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TRANSVERSAL VIBRATION OF A PLATE, PRODUCED BY HEATING

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Introduction

Papers on the vibration of a plate of moderate thickness due to a non-steady-state temperature field are scarce. The basic elements of the theory for plates of moderate thickness have been given by B. A. BOLEY and A. D. BARBER, [1]. In this reference, a rectangular plate is considered, simply supported on the contour, and subject to a uniform step heat input over one face. A numerical analysis is also given.

In the present paper, the equations of thermally excited vibration of a plate are derived. The starting point is the heat equation in three dimensions, coupled (or not coupled) with the deformation field. The member exciting the vibration is the density of moment of a three-dimensional temperature field along the thickness. It is assumed that longitudinal vibrations of the plate are independent of the flexural vibration.

The considerations are confined to harmonic forced vibration. The basic equation is given for an infinite plate on an elastic foundation with a prescribed heat flow across the bounding surfaces harmonically varying in time.

Also, the thermal vibration of a rectangular plate simply supported or simply supported on the contour and having an additional support inside the plate region has been considered. Finally, solution is obtained for a plate of which one side is clamped, the others being simply supported. Thermal vibration of a circular plate is also investigated.

Finally, an approximate solution is given for the problems discussed consisting in the assumption that the moment density of a three-dimensional temperature field may be replaced by the temperature difference between the upper and the lower surface of the plate per unit thickness. With this assumption, the equation of the coupled thermoelastic problem is derived.

1. General Equations

Let us consider a plate of moderate thickness in a non-steady-state temperature field. To determine the displacements and strains, the same assumptions have been made as in the theory of plates of moderate thickness. They are the assumption of plane stress and plane sections after the deformation.

Under heating, stresses and strains appear and the plate undergoes dilatation in its middle plane, and deflection. The stresses σ_{ij} are related with the strains ε_{ij} by the equations:

$$\sigma_{ij} = \frac{2G}{1-\nu}\{(1-\nu)\varepsilon_{ij} + [\nu\varepsilon_{kk} - (1+\nu)\alpha_t T]\delta_{ij}\} \qquad (i,j=1,2), \tag{1.1}$$

where G denotes the shear modulus, ν — Poisson's ratio, α_t — the coefficient of thermal dilatation, T — the temperature. Finally, δ_{ij} is Kronecker delta.

The strains ε_{ij} are connected with the displacements in the following manner :

$$\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon''_{ij} = \frac{1}{2}(u'_{i,j} + u'_{j,i}) - x_3 w_{,ij} \qquad (i,j=1,2), \tag{1.2}$$

where u'_i denotes the displacements due to a uniform tension of the middle surface, $u''_i = -x_3 w_{,i}$ — the displacements due to the deflection of the plate denoted by $u''_3 = w$.

Let us introduce the resultant forces of the stresses N_{ij} acting in the plane of the plate and the moments M_{ij}:

$$N_{ij} = \int_{-h/2}^{h/2} \sigma_{ij}\, dx_3, \quad M_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} x_3 dx_3. \tag{1.3}$$

Introducing (1.1) in (1.3), and integrating over the thickness, we find:

$$N_{ij} = D\{(1-\nu)\varepsilon'_{ij} + [\nu\varepsilon'_{kk} - (1+\nu)\alpha_t\tau_0]\delta_{ij}\}, \tag{1.4}$$

$$M_{ij} = -N\{(1-\nu)w_{,ij} + [\nu w_{,kk} + (1+\nu)\alpha_t\tau]\delta_{ij}\}, \qquad (i,j=1,2), \tag{1.5}$$

where

$$\tau(x_1, x_2; t) = \frac{12}{h^3}\int_{-h/2}^{h/2} T(x_1, x_2, x_3; t)\, x_3 dx_3,$$

$$\tau_0(x_1, x_2; t) = \frac{1}{h}\int_{-h/2}^{h/2} T(x_1, x_2, x_3; t)\, dx_3,$$

and

$$D = \frac{Eh}{1-\nu^2}, \quad N = \frac{Eh^3}{12(1-\nu^2)}.$$

Let us consider the equation of motion in the x_1, x_2 plane:

$$N_{ij,j} = \varrho h \ddot{u}'_i, \tag{1.6}$$

where ϱ is the plate density per unit area of the middle surface.

Introducing the stress function F and expressing thereby the stresses [2], [3]

$$(1.7)\quad N_{ij} = -F_{,ij} + \delta_{ij}\left(\nabla_1^2 - \frac{1}{2c_2^2}\frac{\partial}{\partial t^2}\right)F, \quad c_2^2 = \frac{G}{\varrho h}, \quad \nabla_1^2 = \partial_1^2 + \partial_2^2 \ (i, j = 1, 2),$$

we find for the function F the following wave equation, [3];

$$(1.8)\qquad \Box_2^2\{\Box_1^2 F + E\alpha_t h\tau_0\} = 0,$$

where

$$\Box_\alpha^2 = \nabla_1^2 - \frac{1}{c_\alpha^2}\frac{\partial^2}{\partial t^2}, \quad c_1^2 = \frac{D}{\varrho h} \qquad (\alpha = 1, 2).$$

If, in the equation of motion in the transverse diιection

$$(1.9)\qquad M_{ij,ij} = \varrho h \ddot{w}$$

the moments (1.5) are introduced, we obtain the equation of transversal vibration:

$$(1.10)\qquad \nabla_1^4 w + \frac{\varrho h}{N}\ddot{w} + (1+\nu)\alpha_t \nabla_1^2 \tau = 0.$$

The temperature field will be found from the heat equation

$$(1.11)\qquad \nabla^2 T - \frac{1}{\varkappa}\dot{T} = 0, \quad \nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2$$

wit the prescribed boundary conditions in the two boundary planes and the lateral surface. The temperature function may be resolved into a symmetric and a skew-symmetric function in relation to the middle plane — T_s and T_a, respectively.

