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THE SOMMERFELD RADIATION CONDITIONS FOR COUPLED PROBLEMS OF THERMOELASTICITY. EXAMPLES OF COUPLED STRESS AND TEMPERATURE CONCENTRATION AT CYLINDRICAL AND SPHERICAL CAVITIES

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Introduction

In classical boundary-value problems of steady-state vibration for infinite and semi-infinite regions (an infinite body with a spherical cavity, for instance) we select from the general integral of the (HELMHOLTZ) equation of vibration, after SOMMERFELD, only such solutions as describe waves with a phase propagating to infinity. The conditions concerning the behaviour of the solutions of the equation of vibration at infinity are called the radiation conditions. Among many works devoted to these problems, let us mention the excellent monograph by V. D. KUPRADZE, [1].

The coupled problem of thermoelasticity is described, [2], by equations more general than the classical Lamé's equations of motion; complex vector functions are obtained as solutions of the equations of the Helmholtz type

$$(\nabla^2 + k^2)\, u_i = 0, \quad i = 1, 2, 3, \tag{0.1}$$

where u_i is the displacement vector and k is a complex constant. The mathematical form of the radiation conditions depends on the form of the region considered. We shall consider two semi-bounded elastic regions: an infinite space with a cylindrical and spherical cavity. For these regions we shall seek the solutions of the coupled equations satisfying definite boundary conditions on the spherical surface and the cylindrical surface and the corresponding radiation condition at infinity. From the form of the coupled equations it follows that the rotation part of the displacement vector constituting the solution of the problem must satisfy the classical Sommerfeld condition. The potential part of that vector should satisfy similar conditions with complex parameters k_1 and k_3. It will be shown that the potential wave vanishes at infinity more rapidly in the coupled problem than in the corresponding non-coupled problem, and that the rate of vanishing is determined by the imaginary part of the roots $k_{1,3}$. The radiation conditions for the temperature are determined by two conditions for coupled potentials. The coupled potentials and the temperature are composed of two types of interfering waves [cf. Eq. (2.3)], with two different complex absorptions and two different phases propagating to infinity (cf. P. CHADWICK and I. N. SNEDDON, [8]). In the case of no coupling, we obtain the classical Sommerfeld conditions.

1. Coupled Thermal Stress and Temperature Concentration at a Cylindrical Cavity

Let us consider, in an infinite elastic space referred to a cylindrical system of coordinates (r, θ, z), an infinite cylindrical cavity with axis coinciding with that of z. Let a periodic plane heat source with constant intensity due in the direction of the axis $x_1 = r\cos\theta$ of a Cartesian coordinate system (x_1, x_2, x_3). Thus the cavity is flown past by a homogeneous periodic heat flow uniformly distributed along the z-axis; the problem is thus a plane problem [in the (r, θ)-plane]. Let us assume that the cylinder is impermeable to heat and that its surface is free from stress. Let us assume also that the material and thermal constants are independent of the temperature. With these conditions, the following coupled system of equations must be solved, [3]:

$$\begin{gathered} \square_1^2 \Phi^* = mT^*, \\ \square_{k_1}^2 \square_{k_3}^2 \Phi^* = -\frac{q}{\lambda_0} m\delta(x), \quad \square_{1.2}^2 = \nabla^2 + h_{1.2}^2, \\ \square_2^2 \Psi_i^* = 0, \quad \Psi_{i,i}^* = 0, \quad \square_{k_{1.3}}^2 = \nabla^2 + k_{1.3}^2 \end{gathered} \tag{1.1}$$

The star in (1.1) denotes the amplitude of the solution. The amplitude of the displacement vector u_i^* $(i = 1, 2, 3)$ resolving the problem has the form:

$$u_i^* = \Phi_{,i}^* + \Psi_i^*, \quad u_i = u_i^* e^{i\omega t}, \quad i^2 = -1, \tag{1.2}$$

T^* is the amplitude of the temperature. In addition, the following symbols are used:

$$\begin{gathered} k_{1.3}^2 = \frac{1}{2}\left\{h_1^2 + (1+\varepsilon)h_3^2 \pm \sqrt{[h_1^2 + (1+\varepsilon)h_3^2]^2 - 4h_1^2 h_3^2}\right\}, \\ h_{1.2} = \frac{\omega}{c_{1.2}}, \quad h_3 = \frac{1}{i}\sqrt{\frac{\omega i}{\varkappa}}, \quad \varepsilon = \varkappa m\eta, \quad m = \frac{3\lambda + 2\mu}{\lambda + 2\mu}\alpha_t, \\ \frac{1}{c_1^2} = \frac{\varrho}{\lambda + 2\mu}, \quad \frac{1}{c_2^2} = \frac{\varrho}{\mu}, \end{gathered} \tag{1.3}$$

where ω is the frequency of vibration, $\varkappa$ — the diffusivity constant, λ, μ — Lamé constants, ϱ — the density, α_t — the coefficient of thermal dilatation and η is the coupling quantity of (1.1). The quantity q in (1.1) is the constant intensity of plane heat source, λ_0 — the coefficient of heat conduction and $\delta = \delta(x)$ — the Dirac function.

