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APPLIED MECHANICS

(ELASTICITY)

Generation of Waves in an Infinite Micropolar Elastic Solid Body. II.

by

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1. Introduction

In our previous paper [1] we were concerned with the generation of waves in an infinite micropolar elastic medium under the effect of body forces and body couples.

In what follows we shall consider the case of axi-symmetric deformation of the body. The field of displacements **u** and rotations **ω** is characterized by the axi-symmetry with respect to the z-axis.

The starting point for our considerations will be the equations of motion given in [2]—[4].

$$(\mu+\alpha)\,\nabla^2\mathbf{u}+(\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,\mathbf{u}+2\alpha\,\mathrm{rot}\,\boldsymbol{\omega}+\mathbf{X}=\varrho\ddot{\mathbf{u}}, \tag{1.1}$$

$$(\gamma+\varepsilon)\,\nabla^2\boldsymbol{\omega}+(\gamma+\beta-\varepsilon)\,\mathrm{grad\,div}\,\boldsymbol{\omega}-4\alpha\boldsymbol{\omega}+2\alpha\,\mathrm{rot}\,\mathbf{u}+\mathbf{Y}=J\ddot{\boldsymbol{\omega}}. \tag{1.2}$$

In the above equations the following notations were used: **u** denotes the displacement vector, **ω** stands for the rotation vector, **X** for the vector of body forces, **Y** for the body couples, $\mu, \lambda, \alpha, \beta, \gamma, \varepsilon$ denote the material constants, ϱ — the density and J the rotational inertia. The functions **u**, **ω**, **X**, **Y** are functions of position **x** and time t.

Within the system of cylindrical coordinates (r, φ, z) — assuming the independence of all causes and effects of the angle φ — we arrive at two systems of equations independent of each other, namely

$$\begin{aligned}
&(\mu+\alpha)\left(\nabla^2 u_r-\frac{u_r}{r^2}\right)+(\lambda+\mu-\alpha)\frac{\partial e}{\partial r}-2\alpha\frac{\partial\omega_\varphi}{\partial z}+X_r=\varrho\ddot{u}_r,\\
&(\mu+\alpha)\,\nabla^2 u_z+(\lambda+\mu-\alpha)\frac{\partial e}{\partial z}+\frac{2\alpha}{r^2}\,\frac{\partial}{\partial r}(r\omega_\varphi)+X_z=\varrho\ddot{u}_z,\\
&(\gamma+\varepsilon)\left(\nabla^2\omega_\varphi-\frac{\omega_\varphi}{r^2}\right)-4\alpha\omega_\varphi+2\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)+Y_\varphi=J\ddot{\omega}_\varphi,
\end{aligned} \tag{1.3}$$

and

$$(\gamma+\varepsilon)\left(\nabla^2\omega_r-\frac{\omega_r}{r^2}\right)-4\alpha\omega_r+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial r}-2\alpha\frac{\partial u_\varphi}{\partial z}+Y_r=J\ddot{\omega}_r,$$

$$(1.4)\qquad (\gamma+\varepsilon)\,\nabla^2\,\omega_z-4\alpha\omega_z+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial z}+\frac{2\alpha}{r}\,\frac{\partial}{\partial r}(ru_\varphi)+Y_z=J\ddot{\omega}_z,$$

$$(\alpha+\mu)\left(\nabla^2u_\varphi-\frac{u_\varphi}{r^2}\right)+2\alpha\left(\frac{\partial\omega_r}{\partial z}-\frac{\partial\omega_z}{\partial r}\right)+X_\varphi=\varrho\ddot{u}_\varphi.$$

The following notations were used in Eqs. (1.3) and (1.4)

$$\mathbf{u}\equiv(u_r,u_\varphi,u_z),\quad \boldsymbol{\omega}\equiv(\omega_r,\omega_\varphi,\omega_z),\quad \mathbf{X}\equiv(X_r,X_\varphi,X_z),$$
$$\mathbf{Y}\equiv(Y_r,Y_\varphi,Y_z),\quad e=\frac{1}{r}\,\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z},\quad \varkappa=\frac{1}{r}\,\frac{\partial}{\partial r}(r\omega_r)+\frac{\partial\omega_z}{\partial z}.$$

