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RESEARCH PAPER

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Determination of optimal actuator forces and positions in smart structures using adjoint method*

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Abstract The problem of optimal design of structures with active support is analyzed in the paper. The sensitivity expressions with respect to the generalized force and the position of actuator are derived by the adjoint structure approach. Next, the optimality conditions are formulated by means of an introduced Lagrangian function. The problem of introduction of a new actuator is also considered and the condition of modification is expressed by means of the topological derivative. The obtained sensitivity formula, optimality conditions and modification conditions are applied in the optimization algorithm with respect to the number, positions and generalized forces of the actuators. Numerical examples of optimal control of beams illustrate the procedure proposed in the paper.

Keywords Smart structures · Actuators · Adjoint method · Optimal active support

1 Introduction

The application of so-called smart structures has attracted a lot of interest in recent years. These structures contain actuators and sensors as constitutive elements. The actuators play the role of active support, and they can generate suitable forces or moments in order to adapt the structure to different external loads, including varying loads. Now, the optimal design problem, besides the specification of cross-sectional parameters, material parameters, configuration or shape parameters and topology of structure, can be formulated in a broader sense, including number, positioning and control of these actuators.

The present paper provides an extension of previous optimal design formulations for beam structures, including position and stiffness of supports (cf. Mróz and Rozvany 1975; Mróz 1980; Mróz and Lekszycki 1981; Garstecki and Mróz 1987; Bojczuk and Mróz 1998; Mróz and Bojczuk 2003). Considering the example of a specific beam structure, the problem of optimal active support using the adjoint method is analyzed. Two types of actuators are taken into account, namely force actuators and moment actuators. Force actuators usually contain hydraulic or pneumatic elements, while piezoelectric membranes can execute moment actions. The optimization problem is formulated as the minimization of displacement or stress functional with constraints set on the total generalized force of the actuators and possibly on the orientation of the generalized forces generated by the actuators. It can also be presented as the minimization of the global force of the actuators with the constraint set on the displacement or stress functional. In order to solve these problems an algorithm composed of two steps is proposed. In the first step, using sensitivity expressions and optimality conditions, the standard gradient optimization with respect to the positions and forces of actuators is performed. In the second step, the conditions of topology modification are applied in order to introduce new actuators into the structure.

In numerical examples of optimal design and optimal control of beams, attention is focussed on the applicability of the proposed approach. At first, assuming the potential energy as the measure of global stiffness, the maximization of the global stiffness is analyzed as an example of the self-adjoint problem. Next, as an example of the non-self-adjoint problem, the minimization of the maximal deflection, measured by an appropriate functional, is considered. In both cases the problem of movable force actuators is studied. Here the relation between the optimal position of the actuator and position of the external load for different maximal actuator force is determined and analyzed. Moreover, it is shown that, for single-force loading, one actuator alone is sufficient and usually its position does not coincide with the

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position of the concentrated loading. The problem of active control of forces generated by actuators at fixed positions is also discussed here. The optimal actions and switching conditions of actuators are obtained for a moving load, both in the case of unilateral (with fixed force orientation) and bilateral actuators (with varying orientations). Also, a problem of active control of actuators placed at fixed positions with respect to global force minimization is considered. Finally, the application of moment actuators is analyzed in the paper.

2 Optimal design of active support with force actuators

The analysis is focussed here on optimal design of beam structures under static loading with force actuators. The general optimization problem, with conditions imposed on global resultant force of actuators, can be presented as follows

$$\min_{p_i, s_i} G \quad \text{subject to} \quad \sum_{i=1}^{n} p_i - P_0 \leq 0 \,, \tag{1}$$

where $p_i, s_i (i = 1, 2, \ldots, n)$ are, respectively, the force and position of i-th actuator, n is the number of actuators, P_0 denotes the global maximal actuator force, and G is an arbitrary displacement or stress functional. Here G is assumed to be the functional of generalized displacements in the form of

$$G = \int_0^l F(w)\,\mathrm{d}x \,, \tag{2}$$

where w denotes the generalized displacement and, in the case of a beam, can be treated as the transverse displacement. Moreover, l denotes the length of the beam. In order to derive sensitivity expressions let us introduce an adjoint structure of the same geometry and boundary conditions as the primary structure, but subjected to a distributed load $q^a = \partial F / \partial w$ (Fig. 1), where superscript a denotes the displacement and stress state, and loading of an adjoint structure. Now, the variation of the functional G can be expressed in the general form as follows

$$\delta G = \int_0^l \frac{\partial F}{\partial w} \delta w \,\mathrm{d}x = \int_0^l q^a \delta w \,\mathrm{d}x \,. \tag{3}$$

Consider first the sensitivity of the functional G with respect to variation of the actuators forces. The virtual work equation for primary and adjoint structures can be presented in the form

$$\int_0^l q^a \delta w \,\mathrm{d}x = \int_0^l M^a \delta\kappa \,\mathrm{d}x \,, \tag{4}$$

where M^a is the bending moment and κ denotes the curvature. Similarly, taking into account that

$$\int_0^l q w^a \,\mathrm{d}x - \sum_{i=1}^{n} p_i w_i^a = \int_0^l M \kappa^a \,\mathrm{d}x \,, \tag{5}$$

the complementary virtual work equation can be expressed as follows

$$-\sum_{i=1}^{n} \delta p_i w_i^a = \int_0^l \delta M \kappa^a \,\mathrm{d}x \,, \tag{6}$$

where w_i^a is the deflection at the i-th actuator location. Moreover, the following relation holds

$$\int_0^l \delta M \kappa^a \,\mathrm{d}x = \int_0^l \delta\,(\kappa E I)\ \kappa^a \,\mathrm{d}x = \int_0^l E I \kappa^a \delta\kappa \,\mathrm{d}x$$
$$= \int_0^l M^a \delta\kappa \,\mathrm{d}x \,, \tag{7}$$

where E is the Young's modulus and I denotes the moment of inertia. Using (4), (6) and (7) the variation (3) of G can be written in the form

$$\delta G = -\sum_{i=1}^{n} \delta p_i w_i^a \,. \tag{8}$$

Consider now the sensitivity with respect to translations δs_i of the force actuators. The virtual work equation is expressed here, as previously, by (4). Next, taking into account that the deflection of the adjoint structure at the point corresponding to the new position of the i-th actuator can be expressed as follows

$$w^a\,(s_i + \delta s_i) = w_i^a + \theta_i^a \delta s_i + \ldots \,, \tag{9}$$

where θ_i^a is the adjoint deflection slope at the i-th actuator point, the complementary virtual work equation becomes

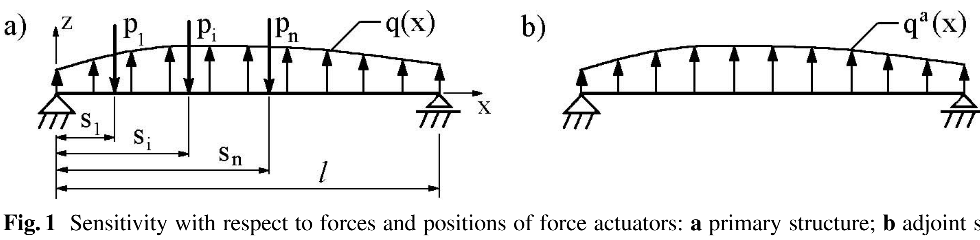

Fig. 1 Sensitivity with respect to forces and positions of force actuators: **a** primary structure; **b** adjoint structure