The roots $k_{1.3}$ are functions of the coupling parameter ε. Denoting $k_{1.3} = k_{1.3}(\varepsilon)$, we have

$$k_{1.3}(0) = h_{1.3}. \tag{1.4}$$

Let us write the roots (1.3) in another form, convenient for discussion of the radiation conditions

$$k_1 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{a_1^2 + b_1^2} + a_1} - i\sqrt{\sqrt{a_1^2 + b_1^2} - a_1}\right), \tag{1.5}$$

where

$$a_1 = h_1^2 + \frac{1}{\sqrt{2}}\sqrt{\sqrt{a_0^2+b_0^2}+a_0}\,,$$

$$b_1 = \omega_0(1+\varepsilon) - \frac{1}{\sqrt{2}}\sqrt{\sqrt{a_0^2+b_0^2}-a_0}\,;$$

(1.6) $$k_3 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{a_3^2+b_3^2}+a_3} - i\sqrt{\sqrt{a_3^2+b_3^2}-a_3}\right),$$

where

$$a_3 = h_1^2 - \frac{1}{\sqrt{2}}\sqrt{\sqrt{a_0^2+b_0^2}+a_0}\,,$$

$$b_3 = \omega_0(1+\varepsilon) + \frac{1}{\sqrt{2}}\sqrt{\sqrt{a_0^2+b_0^2}-a_0}\,,$$

and

$$a_0 = h_1^4 - \omega_0^2(1+\varepsilon)^2, \quad b_0 = 2\omega_0(1-\varepsilon)h_1^2, \quad \omega_0 = \omega/\varkappa\,.$$

For $\varepsilon = 0$ we obtain:

(1.7) $$\frac{1}{\sqrt{2}}\sqrt{\sqrt{a_0^2+b_0^2}+a_0} = h_1^2, \quad \frac{1}{\sqrt{2}}\sqrt{\sqrt{a_0^2+b_0^2}-a_0} = \omega_0\,,$$
$$a_1 = 2h_1^2\,, \quad b_1 = 0\,, \quad a_3 = 0\,, \quad b_3 = 2\omega_0\,.$$

All the roots written above are arithmetic, that is assuming

(1.8) $$k_{1.3} = \alpha_{1.3} - i\beta_{1.3}\,,$$

we have:

$$\alpha_{1.3} > 0, \quad \beta_{1.3} \geqslant 0\,.$$

It is seen that for a vector of zero divergence Ψ_i^*, the radiation condition in the plane (r, θ) has the form:

(1.9) $$\Psi_i^* = e^{-ih_2r}0(r^{-1/2})\,,$$
$$\frac{\partial \Psi_i^*}{\partial r} + ih_2\Psi_i^* = e^{-ih_2r}0(r^{-3/2})\,,$$

where $0(\xi)$ is a quantity as decreasing as ξ for $\xi \to 0$. The quantity h_2 being real, $e^{i\omega t}\Psi_i^*$ corresponds to the solution

$$r^{-1/2}e^{i(\omega t - h_2 r)}\,,$$

which is a wave with the phase propagating to infinity.

For the potential vector, two systems of radiation conditions should be satisfied:

$$\Phi^*(k_{1,3}) = e^{-irk_{1,3}} O(r^{-1/2}),$$

(1.10)

$$\frac{\partial \Phi^*(k_{1,3})}{\partial r} + ik_{1,3}\Phi^*(k_{1,3}) = e^{-irk_{1,3}} O(r^{-3/2}).$$

The conditions (1.10) give solutions of the Helmholtz equations, corresponding also to a wave with the phase propagating to infinity

$$r^{-1/2} e^{-\beta_{1,3} r} e^{i(\omega t - \alpha_{1,3} r)}. \tag{1.11}$$

In addition, we are concerned here with complex absorption of the wave. The potential wave vanishes more rapidly at infinity than in the corresponding non-coupled problem, such vanishing rate being determined by the imaginary part of the roots $k_{1\cdot 3}$.

Let us denote the total solution of our boundary-value problem by $[S]$. This solution may be represented in the form of a sum

$$[S^*] = [S_{\mathrm{I}}^*] + [S_{\mathrm{II}}^*], \tag{1.12}$$

where $[S_{\mathrm{I}}^*]$ corresponds to the infinite plane, thus satisfying some of the boundary conditions on the surface of the cylinder and the radiation conditions at infinity, and $[S_{\mathrm{II}}^*]$ is a correcting solution. This correcting solution $[S_{\mathrm{II}}^*]$ should also satisfy the conditions (1.10). It is evident that the component $[S_{\mathrm{I}}^*]$ can be so selected that the temperature T_{I}^* satisfies the impermeability condition on the surface $r = a$ of the cylinder.

