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APPLIED MECHANICS

Two Cases of Discontinuous Temperature Field in an Elastic Space and Semi-Space

by

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In this paper we shall be concerned with the determination of the state of stress in an elastic space and semi-space, due to two types of discontinuous temperature field. The first is that of a thermal inclusion in the form of a rectangular parallelepiped, the second — in the form of a cylinder. By the term of thermal inclusion we understand a region in which the temperature is constant whereas, in the remaining region the temperature equals zero.

1. Thermal inclusion in the form of a rectangular parallelepiped in an elastic space and semi-space

Let the temperature inside the parallelepiped $-a_i \leqslant \xi_i \leqslant a_i \ (i=1,2,3)$ be constant and equal to unity, the temperature outside being zero. The temperature field may be expressed in terms of Heaviside's function η

$$(1.1) \qquad T = [\eta(x_1+a_1) - \eta(x_1-a_1)]\,[\eta(x_2+a_2) - \eta(x_2-a_2)]\,[\eta(x_3+a_3) - \eta(x_3-a_3)].$$

It is known that the thermal stresses, due to this state, may be expressed by means of the potential of thermoelastic displacement Φ, by using the following equations [1],

$$(1.2) \qquad \sigma_{ij} = 2G\left(\frac{\partial^2}{\partial x_i \partial x_j} - \delta_{ij}\nabla^2\right)\Phi \qquad i,j=1,2,3,$$

where G is the shear modulus and δ_{ij} — Kronnecker's delta. The function Φ is related to the stresses by the equations

$$(1.3) \qquad u_i = \frac{\partial \Phi}{\partial x_i} \qquad i=1,2,3.$$

Expressing the displacements in the displacement equations of the theory of elasticity in terms of the derivatives of Φ, we reduce them to the one single equation [1],

$$(1.4) \qquad \nabla^2\Phi = \vartheta_0 T, \qquad \nabla^2 = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}, \qquad \vartheta_0 = \frac{1+\nu}{1-\nu}\alpha_t,$$

where ν is Poisson's ratio and α_t the coefficient of thermal dilatation.

Let us denote by Φ^* the potential of thermoelastic displacement due to a nucleus of thermoelastic strain. In this case the temperature distribution is

$$T^* = T^*(x_i; \xi_k) = \delta(x_1 - \xi_1)\,\delta(x_2 - \xi_2)\,\delta(x_3 - \xi_3), \tag{1.5}$$

where δ is the Dirac function.

Using the integral representation of the Dirac function, we can express the solution of the following equation

$$\nabla^2 \Phi^* = \vartheta_0 T^* \tag{1.6}$$

by the integral

$$\Phi^* = -\frac{\vartheta_0}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty (\alpha_1^2+\alpha_2^2+\alpha_3^2)^{-1}\cos\alpha_1(x_1-\xi_1)\cos\alpha_2(x_2-\xi_2)\cos\alpha_3(x_3-\xi_3)\,d\alpha_1\,d\alpha_2\,d\alpha_3. \tag{1.7}$$

On the other hand, the function

$$\Phi^* = -\frac{\vartheta_0}{4\pi}R_{123}^{-1} \tag{1.8}$$

is the solution of Eq. (1.6), where

$$R_{123} = [(x_1-\zeta_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2]^{1/2}.$$

Using the theorem of uniqueness of solutions of boundary value problems of the theory of elasticity and setting the right-hand members of Eqs. (1.6) and (1.8), we obtain

$$\int_0^\infty\int_0^\infty\int_0^\infty (\alpha_1^2+\alpha_2^2+\alpha_3^2)^{-1}\cos\alpha_1(x_1-\xi_1)\cos\alpha_2(x_2-\xi_2)\cos\alpha_3(x_3-\xi_3)\times d\alpha_1\,d\alpha_2\,d\alpha_3 = \frac{\pi^2}{4}R_{123}^{-1}. \tag{1.9}$$

Using the function $\Phi^*(x_i, \xi_i)$, we determine Green's functions for stresses by means of the equation

$$\sigma_{ij}^* = 2G\left(\frac{\partial^2}{\partial x_i\,\partial x_j} - \delta_{ij}\nabla^2\right)\Phi \qquad i,j = 1,2,3. \tag{1.10}$$

These functions have the form:

$$\sigma_{ij}^* = -\frac{G\vartheta_0}{2\pi}\left[\frac{\partial^2}{\partial x_i\,\partial x_j}[(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2]^{-1/2} + 4\pi\delta_{ij}\,\delta(x_1-\xi_1)\,\delta(x_2-\xi_2)\,\delta(x_3-\xi_3)\right]. \tag{1.11}$$

For a temperature field expressed by the function (1.1), we obtain the stress components σ_{ij} by integrating the stresses σ_{ij}^* over the region of the parallelepiped under consideration

$$\sigma_{ij} = -\frac{G\vartheta_0}{\pi}\left\{\int_{-a_1}^{a_1}d\xi_1\int_{-a_2}^{a_2}d\xi_2\int_{-a_3}^{a_3}d\xi_3\,\frac{\partial^2}{\partial\xi_i\,\partial\xi_j}[(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2]^{-1/2} + 4\pi\delta_{ij}\int_{-a_1}^{a_1}d\xi_1\int_{-a_2}^{a_2}d\xi_2\int_{-a_3}^{a_3}d\xi_3\,\delta(x_1-\xi_1)\,\delta(x_2-\xi_2)\,\delta(x_3-\xi_3)\right\}. \tag{1.12}$$

