then time differentiation of $\dot{\mathcal{Q}} = -\mathbf{P} \cdot \dot{\mathbf{q}} = -S_{11} \dot{F}_{11}$ yields

$$\dot{\mathbf{q}} \cdot \mathbf{K}^* \cdot \dot{\mathbf{q}} - P_1 \ddot{q}_1 = C^* (\dot{F}_{11})^2 - S_{11} \ddot{F}_{11}, \tag{6.8}$$

in the current configuration. From the second-order formula for the logarithmic strain \mathbf{e} expressed in terms of the Green strain (cf. Hill 1968) and by imposing next the restriction (6.1), we obtain

$$\ddot{q}_{11} = \ddot{e}_{11} = \ddot{F}_{11} + \dot{F}_{i1} \dot{F}_{i1} - 2D_{1i} D_{i1} = \ddot{F}_{11} - D_{11}^2 - 2D_{12}^2 - 2D_{13}^2, \tag{6.9}$$

with the summation for repeated i . Substitution of equations (6.9) and (6.6)₁ into equation (6.8) gives the following identification

$$\mathbf{K}^* = \text{Diag.}(C^* - \sigma, 0, 0, 0, -\sigma, -\sigma), \tag{6.10}$$

where Diag. denotes a diagonal matrix of components that follow in parentheses.

Under the assumption that the elastic stiffness matrix \mathbf{K}^e is positive definite, the matrix (g_{KL}^*) defined by equation (3.35) can be transformed with the help of

equations (6.5) and (6.6) as follows

$$\begin{aligned}
 g_{KL}^* &= h_{KL} + \mathbf{N}_K \cdot (\mathbf{K}^e)^{-1} \cdot \mathbf{N}_L - \mathbf{N}_K \cdot \mathbf{G}^* \cdot \mathbf{N}_L \\
 &= h_{KL} + \mathbf{N}_K \cdot ((\mathbf{K}^e + \mathbf{K}^*)^{-1} (\mathbf{K}^e + \mathbf{K}^*) - (\mathbf{K}^e + \mathbf{K}^*)^{-1} \mathbf{K}^e) (\mathbf{K}^e)^{-1} \cdot \mathbf{N}_L \\
 &= h_{KL} + \mathbf{N}_K \cdot (\mathbf{K}^e)^{-1} \mathbf{H}^* (\mathbf{K}^e)^{-1} \cdot \mathbf{N}_L,
 \end{aligned}
 \tag{6.11}$$

where

$$\mathbf{H}^* \equiv (\mathbf{I} + \mathbf{K}^* (\mathbf{K}^e)^{-1})^{-1} \mathbf{K}^*,
 \tag{6.12}$$

with the usual notation for matrix multiplication.

Suppose for simplicity that \mathbf{L}^e coincides with the elastic stiffness tensor of infinitesimal isotropic elasticity. On using equation (6.10), straightforward matrix transformations then yield

$$\begin{aligned}
 \mathbf{H}^* &= \text{Diag.}(H_1, 0, 0, 0, -H_2, -H_2), \\
 H_1 &\equiv ((\mathbf{C}^* - \boldsymbol{\sigma})^{-1} + \mathbf{E}^{-1})^{-1}, \quad H_2 \equiv (\boldsymbol{\sigma}^{-1} - (2\mathbf{G})^{-1})^{-1},
 \end{aligned}
 \tag{6.13}$$

where E is the Young modulus and G is the elastic shear modulus. On substituting equation (6.13) into equation (6.11) and using the correspondence between $(\mathbf{K}^e)^{-1} \cdot \mathbf{N}_K$ and $\boldsymbol{\mu}_K$ which reproduces that between $\dot{\mathbf{q}}$ and \mathbf{D} in equation (6.6)₁, the matrix (g_{KL}^*) specified for the present problem as (g_{KL}^{uni}) takes the form

$$g_{KL}^* \rightarrow g_{KL}^{\text{uni}} = h_{KL} + H_1(\mu_{11})_K(\mu_{11})_L - 2H_2(\mu_{12})_K(\mu_{12})_L - 2H_2(\mu_{13})_K(\mu_{13})_L.
 \tag{6.14}$$

The corresponding quadratic form reads

$$\begin{aligned}
 \sum_{K,L \in \mathcal{P}^0} g_{KL}^{\text{uni}} z_K z_L &= \sum_{K,L \in \mathcal{P}^0} h_{KL} z_K z_L + H_1 \left(\sum_{K \in \mathcal{P}^0} (\mu_{11})_K z_K \right)^2 \\
 &\quad - 2H_2 \left(\sum_{K \in \mathcal{P}^0} (\mu_{12})_K z_K \right)^2 - 2H_2 \left(\sum_{K \in \mathcal{P}^0} (\mu_{13})_K z_K \right)^2.
 \end{aligned}
 \tag{6.15}$$