Let us denote:

$$T_{\mathrm{I}}^* = T_1^* - T_2^*, \quad \Phi_{\mathrm{I}}^* = \Phi_1^* - \Phi_2^*; \tag{1.13}$$

$F_1^*(k)$ is the solution of the equation

$$\Box_k^2 F_1^*(k) = -\frac{q}{\lambda_0}\delta(x_1) \tag{1.14}$$

for the infinite space, the radiation conditions being satisfied. We obtain

$$F_1^*(k) = -\frac{q}{\lambda_0}\frac{\delta(x_1)}{\Box_k^2} = \frac{q}{2\lambda_0}\frac{e^{-ikx_1}}{ik}. \tag{1.15}$$

Making use of the second of the Eqs. (1.1), we obtain:

$$\Phi_1^* = -\frac{mq}{\lambda_0}\frac{1}{k_1^2-k_3^2}\left(\frac{1}{\Box_{k_3}^2} - \frac{1}{\Box_{k_1}^2}\right)\delta(x_1) = \frac{m}{k_1^2-k_3^2}[F_1^*(k_3) - F_1^*(k_1)]. \tag{1.16}$$

The first of the Eqs. (1.1) yields the temperature:

$$T_1^* = \frac{1}{m}\Box_1^2\Phi_1^* = \frac{1}{k_1^2-k_3^2}[(-k_3^2+h_1^2)F_1^*(k_3) - (-k_1^2+h_1^2)F_1^*(k_1)]. \tag{1.17}$$

The functions Φ_{I}^* and T_{I}^* corresponding to the infinite space without a hole can be found in W. NOWACKI's monograph, [3]. Making use of the expansion

$$(1.18)\quad F_{\mathrm{I}}^*(k) = \frac{q}{2\lambda_0}\sum_{n=0}^{\infty}(-1)^n\delta_n\frac{I_n(ikr)}{ik}\cos n\theta;\qquad \delta_n=\begin{cases}1 & \text{for}\quad n=0,\\ 2 & \text{for}\quad n\geqslant 1,\end{cases}$$

where $I_n(x)$ is modified Bessel function of the first kind, it is seen that the satisfaction of the impermeability condition for temperature for $r=a$ requires the superposition on the components of the sum (1.18) of appropriate components containing modified Bessel functions of the second kind. We find

$$(1.19)\quad \Phi_{\mathrm{I}}^* = \frac{m}{k_1^2-k_3^2}\{[F_1^*(k_3)-F_1^*(k_1)]-[F_2^*(k_3)-F_2^*(k_1)]\},$$

$$(1.20)\quad T_{\mathrm{I}}^* = \frac{1}{k_1^2-k_3^2}\{[(-k_3^2+h_1^2)F_1^*(k_3)-(-k_1^2+h_1^2)F_1^*(k_1)]-$$
$$-[(-k_3^2+h_1^2)F_2^*(k_3)-(-k_1^2+h_1^2)F_2^*(k_1)]\},$$

where $F_2^*(k)$ satisfies the equation

$$(1.21)\qquad \Box_k^2 F_2^* = 0,$$

and has the following sum representation:

$$(1.22)\qquad F_2^*(k) = \frac{q}{2\lambda_0}\sum_{0}^{\infty}(-1)^n\delta_n\frac{I_n'(ika)}{K_n'(ika)}\frac{K_n(ikr)}{ik}\cos n\theta.$$

It is seen that the functions Φ_{I}^* and T_{I}^* satisfy the conditions

$$(1.23)\qquad \left.\frac{\partial\Phi_{\mathrm{I}}^*}{\partial r}\right|_{r=a}=0,\qquad \left.\frac{\partial T_{\mathrm{I}}^*}{\partial r}\right|_{r=a}=0.$$

In the solution $[S_{\mathrm{I}}^*]$, we assume $(\Psi_i^*)_{\mathrm{I}}\equiv 0$. Therefore, the stresses produced by the potential Φ_{I}^* and the temperature T_{I}^* are determined from the equations, [4]:

$$(1.24)\qquad (\sigma_{\mathrm{I}})_{ij} = 2\mu(\Phi_{\mathrm{I},ij}-\Phi_{\mathrm{I},kk}\delta_{ij})+\varrho\delta_{ij}\ddot{\Phi}_{\mathrm{I}},\quad i,j,k=1,2,3.$$

In the case of cylindrical coordinates we find, in view of the conditions (1.23), the following sum representation of the load on the cylinder:

$$(1.25)\quad \begin{aligned} \left.(\sigma_{\mathrm{I}}^*)_{rr}\right|_{r=a} &= \mu C\sum_{n=0}^{\infty}(-1)^n\delta_n(\tfrac{1}{2}a^2h_2^2-n^2)\frac{\chi_n(ik_3a)-\chi_n(ik_1a)}{k_1^2-k_3^2}\cos n\theta,\\ \left.(\sigma_{\mathrm{I}}^*)_{r\theta}\right|_{r=a} &= \mu C\sum_{n=0}^{\infty}(-1)^n\delta_n\frac{\chi_n(ik_3a)-\chi_n(ik_1a)}{k_1^2-k_3^2}(\cos n\theta)_{,\theta},\end{aligned}$$