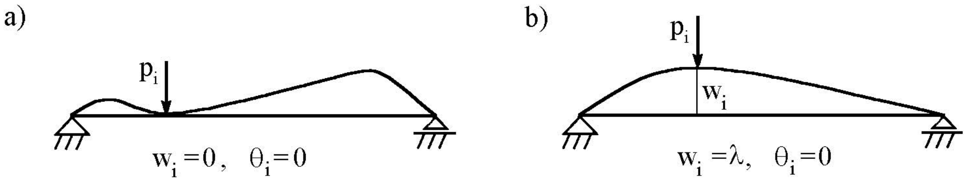

Fig. 2 Interpretation of the optimality criteria for the self-adjoint optimization problem: **a** the case of nonactive constraint imposed on the global force of the actuators; **b** the case with the active constraint

$$-\sum_{i=1}^{n} p_i \theta_i^a \delta s_i = \int_0^l \delta M \kappa^a \, dx \,. \tag{10}$$

Finally, using (4), (7) and (10) in (3), the sensitivity of G with respect to translations of the actuators is

$$\delta G = -\sum_{i=1}^{n} p_i \theta_i^a \delta s_i \,. \tag{11}$$

Using the Lagrangian

$$G^* = G + \lambda \left(\sum_{i=1}^{n} p_i - P_0 \right), \tag{12}$$

where λ ($\lambda \geq 0$) is the Lagrange multiplier and taking into account the sensitivity expressions (8) and (11), we obtain optimality conditions in the form

$$\frac{\partial G^*}{\partial p_i} = -w_i^a + \lambda = 0 \,, \quad i = 1, 2, \ldots, n \,,$$
$$\frac{\partial G^*}{\partial s_i} = -p_i \theta_i^a = 0 \,, \quad i = 1, 2, \ldots, n \,,$$
$$\lambda \left(\sum_{i=1}^{n} p_i - P_0 \right) = 0 \,, \quad \lambda \geq 0 \,. \tag{13}$$

There is a simple geometric interpretation of the optimality criteria (13). When $\sum_{i=1}^{n} p_i < P_0$, then $\lambda = 0$ and the optimal actuator force and location correspond to the conditions $w_i^a = 0$, $\left(w_i^a\right)' = \theta_i^a = 0$. Thus the vanishing deflection and its slope specify the optimal action. When constraint $\sum_{i=1}^{n} p_i = P_0$ occurs, then $\lambda \geq 0$ and conditions $w_i^a = \lambda$, $\theta_i^a = 0$ specify the optimal location.

In the case of the self-adjoint optimization problem, the adjoint system coincides with the primary system. Then, for the nonactive constraint, the deflection and slope vanish at the actuator force location. When the constraint is active i.e. $\sum_{i=1}^{n} p_i = P_0$, we have $w_i = \lambda$, $\theta_i = 0$, Fig. 2.

Moreover, applying the topological derivative, the condition of introduction of a new $(n+1)$-th actuator can be expressed as follows

$$\left. \frac{\partial G^*}{\partial p_{n+1}} \right|_{p_{n+1}=0} < 0 \,, \text{ or } w_{n+1}^a > \lambda \,, \tag{14}$$

where w_{n+1}^a denotes the deflection at the point where a new actuator will be introduced.

3 An alternative formulation of the problem of active support design

Now, instead of (1), the optimization can be presented as follows

$$\min_{p_i, s_i} H \text{ subject to } G - G_0 \leq 0 \,, \text{ where } H = \sum_{i=1}^{n} |p_i| \tag{15}$$

where the notation is the same as in Sect. 2.

Here, using the Lagrangian

$$H^* = H + \lambda (G - G_0) \,, \tag{16}$$

where λ is the Lagrange multiplier and taking into account the sensitivity relations (8), (11), we obtain the optimality conditions in the form

$$\frac{\partial H^*}{\partial p_i} = \operatorname{sgn}(p_i) - \lambda w_i^a = 0 \,, \quad i = 1, 2, \ldots, n \,,$$
$$\frac{\partial H^*}{\partial s_i} = -\lambda p_i \theta_i^a = 0 \,, \quad i = 1, 2, \ldots, n \,,$$
$$\lambda (G - G_0) = 0 \,. \tag{17}$$

Moreover, applying the topological derivative, the condition of introduction of a new $(n+1)$-th actuator can be expressed as follows

$$\left. \frac{\partial H^*}{\partial p_{n+1}} \right|_{p_{n+1}=0} < 0 \,, \text{ or } w_{n+1}^a > \frac{1}{\lambda} \operatorname{sgn}(p_i) \,, \tag{18}$$

where w_{n+1}^a denotes deflection at the point where a new actuator will be introduced.

4 Numerical examples: application of force actuators

Let us consider a beam of length l simply supported at the ends. The beam is loaded by a concentrated force Q moving along the beam. Its variable position is specified by the distance a measured from the left end of the beam.

4.1 Optimal design of moveable actuators for global stiffness maximization

First, we assume that one moveable actuator is used and its position is determined by the distance s, measured from the left end of the beam. Now, the problem of the global stiffness maximization is of the form

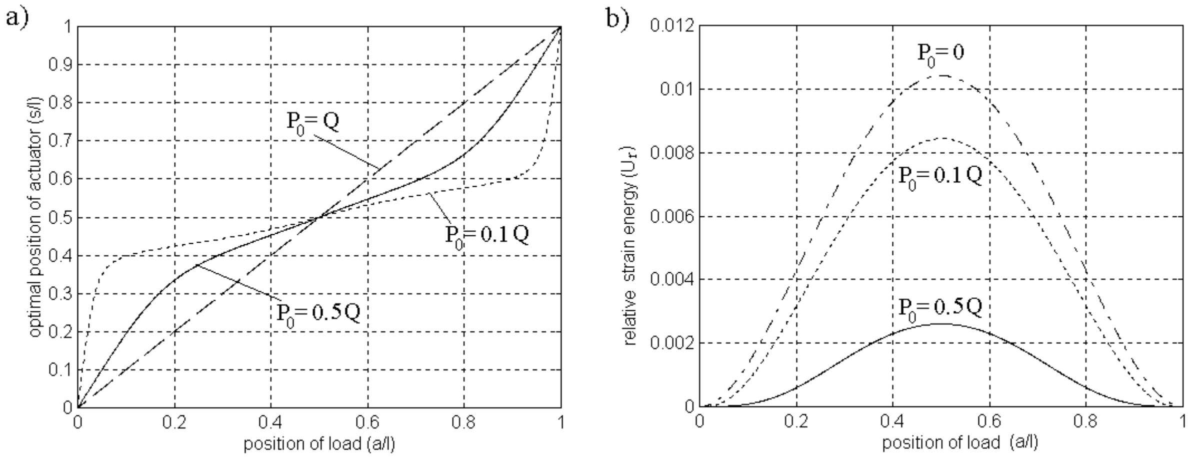

Fig. 3 The problem of the global stiffness maximization for different values of maximal force of the actuator: **a** optimal position of the actuator as a function of load position; **b** the relative strain energy variation for optimal position of actuator

$$\min_{p,s} U \text{ subject to } p - P_0 \le 0\,, \tag{19}$$

where U is the strain energy. Using the proposed approach, the relation between the nondimensional optimal position of the actuator (s/l) and the nondimensional position of the external load (a/l) is obtained. Usually the actuator position does not coincide with the position of the force. The results are shown in Fig. 3a. Here the dashed line, the continuous line and the dotted line correspond to following maximal forces of the actuator: $P_0 = Q$, $P_0 = 0.5Q$ and $P_0 = 0.1Q$. In each case, and for an arbitrary position of load, the maximal admissible force of actuator is utilized.

In order to compare values of the strain energy for the problems analyzed, the relative strain energy U_r corresponding to the unit stiffness ($EI = 1\ \mathrm{N/m^2}$), unit length ($l = 1$ m) and unit external load ($Q = 1$ N), is introduced. The relations between the nondimensional position of the external load (a/l) and the relative strain energy (U_r) are shown in Fig. 3b. Here, the continuous and dotted lines correspond to the following maximal forces of the actuator: $P_0 = 0.5Q$ and $P_0 = 0.1Q$. Moreover, the dot-and-dash line corresponds to the beam without actuators, and the line for the case $P_0 = Q$ covers the horizontal axis.