From condition (3.34) we obtain the following condition for stability of equilibrium against internal structural rearrangements

$$\sum_{K,L \in \mathcal{P}^0} g_{KL}^{\text{uni}} z_K z_L > 0 \quad \text{for every } [z] \neq [0], \quad [z] \geq [0],
 \tag{6.16}$$

under the restriction to uniform deformation in uniaxial tension. Analogously, from inequality (5.24) we obtain the following condition for stability of a process of internal structural rearrangements

$$\sum_{K,L \in \mathcal{P}^0} g_{KL}^{\text{uni}} z_K z_L > 0 \quad \text{for every } [z] \neq [0],
 \tag{6.17}$$

under the restriction to uniform deformation in uniaxial tension and under the assumption $h_{KL} = h_{LK}$. The latter condition is clearly more restrictive in general.

In the case of single slip, when all indices K, L are omitted, the conditions (6.16) and (6.17) coincide and reduce to

$$h + H_1\mu_{11}^2 - 2H_2(\mu_{12}^2 + \mu_{13}^2) > 0. \quad (6.18)$$

If the elastic compliances are neglected then $H_1 = C^* - \sigma$, $H_2 = \sigma$, and inequality (6.18) simplifies to

$$h + C^*\mu_{11}^2 > \sigma(\mu_{11}^2 + 2\mu_{12}^2 + 2\mu_{13}^2). \quad (6.19)$$

From condition (6.18) or (6.19) the critical value of the hardening modulus h can be read off for a given C^* , or conversely. It may be noted that if $\mu_{11} \neq 0$ then for any $h < 0$ the condition (6.19) can be satisfied if the loading device is sufficiently stiff (i.e. C^* sufficiently large), but this is *not* true for the condition (6.18). Obviously, the reduced forms (6.18) or (6.19) of the stability conditions are not limited to crystal plasticity but pertain to any model of classical associative elastoplasticity with a single (averaged) mechanism of plastic deformation, provided the normality rule has been correctly extended to finite strain as in equation (6.4). For a controlled tensile load ($C^* = 0$), the above formulae may be compared with those for dead loading derived by Hill (1967) on a completely different route.

It is beyond the scope of this paper to examine implications of conditions (6.16) and (6.17) in detail for multiple mechanisms. We restrict ourselves to a few remarks on models with double slip. Consider a matrix of effective hardening moduli in the form

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1+q \\ 1+q & 1 \end{pmatrix} h, \quad h > 0, \quad q > 0. \quad (6.20)$$

We do not enter here the discussion to what extent equation (6.20) is justified experimentally; cf. Asaro (1983), Havner (1992). Suppose that q is given, $(\mu_{11})_1$ and $(\mu_{11})_2$ are positive, and that the influence of elastic compliances on the stability conditions is negligible so that we can substitute $H_1 = C^* - \sigma$, $H_2 = \sigma$. If the tensile load is directly controlled ($C^* = 0$) then the critical value of h corresponding to violation of the condition (6.16) is positive and proportional to σ . For a loading device sufficiently stiff (C^* sufficiently large), the condition (6.16) for stability of equilibrium holds for every h (positive by assumption). Suppose that this is so, and eliminate the term proportional to C^* in the quadratic form (6.15) by taking $z_1/z_2 = -(\mu_{11})_2/(\mu_{11})_1 \equiv -r$. From equation (6.20) it follows that the path stability condition (6.17) necessarily fails for *every* positive h if $q > (1-r)^2/2r$.

We arrive thus at the conclusion that under the assumptions introduced, *the process of uniform symmetric double slipping obeying condition (6.20) is never stable under uniaxial tension*, in the sense that the path instability condition (5.28) is always met. According to the interpretation of path instability given in §5*d*, this means that in every stable equilibrium state along the unstable path of double slipping, another rate-solution with only one active slip system exists which is energetically preferable. Thus, an unstable path of double slipping can at every instant be continued further as a (*possibly* stable) path with single slip. The foregoing conclusion remains valid for uniform non-symmetric double

slipping if $q > (1 - r)^2 / 2r$, but should *not* be automatically extended to multifold symmetry with more than two active systems.