Knowing T, the functions τ_0 and τ can be found:

$$(1.12)\qquad \begin{cases} \tau_0 = \dfrac{1}{h}\displaystyle\int_{-h/2}^{h/2} T\,dx_3 = \dfrac{1}{h}\int_{-h/2}^{h/2} T_s\,dx_3, \\ \tau = \dfrac{12}{k^3}\displaystyle\int_{-h/2}^{h/2} Tx_3\,dx_3 = \dfrac{12}{h^3}\int_{-h/2}^{h/2} T_a x_3\,dx_3. \end{cases}$$

On determining the function $\tau_0 = \tau_0(x_1, x_2; t)$ and $\tau = \tau(x_1, x_2; t)$, we solve the Eqs. (1.8) and (1.10) separately. It is assumed [cf. (1.2)] that the longitudinal vibration is independent of the transversal vibration of the plate. This assumption is legitimate if the stresses due to tension are insignificant in relation to those due to bending, which is the case if the edges are free from stresses. Otherwise, the system of equations (1.8) should be taken into consideration, and

$$(1.13)\qquad \nabla_1^4 w + \frac{\varrho h}{N}\ddot{w} + (1+\nu)\alpha_t \nabla_1^2 \tau = \frac{1}{N} N_{ij} w_{,ij}.$$

In further considerations we shall confine ourselves to the solution of the Eq. (1.10) by investigating the transversal vibration as independent of the longitudinal vibration. The solution of an equation analogous to (1.8), for plane strain, has been discussed in detail in [4].

Let us consider the equation of transversal vibration of a plate resting on a Winklerian foundation

$$\nabla_1^4 w + \frac{\varrho h}{N}\ddot{w} + kw + (1+\nu)\alpha_t \nabla_1^2 \tau = 0 \qquad (k = c/N), \tag{1.14}$$

where c is the foundation coefficient, and let us suppose the solution of (1.14) to be composed of two parts:

$$w = w_q + w_d, \tag{1.15}$$

where w_q is the deflection in the quasi-static problem and w_d is a deflection due to the inertia forces. The Eq. (1.14) will be split up into the system of two equations:

$$\nabla_1^4 w_q + kw_q + (1+\nu)\alpha_t \nabla_1^2 \tau = 0, \tag{1.16}$$

$$\nabla_1^4 w_d + kw_d + \frac{\varrho h}{N}\ddot{w}_d = -\frac{\varrho h}{N}\ddot{w}_q. \tag{1.17}$$

Let us represent the solution of the Eq. (1.16) in the form of the integral

$$w_q(x_1, x_2; t) = -(1+\nu)\alpha_t \iint\limits_{(\Gamma)} \tau(\xi_1, \xi_2; t)\nabla_1^2 G(x_1, x_2; \xi_1, \xi_2)\, d\xi_1 d\xi_2, \tag{1.18}$$

where G is the Green's function satisfying the equation

$$(\nabla_1^4 + k)G = \delta(x_1 - \xi_1)\delta(x_2 - \xi_2) \tag{1.19}$$

with the same boundary conditions as the function w_q. Γ is the region of the plate.

Applying the Green formulae to (1.18), we find:

$$w_q(x_1, x_2; t) = -(1+\nu)\alpha_t \Big\{ \iint\limits_{(\Gamma)} G(x_1, x_2; \xi_1, \xi_2)\nabla_1^2 \tau(\xi_1, \xi_2; t)\, d\xi_1 d\xi_2 + \int\limits_{(s)} \left(\tau \frac{\partial G}{\partial n} - G\frac{\partial \tau}{\partial n}\right) ds \Big\} \tag{1.20}$$

where (s) is the contour of Γ.

Let us observe that the line integral vanishes if the plate is clamped along the contour ($G = \partial G/\partial n = 0$), and the surface integral vanishes if τ depends on the time t only. In this particular case we have $w_q \equiv 0$ at every point of the plate, therefore, also $w_d \equiv 0$ at every point of the plate. If, then, $\tau = \tau(t)$, and the plate is clamped along the contour, the deflection will be zero and the moments will be found from the equations:

$$M_{ij} = -N(1+\nu)\alpha_t \tau(t)\delta_{ij}. \tag{1.21}$$

Further considerations will be confined to the case of harmonic heating. Therefore, in view of the fact that $T(x_1, x_2, x_3; t) = e^{i\omega t}U(x_1, x_2, x_3)$, we have $w(x_1, x_2; t) = e^{i\omega t}W(x_1, x_2)$, $\tau(x_1, x_2; t) = e^{i\omega t}\theta(x_1, x_2)$, and the Eqs. (1.10) and (1.11) take the form:

$$[(\partial_1^2+\partial_2^2)^2-\beta^2+k]W+(1+\nu)\,\alpha_t(\partial_1^2+\partial_2^2)\theta = 0, \tag{1.22}$$

$$(\partial_1^2+\partial_2^2+\partial_3^2)U-i\eta U = 0, \quad \beta^2 = \frac{\omega^2\varrho h}{N}, \quad \eta = \frac{\omega}{\varkappa}. \tag{1.23}$$

The solution procedure for (1.22) and (1.23) will be as follows. These equations are subject to a double transformation in relation to x_1, x_2 proper for the given region, thus transforming the Eq. (1.22) into an algebraic equation and the Eq. (1.23) into an ordinary differential equation for x_3. Performing the double transformation on the equation $\theta=(12/h^3)\int_{-h/2}^{h/2} Ux_3dx_3$ we obtain the transform of the function θ appearing in the Eq. (1.22). The inverse transformation yields the solution of the problem.