Next, let us consider the problem of introduction of a second actuator. Here, the condition (14) takes the form $w_2 > w_1$, where w_1 denotes the deflection at the optimal position of the first actuator, while w_2 is the deflection at the point at which a new actuator could be added. However, this condition is not satisfied at any point. So, for the single-force loading, only one actuator is sufficient.

4.2 Optimal control of two fixed unilateral actuators for global stiffness maximization

We consider at first optimal control of two actuators, nonsymmetrically placed at fixed positions $s_1 = 0.25l$ and $s_2 = 0.5l$, which can only generate forces of fixed orientation. Now, the problem of the global stiffness maximization can be presented in the form

$$\min_{p_1,p_2} U \text{ subject to } p_1 + p_2 - P_0 \le 0\,, \quad p_1 \ge 0\,, \quad p_2 \ge 0\,. \tag{20}$$

Here, two cases of the global, maximal force of two actuators are analyzed, namely $P_0 = 0.5Q$ and $P_0 = Q$. The results are shown in nondimensional coordinates (a/l) and (p_i/Q), respectively, in Figs. 4a and 4b. The continuous line presents the optimal control of the first actuator, while the dashed line corresponds to the optimal control of the second actuator. In the first case the maximal admissible force is used for $0.12 \le (a/l) \le 0.82$, while in the second case only for $0.25 \le (a/l) \le 0.50$.

Next, the problem (20) of optimal control of two actuators, symmetrically placed at fixed positions $s_1 = 0.3l$ and $s_2 = 0.7l$, is considered. The solutions for two cases, namely $P_0 = 0.5Q$ and $P_0 = Q$, are shown, respectively in Fig. 5a and 5b. The maximal admissible force is used for $0.13 \le (a/l) \le 0.87$ in the first case, and for $0.3 \le (a/l) \le 0.7$ in the second case.

The variation of the relative strain energy U_r in function of the nondimensional position of the external load (a/l), respectively for nonsymmetric ($s_1 = 0.25l$, $s_2 = 0.5l$) and symmetric ($s_1 = 0.3l$, $s_2 = 0.7l$) positions of the actuators, is presented in Fig. 6a and 6b. Here, the continuous and dotted lines correspond to the cases $P_0 = 0.5Q$ and $P_0 = Q$.

4.3 Optimal control of two fixed bilateral actuators for global stiffness maximization

We consider optimal control of two actuators, placed at fixed positions, which can generate forces of arbitrary orientation. In this case, the problem of the global stiffness maximization can be presented in the form

$$\min_{p_1,p_2} U \text{ subject to } |p_1| + |p_2| - P_0 \le 0\,. \tag{21}$$

As previously, the case of nonsymmetric location of actuators at positions $s_1 = 0.25l$ and $s_2 = 0.5l$, and the case of symmetric location of actuators at points $s_1 = 0.3l$ and

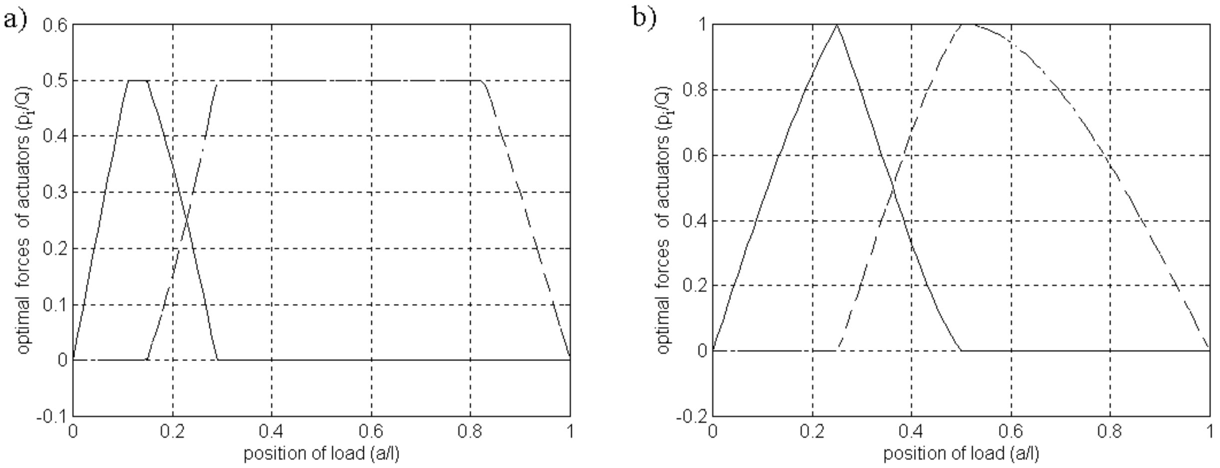

Fig. 4 Optimal control of two unilateral, nonsymmetrically placed actuators for problem of stiffness maximization: **a** the case $P_0 = 0.5Q$; **b** the case $P_0 = Q$

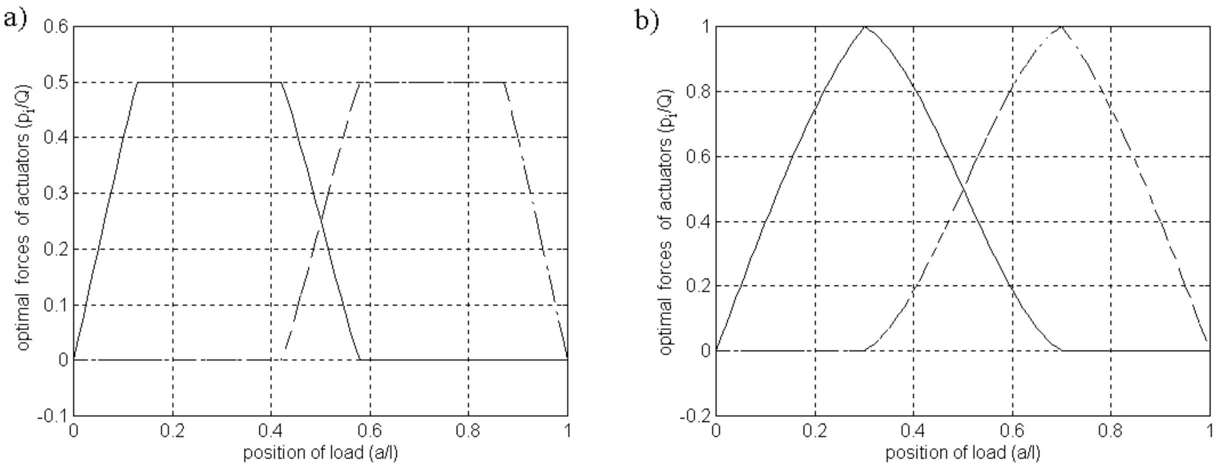

Fig. 5 Optimal control of two unilateral, symmetrically placed actuators for problem of stiffness maximization: **a** the case $P_0 = 0.5Q$; **b** the case $P_0 = Q$

$s_2 = 0.7l$, are examined for two values of the maximal force, namely $P_0 = 0.5Q$ and $P_0 = Q$. The optimal control is presented in Figs. 7 and 8. In some intervals the actuator forces are negative; this ensures better control than in the case studied in Sect. 4.2.

The variation of the relative strain energy U_r dependent on the nondimensional position of the external load (a/l), respectively for nonsymmetric $(s_1 = 0.25l,\ s_2 = 0.5l)$ and symmetric $(s_1 = 0.3l, s_2 = 0.7l)$ positions of the actuators, is presented in Fig. 9a and 9b. Here, the continuous and dotted lines correspond to the cases $P_0 = 0.5Q$ and $P_0 = Q$.