The displacements and rotations appearing in Eqs. (1.3) will be now expressed by the potentials Φ,Ψ,Γ

$$(1.5)\quad u_r=\frac{\partial\Phi}{\partial r}+\frac{\partial^2\Psi}{\partial r\partial z},\quad u_z=\frac{\partial\Phi}{\partial z}-\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right),\quad \omega_\varphi=-\frac{\partial\Gamma}{\partial r},$$

while the body forces and body couples will be decomposed into the potential and solenoidal parts

$$(1.6)\qquad X_r=\varrho\left(\frac{\partial\vartheta}{\partial r}-\frac{\partial\chi_\varphi}{\partial z}\right),\quad X_z=\varrho\left(\frac{\partial\vartheta}{\partial z}+\frac{1}{r}\,\frac{\partial}{\partial r}\,(r\chi_\varphi)\right).$$

Substituting (1.5) and (1.6) into Eqs. (1.3) we obtain the following system of wave equations

$$\left(\nabla^2-\frac{1}{c_1^2}\,\partial_t^2\right)\Phi+\frac{1}{c_1^2}\,\vartheta=0,$$
$$(1.7)\qquad -\frac{\partial}{\partial r}\left[\left(\nabla^2-\frac{1}{c_2^2}\,\partial_t^2\right)\Psi+p\Gamma\right]+\frac{1}{c_2^2}\,\chi_\varphi=0,$$
$$-\frac{\partial}{\partial r}\left[\left(\nabla^2-\nu^2-\frac{1}{c_4^2}\,\partial_t^2\right)\Gamma-s\nabla^2\,\Psi\right]+\frac{Y_\varphi}{Jc_4^2}=0.$$

where

$$c_1=\left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2},\quad c_2=\left(\frac{\mu+\alpha}{\varrho}\right)^{1/2},\quad c_4=\left(\frac{\gamma+\varepsilon}{J}\right)^{1/2},$$
$$p=\frac{2\alpha}{\mu+\alpha},\quad s=\frac{2\alpha}{\gamma+\varepsilon},\quad \nu^2=\frac{4\alpha}{\gamma+\varepsilon}.$$

The displacements and rotations appearing in Eqs. (1.4) will be expressed by the potentials φ,ψ,Ω

$$(1.8)\quad \omega_r=\frac{\partial\varphi}{\partial r}+\frac{\partial^2\psi}{\partial r\partial z},\quad \omega_z=\frac{\partial\varphi}{\partial z}-\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right),\quad u_\varphi=-\frac{\partial\Omega}{\partial r},$$

The body couples will be decomposed into the potential and solenoidal parts

$$Y_r = J\left(\frac{\partial\sigma}{\partial r} - \frac{\partial\eta_\varphi}{\partial z}\right), \quad Y_z = J\left(\frac{\partial\sigma}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\eta_\varphi)\right). \tag{1.9}$$

Introducing (1.8) and (1.9) jnto Eqs. (1.4), we obtain the following system of wave equations:

$$\begin{aligned} &\left(\nabla^2 - \nu_0^2 - \frac{1}{c_3^2}\partial_t^2\right)\varphi + \frac{1}{c_3^2}\sigma = 0, \\ &-\frac{\partial}{\partial r}\left[\left(\nabla^2 - \nu^2 - \frac{1}{c_4^2}\partial_t^2\right)\psi + s\Omega\right] + \frac{1}{c_4^2}\eta_\varphi = 0, \\ &-\frac{\partial}{\partial r}\left[\left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right)\Omega - p\nabla^2\psi\right] + \frac{1}{\varrho c_2^2}X_\varphi = 0. \end{aligned} \tag{1.10}$$

Here we have

$$\nu_0^2 = \frac{4\alpha}{2\gamma+\beta}, \quad c_3 = \left(\frac{\beta+2\gamma}{J}\right)^{1/2}.$$

In the next section we shall give the general solution of wave equations (1.7) and (1.10) reccurring to the Fourier—Hankel integral transformation.