2. The Infinite Plate Resting on an Elastic Foundation

Let us assume that the thermal boundary conditions for the bounding planes are:

$$\lambda\,\frac{\partial T}{\partial x_3}\bigg|_{x_3=h/2} = q(x_1, x_2)\,e^{i\omega t}, \quad \lambda\,\frac{\partial T}{\partial x_3}\bigg|_{x_3=-h/2} = -p(x_1, x_2)\,e^{i\omega t}. \tag{2.1}$$

Let us perform on (1.23) the double Fourier exponential transformation. We find:

$$[d^2/dx_3^2-(\alpha_1^2+\alpha_2^2+i\eta)]U^* = 0, \tag{2.2}$$

where

$$U^*(\alpha_1, \alpha_2; x_3) = \frac{1}{2\pi}\int\int_{-\infty}^{\infty} U(x_1, x_2, x_3)\exp[i(\alpha_1x_1+\alpha_2x_2)]\,dx_1dx_2.$$

Taking into account the boundary conditions (2.1) which, on performing the Fourier transformation, take the form

$$\lambda\frac{dU^*}{dx_3}\bigg|_{x_3=h/2} = q^*(\alpha_1, \alpha_2), \quad \lambda\frac{dU^*}{dx_3}\bigg|_{x_3=-h/2} = -p^*(\alpha_1, \alpha_2), \tag{2.3}$$

we obtain the solution of (2.2). On performing the inverse Fourier transformation, we obtain:

$$U(x_1, x_2, x_3) = \frac{1}{4\pi\lambda}\int\int_{-\infty}^{\infty}\frac{1}{\gamma}\left[(q^*+p^*)\frac{\operatorname{ch}\gamma x_3}{\operatorname{sh}\vartheta}+(q^*-p^*)\frac{\operatorname{sh}\gamma x_3}{\operatorname{ch}\vartheta}\right]\times \tag{2.4}$$
$$\times\exp[-i(\alpha_1x_1+\alpha_2x_2)]\,d\alpha_1d\alpha_2,$$

where

$$\gamma = \sqrt{\alpha_1^2+\alpha_2^2+i\eta}\,, \quad \vartheta = \gamma h/2$$

and

$$(p^*, q^*) = \frac{1}{2\pi}\int\int_{-\infty}^{\infty} (p, q)\exp[i(\alpha_1 x_1 + \alpha_2 x_3)]\,dx_1\,dx_2\,.$$

The transform of θ^* will be obtained from:

$$\theta^*(\alpha_1, \alpha_2) = \frac{12}{h^3}\int_{-h/2}^{h/2} U^*(\alpha_1, \alpha_2; x_3)x_3 dx_3 = \frac{3}{2\lambda}\,\frac{q^*-p^*}{\vartheta^3}(\vartheta - \operatorname{th}\vartheta). \tag{2.5}$$

Let us perform the double Fourier exponential transformation on the Eq. (1.22). We have:

$$[(\alpha_1^2+\alpha_2^2)^2+k-\beta^2]W^* - (\alpha_1^2+\alpha_2^2)(1+\nu)\alpha_t\theta^* = 0\,. \tag{2.6}$$

Solving this algebraic equation for W^*, and performing the inverse Fourier transformation, we obtain the solution of the problem in the form of the integral:

$$w(x_1, x_2; t) = \tag{2.7}$$

$$= \frac{(1+\nu)\alpha_t e^{i\omega t}}{2\pi}\int\int_{-\infty}^{\infty}\frac{(\alpha_1^2+\alpha_2^2)\theta^*(\alpha_1,\alpha_2)}{[(\alpha_1^2+\alpha_2^2)^2+k-\beta^2]}\exp[-i(\alpha_1 x_1+\alpha_2 x_2)]\,d\alpha_1\,d\alpha_2\,,$$

where θ^* is given by the Eq. (2.5). From the Eq. (2.7) it follows that $w(x_1, x_2; t) \equiv 0$ if $p = q$, as was to be expected. The Eq. (2.7) contains a number of particular cases. Thus, for $\beta = 0$, that is if the inertia term is rejected, the Eq. (2.7) will determine the quasi-static deflection of the plate, and for $\omega = 0$ we shall obtain the case of static deflection.

If the temperature field depends on x_1 and t only, the solution will be the simple integral:

$$w(x_1, x_2; t) = \frac{(1+\nu)\alpha_t e^{i\omega t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\theta^*(\alpha_1)\alpha_1^2}{\alpha_1^4+k-\beta^2}\exp(-i\alpha_1 x_1)\,d\alpha_1\,, \tag{2.8}$$

where

$$\theta^*(\alpha_1) = \frac{3}{2}\,\frac{1}{\lambda\vartheta_0^3}(q^*-p^*)(\vartheta_0 - \operatorname{th}\vartheta)\,, \quad \gamma_0 = \sqrt{\alpha_1^2+i\eta}\,, \quad \vartheta_0 = \gamma_0 h/2$$

and

$$(p^*, q^*) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(p, q)\exp(i\,\alpha_1 x_1)\,dx_1\,.$$

Finally, if the temperature is axially symmetric, then, applying to the Eqs. (1.22) and (1.23) the Hankel integral transformation, we obtain the solution in the following form:

$$w(r,t) = (1+\nu)\alpha_t e^{i\omega t} \int_0^\infty \frac{\alpha^3 \theta^*(\alpha)}{\alpha^4 + k - \beta^2} J_0(\alpha r)\, d\alpha\,, \tag{2.9}$$

where

$$\theta^*(\alpha) = \frac{3}{2\lambda\bar{\vartheta}^3}(q^* - p^*)(\bar{\vartheta} - \operatorname{th}\bar{\vartheta}), \quad \bar{\gamma} = \sqrt{\alpha^2 + i\eta}\,, \quad \bar{\vartheta} = \bar{\gamma} h/2$$

and

$$(q^*, p^*) = \int_0^\infty (q, p)\, r J_0(\alpha r)\, dr.$$

Let us assume, finally, that the temperature field depends on x_3 and t only. In this case we have:

$$U(x_3) = \frac{1}{2\lambda\zeta}\left[(q+p)\frac{\operatorname{ch}\zeta x_3}{\operatorname{sh}\delta} + (q-p)\frac{\operatorname{sh}\zeta x_3}{\operatorname{ch}\delta}\right], \quad \zeta = \sqrt{i\eta}, \quad \delta = \zeta h/2 \tag{2.10}$$

and

$$\theta = \frac{12}{h^3}\int_{-h/2}^{h/2} U(x_3)\, x_3\, dx_3 = \frac{3}{2\lambda\delta^3}(q-p)(\delta - \operatorname{th}\delta). \tag{2.11}$$

Since the deflection of the plate does not depend on x_1, x_2, therefore $w = 0$ at every point of the plate.