4.4 Optimal design and optimal control of actuators with respect to minimization of the maximal deflection

Assume now that, as in Sect. 4.1, one moveable actuator is used and its position is determined by the distance s, measured from the left end of the beam. Now, the problem of the minimization of the maximal deflection is of the form

$$\min_{p,s} G \quad \text{subject to} \quad p - P_0 \le 0\,, \quad \text{where} \quad G = \int_0^l \left[\frac{w(x)}{w_0}\right]^n \mathrm{d}x\,. \tag{22}$$

Here, the value $n = 20$ is assumed and w_0 denotes the scaling factor. It is chosen as the deflection at the center of the beam loaded only by the force Q also applied in the beam center.

The problem (22) is an example of a non-self-adjoint problem. Now, the adjoint structure is subjected to the distributed load

$$q^a(x) = \partial F \big/ \partial w = \frac{n}{w_0}(w/w_0)^{n-1}\,. \tag{23}$$

The relations between the nondimensional optimal position of the actuator (s/l) and nondimensional position of the external load (a/l) are shown in Fig. 10a. Here the dashed, continuous and dotted lines correspond to the following

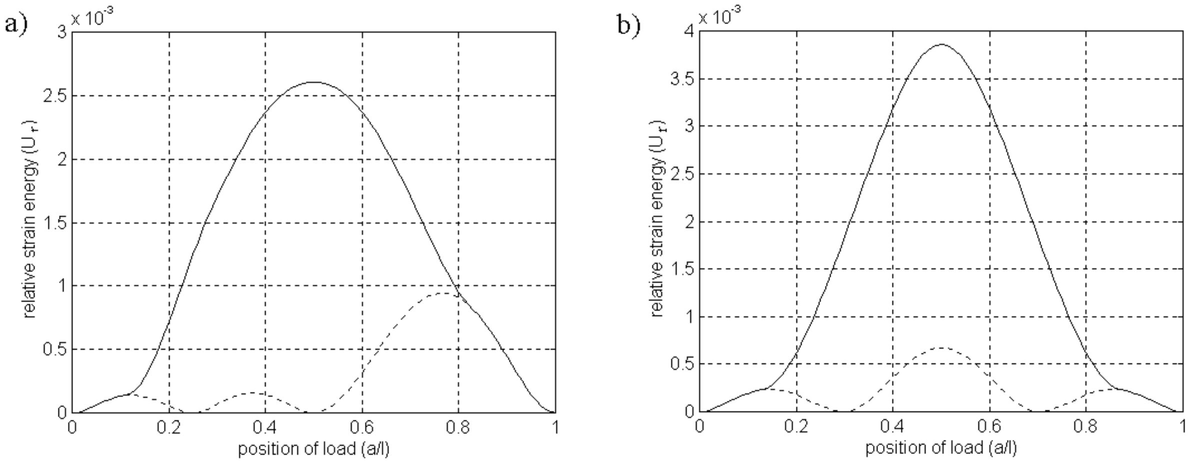

Fig. 6 The relative strain energy variation for problem of optimal control of two unilateral actuators: **a** the case of actuators placed at positions $s_1 = 0.25l$, $s_2 = 0.5l$; **b** the case of actuators placed at positions $s_1 = 0.3l$, $s_2 = 0.7l$

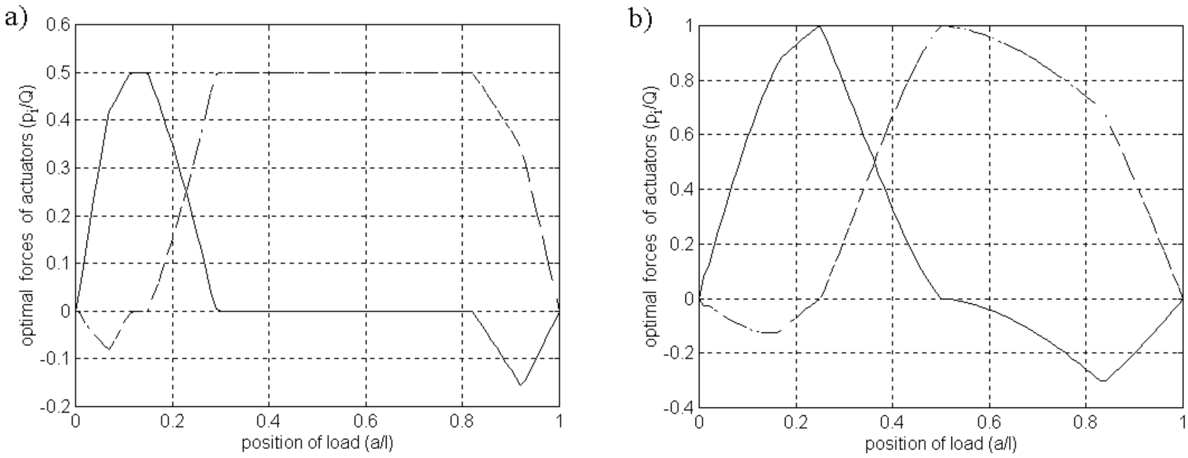

Fig. 7 Optimal control of two bilateral, nonsymmetrically placed actuators for problem of stiffness maximization: **a** the case $P_0 = 0.5Q$; **b** the case $P_0 = Q$

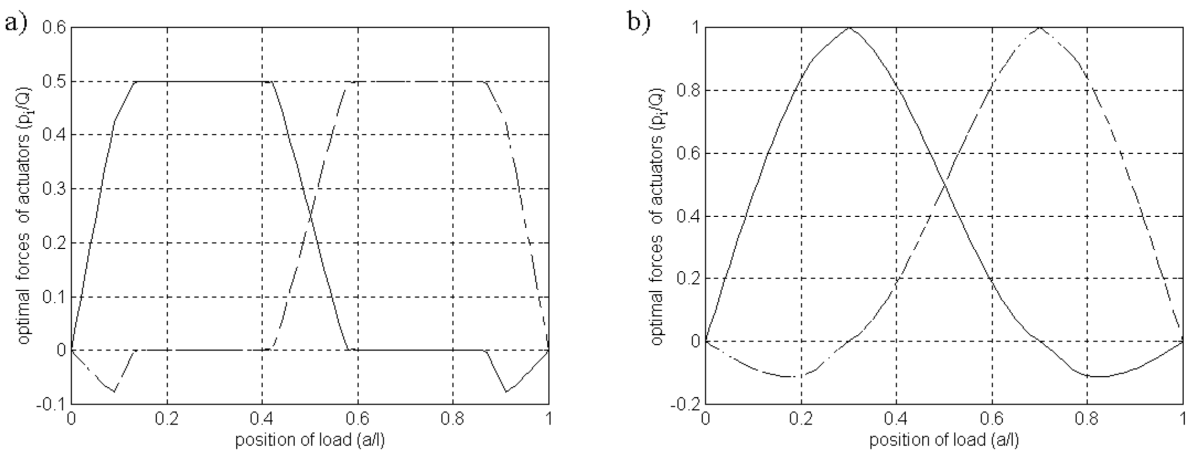

Fig. 8 Optimal control of two bilateral symmetrically placed actuators for problem of stiffness maximization: **a** the case $P_0 = 0.5Q$; **b** the case $P_0 = Q$

maximal forces of the actuator: $P_0 = Q$, $P_0 = 0.5Q$ and $P_0 = 0.1Q$.

Moreover, the variation of the displacement functional value (G) with respect to the nondimensional position of the external load (a/l) is shown in Fig. 10b. Here, the dot-and-dash curve is plotted for the beam without the actuator, the dotted curve corresponds to the case $P_0 = 0.1Q$, while the respective curve for the case $P_0 = Q$ coincides with the horizontal axis. In the case $P_0 = 0.5Q$, the displacement functional takes very small values, so the respective response curve lies close to the horizontal axis. Here, the maximal value of the displacement functional determined at the point $a/l = 0.5$ equals $G = 0.157 \times 10^{-6}$. However, the shapes of the curves are similar to those of the respective curves for the problem of the stiffness maximization (cf. Fig. 3b).