The second integral in Eq. (1.12) vanishes if the point x_i does not belong to the region of the parallelepiped considered. By integrating, we obtain

$$
\left\{
\begin{aligned}
\sigma_{12} = {} & A_0 \ln [(a_3 - x_3 + r_{123})(a_3 - x_3 + r_{-1,-2,3})(a_3 + x_3 - \\
& - r_{-1,2,-3})(a_3 + x_3 - r_{1,-2,-3})] : [a_3 - x_3 + r_{-1,23})(a_3 - x_3 + \\
& + r_{1,-2,3})(a_3 + x_3 - r_{1,2,-3})(a_3 + x_3 - r_{-1,-2,-3})]' \\
\sigma_{23} = {} & A_0 \ln [(a_1 - x_1 + r_{123})(a_1 - x_1 + r_{1,-2,-3})(a_1 + x_1 - \\
& - r_{-1,-2,3})(a_1 + x_1 - r_{-1,2,-3})] : [a_1 - x_1 + r_{1,-2,3})(a_1 - x_1 + \\
& - r_{12,-3})(a_1 + x_1 - r_{-1,23})(a_1 + x_1 - r_{-1,-2,-3})], \\
\sigma_{13} = {} & A_0 \ln [(a_2 - x_2 + r_{123})(a_2 - x_2 + r_{-1,2,-3})(a_2 + x_2 - \\
& - r_{1,-2,-3})(a_2 + x_2 - r_{-1,-2,3})] : [(a_2 - x_2 + r_{12,-3})(a_2 - x_2 + \\
& + r_{-1,23})(a_2 + x_2 - r_{-1,-2,3})(a_2 + x_2 - r_{-1,-2,-3})], \\
\sigma_{11} = {} & A_0 \Big[- \operatorname{tg}^{-1} \frac{a_2 - x_2}{x_1 - a_1} \frac{a_3 - x_3}{r_{123}} + \operatorname{tg}^{-1} \frac{a_2 - x_2}{x_1 - a_1} \frac{a_3 + x_3}{r_{12,-3}} + \\
& + \operatorname{tg}^{-1} \frac{x_2 + a_2}{x_1 - a_1} \frac{a_3 - x_3}{r_{1,-2,3}} - \operatorname{tg}^{-1} \frac{x_2 + a_2}{a_1 - x_1} \frac{a_3 + x_3}{r_{1,-2,-3}} + \\
& + \operatorname{tg}^{-1} \frac{x_2 - a_2}{x_1 + a_1} \frac{a_3 - x_3}{r_{-1,23}} - \operatorname{tg}^{-1} \frac{a_2 - x_2}{x_1 + a_1} \frac{a_3 + x_3}{r_{-1,2,-3}} - \\
& - \operatorname{tg}^{-1} \frac{x_2 + a_2}{x_1 + a_1} \frac{a_3 - x_3}{r_{-1,-2,3}} + \operatorname{tg}^{-1} \frac{x_2 + a_2}{x_1 + a_1} \frac{[-(x_3 + a_2)]}{r_{-1,-2,-3}} \Big] \\
\sigma_{22} = {} & A_0 \Big[- \operatorname{tg}^{-1} \frac{x_3 - a_3}{x_2 - a_2} \frac{a_1 - x_1}{r_{123}} + \operatorname{tg}^{-1} \frac{a_3 - x_3}{x_2 - a_2} \frac{a_1 + x_1}{r_{-1,23}} + \\
& + \operatorname{tg}^{-1} \frac{x_3 + a_3}{x_2 - a_2} \frac{a_1 - x_1}{r_{12,-3}} - \operatorname{tg}^{-1} \frac{x_3 + a_3}{a_2 - x_2} \frac{a_1 + x_1}{r_{-1,2,-3}} + \\
& + \operatorname{tg}^{-1} \frac{x_3 - a_3}{x_2 + a_2} \frac{a_1 - x_1}{r_{1,-2,3}} - \operatorname{tg}^{-1} \frac{a_3 - x_3}{x_2 + a_2} \frac{a_1 + x_1}{r_{-1,2,3}} - \\
& - \operatorname{tg}^{-1} \frac{x_3 + a_3}{x_2 + a_2} \frac{a_1 - x_1}{r_{1,-2,-3}} + \operatorname{tg}^{-1} \frac{x_3 + a_3}{x_2 + a_2} \frac{[-(x_1 + a_1)]}{r_{-1,-2,-3}} \Big], \\
\sigma_{33} = {} & A_0 \Big[- \operatorname{tg}^{-1} \frac{x_1 - a_1}{x_3 - a_3} \frac{a_2 - x_2}{r_{123}} + \operatorname{tg}^{-1} \frac{a_1 - x_1}{x_3 - a_3} \frac{a_2 + x_2}{r_{1,-2,3}} + \\
& + \operatorname{tg}^{-1} \frac{x_1 + a_1}{x_3 - a_3} \frac{a_2 - x_2}{r_{-1,23}} - \operatorname{tg}^{-1} \frac{x_1 + a_1}{a_3 - x_3} \frac{a_2 + x_2}{r_{-1,-2,3}} + \\
& + \operatorname{tg}^{-1} \frac{x_1 - a_1}{x_3 + a_3} \frac{a_2 - x_2}{r_{12,-3}} - \operatorname{tg}^{-1} \frac{a_1 - x_1}{x_3 + a_3} \frac{a_2 + x_2}{r_{1,-2,-3}} - \\
& - \operatorname{tg}^{-1} \frac{x_1 + a_1}{x_3 + a_3} \frac{a_2 - x_2}{r_{-1,2,-3}} + \operatorname{tg}^{-1} \frac{x_1 + a_1}{x_3 + a_3} \frac{[-(x_2 + a_2)]}{r_{-1,-2,-3}} \Big].
\end{aligned}
\right. \tag{1.13}
$$