2. General solution of wave equations

The Fourier—Hankel integral transformation used in solving the system of wave equations (1.7) has the form [5]

$$\begin{aligned} \tilde{\Phi}(\eta, \zeta, \omega) &= \frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{i(\zeta z+\omega t)}\,dz\,dt \int_0^\infty r\,\Im_0(\eta r)\,\Phi(r, z, t)\,dr, \\ \Phi(r, z, t) &= \frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\zeta\,d\omega \int_0^\infty \eta\,\Im_0(\eta r)\,\tilde{\Phi}(\eta, \zeta, \omega)\,d\eta. \end{aligned} \tag{2.1}$$

Similar expressions may be obtained for the functions Ψ and Γ. Performing integral transformation on Eq. (1.7) we obtain a system of algebraic equations. The solution of this system gives the transforms $\tilde{\Phi}$, $\tilde{\Psi}$, $\tilde{\Gamma}$

$$\begin{aligned} \tilde{\Phi} &= \frac{1}{c_1^2}\frac{\tilde{\vartheta}}{\alpha^2 - \sigma_1^2}, \\ \tilde{\Psi} &= \frac{1}{\eta\Delta}\left[\frac{(\alpha^2+\nu^2-\sigma_4^2)}{c_2^2}\tilde{\chi}_\varphi + \frac{p\tilde{Y}_\varphi}{Jc_4^2}\right], \\ \tilde{\Gamma} &= \frac{1}{\eta\Delta}\left[\frac{\alpha^2 s\tilde{\chi}_\varphi}{c_2^2} + \frac{(\alpha^2-\sigma_2^2)}{Jc_4^2}\tilde{Y}_\varphi\right], \quad \alpha^2 = \zeta^2+\eta^2. \end{aligned} \tag{2.2}$$

In the above equation there is

$$\Delta = (\alpha^2 - \lambda_1^2)(\alpha^2 - \lambda_2^2),$$

where

$$\lambda_{1,2}^2 = \frac{1}{2}\left(\sigma_2^2+\sigma_4^2+\xi^2-\nu^2 \mp \sqrt{(\sigma_2^2-\sigma_4^2-\xi^2+\nu^2)^2+4\sigma_2^2\xi^2}\right),$$

$$\xi^2 = \frac{4\alpha^2}{(\alpha+\mu)(\gamma+\varepsilon)}, \quad \sigma_2 = \frac{\omega}{c_2}, \quad \sigma_4 = \frac{\omega}{c_4}.$$

Let us perform the Fourier–Hankel integral transformation on the relations (1.5) and (1.6). We assume that

$$(2.3) \quad \begin{aligned} (\tilde{u}_r, \tilde{X}_r, \tilde{\chi}_\varphi) &= \frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{i(\xi z+\omega t)}\, dz\, dt \int_0^\infty r\, \Im_1(\eta r)\,(u_r, X_r, \chi_\varphi)\, dr, \\ (\tilde{\vartheta}, \tilde{X}_z) &= \frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{i(\xi z+\omega t)}\, dz\, dt \int_0^\infty r\, \Im_0(\eta r)\,(\vartheta, X_z)\, dr. \end{aligned}$$

Under these assumptions we get

$$(2.4) \quad \tilde{u}_r = -\eta\tilde{\Phi}+i\zeta\eta\tilde{\Psi}, \quad \tilde{u}_z = -i\zeta\tilde{\Phi}+\eta^2\tilde{\Psi}, \quad \tilde{\omega}_\varphi = \eta\tilde{\Gamma},$$

$$(2.5) \quad X_r = -\varrho\eta\tilde{\vartheta}+\varrho i\zeta\tilde{\chi}_\varphi, \quad X_z = -\varrho i\zeta\tilde{\vartheta}+\varrho\eta\tilde{\chi}_\varphi.$$