In this case, the moments are obtained from the equations:

$$M_{ij} = -N(1+\nu)\alpha_t e^{i\omega t}\theta\delta_{ij}\,. \tag{2.12}$$

3. The Rectangular Plate

Let us consider a rectangular plate simply supported on the edge. Let the boundary conditions be (2.1) in the planes $x_3 = \pm h/2$ and $T = 0$ on the edges $x_1 = 0,\ a_1$; $x_2 = 0,\ a_2$. Applying to the Eq. (1.20) the double sine transformation, we obtain the solution of this equation in the form of the double series:

$$U(x_1, x_2, x_3) = \frac{2}{a_1 a_2 \lambda}\sum_{n,m}\frac{1}{\gamma_{nm}}\left[(q^*_{nm} - p^*_{nm})\frac{\operatorname{sh}\gamma_{nm}x_3}{\operatorname{ch}\vartheta_{nm}} + (q^*_{nm} + p^*_{nm})\frac{\operatorname{ch}\gamma_{nm}x_3}{\operatorname{sh}\vartheta_{nm}}\right]\sin\alpha_n x_1 \sin\beta_m x_2, \tag{3.1}$$

where

$$\gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2 + i\eta}, \quad \vartheta_{nm} = \gamma_{nm} h/2, \quad \alpha_n = \frac{n\pi}{a_1}, \quad \beta_m = \frac{m\pi}{a_2}$$

and

$$(q^*_{nm}, p^*_{nm}) = \int_0^{a_1} dx_1 \int_0^{a_2} dx_2 (q, p) \sin \alpha_n x_1 \sin \beta_m x_2 .$$

Let us introduce the function $\theta(x_1, x_2) = \frac{12}{h^3} \int_{-h/2}^{h/2} U x_3 \, dx_3$ and the transform

$$\theta^*_{nm} = \frac{3}{2} \frac{q^*_{nm} - p^*_{nm}}{\lambda \vartheta^3_{nm}} (\vartheta_{nm} - \operatorname{th} \vartheta_{nm}) . \tag{3.2}$$

Performing on the Eq. (1.22) the sine transformation, taking (3.2) into consideration and performing the inverse Fourier transformation, we obtain:

$$w(x_1, x_2; t) = \frac{4}{a_1 a_2} e^{i\omega t} (1+\nu) \alpha_t \sum_{n, m} \frac{\theta^*_{nm} (\alpha_n^2 + \beta_m^2)}{(\alpha_n^2 + \beta_m^2)^2 + k - \beta^2} \sin \alpha_n x_1 \sin \beta_m x_2 . \tag{3.3}$$

From the solution (3.3), a number of particular cases may be obtained. Thus, if the inertia terms are rejected $\beta = 0$, we find the solution of the quasi-static problem, and for $w = 0$ the solution of the static problem. Finally, for $k = 0$ we are concerned with a plate non-resting on an elastic foundation.

If the temperature field does not depend on x_2, the solution of (1.22) takes the form:

$$w_1(x_1, t) = \frac{2}{a_1} e^{i\omega t} (1+\nu) \alpha_t \sum_{n=1}^{\infty} \frac{\theta^*_n \alpha_n^2}{\alpha_n^4 + k - \beta^2} \sin \alpha_n x_1 , \tag{3.4}$$

where

$$\theta^*_n = \frac{3}{2} \frac{q^*_n - p^*_n}{\lambda \vartheta^3_n} (\vartheta_n - \operatorname{th} \vartheta_m), \quad \vartheta_n = \gamma_n h/2 , \quad \gamma_n = \sqrt{\alpha_n^2 + i\eta}$$

and

$$(q^*_n, p^*_n) = \int_0^{a_1} (q, p) \sin \alpha_n x_1 \, dx_1 .$$

Let us consider a rectangular plate performing forced vibration due to a temperature field varying harmonically in time. Let, in addition, a load

$$R(x_2, t) = \frac{2}{a_2} e^{i\omega t} \sum_m R^*_m \sin \beta_m x_2$$

act along the line $x_1 = \xi_1$, varying in time with the same frequency as the temperature field. The resultant action of the temperature field and the load $R(x_2; t)$ will produce a deflection w, which must satisfy the differential equation

$$(3.5)\qquad [(\partial_1^2+\partial_2^2)^2+k-\beta^2]\,W+(1+\nu)\alpha_t(\partial_1^2+\partial_2^2)\theta = \frac{1}{N}\delta(x_1-\xi_1)\,r(x_2),$$

$$R(x_2\,;t) = e^{i\omega t}\,r(x_2)\,.$$

The solution of (3.5) may be represented by the double series:

$$(3.6)\qquad w(x_1, x_2; t) = \frac{4}{a_1 a_2}e^{i\omega t}\sum_{n,m}\sin\alpha_n x_1 \sin\beta_m x_2 \times$$

$$\times\frac{\theta^*_{nm}(\alpha_n^2+\beta_m^2)(1+\nu)\alpha_t+(R^*_m/N)\sin\alpha_n\xi_1}{[(\alpha_n^2+\beta_m^2)^2+k-\beta^2]}\;;$$

let us select the load

$$R(x_1, x_2\,; t) = R(x_2; t)\,\delta(x_1-\xi_1)\,,$$

so that the deflection w along the line $x_1 = \xi_1$ be zero. From the condition $w(\xi_1, x_2; t) = 0$ we find R^*_m

$$(3.7)\qquad \sum_{n=1}^{\infty}\frac{\sin\alpha_n\xi_1}{(\alpha_n^2+\beta_m^2)^2+k-\beta_m^2}\left[\frac{R^*_m}{N}\sin\alpha_n\xi_1+(1+\nu)\alpha_t\theta^*_{nm}(\alpha_n^2+\beta_m^2)\right]=0\,.$$

Substituting R^*_m from (3.7) into (3.6), we find an expression for the forced vibration of a rectangular plate simply supported on the contour and, additionally, on the line $x_1 = \xi_1$. If $\xi_1 = a_1/2$, and if the functions $q(x_1, x_2)$, $p(x_1, x_2)$ are symmetric in relation to $x_1 = a_1/2$, we obtain the particular case of a plate simply supported on the edges $x_2 = 0$, a_2 and clamped along the edge $x_1 = a_1/2$.