where

$$r_{\pm 1, \pm 2, \pm 3} = [(x_1 \pm a_1)^2 + (x_2 \pm a_2)^2 + (x_3 \pm a_3)^2]^{1/2},$$

$$\mathrm{tg}^{-1} = \mathrm{arc\,tg}, \qquad A_0 = -\frac{G\vartheta_0}{2\pi}.$$

Passing from the inside to the outside of the parallelepiped, we obtain discontinuities (jumps) of stresses with equal subscripts, non-perpendicular to the walls of the parallelepiped. If the vertices are approached the transverse stresses increase indefinitely.

Let the temperature be constant and equal to unity inside the parallelepiped $a_i < \xi_i < a_i'$ $(i = 1, 2, 3)$, which is inside the elastic semi-space, and let $a_i' > a_i > 0$. Let the temperature outside that parallelepiped be zero.

The stresses due to such a discontinuous temperature field are obtained from the equation

$$\sigma_{ij} = \int_{a_1}^{a_1'} \int_{a_2}^{a_2'} \int_{a_3}^{a_3'} \sigma_{ij}^* \, d\xi_1 \, d\xi_2 \, d\xi_3, \tag{1.14}$$

where

$$\sigma_{ij} = \bar{\sigma}_{ij}^* + \bar{\bar{\sigma}}_{ij}^*.$$

The symbols σ_{ij}^* denote the stresses at a point x_i $(i = 1, 2, 3)$, due to the action of a nucleus of thermoelastic strain acting at the point ξ_i $(i = 1, 2, 3)$ of the elastic semi-space. These stresses take the following values, [2],

$$\bar{\sigma}_{ij}^* = A_0 \Big\{ \frac{\partial^2}{\partial x_i \, \partial x_j} (R_{123}^{-1} - R_{12,-3}^{-1}) + 4\pi \delta_{ij} \delta\,(x_1 - \xi_1)\, \delta\,(x_2 - \xi_2) \times \times [\delta\,(x_3 - \xi_3) - \delta\,(x_3 + \xi_3)] \Big\}, \tag{1.15}$$

where

$$R_{12,\mp 3} = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 \pm \xi_3)^2]^{1/2}, \qquad A_0 = -\frac{G\vartheta_0}{2\pi}.$$

The stresses $\bar{\bar{\sigma}}_{ij}^*$ may be represented by the relations

$$\left\{ \begin{aligned} \bar{\bar{\sigma}}_{11}^* &= 2A_0 \left[x_3 \frac{\partial^3}{\partial x_1^2 \partial x_3} + 2 \frac{\partial^2}{\partial x_1^2} + 2\nu \frac{\partial^2}{\partial x_2^2} \right] R_{1,2,-3}^{-1}, \\ \bar{\bar{\sigma}}_{22}^* &= 2A_0 \left[x_3 \frac{\partial^3}{\partial x_2^2 \partial x_3} + 2 \frac{\partial^2}{\partial x_2^2} + 2\nu \frac{\partial^2}{\partial x_1^2} \right] R_{1,2,-3}^{-1}, \\ \bar{\bar{\sigma}}_{33}^* &= 2A_0 \, x_3 \frac{\partial^3}{\partial x_3^3} R_{1,2,-3}^{-1}, \\ \bar{\bar{\sigma}}_{13}^* &= 2A_0 \frac{\partial}{\partial x_3} \left(x_3 \frac{\partial^2}{\partial x_1 \partial x_3} R_{1,2,-3}^{-1} \right), \\ \bar{\bar{\sigma}}_{23}^* &= 2A_0 \frac{\partial}{\partial x_3} \left(x_3 \frac{\partial^2}{\partial x_2 \partial x_3} R_{1,2,-3}^{-1} \right), \\ \bar{\bar{\sigma}}_{12}^* &= 2A_0 \left[x_3 \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} + 2(1-\nu) \frac{\partial^2}{\partial x_1 \partial x_2} \right] R_{1,2,-3}^{-1}. \end{aligned} \right. \tag{1.16}$$