As the second problem, analogously as in Sect. 4.2, we consider optimal control of two unilateral actuators placed at fixed positions $s_1 = 0.25l$ and $s_2 = 0.5l$. The results obtained in the cases when the global, maximal force of two actuators equals $P_0 = 0.5Q$ and $P_0 = Q$, are shown respectively in Figs. 11a and 11b. Here, the continuous line corresponds to the first actuator, while the dashed line corresponds to the second actuator.

4.5 Optimal control of two fixed actuators with respect to global force minimization

Consider now the optimal control of two actuators, nonsymmetrically placed at fixed positions $s_1 = 0.25l$ and $s_2 = 0.5l$. Now, the problem (15) of the global force minimization can be presented in the form

$$\min_{p_1, p_2} (|p_1| + |p_2|) \quad \text{subject to } U - U_0 \leq 0\,, \tag{24}$$

where U_0 is the upper bound on the strain energy U.

Let us denote by $U_0^{\max}$ the maximal value of the strain energy, which corresponds to the case when the moveable force Q is applied at the beam center. We assume two values of U_0, namely $U_0 = 0.5U_0^{\max}$ and $U_0 = 0.1U_0^{\max}$. The optimal nondimensional actuator forces (p_i/Q) as a function of the nondimensional position of the external load (a/l) for two cases of U_0 are shown respectively in Figs. 12a and 12b. Here, the continuous line presents the optimal control of the first actuator, while the dashed curve corresponds to the optimal control of the second actuator.

In the case of the optimal control presented in Sects. 4.1, 4.2, 4.3 and 4.4, both actuators apply varying forces in a continuous way, even when the objective function (global strain energy or maximal displacement) is relatively small. However, in the case considered here, the actuator(s) are

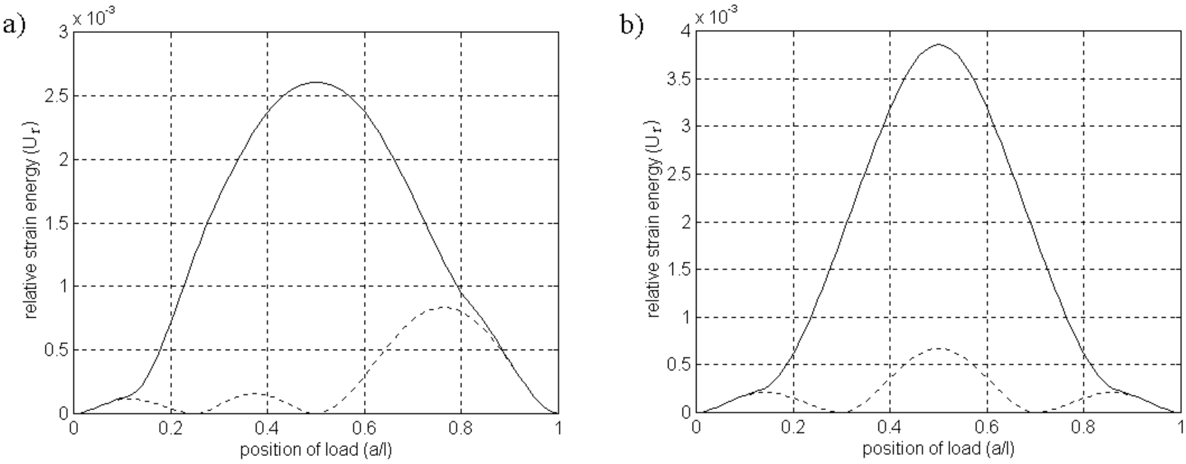

Fig. 9 The relative strain energy variation for problem of optimal control of two bilateral actuators: **a** the case of actuators placed at positions $s_1 = 0.25l$, $s_2 = 0.5l$; **b** the case of actuators placed at positions $s_1 = 0.3l$, $s_2 = 0.7l$

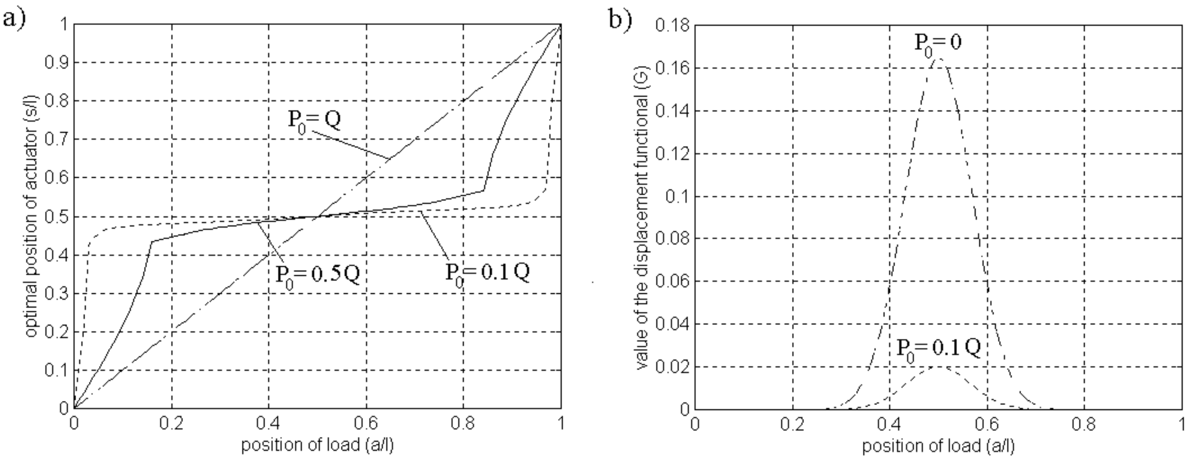

Fig. 10 The problem of minimization of maximal deflection for different values of maximal force of the actuator: **a** optimal position of actuator dependent on load position; **b** variation of the displacement functional value for optimal position of actuator

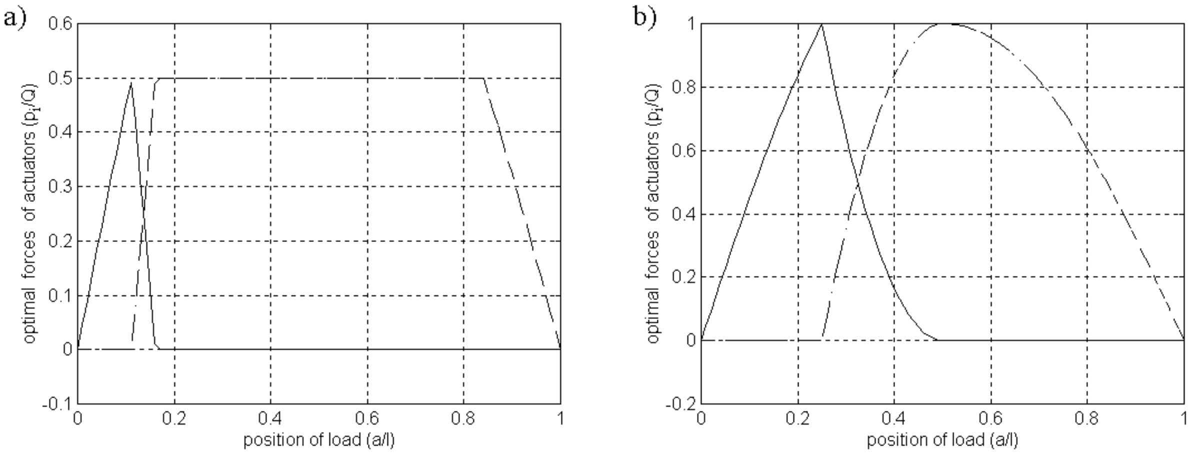

Fig. 11 Optimal control of two unilateral actuators for problem of minimization of maximal deflection: **a** the case $P_0 = 0.5Q$; **b** the case $P_0 = Q$