Let the rectangular plate of side lengths a_1 and a_2 be subject to a temperature field varying harmonically with the time and, in addition, to the moments:

$$M(x_2; t) = e^{i\omega t}\frac{2}{a_2}\sum_m M^*_m \sin\beta_m x_2\,,$$

along the line $x_1 = \xi_1$. The deflection produced by these actions takes the form, [5]:

$$(3.8)\qquad w_1(x_1, x_2\,; t) = \frac{4}{a_1 a_2}e^{i\omega t}\sum_{n,m}\sin\alpha_n x_1\sin\beta_m x_2\times$$

$$\times\frac{\theta^*_{nm}(\alpha_n^2+\beta_m^2)(1+\nu)\alpha_t+M^*_m\alpha_n\cos\alpha_n\xi_1}{[(\alpha_n^2+\beta_m^2)^2+k-\beta^2]}\,.$$

Let us shift moments $M(x_2; t)$ to the edge $x_1 = 0$, and require that:

$$\partial w/\partial x_1|_{x_1=0} = 0\,.$$

We obtain the equation

$$(3.9)\qquad \sum_{n=1}^{\infty}\frac{\alpha_n}{(\alpha_n^2+\beta_m^2)^2+k-\beta^2}[\alpha_n M_m^* + (1+\nu)\alpha_t \theta_{nm}^*(\alpha_n^2+\beta_m^2)] = 0\,.$$

From the Eq. (3.9) we find the quantities M_m^*, which enable the determination of the clamping moment:

$$(3.10)\qquad m(0, x_2; t) = \frac{2}{a_2} e^{i\omega t}\sum_{m=1}^{\infty} M_m^* \sin\beta_m x_2\,.$$

The deflection of the plate acted on by a temperature field harmonically varying with the time, simply supported on the edges $x_1 = a_1$; $x_2 = 0$, a_2 and clamped along the edge $x_1 = 0$, is found from the equation:

$$(3.11)\qquad w(x_1, x_2; t) = \frac{4e^{i\omega t}}{a_1 a_2}\sum_{n,m}\sin\alpha_n x_1 \sin\beta_m x_2 \frac{\theta_{nm}^*(\alpha_n^2+\beta_m^2)(1+\nu)\alpha_t + \alpha_n M_m^*}{[(\alpha_n^2+\beta_m^2)^2+k-\beta^2]}\,.$$

This equation has been obtained from (3.8) by substituting in the latter $\xi_1 = 0$. The solution method just proposed may be generalized to the case of two, three or four edges clamped, by proceeding in a manner analogous to that in which was solved the problem of vibration forced by a harmonically variable load, [5].

4. The Circular Plate

Let us consider a circular plate performing harmonic vibration forced by a temperature field varying harmonically in time.

Let us assume the following thermal boundary conditions:

$$(4.1)\qquad \lambda\frac{\partial T}{\partial x_3}\bigg|_{x_3=h/2} = q(r)e^{i\omega t}, \qquad \lambda\frac{\partial T}{\partial x_3}\bigg|_{x_3=-h/2} = -p(r)e^{i\omega t}; \qquad T(a, t) = 0.$$

The Eq. (1.23), which in the axially symmetric case under consideration takes the form

$$(4.2)\qquad \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial x_3^2}-i\eta\right)U = 0\,,$$

will be solved by means of the finite Hankel transformation. This transformation is defined thus

$$(4.3)\qquad f^*(\alpha_i) = \int_0^a r f(r) J_0(\alpha r)\,dr,$$

$$(4.4)\qquad f(r) = \frac{2}{a^2}\sum_i f^*(\alpha_i)\frac{J_0(\alpha_i r)}{[J_0'(a\alpha_i)]^2},$$

where a is the radius of the plate and $a\alpha_i$ are the roots of the Bessel function of the first kind and zero order $J_0(a\alpha_i) = 0$ and $J_0'(r) = dJ_0(r)/dr$.

The solution of (4.2) with the conditions (4.1) may be represented in the form of the series:

$$U(r, x_3) = \frac{1}{\lambda a^2} \sum_i \frac{1}{\gamma_i}\left[(q^* + p^*)\frac{\operatorname{ch}\gamma_i x_3}{\operatorname{sh}\vartheta_i} + (q^* - p^*)\frac{\operatorname{sh}\gamma_i x_3}{\operatorname{ch}\vartheta_i}\right]\frac{J_0(\alpha_i r)}{[J_0'(\alpha_i a)]^2}, \tag{4.5}$$

$$\gamma_i = \sqrt{\alpha_i^2 + i\eta}, \quad \vartheta_i = \gamma_i h/2,$$

where

$$(q^*, p^*) = \int_0^a (q, p)\, r J_0(\alpha_i r)\, dr.$$

Next, we calculate the transform of the function:

$$\theta^*(\alpha_i) = \frac{3}{2\lambda}\frac{q^* - p^*}{\vartheta_i^3}(\vartheta_i - \operatorname{th}\vartheta_i). \tag{4.6}$$

The equation of the deflection amplitude of the plate (1.22) takes the form:

$$\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 + k - \beta^2\right] W(r) + (1+\nu)\,\alpha_t\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\theta(r) = 0. \tag{4.7}$$