After carrying out the integration prescribed by Eq. (1.14), we obtain for stresses the following equations:

$$
\begin{aligned}
\text{(1.17a)}\quad \sigma_{12} = A_0 \{ & \ln [(a_3' - x_3 + r_{1'2'3'})(a_3' - x_3 + r_{123'})(a_3 - x_3 + r_{12'3})(a_3 - x_3 + \\
& + r_{1'23})] : [(a_3 - x_3 + r_{1'2'3})(a_3 - x_3 + r_{123})(a_3' - x_3 + r_{12'3'})(a_3' - x_3 + r_{1'23'})] + \\
& + (3 - 4\nu) \ln [(a_3' + x_3 + r_{1'2',-3'})(a_3' + x_3 + r_{12,-3'})(a_3 + x_3 + r_{12',-3})(a_3 + x_3 + \\
& + r_{1'2,-3})] : [(a_3 + x_3 + r_{1'2',-3})(a_3 + x_3 + r_{12,-3})(a_3' + x_3 + r_{12',-3'})(a_3' + x_3 + \\
& + r_{1'2,-3})] + 2 x_3 (r_{1'2'3'} - r_{1'2'3} - r_{1'23'} + r_{1'23} - r_{123'} + r_{12'3} + r_{123'} - r_{123})\},
\end{aligned}
$$

$$
\begin{aligned}
\text{(1.17b)}\quad \sigma_{13} = A_0 \Big\{ & \ln [(a_2' - x_2 + r_{1'2'3'})(a_2' - x_2 + r_{12'3})(a_2 - x_2 + r_{1'23})(a_2 - x_2 + \\
& + r_{123'})] : [(a_2 - x_2 + r_{1'23'})(a_2 - x_2 + r_{123})(a_2' - x_2 + r_{1'2'3})(a_2' - x_2 + r_{12'3'})] - \\
& - \ln [(a_2' - x_2 + r_{1'2',-3'})(a_2' - x_2 + r_{12',-3})(a_2 - x_2 + r_{1'2,-3})(a_2 - x_2 + r_{12,-3'})] : \\
& : [(a_2 - x_2 + r_{1'2,-3'})(a_2 - x_2 + r_{12,-3})(a_2' - x_2 + r_{1'2',-3})(a_2' - x_2 + r_{12',-3})] + \\
& - 2 x_3 \Big(- \frac{(a_2' - x_2)(x_3 + a_3')}{r^2_{1',-3'} \cdot r_{1'2',-3'}} + \frac{(a_2 - x_2)(x_3 + a_3')}{r^2_{1',-3'} \cdot r_{1'2,-3'}} + \frac{(a_2' - x_2)(x_3 + a_3)}{r^2_{1',-3} \cdot r_{1'2,-3}} - \\
& - \frac{(a_2 - x_2)(x_3 + a_3)}{r^2_{1,-3} \cdot r_{12,-3}} + \frac{(a_2' - x_2)(x_3 + a_3')}{r^2_{1,-3'} \cdot r_{12',-3'}} - \frac{(a_2 - x_2)(x_3 + a_3')}{r^2_{1,-3'} \cdot r_{12,-3'}} - \\
& - \frac{(a_2' - x_2)(x_3 + a_3)}{r^2_{1,-3}\, r_{12',-3}} + \frac{(a_2 - x_2)(x_3 + a_3)}{r^2_{1,-3}\, r_{12,-3}} \Big) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
\text{(1.17c)}\quad \sigma_{23} = A_0 \Big\{ & \ln [(a_1' - x_1 + r_{1'2'3'})(a_1' - x_1 + r_{1'23})(a_1 - x_1 + r_{12'3})(a_1 - x_1 + \\
& + r_{123'})] : [(a_1 - x_1 + r_{12'3'})(a_1 - x_1 + r_{123})(a_1' - x_1 + r_{1'2'3'})(a_1' - x_1 + r_{1'23'})] - \\
& - \ln [(a_1' - x_1 + r_{1'2',-3'})(a_1' - x_1 + r_{1'2,-3})(a_1 - x_1 + r_{12',-3})(a_1 - x_1 + r_{12,-3'})] : \\
& : [(a_1 - x_1 + r_{12'-3'})(a_1 - x_1 + r_{1'2,-3})(a_1' - x_1 + r_{1'2',-3})(a_1' - x_1 + r_{1'2,-3'})] + \\
& - 2 x_3 \Big(- \frac{(a_1' - x_1)(x_3 + a_3')}{r^2_{2,'-3'} \cdot r_{1'2'-3'}} + \frac{(a_1 - x_1)(a_3' + x_3)}{r^2_{2',-3'} \cdot r_{12',-3'}} + \frac{(a_1' - x_1)(x_3 + a_3)}{r^2_{2',-3} \cdot r_{1'2',-3}} - \\
& - \frac{(a_1 - x_1)(x_3 + a_3)}{r^2_{2',-3} \cdot r_{12',-3}} + \frac{(a_1' - x_1)(x_3 + a_3')}{r^2_{2,-3'} \cdot r_{1'2,-3'}} - \frac{(a_1 - x_1)(x_3 + a_3')}{r^2_{2,-3'} \cdot r_{12,-3'}} - \\
& - \frac{(a_1' - x_1)(x_3 + a_3)}{r^2_{2,-3}\, r_{1'2,-3}} + \frac{(a_1 - x_1)(x_3 + a_3)}{r^2_{2,-3}\, r_{12,-3}} \Big) \Big\}.
\end{aligned}
$$