where

$$\chi_n(z) = \frac{1}{z^2 K_n'(z)}, \qquad C = \frac{qm}{a\lambda_0}.$$

To suppress the stresses on the boundary of the cylinder we add to the state $[S_{\mathrm{I}}^*]$ the state $[S_{\mathrm{II}}^*]$, and solve the following system of equations:

$$\begin{aligned} &\square_1^2 \Phi_{\mathrm{II}}^* = m T_{\mathrm{II}}^*, \quad \square_2^2 (\Psi_{\mathrm{II}}^*)_i = 0, \\ &\square_{k_1}^2 \square_{k_3}^2 \Phi_{\mathrm{II}}^* = 0, \quad (\Psi_{\mathrm{II}}^*)_{i,i} = 0, \end{aligned} \tag{1.26}$$

with the boundary conditions for stresses determined by (1.25) and with thermal insulation for $r = a$, and appropriate radiation conditions.

The state $[S_{\mathrm{II}}^*]$ may be assumed in the form

$$[S_{\mathrm{II}}^*] = \sum_0^\infty (x_n^c [\overset{1}{S_n^*}] + y_n^c [\overset{2}{S_n^*}]), \tag{1.27}$$

where $[\overset{1}{S_n^*}]$ is the potential state, and has the form:

$$\Phi_{\mathrm{II}}^* = -k_1^{-2} H_n^{(2)}(rk_1) \cos n\theta + \hat{\beta}_n [-k_3^{-2} H_n^{(2)}(rk_3) \cos n\theta], \tag{1.28}$$

where $H_n^{(2)}(x)$ is Hankel function of the second kind selected in agreement with the radiation condition (1.10). The function Φ_{II}^* satisfies the second of the Eqs. (1.26), because

$$\square_k^2 H_n^{(2)}(kr) \cos n\theta = 0.$$

The coefficient $\hat{\beta}_n$ in Eq. (1.28) is so chosen that the temperature obtained from the first of the Eqs. (1.26) satisfies, for $r = a$, the condition

$$\left.\frac{\partial T_{\mathrm{II}}^*}{\partial r}\right|_{r=a} = 0. \tag{1.29}$$

We obtain, therefore,

$$\begin{aligned} T_{\mathrm{II}}^* = \frac{1}{m} \Bigg[& -k_1^{-2}(h_1^2 - k_1^2)[H_n^{(2)}(rk_1) + \\ & + \frac{k_1^{-1}(h_1^2 - k_1^2) H_n'^{(2)}(ak_1)}{k_3^{-1}(h_1^2 - k_3^2) H_n'^{(2)}(ak_3)} k_3^{-2}(h_1^2 - k_3^2) H_n^{(2)}(rk_3)\Bigg] \cos n\theta, \end{aligned} \tag{1.30}$$

$$\Phi_{\mathrm{II}}^* = \left[-k_1^{-2} H_n^{(2)}(rk_1) + \frac{k_1^{-1}(h_1^2 - k_1^2) H_n'^{(2)}(k_1 a)}{k_3^{-1}(h_1^2 - k_3^2) H_n'^{(2)}(k_3 a)} k_3^{-2} H_n^{(2)}(rk_3) \right] \cos n\theta. \tag{1.31}$$

It is seen that the functions (1.30) and (1.31) just obtained satisfy the radiation conditions (1.10). It is also seen that for $\varepsilon = 0$, the Eqs. (1.30), (1.31) and (1.10) become those of the corresponding non-coupled problem.

To satisfy the boundary conditions for stress, let us calculate:

$$(1.32)\qquad \begin{aligned}(\overset{1}{\sigma}{}^*_{\rm II})_{rr} &= \mu[-2r^{-2}(r\Phi^*_{{\rm II},r}+\Phi^*_{{\rm II},\theta\theta})-h_2^2\Phi^*_{\rm II}] = \mu\bar{A}^c_n(\omega r)\cos n\theta\,,\\ (\overset{1}{\sigma}{}^*_{\rm II})_{r\theta} &= \mu[-2r^{-2}(\Phi^*_{\rm II}-r\Phi^*_{{\rm II},r})_{,\theta}] = \mu\bar{C}^c_n(\omega r)(\cos n\theta)_{,\theta}\,,\end{aligned}$$

where

$$(1.33)\qquad \bar{A}^{(c)}_n(\omega r) = r^{-2}\Big\{-k_1^{-2}[(2n^2-h_2^2r^2)H^{(2)}_n(k_1r)-2rk_1H'^{(2)}_n(rk_1)]+ + \frac{k_1^{-1}(h_1^2-k_1^2)H'^{(2)}_n(ak_1)}{k_3^{-1}(h_1^2-k_3^2)H'^{(2)}_n(ak_3)}k_3^{-2}[(2n^2-h_2^2r^2)H^{(2)}_n(k_3r)-2rk_3H'^{(2)}_n(k_3r)]\Big\},$$

$$(1.34)\qquad \bar{C}^c_n(\omega r) = 2r^{-2}\Big\{k_1^{-2}[H^{(2)}_n(k_1r)-k_1rH'^{(2)}_n(k_1r)]- - \frac{k_1^{-1}(h_1^2-k_1^2)H'^{(2)}_n(k_1a)}{k_3^{-1}(h_1^2-k_3^2)H'^{(2)}_n(k_3a)}k_3^{-2}[H^{(2)}_n(k_3r)-k_3rH'^{(2)}_n(k_3r)]\Big\}.$$

The remaining stresses, which are of the potential origin, are computed according to (1.24) from $\Phi^*_{\rm II}$, (1.31).