Applying to (4.7) the finite Hankel transformation, the solution is found in the form of the series:

$$w(r, t) = \frac{2(1+\nu)\alpha_t e^{i\omega t}}{a^2} \sum_i \frac{\alpha_i^2 \theta^*(\alpha_i)}{\alpha_i^4 + k - \beta^2}\,\frac{J_0(\alpha_i r)}{[J_0'(\alpha_i a)]^2}. \tag{4.8}$$

The solution obtained may be treated as approximate because the boundary conditions

$$w(a) = 0, \quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) w(a) = 0 \tag{4.9}$$

are not accurate. For, the second condition should read:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\nu}{r}\frac{\partial}{\partial r}\right) w(a) = 0. \tag{4.10}$$

5. A Simplification of the Vibration Problem of Plates

The procedure of the foregoing sections was that of solving in an accurate manner the heat equation, and then determining the mean temperatures along the thickness of the plate, described by the functions τ_0 and τ determined by the Eqs. (1.12). A considerable simplification of the solution may be achieved by applying to the heat equation the method proposed by K. MARGUERRE, [6], and consisting in approximate integration of the heat equation (1.11).

Let us multiply the Eq. (1.11) by $1/h$, and integrate with respect to x_3 from $-h/2$ to $h/2$.

Then, we obtain the equation:

$$\left(\nabla_1^2 - \frac{1}{\varkappa}\partial_t\right)\tau_0 + \frac{1}{h}\left[\frac{\partial T}{\partial x_3}\right]_{x_3=-h/2}^{x_3=h/2} = 0. \tag{5.1}$$

Bearing in mind the boundary conditions

$$\lambda \frac{\partial T}{\partial x_3}\bigg|_{x_3=h/2} = \bar{q}, \qquad \lambda \frac{\partial T}{\partial x_3}\bigg|_{x_3=-h/2} = -\bar{p}, \tag{5.2}$$

the Eq. (5.1) is reduced to

$$\left(\nabla_1^2 - \frac{1}{\varkappa}\partial_t\right)\tau_0 + \frac{1}{h\lambda}(\bar{q}+\bar{p}) = 0. \tag{5.3}$$

This equation, together with (1.8)

$$\Box_2^2(\Box_1^2 F + E\alpha_t h\tau_0) = 0, \tag{5.4}$$

determines the vibration of the plate due to a time-variable temperature field.

Let us multiply the Eq. (1.11) by $(12/h^3)x_3$ and integrate with respect to x_3 from $-h/2$ to $h/2$. Bearing in mind (1.12), we obtain the equation:

$$\left(\nabla_1^2 - \frac{1}{\varkappa}\partial_t\right)\tau + \frac{12}{h^3}\left[x_3\frac{\partial T}{\partial x_3} - T\right]_{-h/2}^{h/2} = 0, \tag{5.5}$$

or

$$\left(\nabla_1^2 - \frac{1}{\varkappa}\partial_t\right)\tau + \frac{12}{\lambda h^2}\left[\frac{1}{2}(\bar{q}-\bar{p}) - \frac{\lambda}{h}(T_1 - T_2)\right] = 0, \tag{5.6}$$

where

$$T_1 = T(x_1, x_2; h/2; t), \qquad T_2 = T(x_1, x_2; -h/2; t).$$

For a sufficiently thin plate, the temperature may be assumed to vary linearly in the x_3-direction, that is:

$$\tau = \frac{1}{h}(T_1 - T_2).$$

With this simplifying assumption, the Eq. (5.6) takes the form:

$$\left(\nabla_1^2 - \frac{1}{\varkappa}\partial_t - \varepsilon\right)\tau = \frac{6}{\lambda h^2}(\bar{p}-\bar{q}), \qquad \varepsilon = \frac{12}{h^2}. \tag{5.7}$$

The above equation, together with (1.14),

$$\nabla_1^4 w + \frac{\varrho h}{N}\ddot{w} + kw + (1+\nu)\alpha_t \nabla_1^2 \tau = 0 \tag{5.8}$$

describes the flexural vibration of the plate.

If the temperature varies harmonically in time, $T = Ue^{i\omega t}$ we have

$$w = We^{i\omega t}, \quad \tau = \theta e^{i\omega t}, \quad \bar{q} = qe^{i\omega t}, \quad \bar{p} = pe^{i\omega t} \tag{5.9}$$

and the Eqs. (5.7) and (5.8) take the form:

$$(\nabla_1^2 - i\eta - \varepsilon)\theta + \frac{6}{\lambda h^2}(q-p) = 0, \tag{5.10}$$

$$(\nabla_1^4 + k - \beta^2)W + (1+\nu)\alpha_t \nabla_1^2 \theta = 0. \tag{5.11}$$

Let us consider a rectangular plate simply supported on the edges. Performing on the Eqs. (5.10) and (5.11) the Fourier sine transformation, and assuming the notations of Sec. 3, we find the following system of equations:

$$(\alpha_n^2 + \beta_m^2 + i\eta + \varepsilon)\theta_{nm}^* = \frac{6}{\lambda h^2}(q_{nm}^* - p_{nm}^*), \tag{5.12}$$

$$[(\alpha_n^2 + \beta_m^2)^2 + k - \beta^2]W_{nm}^* = (1+\nu)\alpha_t(\alpha_n^2 + \beta_m^2)\theta_{nm}^*. \tag{5.13}$$

Eliminating from these equations θ_{nm}^*, and performing the inverse sine transformation, we obtain finally:

$$w(x_1, x_2; t) = \frac{24(1+\nu)\alpha_t e^{i\omega t}}{a_1 a_2 \lambda h^2} \sum_{n,m} \frac{\Delta_{nm}(q_{nm}^* - p_{nm}^*)\sin\alpha_n x_1}{(\Delta_{nm}^2 + k - \beta^2)(\Delta_{nm} + i\eta + \varepsilon)} \sin\beta_m x_2, \tag{5.14}$$

$$\Delta_{nm} = \alpha_n^2 + \beta_m^2,$$

where

$$(q_{nm}^*, p_{nm}^*) = \int_0^{a_1} dx_1 \int_0^{a_2} dx_2 (q, p) \sin\alpha_n x_1 \sin\beta_m x_2. \tag{5.15}$$