$$
\begin{aligned}
(1.17\text{d})\quad \sigma_{11} = A_0\Big[& -t^{-1}\frac{x_2-a_2'}{x_1-a_1'}\,\frac{a_3'-x_3}{r_{1'2'3'}} + t^{-1}\frac{x_2-a_2'}{x_1-a_1'}\,\frac{a_3-x_3}{r_{1'2'3}} + \\
& + t^{-1}\frac{x_2-a_2}{x_1-a_1'}\,\frac{a_3'-x_3}{r_{1'23'}} - t^{-1}\frac{x_2-a_2}{x_1-a_1'}\,\frac{a_3-x_3}{r_{1'13}} + t^{-1}\frac{x_2-a_2'}{x_1-a_1}\,\frac{a_3'-x_3}{r_{12'3'}} - \\
& - t^{-1}\frac{x_2-a_2'}{x_1-a_1}\,\frac{a_3-x_3}{r_{12'2}} - t^{-1}\frac{x_2-a_2}{x_1-a_1}\,\frac{a_3'-x_3}{r_{123'}} + t^{-1}\frac{x_2-a_2}{x_1-a_1}\,\frac{a_3-x_3}{r_{123}} + \\
& + 3\Big(-t^{-1}\frac{x_2-a_2'}{x_1-a_1'}\,\frac{a_3'+x_3}{r_{1'2',-3'}} + t^{-1}\frac{x_2-a_2'}{x_1-a_1'}\,\frac{a_3+x_3}{r_{1'2',-3}} + t^{-1}\frac{x_2-a_2}{x_1-a_1'}\,\frac{a_3'+x_3}{r_{1'2,-3'}} - \\
& - t^{-1}\frac{x_2-a_2}{x_1-a_1'}\,\frac{a_3+x_3}{r_{1'2,-3}} + t^{-1}\frac{x_2-a_2'}{x_1-a_1}\,\frac{a_3'+x_3}{r_{12',-3'}} - t^{-1}\frac{x_2-a_2'}{x_1-a_1}\,\frac{a_3+x_3}{r_{12',-3}} - \\
& - t^{-1}\frac{x_2-a_2}{x_1-a_1}\,\frac{a_3'+x_3}{r_{12,-3'}} + t^{-1}\frac{x_2-a_2}{x_1-a_1}\,\frac{a_3+x_3}{r_{12,-3}}\Big) - 4\nu\Big(t^{-1}\frac{x_3+a_3'}{x_2-a_2'}\,\frac{x_1-a_1'}{r_{1'2',-3'}} + \\
& + t^{-1}\frac{x_3+a_3}{x_2-a_2'}\,\frac{x_1-a_1}{r_{12',-3}} + t^{-1}\frac{x_3+a_3'}{x_2-a_2'}\,\frac{x_1-a_1'}{r_{1'2',-3'}} - t^{-1}\frac{x_3+a_3}{x_2-a_2'}\,\frac{x_1-a_1}{r_{12',-3}} + \\
& + t^{-1}\frac{x_3+a_3'}{x_2-a_2}\,\frac{x_1-a_1'}{r_{1'2,-3'}} - t^{-1}\frac{x_3+a_3}{x_2-a_2}\,\frac{x_1-a_1}{r_{12,-3}} - t^{-1}\frac{x_3+a_3'}{x_2-a_2}\,\frac{x_1-a_1'}{r_{1'2,-3'}} + \\
& + t^{-1}\frac{x_3+a_3}{x_2-a_2}\,\frac{x_1-a_1}{r_{12,-3}}\Big) - 2x_3\Big(-\frac{(a_2'-x_2')(x_1-a_1')}{r^2_{1',-3'}\cdot r_{1'2',-3'}} + \frac{(a_2-x_2)(x_1-a_1')}{r^2_{1',-3'}\cdot r_{1'2,-3'}} + \\
& + \frac{(a_2'-x_2)(x_1-a_1')}{r^2_{1',-2}\cdot r_{12,-3'}} - \frac{(a_2-x_2)(x_1-a_1')}{r^2_{1',-3}\cdot r_{1'2,-3}} + \frac{(a_2'-x_2)(x_1-a_1)}{r^2_{1,-3'}\cdot r_{12',-3'}} - \\
& - \frac{(a_2-x_2)(x_1-a_1)}{r^2_{1,-3'}\cdot r_{12,-3'}} - \frac{(a_2'-x_2)(x_1-a_1)}{r^2_{1,-3}\cdot r_{12',-3}} + \frac{(a_2-x_2)(x_1-a_1)}{r^2_{1,-3}\cdot r_{12',-3}}\Big)\Big].
\end{aligned}
$$

A corresponding change of subscripts 1, 2, 3 in Eq. (17d) yields the two stress components σ_{22} and σ_{33}. The notations used are as follows:

$$r^2_{1.-3} = (x_1-a_1)^2 + (x_3+a_3)^2$$

$$r^2_{2,-3'} = (x_2-a_2)^2 + (x_3+a_3')^2 \text{ etc.} \qquad t^{-1} = \text{arc tg}.$$

2. Cylindrical thermal inclusion in an elastic space and semi-space

Let us consider a discontinuous temperature field expressed by the equation

$$
(2.1)\qquad \begin{cases} T = \eta(a-r)\,[\eta(z+c)-\eta(z-c)], \\ a \geqslant 0, \quad c \geqslant 0, \quad z = x_3, \quad r = (x_1^2+x_2^2)^{1/2}. \end{cases}
$$

The temperature within the cylinder of a height of $2c$ and of radius a is equal to unity. The temperature outside the cylinder is equal to zero. The temperature field (2.1) may be expressed by the integral

$$T(r,z) = \frac{2a}{\pi} \int_0^\infty \int_0^\infty J_0(\alpha r) J_1(\alpha a) \frac{\cos \gamma z \sin \gamma c}{\gamma} d\alpha d\gamma. \tag{2.2}$$