The two load components $(\overset{2}{\sigma}{}^*_{\rm II})_{rr}$ and $(\overset{2}{\sigma}{}^*_{\rm II})_{r\theta}$ taken from the set of correction components, originating from the rotational vector have the form [5]:

$$(1.35)\qquad [\overset{2}{S}{}^*_n]:\ \begin{cases}(\overset{2}{\sigma}{}^*_{\rm II})_{rr} = \mu B^{(c)}_n(\omega r)\cos n\theta\,,\\ (\overset{2}{\sigma}{}^*_{\rm II})_{r\theta} = \mu D^{(c)}_n(\omega r)(\cos n\theta)_{,\theta}\,,\end{cases}$$

where

$$(1.36)\qquad \begin{aligned}B^{(c)}_n(\omega r) &= 2r^{-2}n^2[H^{(2)}_n(h_2r)-h_2rH'^{(2)}_n(h_2r)]\,,\\ D^{(c)}_n(\omega r) &= r^{-2}\{[(h_2r)^2-2n^2]H^{(2)}_n(h_2r)+2h_2rH'^{(2)}_n(h_2r)\}\,.\end{aligned}$$

The condition of vanishing of the cylindrical surface $r=a$ yields the coefficients $\{x^c_n\}$, $\{y^c_n\}$:

$$(1.37)\qquad \begin{aligned}x^c_n &= C(\bar{\Delta}^c_n)^{-1}(-1)^n\delta_n\frac{\chi_n(ik_3a)-\chi_n(ik_1a)}{k_1^2-k_3^2}[B^c_n(\omega a)-(\tfrac{1}{2}a^2h_2^2-n^2)D^c_n(\omega a)]\,,\\ y^c_n &= C(\bar{\Delta}^c_n)^{-1}(-1)^n\delta_n\frac{\chi_n(ik_3a)-\chi_n(ik_1a)}{k_1^2-k_3^2}[\bar{A}^c_n(\omega a)-(\tfrac{1}{2}a^2h_2^2-n^2)\bar{C}^c_n(\omega a)]\,,\end{aligned}$$

$$\bar{\Delta}^c_n = \bar{A}^c_n(\omega a)D^c_n(\omega a)-B^c_n(\omega a)\bar{C}^c_n(\omega a)\,.$$

The equation for the temperature field in the second state has the form:

$$(1.38)\qquad T^*_{\rm II} = \frac{1}{m}\sum_{n=0}^{\infty}x^c_n\Big[-k_1^{-2}(h_1^2-k_1^2)H^{(2)}_n(k_1r)+ + \frac{k_1^{-1}(h_1^2-k_1^2)H'^{(2)}_n(k_1a)}{k_3^{-1}(h_1^2-k_3^2)H'^{(2)}_n(k_3a)}k_3^{-2}(h_1^2-k_3^2)H^{(2)}_n(k_3r)\Big]\cos n\theta\,.$$

2. Coupled Stress and Temperature Concentration at a Spherical Cavity

The case of the sphere is analogous to that of the cylinder. Let the plane of the heat wave in the infinite body be normal to the axis of $x_3 = R\cos\theta$ of spherical coordinates (R, θ, φ). Let us consider a spherical cavity in the space with its centre located at the origin. Let us assume also that the spherical surface is impermeable to heat and free from stress. Thus, the cavity is flown past by a periodic uniform heat flow, the process being identical for every meridian. The problem is therefore axially symmetric, the symmetry axis coinciding with the x_3-axis. Similarly to the case of the cylinder, we assume that the material constants are independent of temperature. With these assumptions, our aim is to solve the system of equations (1.1). The radiation conditions have the following form. For the rotational vector:

$$\begin{gathered}\Psi_i^* = e^{-iRh_2} 0(R^{-1}),\\ \frac{\partial \Psi_i^*}{\partial R} + ih_2 \Psi_i^* = e^{-iRh_2} 0(R^{-2});\end{gathered} \tag{2.1}$$

and for the potentials coupled with the temperature

$$\begin{gathered}\Phi^*(k_{1.3}) = e^{-iRk_{1.3}} 0(R^{-1}),\\ \frac{\partial \Phi^*(k_{1.3})}{\partial R} + ik_{1.3}\Phi^* = e^{-iRk_{1.3}} 0(R^{-2}).\end{gathered} \tag{2.2}$$