The above analysis illustrates how the energy criterion of path stability can be used to deal with the long-standing difficulty in crystal plasticity connected with non-uniqueness of the set of active slip systems. A related approach was proposed by Franciosi & Zaoui (1991); other existing approaches are not reviewed here.

(b) *Stress-induced martensitic transformation*

Consider a material sample undergoing elastic deformation and stress-induced, diffusionless phase transformation (or mechanical twinning) with coherent interfaces. We assume that the local strain and stress are uniform on every material plane orthogonal in the reference configuration to a fixed vector, and may suffer jumps across such planes. For instance, this can correspond to formation of successive parallel martensitic plates in an austenitic single crystal of a shape memory alloy (cf. e.g. Huo & Müller 1993; Patoor *et al.* 1996). This process is modelled at some selected stage as propagation of a single (equivalent) planar interface between two phases (figure 2). Of course, the figure is only schematic: a fine phase mixture is actually considered such that the material sample undergoes *macroscopically uniform* deformation. The material before transformation (parent phase) and after transformation (martensite) is treated as one elastic material with a local free energy function that fails to be strongly elliptic at some unobservable strains (cf. Knowles & Sternberg 1978). Thermal effects are disregarded in this example of so-called pseudoelasticity. The phase transformation criterion, as a specific form of the activity rule (2.20) with a single mechanism, is adopted in the form

$$\dot{\eta} \geq 0, \quad f \equiv \pi - \pi_c \leq 0, \quad f\dot{\eta} = 0, \quad \pi_c = \text{const} \geq 0, \quad (6.21)$$

where η is the volume fraction of martensite evaluated in the reference configuration, π is the thermodynamic driving force associated with phase transformation and π_c is its threshold value, assumed here constant.

Not entering into details, we take the resulting *macroscopic* constitutive rate-equations in the form derived by Petryk (1998); see also the references therein. These equations have a structure analogous to that of single-mode plasticity with normality at finite deformation, viz.

$$\dot{\mathbf{S}} = \mathbf{C}^e \cdot \dot{\mathbf{F}} - A\dot{\eta}, \quad \dot{f} = A \cdot \dot{\mathbf{F}} - g\dot{\eta}, \quad (6.22)$$

where \mathbf{F} , \mathbf{S}^T and \mathbf{C}^e are the *overall* deformation gradient, nominal stress and related elastic stiffness moduli tensors, respectively, for the whole material sample, and

$$A = \mathbf{C}^e \cdot \Delta \mathbf{F}^{\text{loc}} - \Delta \mathbf{S}^{\text{loc}}, \quad g = \Delta \mathbf{F}^{\text{loc}} \cdot \mathbf{C}^e \cdot \Delta \mathbf{F}^{\text{loc}}, \quad \Delta \mathbf{S}^{\text{loc}} \cdot \Delta \mathbf{F}^{\text{loc}} = 0. \quad (6.23)$$

Here, $\Delta \mathbf{F}^{\text{loc}}$ and $\Delta \mathbf{S}^{\text{loc}}$ are the jumps in the *local* deformation gradient and nominal stress, respectively, due to instantaneous phase transformation. These jumps, regarded here as known, are to be found from the local compatibility conditions and a free energy function determined with the help of a crystallographic analysis.

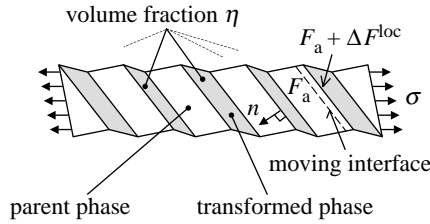


Figure 2. Schematic view of a single crystal undergoing martensitic phase transformation induced by overall uniaxial tension. Boundary effects are neglected.

Since, basic constitutive equations are expressed in terms of non-symmetric tensors, it is now convenient to identify components of \mathbf{F} itself with elements of \mathbf{q} as generalized strains. For macroscopically uniaxial tension⁵ under the constraint (6.1), we may assume the correspondence $(\dot{\mathbf{q}}, \dot{\mathbf{Q}}, \mathbf{K}^e) \leftrightarrow (\dot{\mathbf{F}}, \dot{\mathbf{S}}, \mathbf{C}^e)$ with a single index $\zeta=1, \dots, 6$ of a quantity from the former triple corresponding to a double index $(ij)=(11), (12), \dots, (33)$, $i \leq j$, of the associated quantity from the latter triple. For invariant f and η , we have the analogous correspondence between N_ζ and Λ_{ij} with $i \leq j$. Finally, for *this* choice of \mathbf{q} , the matrix \mathbf{K}^* has only one non-zero element $K_{11}^* = C^*$.