The potential of thermoelastic displacement is used in the form of the integral representation

$$\Phi(r,z) = -\frac{2a\vartheta_0}{\pi} \int_0^\infty \int_0^\infty \gamma^{-1}(\alpha^2+\gamma^2) J_1(\alpha a) J_0(\alpha r) \sin \gamma c \cos \gamma z \, d\alpha d\gamma, \tag{2.3}$$

where

$$\nabla^2 \Phi = \vartheta_0 T.$$

Using Eqs. (1.2), we obtain

$$\sigma_{zz} = -2G\vartheta_0 a \begin{cases} \int_0^\infty e^{\alpha z} \sinh \alpha c J_1(\alpha a) J_0 \alpha r) \, d\alpha & z < -c, \\ \int_0^\infty (1 - e^{-\alpha c} \cosh \alpha z) J_1(\alpha a) J_0(\alpha r) \, d\alpha & |z| < c, \\ \int_0^\infty e^{-\alpha z} \sinh \alpha c J_1(\alpha a) J_0(\alpha r) \, d\alpha & z > c. \end{cases} \tag{2.4}$$

$$\sigma_{rr} = 2G\vartheta_0 a \begin{cases} \int_0^\infty e^{\alpha z} \sinh \alpha c J_1(\alpha a) \left[J_0(\alpha r) - \frac{J_1(\alpha r)}{\alpha r} \right] d\alpha & z > -c, \\ -\int_0^\infty e^{-\alpha c} \cosh \alpha z J_1(\alpha a) \left[J_0(\alpha r) - \frac{J_1(\alpha r)}{\alpha r} \right] d\alpha - \\ \qquad - \int_0^\infty J_1(\alpha a) \frac{J_1(\alpha r)}{\alpha r} d\alpha & |z| < c, \\ \int_0^\infty e^{-\alpha z} \sinh \alpha c J_1(\alpha a) \left[J_0(\alpha r) - \frac{J_1(\alpha r)}{\alpha r} \right] d\alpha & z > c, \end{cases} \tag{2.5}$$

$$\sigma_{\varphi\varphi} = 2G\vartheta_0 a \begin{cases} \frac{1}{r} \int_0^\infty \alpha^{-1} e^{\alpha z} \sinh \alpha c J_1(\alpha a) J_1(\alpha r) \, d\alpha & z < -c, \\ -\int_0^\infty e^{-\alpha c} \cosh \alpha z J_1(\alpha a) \frac{J_1(\alpha r)}{\alpha r} d\alpha - \\ \qquad - \int_0^\infty J_1(\alpha e) \left[J_0(\alpha r) - \frac{J_1(\alpha r)}{\alpha r} \right] d\alpha & |z| < c, \\ \frac{1}{r} \int_0^\infty \alpha^{-1} e^{-\alpha z} \sinh \alpha c \, J_1(\alpha a) J_1(\alpha r) \, d\alpha & z > c. \end{cases} \tag{2.6}$$

$$
(2.7)\qquad \sigma_{rz} = -2G\vartheta_0 a \begin{cases} -\int\limits_0^\infty e^{\alpha z} \sinh \alpha c J_1(\alpha a) J_1(\alpha r)\, d\alpha & z < -c, \\ \int\limits_0^\infty e^{-\alpha c} \sinh \alpha z J_1(\alpha a) J_1(\alpha r)\, d\alpha & |z| < c, \\ \int\limits_0^\infty e^{-\alpha z} \sinh \alpha c J_1(\alpha a) J_1(\alpha r)\, d\alpha & z > c. \end{cases}
$$

It follows from these expressions that

$$
\sigma_{rr} + \sigma_{\varphi\varphi} + \sigma_{zz} = 2\,G\vartheta_0\, a \begin{cases} 0 & \text{if} \quad z < -c, \\ -\dfrac{2}{a}\,\eta(a-r) & \text{if} \quad |z| < c, \\ 0 & \text{if} \quad z > c. \end{cases}
$$

Let the temperature inside the semi-infinite cylinder of radius a be $T=1$, the temperature outside the cylinder being equal to zero. We assume that the plane $z=0$ limiting the elastic semi-space $z \geqslant 0$ is free from stresses. The temperature field may be expressed in terms of Heaviside's function, by the equation

$$
(2.8)\qquad T = \eta(a-r)\,[\eta(z) - \eta(-z)],
$$

or by the integral representation

$$
(2.9)\qquad T(r,z) = \frac{2\,a}{\pi} \int\limits_0^\infty \int\limits_0^\infty \gamma^{-1} \sin \gamma z J_1(\alpha a)\, J_0(\alpha r)\, d\alpha\, d\gamma.
$$

The potential of thermoelastic displacement will be expressed by Fourier-Hankel's integral

$$
(2.10)\qquad \Phi = -\frac{2\,\vartheta_0 a}{\pi} \int\limits_0^\infty \int\limits_0^\infty \gamma^{-1} (\alpha^2+\gamma^2)^{-1} \sin \gamma z J_1(\alpha a)\, J_0(\alpha r)\, d\alpha\, d\gamma,
$$

or

$$
\Phi = \vartheta_0\, a \int\limits_0^\infty \alpha^{-2} (e^{-\alpha z} - 1)\, J_1(\alpha a)\, J_0(\alpha r)\, d\alpha \qquad z \geqslant 0.
$$

Using the function Φ, we shall determine the stresses $\bar{\sigma}_{ij}$ from Eqs. (1.2). It may easily be verified that not all the conditions of no-stress in the plane $z=0$ are satisfied. We, namely, have $\bar{\sigma}_{zz}(r,0)=0$, $\bar{\sigma}_{rz}(r,0) \neq 0$. To the state of stress $\bar{\bar{\sigma}}_{ij}$, we should therefore add a state $\bar{\bar{\sigma}}_{ij}$, such that the following conditions be satisfied in the plane $z=0$.