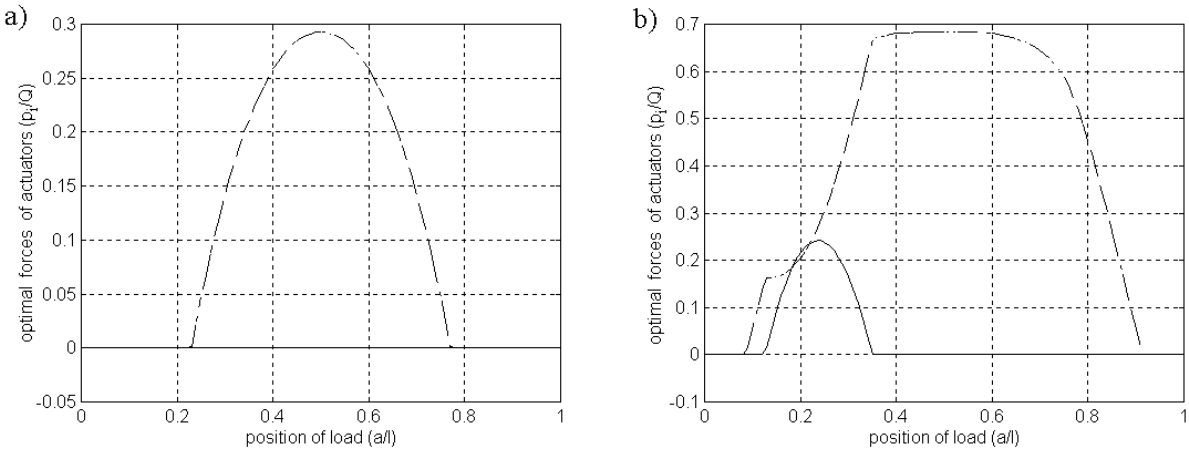

Fig. 12 Optimal control of two actuators for problem of minimization of their global force: **a** the case $U_0 = 0.5U_0^{\max}$; **b** the case $U_0 = 0.1U_0^{\max}$

switched on only when the strain energy attains its upper limit. So, when $U_0 = 0.5U_0^{\max}$, they perform action for $0.22 \leq (a/l) \leq 0.78$, and when $U_0 = 0.1U_0^{\max}$ they act for $0.08 \leq (a/l) \leq 0.93$. In fact, the variation of actuator force results from the constraint $U = U_0$ in (24). It seems that this second way of optimal control is more efficient.

5 Optimal design of active support with moment actuators

In this section we shall discuss the optimal design of beam structures with moment actuators. The general optimization problem, with the condition imposed on the global resultant moment of actuators, can be presented analogously to (1), namely

$$\min_{m_i, s_i} G \quad \text{subject to} \quad \sum_{i=1}^{n} m_i - M_0 \leq 0\,, \tag{25}$$

where m_i $(i = 1, 2, \ldots, n)$ is the moment of the i-th moment actuator and M_0 denotes the global maximal moment of these actuators.

In order to derive sensitivity expressions let us introduce, analogously to Sect. 2, an adjoint structure of the same geometry and boundary conditions as the primary structure,

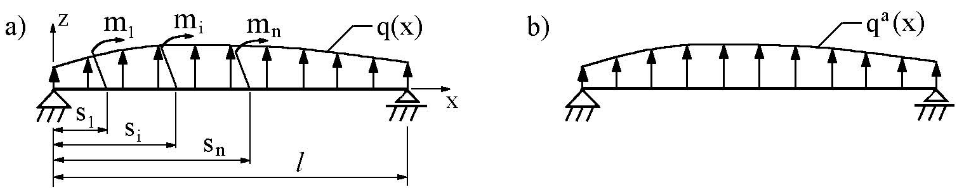

Fig. 13 Sensitivity with respect to moments and positions of the actuators: **a** primary structure; **b** adjoint structure

and subjected to a generalized load $q^a = \partial F/\partial w$ (Fig. 13). Now, as previously, the variation of the functional G is expressed by (3).

At first, consider the sensitivity of the functional G with respect to the variation of the actuators moments. The virtual work equation for the primary and adjoint structures is presented by (4). Moreover, we can write for these structures

$$\int_0^l q w^a \,\mathrm{d}x - \sum_{i=1}^n m_i \theta_i^a = \int_0^l M \kappa^a \,\mathrm{d}x \,. \tag{26}$$

So, the complementary virtual work equation takes the form

$$-\sum_{i=1}^n \delta m_i \theta_i^a = \int_0^l \delta M \kappa^a \,\mathrm{d}x \,, \tag{27}$$

where, as previously, θ_i^a denotes the deflection slope of the adjoint structure at the i-th actuator position. Using (4), (7) and (27) the variation (3) of G can be written in this case as follows

$$\delta G = -\sum_{i=1}^n \delta m_i \theta_i^a \,. \tag{28}$$

Consider now the sensitivity with respect to translations δs_i of the moment actuators. The virtual work equation is expressed here, as previously, by (4). Next, for the modified primary structure and for the adjoint structure, we have

$$\int_0^l q w^a \,\mathrm{d}x - \sum_{i=1}^n m_i \theta^a (s_i + \delta s_i) = \int_0^l (M + \delta M)\ \kappa^a \,\mathrm{d}x \,. \tag{29}$$

The deflection slope of the adjoint structure at the point corresponding to the new position of the i-th actuator, can be expressed in the form

$$\theta^a (s_i + \delta s_i) = \theta_i^a + \kappa_i^{a(-)} \delta s_i + \ldots \,, \tag{30}$$

or in the form

$$\theta^a (s_i + \delta s_i) = \theta_i^a + \kappa_i^{a(+)} \delta s_i + \ldots \,, \tag{31}$$

where $\kappa_i^{a(-)}$, $\kappa_i^{a(+)}$ are the curvatures at this point, respectively, from the left and right hand sides. So, here the curvature discontinuity may appear and this effect can be caused by action of the concentrated moment in the adjoint structure. For example it takes place for the strain energy functional U, where the adjoint structure is identical to that for the primary structure. In order to take into account this effect and avoid uncertainty in the deflection slope, the concentrated moment is presented as the couple of forces, f_i (cf. Mróz 1980). Then $m_i = f_i \times 2a_i$, where $2a_i$ denotes the distance between the forces f_i and it is assumed that a_i is a small quantity (Fig. 14). Now (26), (29) can be rewritten in the form

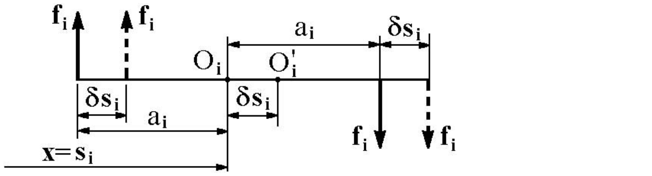

Fig. 14 The concentrated moment, m_i, as the couple of the forces, f_i, and translation, δs_i

$$\int_0^l q w^a \,\mathrm{d}x - \sum_{i=1}^n \left[f_i w^a (s_i + a_i) - f_i w^a (s_i - a_i) \right]$$
$$= \int_0^l M \kappa^a \,\mathrm{d}x \,, \tag{32}$$

and

$$\int_0^l q w^a \,\mathrm{d}x - \sum_{i=1}^n \left[f_i w^a (s_i + \delta s_i + a_i) \right.$$
$$\left. - f_i w^a (s_i + \delta s_i - a_i) \right] = \int_0^l (M + \delta M)\ \kappa^a \,\mathrm{d}x \,. \tag{33}$$