In this case we also have solutions with complex absorption of the type:

$$\frac{1}{R} e^{-\beta_{1.3}R} e^{i(\omega t - R\alpha_{1.3})}. \tag{2.3}$$

Let us denote the total solution of the above problem for the sphere by $[{}_sS]$. This solution may be represented in the form of the sum:

$$[{}_sS^*] = [{}_sS_{\mathrm{I}}^*] + [{}_sS_{\mathrm{II}}^*], \tag{2.4}$$

where $[{}_sS_{\mathrm{I}}^*]$ originates from the potential Φ_{I}^* Eq. (1.19). The functions $F_1^*(k)$ and $F_2^*(k)$ can be represented in the form of the sums:

$$\begin{gathered}F_1^*(k) = \frac{q}{2\lambda_0}\sqrt{2\pi}\sum_{n=0}^{\infty}(-1)^n\left(n+\frac{1}{2}\right)\frac{1}{ik}\frac{I_{n+\frac12}(ikR)}{(ikR)^{1/2}}\mathcal{P}_n(\cos\theta),\\ F_2^*(k) = \frac{q}{2\lambda_0}\sqrt{2\pi}\sum_{n=0}^{\infty}(-1)^n\left(n+\frac{1}{2}\right)\frac{[I]'_{n+\frac12}(ika)}{[K]'_{n+\frac12}(ika)}\frac{1}{ik}\frac{K_{n+\frac12}(ikR)}{(ikR)^{1/2}}\mathcal{P}_n(\cos\theta),\end{gathered} \tag{2.5}$$

where

$$\begin{gathered}[I]'_n(z) = I'_n(z)z - \frac{1}{2}I_n(z),\\ [K]'_n(z) = K'_n(z)z - \frac{1}{2}K_n(z).\end{gathered} \tag{2.6}$$

It can be verified, by means of the relations (2.5), (2.6), and (1.19), that

$$\left.\frac{\partial T_{\mathrm{I}}^*}{\partial R}\right|_{R=a} = 0, \qquad \left.\frac{\partial \Phi_{\mathrm{I}}^*}{\partial R}\right|_{R=a} = 0. \tag{2.7}$$

In the case of spherical coordinates, we obtain from (1.24) and (2.7) the following sum representations of the load on the sphere $R=a$ originating from the potential Φ_{I}^*:

$$\begin{aligned}(\sigma_{\mathrm{I}}^*)_{RR}\Big|_{R=a} &= \mu S \sum_n (-1)^n \left(n+\frac{1}{2}\right)\left[\frac{1}{2}a^2h_2^2 - n(n+1)\right]\times \\ &\qquad \times \frac{\Psi_{n+\frac{1}{2}}(ik_3a)-\Psi_{n+\frac{1}{2}}(ik_1a)}{k_1^2-k_3^2}\,\mathcal{P}_n(\cos\theta), \\ (\sigma_{\mathrm{I}}^*)_{R\theta}\Big|_{R=a} &= \mu S \sum_n (-1)^n\left(n+\frac{1}{2}\right)\frac{\Psi_{n+\frac{1}{2}}(ik_3a)-\Psi_{n+\frac{1}{2}}(ik_1a)}{k_1^2-k_3^2}[\mathcal{P}_n(\cos\theta)]_{,\theta},\end{aligned} \tag{2.8}$$

where

$$\Psi_n(z) = \frac{1}{z^{3/2}[K]_n'(z)}, \qquad S = \frac{qm\sqrt{2\pi}}{a\lambda_0}.$$

The correcting solution $[{}_sS_{\mathrm{II}}^*]$ has the form:

$$[{}_sS_{\mathrm{II}}^*] = \sum_{n=0}^{\infty}\left(x_n^s\,[{}_s\overset{1}{S}{}_n^*] + y_n^s\,[{}_s\overset{2}{S}{}_n^*]\right), \tag{2.9}$$

where $[{}_s\overset{1}{S}{}_n^*]$ is the state originating from the potential

$$\begin{aligned}{}_s\Phi_{\mathrm{II}}^* = \Big\{ &-k_1^{-2}R^{-1/2}H_{n+\frac{1}{2}}^{(2)}(k_1R) + \\ &+\frac{k_1^{-2}(h_1^2-k_1^2)[H]'^{(2)}_{n+\frac{1}{2}}(k_1a)}{k_3^{-2}(h_1^2-k_3^2)[H]'^{(2)}_{n+\frac{1}{2}}(k_3a)}k_3^{-2}R^{-1/2}H_{n+\frac{1}{2}}^{(2)}(k_3R)\Big\}\,\mathcal{P}_n(\cos\theta).\end{aligned} \tag{2.10}$$