(2.11) $$\bar{\sigma}_{zz}(r,0) + \bar{\bar{\sigma}}_{zz}(r,0) = 0, \quad \bar{\sigma}_{rz}(r,0) + \bar{\bar{\sigma}}_{rz}(r,0) = 0.$$

The stresses $\bar{\bar{\sigma}}_{ij}$ will be expressed in terms of Love's function φ by means of the equations, [3].

(2.12) $$\begin{cases} \bar{\sigma}_{rr} = \dfrac{2G}{1-2\nu}\dfrac{\partial}{\partial z}\left(\nu\nabla^2 - \dfrac{\partial^2}{\partial r^2}\right)\varphi, \\ \bar{\sigma}_{\varphi\varphi} = \dfrac{2G}{1-2\nu}\dfrac{\partial}{\partial z}\left(\nu\nabla^2 - \dfrac{1}{r}\dfrac{\partial}{\partial r}\right)\varphi, \\ \bar{\sigma}_{zz} = \dfrac{2G}{1-2\nu}\dfrac{\partial}{\partial z}\left[(2-\nu)\nabla^2 - \dfrac{\partial^2}{\partial z^2}\right]\varphi, \\ \bar{\sigma}_{rz} = \dfrac{2G}{1-2\nu}\dfrac{\partial}{\partial \nu}\left[(1-\nu)\nabla^2 - \dfrac{\partial^2}{\partial z^2}\right]\varphi. \end{cases}$$

Love's biharmonic function φ is assumed in the form:

(2.13) $$\varphi = \int_0^\infty \alpha^{-2}(A + B\alpha z)\, e^{-\alpha z} J_0(\alpha r)\, d\alpha.$$

We determine the quantities A, B from the boundary conditions (2.11). Adding the stresses $\bar{\sigma}_{ij}$ and $\bar{\bar{\sigma}}_{ij}$, we obtain finally the stresses σ_{ij}.

(2.14) $$\begin{cases} \sigma_{rr} = 2G\vartheta_0 a\left\{\displaystyle\int_0^\infty e^{-\alpha z} J_1(\alpha a)\left[(1-\alpha z)J_0(\alpha r) + \right.\right. \\ \qquad\left.\left. + (2\nu - 1 + \alpha z)\dfrac{J_1(\alpha r)}{\alpha r}\right] d\alpha - \displaystyle\int_0^\infty J_1(\alpha a)\dfrac{J_1(\alpha r)}{\alpha r}\, d\alpha\right\}, \\ \sigma_{\varphi\varphi} = 2G\vartheta_0 a\left\{\displaystyle\int_0^\infty e^{-\alpha z} J_1(\alpha a)\left[2\nu J_0(\alpha r) - (2\nu - 1 + \alpha z)\dfrac{J_1(\alpha r)}{\alpha r}\right] d\alpha - \right. \\ \qquad\left. - \displaystyle\int_0^\infty J_1(\alpha a)\left[J_0(\alpha r) - \dfrac{J_1(\alpha r)}{\alpha r}\right] d\alpha\right\}, \\ \sigma_{zz} = 2G\vartheta_0 a\left\{\displaystyle\int_0^\infty e^{-\alpha z}(1+\alpha z) J_1(\alpha a) J_0(\alpha r)\, d\alpha - \int_0^\infty J_1(\alpha a) J_0(\alpha r)\, d\alpha\right\}, \\ \sigma_{rz} = 2G\vartheta_0 a z \displaystyle\int_0^\infty \alpha e^{-\alpha z} J_1(\alpha a) J_1(\alpha r)\, d\alpha. \end{cases}$$

The Eqs. (2.4) - (2.7) and (2.14) may be transformed by introducing elliptic integrals or Legendre's functions. Let us observe that, in order to express the first three stresses in Eq. (2.14) in terms of known special functions, it suffices to know the three integrals:

$$(2.15)\quad \begin{cases} I_1 = \int_0^\infty e^{-\alpha z} J_0(\alpha a)\, J_0(\alpha r)\, d\alpha = \pi^{-1}(ar)^{-1/2}\, Q_{-1/2}\left(\dfrac{r^2+z^2+a^2}{2\,ar}\right), \\ I_2 = \int_0^\infty e^{-\alpha z} J_1(\alpha a)\, J_1(\alpha r)\, d\alpha = \pi^{-1}(ar)^{-1/2}\, Q_{1/2}\left(\dfrac{r^2+z^2+a^2}{2\,ar}\right), \\ I_3(a,r) = \int_0^\infty e^{-\alpha z} J_1(\alpha a)\, J_0(\alpha r)\, d\alpha = \pi^{-1} a^{-1}\,[K'\,E(k,\Theta) + \\ \qquad + (E'-K')\,F(k,\Theta) - z\,[(a+r)^2+z^2]^{-1/2}\,K'. \end{cases}$$