Using, for the displacements from (32) and (33), the second-order Taylor series expansion at the point $x = s_i$, we have

$$w^a (s_i - a_i) = w_i^a - \theta_i^a a_i + \frac{1}{2} \kappa_i^{a(-)} a_i^2 + \ldots \,,$$
$$w^a (s_i + a_i) = w_i^a + \theta_i^a a_i + \frac{1}{2} \kappa_i^{a(+)} a_i^2 + \ldots \,,$$
$$w^a (s_i + \delta s_i - a_i) = w_i^a - \theta_i^a (a_i - \delta s_i)$$
$$+ \frac{1}{2} \kappa_i^{a(-)} (a_i - \delta s_i)^2 + \ldots \,,$$
$$w^a (s_i + \delta s_i + a_i) = w_i^a + \theta_i^a (a_i + \delta s_i)$$
$$+ \frac{1}{2} \kappa_i^{a(+)} (a_i + \delta s_i)^2 + \ldots \,. \tag{34}$$

Subtracting (33) and (32) and taking into account (34), we get

$$-\sum_{i=1}^n \left[f_i a_i \left(\kappa_i^{a(-)} + \kappa_i^{a(+)} \right) \delta s_i \right.$$
$$\left. + \frac{1}{2} f_i \left(\kappa_i^{a(+)} - \kappa_i^{a(-)} \right) (\delta s_i)^2 \right] = \int_0^l \delta M \kappa^a \,\mathrm{d}x \,. \tag{35}$$

Moreover, taking into account that $f_i a_i = \frac{1}{2} m_i$ and neglecting small quantities of higher order, the complementary virtual work equation (35) can be rewritten as follows

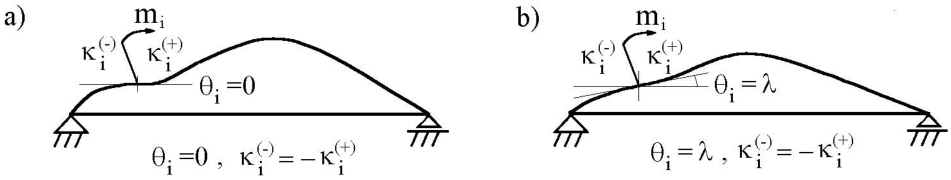

Fig. 15 Interpretation of the optimality criteria for the self-adjoint optimization problem: **a** the case of the nonactive constraint imposed on the global moment of the actuators; **b** the case with the active constraint

$$-\frac{1}{2}\sum_{i=1}^{n} m_i \left(\kappa_i^{a(-)} + \kappa_i^{a(+)}\right) \delta s_i = \int_0^l \delta M \kappa^a \, \mathrm{d}x \,. \tag{36}$$

Finally, using (4), (7) and (36) in (3), the sensitivity of G with respect to translations of the actuators is

$$\delta G = -\frac{1}{2}\sum_{i=1}^{n} m_i \left(\kappa_i^{a(-)} + \kappa_i^{a(+)}\right) \delta s_i \,. \tag{37}$$

Now, using the Lagrangian

$$G^* = G + \lambda \left(\sum_{i=1}^{n} m_i - M_0 \right), \tag{38}$$

where λ ($\lambda \geq 0$) is the Lagrange multiplier and taking into the account sensitivity relations (28) and (37), we obtain the optimality conditions in the form

$$\begin{aligned} &\frac{\partial G^*}{\partial m_i} = -\theta_i^a + \lambda = 0\,, && i = 1, 2, \ldots, n\,, \\ &\frac{\partial G^*}{\partial s_i} = -\frac{1}{2} m_i \left(\kappa_i^{a(-)} + \kappa_i^{a(+)}\right) = 0\,, && i = 1, 2, \ldots, n\,, \\ &\lambda \left(\sum_{i=1}^{n} m_i - M_0 \right) = 0\,, && \lambda \geq 0\,. \end{aligned} \tag{39}$$

Here again, we have a simple geometric interpretation of the optimality criteria (39). When $\sum_{i=1}^{n} m_i < M_0$, then $\lambda = 0$ and the optimal actuator moment and location correspond to the conditions $\left(w_i^a\right)' = \theta_i^a = 0$, $\kappa_i^{a(-)} + \kappa_i^{a(+)} = 0$. Thus the vanishing deflection slope and average curvature specify the optimal action. When constraint $\sum_{i=1}^{n} m_i = M_0$ occurs, then $\lambda \geq 0$ and conditions $\theta_i^a = \lambda$, $\kappa_i^{a(-)} + \kappa_i^{a(+)} = 0$ specify the optimal location.

In the case of the self-adjoint optimization problem, the adjoint system coincides with the primary system. Then, for the nonactive constraint, the deflection slope and average curvature vanish at the actuator force location. When the constraint is active i.e. $\sum_{i=1}^{n} m_i = M_0$, we have $\theta_i = \lambda$, $\kappa_i^{(-)} + \kappa_i^{(+)} = 0$, Fig. 15.

Moreover, applying the topological derivative, the condition for introduction of a new $(n+1)$-th moment actuator can be expressed as follows

$$\left.\frac{\partial G^*}{\partial m_{n+1}}\right|_{m_{n+1}=0} < 0\,, \ \text{ or } \theta_{n+1}^a > \lambda\,, \tag{40}$$

where θ_{n+1}^a denotes the deflection slope at the point where a new actuator will be introduced.

6 Numerical examples: application of moment actuators

Let us consider a beam of length l simply supported at the ends. The beam is loaded by the single moving force Q, whose variable position is specified by distance a measured from the left end of the beam.

6.1 Optimal design of moveable moment actuator for global stiffness maximization

Assume that one moveable moment actuator is used and its position is determined by the distance s, measured from the left end of the beam. The problem of the global stiffness maximization is of the form

$$\min_{m,s} U \ \text{ subject to } \ |m| - M_0 \leq 0\,, \tag{41}$$

where U is the strain energy. Using the proposed approach, the optimal position and optimal moment of the actuator as a function of the external load position is obtained. The results are shown respectively in Figs. 16a and 16b. The actuator position does not coincide with the force position. Moreover, when $M_0 \geq 0.3Ql$, the condition imposed on the global moment of the actuator is not active.

In the second case we assume that the maximal moment of actuator is $M_0 = 0.1Ql$. The diagrams of the optimal position and optimal moment of actuator as a function of the external load position are presented respectively in Figs. 17a and 17b. Here, the moment actuator lies closer to the beam ends than previously, and the value of moment is truncated by M_0.

6.2 Optimal design of two actuators for global stiffness maximization

Assume now that two moveable moment actuators are used and their positions are determined by the distances s_1 and s_2 measured from the left end of the beam. Now, the problem of the global stiffness maximization is of the form

$$\min_{m_1,m_2,s_1,s_2} U \ \text{ subject to } \ |m_1| + |m_2| - M_0 \leq 0\,, \tag{42}$$