The function $H_{n+\frac{1}{2}}^{(2)}(kR)$ is Hankel function of the second kind, and the order $n+1/2$ selected to satisfy the radiation conditions (2.2). It is seen that in the neighbourhood of $R=\infty$ the potential state ${}_s\Phi_{\mathrm{II}}^*$ is composed of two interfering waves (2.3) with two different complex absorptions and two different phases propagating to infinity

$$(\omega t - R\alpha_1) \quad \text{and} \quad (\omega t - R\alpha_3).$$

The temperature field is composed of two wave types with the above phases and analogous dampings:

$$\begin{aligned}{}_sT_{\mathrm{II}}^* = \frac{1}{m}\sum_n x_n^s\Big\{&-k_1^{-2}(h_1^2-k_1^2)R^{-1/2}H_{n+\frac{1}{2}}^{(2)}(k_1R) + \\ &+\frac{k_1^{-2}(h_1^2-k_1^2)[H]'^{(2)}_{n+\frac{1}{2}}(k_1a)}{k_3^{-2}(h_1^2-k_3^2)[H]'^{(2)}_{n+\frac{1}{2}}(k_3a)}(h_1^2-k_3^2)k_3^{-2}R^{-1/2}H_{n+\frac{1}{2}}^{(2)}(k_3R)\Big\}\,\mathcal{P}_n(\cos\theta).\end{aligned} \tag{2.11}$$

The load components of the sphere $R=a$ originating from the potential ${}_s\Phi^*_{\mathrm{II}}$ can be obtained from the equations:

$$(2.12)\qquad \begin{aligned}({}_s\overset{1}{\sigma}{}^*_{\mathrm{II}})_{RR} &= \mu[-4R^{-1}{}_s\Phi^*_{\mathrm{II},R}-2R^{-2}\operatorname{cosec}\theta\,(\sin\theta\,{}_s\Phi^*_{\mathrm{II},\theta})_{,\theta}-h_2^2\,{}_s\Phi^*_{\mathrm{II}}] = \\ &= \mu\bar{A}^s(\omega R)\,\mathcal{P}_n(\cos\theta),\\ ({}_s\overset{1}{\sigma}{}^*_{\mathrm{II}})_{R\theta} &= \mu[2R^{-1}{}_s\Phi^*_{\mathrm{II},R}-2R^{-2}{}_s\Phi^*_{\mathrm{II}}]_{,\theta} = \mu\bar{C}^s_n(\omega R)[\mathcal{P}_n(\cos\theta)]_{,\theta},\end{aligned}$$

where

$$(2.13)\qquad \begin{aligned}\bar{A}^s_n(\omega R) &= R^{-5/2}\Big\{-k_1^{-2}[(2n(n+1)-R^2h_2^2)\,H^{(2)}_{n+\frac12}(k_1R)-\\ &\quad -4[H]'^{(2)}_{n+\frac12}(k_1R)]+\frac{k_1^{-2}(h_1^2-k_1^2)\,[H]'^{(2)}_{n+\frac12}(k_1a)}{k_3^{-2}(h_1^2-k_3^2)\,[H]'^{(2)}_{n+\frac12}(k_3a)}\times\\ &\quad \times k_3^{-2}[(2n(n+1)-R^2h_2^2)\,H^{(2)}_{n+\frac12}(k_3R)-4[H]'^{(2)}_{n+\frac12}(k_3R)]\Big\},\\ \bar{C}^s_n(\omega R) &= 2R^{-5/2}\Big\{-k_1^{-2}[[H]'^{(2)}_{n+\frac12}(k_1R)-H^{(2)}_{n+\frac12}(k_1R)]+\\ &\quad +\frac{k_1^{-2}(h_1^2-k_1^2)[H]'^{(2)}_{n+\frac12}(k_1a)}{k_3^{-2}(h_1^2-k_3^2)[H]'^{(2)}_{n+\frac12}(k_3a)}\,k_3^{-2}[[H]'^{(2)}_{n+\frac12}(k_3R)-H^{(2)}_{n+\frac12}(k_3R)]\Big\}.\end{aligned}$$

The load components $({}_s\overset{2}{\sigma}{}^*_{\mathrm{II}})_{RR}$ and $({}_s\overset{2}{\sigma}{}^*_{\mathrm{II}})_{R\theta}$ taken from the set $[{}_s\overset{2}{S}{}^*_{\mathrm{II}}]$ of the correcting components originating from the rotational vector have the form, [6]:

$$(2.14)\qquad [{}_s\overset{2}{S}{}^*_n]:\quad \begin{cases}({}_s\overset{2}{\sigma}{}^*_{\mathrm{II}})_{RR} = \mu(n+1)nB^s_n(\omega R)\mathcal{P}_n(\cos\theta),\\ ({}_s\overset{2}{\sigma}{}^*_{\mathrm{II}})_{R\theta} = \mu D^s_n(\omega R)\,[\mathcal{P}_n(\cos\theta)]_{,\theta},\end{cases}$$

where

$$\begin{aligned}B^s_n(\omega R) &= R^{-5/2}[2(h_2R)H'^{(2)}_{n+\frac12}(h_2R)-3H^{(2)}_{n+\frac12}(h_2R)],\\ D^s_n(\omega R) &= -R^{-5/2}\{[(h_2R)^2-2(n-1)(n+2)]H^{(2)}_{n+\frac12}(h_2R)+\\ &\quad +2h_2RH'^{(2)}_{n+\frac12}(h_2R)-3H^{(2)}_{n+\frac12}(h_2R)\},\end{aligned}$$