From (3.3) and (5.14) it follows that the surfaces described by these equations approach each other if

$$\frac{3(\vartheta_{nm} - \operatorname{th}\vartheta_{nm})}{\vartheta_{nm}^3} \approx \frac{1}{1 + \frac{1}{3}\theta_{nm}^2}. \tag{5.16}$$

If $h \to 0$, then $\vartheta_{nm} \to 0$ and

$$\frac{3(\vartheta_{nm} - \operatorname{th}\vartheta_{nm})}{\vartheta_{nm}^3} \to 1 \quad \text{and} \quad \frac{1}{1 + \frac{1}{3}\vartheta_{nm}^2} \to 1.$$

In this case, (5.16) is satisfied.

In the particular case where $p = 0$, $q = \delta(x_1 - \xi_1)\delta(x_2 - \xi_2)$, the deflection surface (5.14) may be represented in the form of a simple series:

$$w(x_1, x_2; t) = -(1+\nu)\alpha_t \frac{e^{i\omega t}\varepsilon}{2\lambda a_1 a_2} \nabla_1^2 [\Phi(x_1, \xi_1; x_2 - \xi_2) - \Phi(x_1, \xi_1; x_2 + \xi_2)], \tag{5.17}$$

where

$$\Phi(x_1, \xi_1; \mu) = \sum_{m=1}^{\infty} \cos \beta_m \mu [\psi_m(x_1-\xi_1) - \psi_m(x_1+\xi_1)],$$

and

$$\psi_m(\hat{\mu}) = \frac{1}{\hat{a}_1^2 - \hat{a}_2^2} \left\{ \frac{1}{\hat{a}_2^2 - \hat{a}_3^2} [\chi_m(\hat{\mu}, \hat{a}_3) - \chi_m(\hat{\mu}, \hat{a}_2)] - \frac{1}{\hat{a}_1^2 - \hat{a}_3^2} [\chi_m(\hat{\mu}, \hat{a}_3) - \chi_m(\hat{\mu}, a_1)] \right\},$$

with the notations:

$$\chi_m(\hat{\mu}, \hat{a}) = \frac{1}{2(\hat{a}^2+\beta_m^2)} \left[\frac{a_1\sqrt{\hat{a}^2+\beta_m^2}\, \mathrm{ch}\,(a_1-\hat{\mu})\sqrt{\hat{a}^2+\beta_m^2}}{\mathrm{sh}\, a_1\sqrt{\hat{a}^2+\beta_m^2}} - 1 \right],$$

$$\hat{a}_1^2 = i\eta + \varepsilon, \quad \hat{a}_2^2 = i\sqrt{k-\beta^2}, \quad \hat{a}_3^2 = -i\sqrt{k-\beta^2}.$$

6. Coupled Temperature and Strain Field

Let us consider a plate in a non-steady-state temperature field, assuming that the temperature field and the strain field are coupled. The stress-strain relations (1.1) to (1.5) of Sec. 1 remain valid. Let us assume in addition the functions τ_0 and τ of that section. The Eq. (1.11) will be replaced by the generalized heat equation, [7], [8],

$$\left(\nabla^2 - \frac{1}{\varkappa}\partial_t\right) T - \zeta^* \partial_t e = 0, \tag{6.1}$$

where

$$e = u'_{1,1} + u'_{2,2} - x_3 w_{,kk} \tag{6.2}$$

is the dilatation, and ζ^* the coefficient determining the coupling of the temperature and strain field.

Let us multiply (6.1) by $1/h$, and integrate with respect to x_3 from $-h/2$ to $h/2$. Bearing in mind (1.3) and (1.5), we find the following equation

$$\left(\nabla_1^2 - \frac{1}{\varkappa}\partial_t\right)\tau_0 - \zeta^*\partial_t(u'_{1,1} + u'_{2,2}) + \frac{1}{h}\left[\frac{\partial T}{\partial x_3}\right]_{-h/2}^{h/2} = 0. \tag{6.3}$$

Since

$$u'_{1,1} + u'_{2,2} = \varepsilon'_{kk} = \frac{N_{11}+N_{22}}{D(1+\nu)} + 2a_t\tau_0 \tag{6.4}$$

and

$$N_{kk} = \Box_2^2 F,$$

therefore, bearing in mind the boundary conditions (5.2), the Eq. (6.3) is reduced to:

$$\left(\nabla_1^2-\frac{1}{\varkappa_1}\partial_t\right)\tau_0-\beta^*\Box_2^2\partial_t F+\frac{1}{h\lambda}(\overline{q}+\overline{p})=0\,, \tag{6.5}$$

$$\beta^*=\frac{\zeta^*}{D(1+\nu)},\qquad \frac{1}{\varkappa_1}=\frac{1}{\varkappa}-2a_t\zeta^*\,.$$

To this the following equation should be joined.

$$\Box_2^2(\Box_1^2 F+Ea_t h\tau_0)=0\,. \tag{6.6}$$

From these two equations, the function τ_0 should be eliminated. From the result thus obtained we shall find the function F, and from the Eqs. (1.7) the forces N_{ij}.

The problem of forced longitudinal vibration described by the Eqs. (6.6) and (6.5) due to a non-steady-state temperature field will not be treated here. It is discussed in Refs. [3], [4].

Let us consider in greater detail the problem of forced flexural vibration.