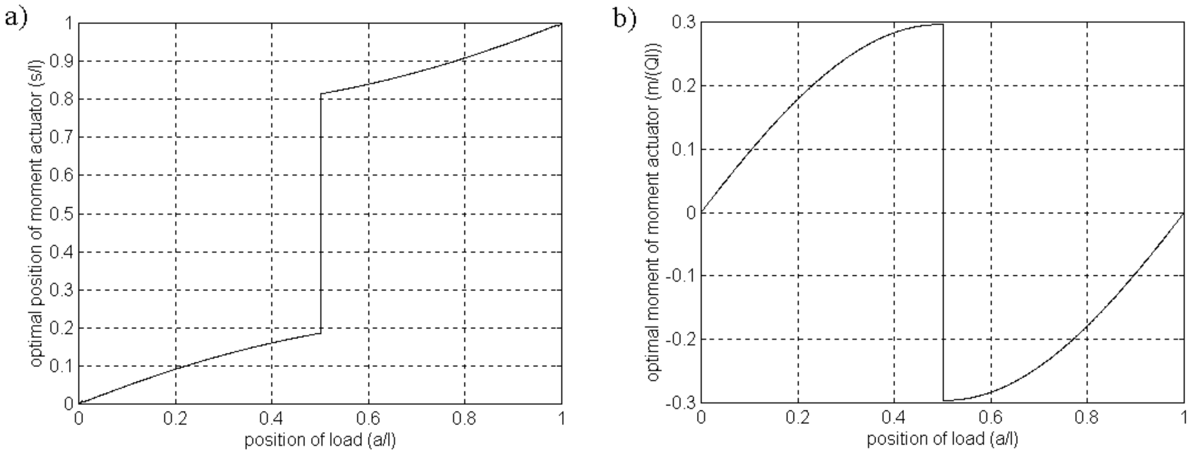

Fig. 16 Optimization of moment actuator: **a** optimal position of the actuator as a function of the load position; **b** optimal moment of the actuator as a function of load position

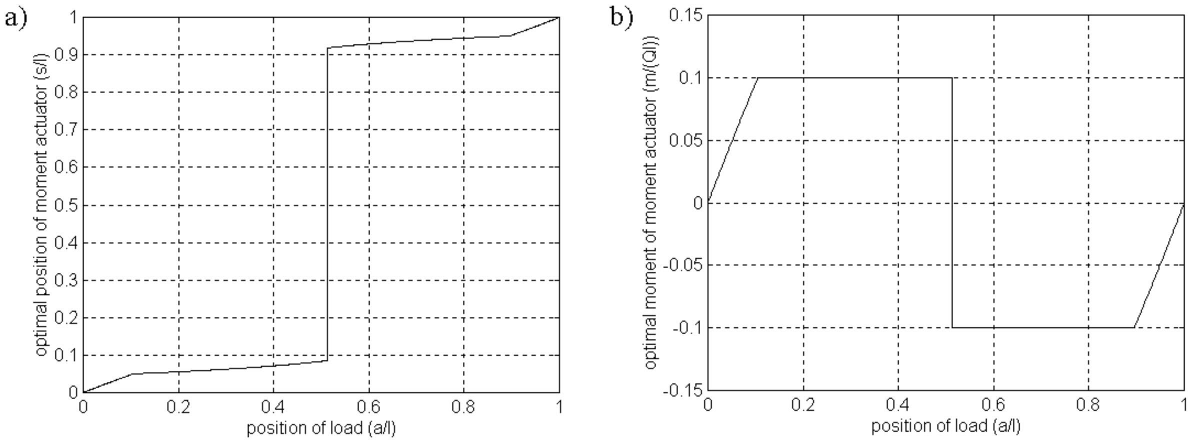

Fig. 17 Optimization of the moment actuator for $M_0 = 0.1Ql$: **a** optimal position of the actuator as a function of load position; **b** optimal moment of the actuator as a function of load position

where U is the strain energy. The optimal positions and optimal moments of the actuators as a function of the external load position are shown respectively in Figs. 18a and 18b. Here, the continuous line corresponds to the first actuator, while the dashed line corresponds to the second actuator. When $M_0 \geq 0.35Ql$, the condition im-

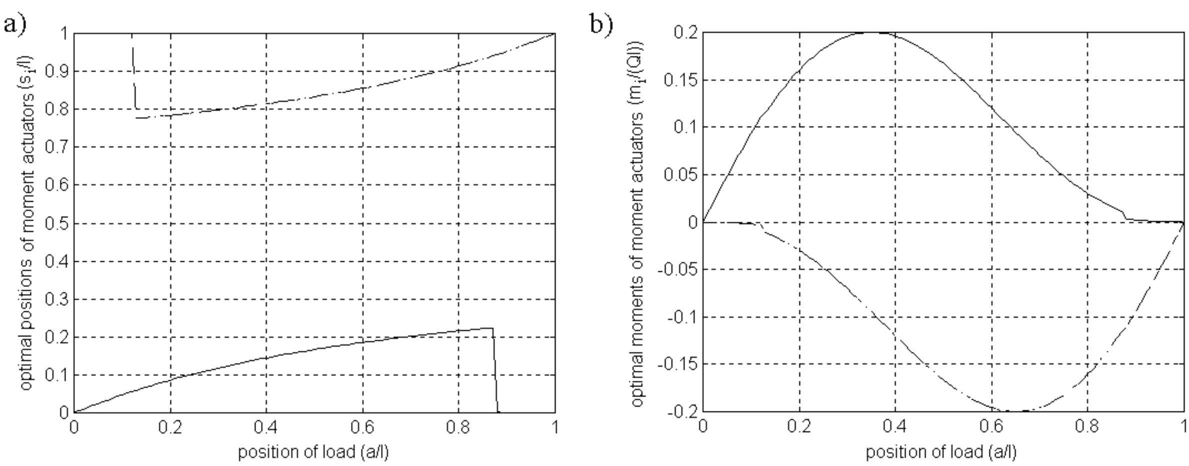

Fig. 18 Optimization of two moment actuators: **a** optimal positions of the actuators as a function of load position; **b** optimal moments of the actuators as a function of load position

posed on the global moment of the actuators is not active.

In order to compare values of the strain energy for different problems with moment actuators, the relative strain energy U_r is used (see Sect. 4.1). The relations between the nondimensional position of the external load (a/l) and the relative strain energy (U_r) are shown in Fig. 19. Here, the dotted line corresponds to the problem (42) of optimal control of two actuators, while the continuous line describes optimal action of only one actuator working without any constraint on its maximal moment. Moreover, the dot-and-dash line corresponds to the beam without actuators.

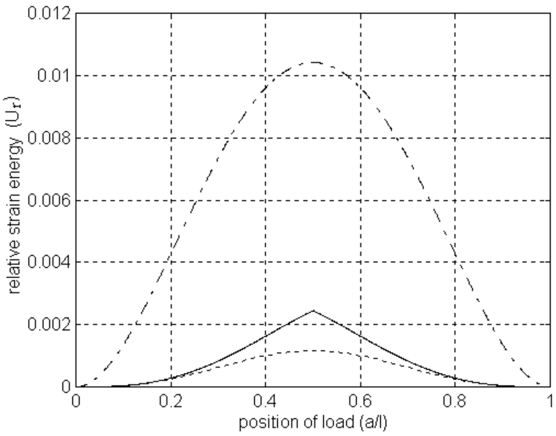

Fig. 19 Comparison of variations of the relative strain energy for different problems with moment actuators

7 Conclusions

The approach presented provides a useful tool for the determination of the number, positions and generalized forces of actuators in the case of problems with fixed load and also for control of these actions in the case of problems with varying load. Application of the active support changes essentially the structure response and enables significant increase of structure stiffness or decrease of maximal deflection. The method can be extended for other problems of active control and optimal design, such as reduction of stresses, or control of vibrations.

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References

Bojczuk D, Mróz Z (1998) On optimal design of supports in beam and frame structures. Struct Optim 16:47–57

Garstecki A, Mróz Z (1987) Optimal design of supports of elastic structures subject to loads and initial distortions. Mech Struct Mach 15:47–68

Mróz Z (1980) On optimal force action and reaction on structures. SM Archives 5:475–496

Mróz Z, Bojczuk D (2003) Finite topology variations in optimal design of structures. Struct Multidisc Optim 25:153–173

Mróz Z, Lekszycki T (1981) Optimal support reaction in elastic frame structures. Comp Struct 14:179–185

Mróz Z, Rozvany GIN (1975) Optimal design of structures with variable support positions. J Opt Theor Appl 15:85–101