The condition of zero load on the spherical surface $R=a$ has the form:

$$(2.15)\qquad \begin{aligned}x^s_n &= S(\Delta^s_n)^{-1}(-1)^n(n+\tfrac12)\frac{\Psi_{n+\frac12}(ik_3a)-\Psi_{n+\frac12}(ik_1a)}{k_1^2-k_3^2}\times\\ &\quad \times\{n(n+1)\,B^s_n(\omega a)-[\tfrac12a^2h_2^2-n(n+1)]\,D^s_n(\omega a)\},\\ y^s_n &= S(\Delta^s_n)^{-1}(-1)^n(n+\tfrac12)\frac{\Psi_{n+\frac12}(ik_3a)-\Psi_{n+\frac12}(ik_1a)}{k_1^2-k_3^2}\times\\ &\quad \times\{\bar{A}^s_n(\omega a)-[\tfrac12a^2h_2^2-n(n+1)]\,\bar{C}^s_n(\omega a)\},\\ \Delta^s_n &= \bar{A}^s_n(\omega a)\,D^s_n(\omega a)-n(n+1)B^s_n(\omega a)\,\bar{C}^s_n(\omega a).\end{aligned}$$

The above problems of the coupled stress and temperature concentration at spherical and cylindrical cavities include steady-state stress concentration problems treated by A. L. FLORENCE and J. N. GOODIER in Ref. [7].

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Streszczenie

WARUNKI WYPROMIENIOWANIA SOMMERFELDA DLA SPRZĘŻONYCH ZAGADNIEŃ TERMOSPRĘŻYSTOŚCI. PRZYKŁADY SPRZĘŻONEJ KONCENTRACJI NAPRĘŻEŃ I TEMPERATURY DOOKOŁA OTWORÓW WALCOWYCH I KULISTYCH

W pracy rozważono dwa półograniczone obszary sprężyste: przestrzeń z otworem walcowym i z otworem kulistym. Dla tych obszarów poszukiwane są rozwiązania równań sprzężonych, spełniających określone warunki brzegowe na powierzchni walcowej i kulistej oraz odpowiednie warunki wypromieniowania w nieskończoność. Z postaci równań sprzężonych wynika, że część rotacyjna wektora przemieszczenia rozwiązującego problem powinna spełniać klasyczny warunek Sommerfelda, natomiast część potencjalna spełniać powinna podobne dwa warunki z parametrami zespolonymi k_1 i k_3. Wykazano, że w nieskończoności fala potencjalna zanika szybciej w zagadnieniu sprzężonym niż w odpowiednim problemie niesprzężonym i szybkość zanikania jest określona wielkością części czysto urojonej pierwiastków $k_{1,3}$. Warunek wypromieniowania dla temperatury jest określony przez dwa warunki dla sprzężonych potencjałów. Sprzężony potencjał i temperatura składają się z dwóch typów interferujących się fal [por. wz. (2.3)] o dwóch rozmaitych pochłanianiach zespolonych i dwóch różnych fazach biegnących do nieskończoności. W przypadku braku sprzężenia otrzymuje się klasyczne warunki wypromieniowania z teorii drgań ustalonych.

Резюме

УСЛОВИЯ ИЗЛУЧЕНИЯ ЗОММЕРФЕЛЬДА ДЛЯ СОПРЯЖЕННЫХ ЗАДАЧ ТЕРМОУПРУГОСТИ. ПРИМЕРЫ СОПРЯЖЕННОЙ КОНЦЕНТРАЦИИ НАПРЯЖЕНИЙ И ТЕМПЕРАТУРЫ ВОКРУГ ЦИЛИНДРИЧЕСКИХ И ШАРООБРАЗНЫХ ОТВЕРСТИЙ

Рассматриваются две полуограниченные упругие области, а именно: пространство с цилиндрическим и шарообразным отверстием и для этих областей даются решения сопряженных уравнений, удовлетворяющих определенным краевым условиям на цилиндрической и шарообразной поверхностях и соответствующим условиям излучения в бесконечность. Из формы сопряженных уравнений вытекает, что вращательная часть вектора перемещения, решающего задачу, должна удовлетворять классическому условию Зоммерфельда, тогда как потенциальная часть должна удовлетворять двум подобным условиям с комплексивными параметрами k_1 и k_3. В работе показывается, что потенциальная волна в сопряженной задаче затухает скорее, чем в соответствующей несопряженной задаче и скорость затухания определяется величиной части чистомнимой корней $k_{1,3}$. Условие излучения для температуры определяется двумя условиями для сопряженных потенциалов. Сопряженный потенциал и температура, составляются из двух типов интерферирующихся волн [ср. формулу (2.3)] с двумя различными комплексными поглощениями и с двумя различными фазами бегущими в бесконечность. При отсутствии сопряжения получаются классические условия излучения из теории установившихся колебаний.

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