In the last equation $F(k,\Theta)$ and $E(k,\Theta)$ denote incomplete, respectively, elliptic integrals of the first and second kind related to the complementing modulus $k=[(a-r)^2+z^2]^{1/2}\,[(a+r)^2+z^2]^{-1/2}$ and the argument $\Theta = \sin^{-1} z\,[(a-r)^2+z^2]^{-1/2}$, $0 \leqslant \Theta \leqslant \pi$. K' and E' denote complete elliptic integrals of the first and second kind related to the complementing modulus $k'=(1-k^2)^{1/2}$. Using these three formulae, it is easy to verify that the third of the Eqs. (2.14) may be represented in the form

$$(2.16)\quad \sigma_{zz} = 2\,G\vartheta_0\, a\left[I_3(a,r) - z\frac{\partial}{\partial a} I_1 - \frac{\eta(a-r)}{a}\right].$$

Since

$$\int_0^\infty e^{-\alpha z}\, \alpha^{-1} J_1(\alpha a)\, J_0(\alpha r)\, d\alpha = \frac{1}{2}\,[aI_3(r,a) + rI_3(a,r) - zI_2],$$

we obtain

$$(2.17)\quad \begin{cases} \sigma_{rr} = 2\,G\vartheta_0\, a\left\{I_3(a,r) + z\dfrac{\partial}{\partial a} I_1 - \dfrac{1-2\nu}{2r}\,[aI_3(r,a)+rI_3(a,r)-zI_2] + \right. \\ \qquad \left. + \dfrac{z}{r} I_2 - \dfrac{1}{2a}\left|\begin{matrix} 1 & \text{if} & r<a \\ \dfrac{a^2}{r^2} & \text{if} & r>a \end{matrix}\right|\right\}, \\ \sigma_{\varphi\varphi} = 2\,G\vartheta_0\, a\left\{2\,\nu I_3(a,r) + \dfrac{1-2\nu}{2r}\,[aI_3(r,a)+rI_3(a,r)-zI_2] - \right. \\ \qquad \left. - \dfrac{z}{r} I_2 - \dfrac{1}{2a}\left|\begin{matrix} 1 & \text{if} & r<a \\ -\dfrac{a^2}{r^2} & \text{if} & r>a \end{matrix}\right|\right\}. \end{cases}$$

The stress σ_{rz} may be represented in the form

$$(2.18)\quad \sigma_{rz} = -2\,G\vartheta_0\, z^2\,\pi^{-1} a^{-1/2} r^{-3/2}\, \Theta'_{1/2}\left(\frac{r^2+a^2+z^2}{2\,ar}\right),$$

where $Q_{1/2}\,[(r^2+a^2+z^2)/2\,ar]$ is Legendre's function of the second kind with script 1/2.

The prime in Legendre's function of Eq. (2.18) denotes the differential after the argument in parentheses.

The state of stress at infinity ($z \to \infty$) is a plane state of deformation since $\sigma_{rz} \to 0$. For the remaining stress components we have

$$
(2.19) \quad \begin{cases}
\sigma_{rr}(r, \infty) = -2 G\vartheta_0 a \int\limits_0^\infty J_1(\alpha a) J_1(\alpha r)(\alpha r)^{-1} d\alpha = \\
\qquad\qquad = -G\vartheta_0 \begin{cases} 1 & \text{for} \quad 0 < r < a \\ \dfrac{a^2}{r^2} & \text{for} \quad a < r < \infty \end{cases} \\
\sigma_{\varphi\varphi}(r, \infty) = -2 G\vartheta_0 a \int\limits_0^\infty J_1(\alpha a) \left[J_0(\alpha r) - \dfrac{J_1(\alpha r)}{\alpha r} \right] d\alpha = \\
\qquad\qquad = -G\vartheta_0 \begin{cases} 1 & \text{for} \quad 0 < r < a \\ -\dfrac{a^2}{r^2} & \text{for} \quad a < r < \infty \end{cases} \\
\sigma_{zz}(r, \infty) = -2 G\vartheta_0 a \int\limits_0^\infty J_0(\alpha r) J_1(\alpha a) d\alpha = \\
\qquad\qquad = -2 G\vartheta_0 \begin{cases} 1 & \text{for} \quad 0 < r < a \\ 0 & \text{for} \quad a < r < \infty. \end{cases}
\end{cases}
$$

Let us observe that

$$
\sigma_{rr}(r, \infty) + \sigma_{\varphi\varphi}(r, \infty) + \sigma_{zz}(r, \infty) = -4 G\vartheta_0 \begin{cases} 1 \\ 0 \end{cases}.
$$

For $z = 0$, we obtain

$$
(2.20) \quad \begin{cases}
\sigma_{rr}(r, 0) = 2 G\vartheta_0 \begin{cases} \nu & \text{for} \quad 0 < r < a \\ -(1-\nu)\dfrac{a^2}{r^2} & \text{for} \quad a < r < \infty \end{cases} \\
\sigma_{\varphi\varphi}(r, 0) = 2 G\vartheta_0 \begin{cases} \nu \\ (1-\nu)\dfrac{a^2}{r^2} \end{cases} \\
\sigma_{zz}(r, 0) = 0, \quad \sigma_{rz}(r, 0) = 0 \quad \begin{matrix} \text{for} \quad 0 < r < a \\ \text{for} \quad a < r < \infty. \end{matrix}
\end{cases}
$$

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