For the assumed single mode of martensitic transformation, the stability conditions (3.34) for equilibrium and (5.24) for the transformation process coincide and reduce to

$$g - \mathbf{N} \cdot (\mathbf{K}^e + \mathbf{K}^*)^{-1} \cdot \mathbf{N} > 0. \tag{6.24}$$

The influence of constitutive quantities on fulfillment of equation (6.24) is not transparent and requires further analysis. We will proceed by removing the constraint imposed on \mathbf{F} by equation (6.1) and identifying the whole \mathbf{F} with a nine-dimensional vector $\dot{\mathbf{q}}$ in the stability conditions. Accordingly, a full correspondence $(\dot{\mathbf{q}}, \dot{\mathbf{Q}}, \mathbf{K}^e) \leftrightarrow (\dot{\mathbf{F}}, \dot{\mathbf{S}}, \mathbf{C}^e)$ is now assumed, where a single index $\zeta=1, \dots, 9$ of a quantity from the former triple corresponds to a double index $(ij)=(11), \dots, (33)$ of the associated quantity from the latter triple, with the analogous correspondence between N_ζ and Λ_{ij} . The matrix \mathbf{K}^* is taken to correspond in analogous manner to the fourth-order tensor \mathbf{C}^* of only one non-zero component $C_{1111}^* = C^*$.

A difficulty now arises since \mathbf{C}^e is not invertible in the state of uniaxial tension. It can be overcome in a simple although somewhat artificial way by superimposing an arbitrarily small tensile stress in x_2 direction, say, kept fixed as a nominal stress. We assume thus that the current state of *overall* Cauchy stress is

$$\sigma_{11} = \sigma > 0, \quad \sigma_{22} = \varepsilon \sigma \text{ with } 1 \gg \varepsilon > 0, \quad \text{other } \sigma_{ij} = 0. \tag{6.25}$$

For that (predominantly tensile) overall stress, we may assume that the macroscopic elastic stiffness tensor \mathbf{C}^e for the phase mixture is positive definite,

⁵It is emphasized that the local stress field inside the specimen may be non-uniform with self-equilibrated fluctuations in the direction of \mathbf{n} . Other fluctuations near specimen boundaries are neglected, assuming that the phase mixture is sufficiently fine.

which is the macroscopic condition for elastic stability under all-round dead loading (Hill 1967), i.e. for $C^*=0$. The condition (3.28) for elastic stability is then fulfilled also for $C^*>0$.

The stability conditions (3.34) and (5.24) coincide and reduce now to

$$g - \mathbf{A} \cdot (\mathbf{C}^e + \mathbf{C}^*)^{-1} \cdot \mathbf{A} > 0. \quad (6.26)$$

This can be transformed further as follows. From equation (6.23) we have

$$g = -\Delta \mathbf{S}^{\text{loc}} \cdot (\mathbf{C}^e)^{-1} \cdot \Delta \mathbf{S}^{\text{loc}} + \mathbf{A} \cdot (\mathbf{C}^e)^{-1} \cdot \mathbf{A}. \quad (6.27)$$

Substitute this into condition (6.26) and rearrange analogously as in the transition from equation (6.11) to (6.14). With the use of the notation

$$M_{11} = (\Delta \mathbf{F}^{\text{loc}} - (\mathbf{C}^e)^{-1} \cdot \Delta \mathbf{S}^{\text{loc}})_{11}, \quad H \equiv ((C^*)^{-1} + ((\mathbf{C}^e)^{-1})_{1111})^{-1}, \quad (6.28)$$

we obtain in that way from condition (6.26) the following condition for stability against uncontrolled phase transformation

$$HM_{11}^2 - \Delta \mathbf{S}^{\text{loc}} \cdot (\mathbf{C}^e)^{-1} \cdot \Delta \mathbf{S}^{\text{loc}} > 0, \quad (6.29)$$

in the state (6.25) of ‘almost’ uniaxial overall stress and under the restriction to parallel interfaces at a micro-scale.

For macroscopic dead loading we have $H=0$. Then, the tensile axis has no longer a privileged meaning in the stability condition (6.29) which can thus be immediately extended to any other state of stress such that \mathbf{C}^e is positive definite. For other \mathbf{C}^e the energy condition (3.26) for stability fails in the purely elastic regime. We arrive thus at the following conclusion: irrespective of the actual jumps on micro-scale in local strain, stress and elastic properties across a moving interface, the stability condition (3.26) cannot be satisfied under dead loading at $f=0$ for the assumed criterion (6.21) of phase transformation.

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