Let us multiply (6.1) by $(12/h^3)x_3$, and integrate with respect to x_3 from $-h/2$ to $h/2$. Bearing in mind (1.2), we find:

$$\left(\nabla_1^2-\frac{1}{\varkappa}\partial_t\right)\tau+\zeta^*\partial_t\nabla_1^2 w+\frac{12}{h^3}\left[x_3\frac{\partial T}{\partial x_3}-T\right]_{-h/2}^{h/2}=0\,. \tag{6.7}$$

In view of the boundary conditions (5.2), the Eq. (6.7) is reduced to:

$$\left(\nabla_1^2-\frac{1}{\varkappa}\partial_t\right)\tau+\zeta^*\partial_t\nabla_1^2 w+\frac{6}{h^2\lambda}(\overline{q}-\overline{p})-\frac{12}{h^3}(T_1-T_2)=0\,. \tag{6.8}$$

Assuming that the temperature varies linearly along the thickness $\tau=\dfrac{1}{h}(T_1-T_2)$, this approximation being better for thinner plates, we have:

$$\left(\nabla_1^2-\frac{1}{\varkappa}\partial_t-\varepsilon\right)\tau+\zeta^*\partial_t\nabla_1^2 w+\frac{6}{h^2\lambda}(\overline{q}-\overline{p})=0. \tag{6.8'}$$

This, and the Eq. (1.14)

$$\nabla_1^4 w+\frac{\varrho h}{N}\ddot{w}+kw+(1+\nu)a_t\nabla_1^2\tau=0 \tag{6.9}$$

determine the forced vibration of the plate, bearing in mind that for $\zeta^*\to 0$ (that is for the non-coupled problem) the Eqs. (6.5) and (6.8) become (5.3) and (5.7).

Let us consider a rectangular plate simply supported on the contour. Introducing the notations (5.9), the Eqs. (6.8') and (6.9) are reduced to the form:

$$(\nabla_1^2-i\eta-\varepsilon)\theta+i\omega\zeta^*\nabla_1^2 W+\frac{6}{\lambda h^2}(q-p)=0, \tag{6.10}$$

$$(\nabla_1^4+k-\beta^2)W+(1+\nu)a_t\nabla_1^2\tau=0. \tag{6.11}$$

Performing on the Eqs. (6.10) and (6.11) the double Fourier sine transformation, we obtain the following system of algebraic equations:

$$(6.12)\qquad (\Delta_{nm}+i\eta+\varepsilon)\theta^*_{nm}+i\omega\zeta^*\Delta_{nm}W^*_{nm}=\frac{6}{\lambda h^2}(q^*_{nm}-p^*_{nm}),$$

$$(6.13)\qquad (\Delta^2_{nm}+k-\beta^2)W^*_{nm}-(1+\nu)\alpha_t\Delta_{nm}\theta^*_{nm}=0.$$

Eliminating from these equations θ^*_{nm}, and performing the inverse sine transformation, we obtain the following expression for the deflection:

$$(6.14)\qquad w(x_1,x_2;t)=\frac{24(1+\nu)\alpha_t e^{i\omega t}}{a_1a_2\lambda h^2}\sum_{n,m}\sin\alpha_n x_1\sin\beta_m x_2\times$$
$$\times\frac{\Delta_{nm}(q^*_{nm}-p^*_{nm})}{(\Delta^2_{nm}+k-\beta^2)(\Delta_{nm}+i\eta+\varepsilon)+(1+\nu)\alpha_t i\omega\zeta^*\Delta^2_{nm}},$$

where

$$\Delta_{nm}=\alpha_n^2+\beta_m^2,$$

and the quantities q^*_{nm}, p^*_{nm} are given by the equations (5.15). Knowing the deflection, it is easy to determine the bending moments and torques from the Eqs. (1.5).

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Streszczenie

DRGANIA POPRZECZNE PŁYTY WYWOŁANE JEJ OGRZANIEM

W pracy wyprowadzono równanie termosprężystych drgań płyty wychodząc z trójwymiarowej teorii termosprężystości. Członem indukującym drgania jest gęstość momentu trójwymiarowej temperatury wzdłuż grubości płyty. Założono przy tym, że drgania podłużne są niezależne od drgań giętnych płyty. Podano rozwiązanie podstawowe zagadnienia dla drgań okresowych płyty nieograniczonej na sprężystym podłożu, gdy w płaszczyznach ograniczających płytę dany jest przepływ ciepła. Rozważono również płytę swobodnie podpartą lub swobodnie podpartą na obwodzie i dodatkowo podpartą wzdłuż linii równoległej do brzegu i położonej wewnątrz obszaru płyty. Wreszcie podano rozwiązanie dla płyty, której jeden brzeg jest utwierdzony, zaś pozostałe są swobodnie podparte. Badano również przypadek płyty kołowej.

Praca podaje porównanie przedstawionej teorii z przypadkiem, gdy temperaturę indukującą drgania giętne zastąpiono przez różnicę temperatur górnej i dolnej powierzchni płyty, odniesioną do jednostki grubości płyty.

Резюме

ПОПЕРЕЧНЫЕ КОЛЕБАНИЯ ПЛАСТИНКИ, ВЫЗВАННЫЕ ЕЕ НАГРЕВОМ

Выводятся уравнения термоупругих колебаний пластинки, исходя из трехмерной теории термоупругости. Членом, вызывающим колебания, является плотность момента трехмерной температуры вдоль толщины пластинки. При этом предполагается, что продольные колебания не зависят от изгибных колебаний пластинки. Дается основное решение задачи для периодических колебаний бесконечной пластинки на упругом основании при заданном в ограничивающих плоскостях потоке тепла. Рассматривается также свободно опертая пластинка или свободно опертая по окружности и добавочно опертая вдоль линии параллельной краю и расположенной внутри области пластинки. Наконец дается решение для пластинки, один край которой закреплен, а остальные свободно оперты. Исследуется также случай круговой пластинки.

В работе приводится сравнение представленной теории со случаем, когда температура, вызывающая изгибные колебания, заменяется разницей температур верхней и нижней поверхности пластинки, отнесенной к единице толщины пластинки.

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