From the relations (2.5) we have

$$(2.6) \quad \tilde{\vartheta} = \frac{1}{\varrho\boldsymbol{\alpha}^2}(i\zeta\tilde{X}_z-\eta\tilde{X}_r), \quad \tilde{\chi}_\varphi = \frac{1}{\varrho\boldsymbol{\alpha}^2}(\eta\tilde{X}_z-i\zeta\tilde{X}_r).$$

Introducing (2.2) into (2.4) and making use of (2.6) we obtain the transforms of the quantitites $\tilde{u}_r$, $\tilde{u}_z$, $\tilde{\omega}_\varphi$ expressed by the transforms $\tilde{X}_r$, $\tilde{X}_z$, $\tilde{Y}_\varphi$. Performing the inverse Fourier–Hankel transformation we obtain finally

$$(2.7) \quad \begin{aligned} u_r = -\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\xi z+\omega t)}\, d\zeta\, d\omega \int_0^\infty \eta\, \Im_1(\eta r) \Bigg\{ &\frac{\eta(i\zeta\tilde{X}_z-\eta\tilde{X}_r)}{\varrho c_1^2(\boldsymbol{\alpha}^2-\sigma_1^2)\boldsymbol{\alpha}^2} - \\ &-\frac{i\zeta}{\Delta}\left[\frac{(\boldsymbol{\alpha}^2+\nu^2-\sigma_4^2}{\varrho c_2^2\boldsymbol{\alpha}^2}(\eta\tilde{X}_z-i\zeta\tilde{X}_r)+\frac{p\tilde{Y}_\varphi}{Jc_4^2}\right]\Bigg\}\, d\eta, \end{aligned}$$

$$(2.8) \quad \begin{aligned} u_z = -\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\xi z+\omega t)}\, d\zeta\, d\omega \int_0^\infty \eta\, \Im_0(\eta r) \Bigg\{ &\frac{i\zeta(i\zeta\tilde{X}_z-\eta\tilde{X}_r)}{\varrho c_1^2\boldsymbol{\alpha}^2(\boldsymbol{\alpha}^2-\sigma_1^2)} - \\ &-\frac{\eta}{\Delta}\left[\frac{(\boldsymbol{\alpha}^2+\nu^2-\sigma_4^2)}{\varrho\boldsymbol{\alpha}^2 c_2^2}(\eta\tilde{X}_z-i\zeta\tilde{X}_r)+\frac{p\tilde{Y}_\varphi}{Jc_4^2}\tilde{Y}_\varphi\right]\Bigg\}\, d\eta, \end{aligned}$$

$$(2.9) \quad \omega_\varphi = \frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\xi z+\omega t)}\, d\zeta\, d\omega \int_0^\infty \frac{\eta\, \Im_1(\eta r)}{\Delta}\left[\frac{s(\eta\tilde{X}_z-i\zeta\tilde{X}_r)}{\varrho c_2^2}+\frac{(\boldsymbol{\alpha}^2-\sigma_2^2)\tilde{Y}_\varphi}{Jc_4^2}\right] d\eta.$$

Knowing the displacements and rotations we are able to determine the components of the tensor of stresses σ_{ji} and the tensor of couple stresses μ_{ji} from the formulae as below

$$
\begin{aligned}
&\sigma_{rr} = 2\mu \frac{\partial u_r}{\partial r} + \lambda e, \qquad \sigma_{\varphi\varphi} = 2\mu \frac{u_r}{r} + \lambda e, \qquad \sigma_{zz} = 2\mu \frac{\partial u_z}{\partial z} + \lambda e, \\
&\sigma_{rz} = \mu \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) - a \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) + 2a\omega_\varphi, \\
&\sigma_{zr} = \mu \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) + a \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) - 2a\omega_\varphi, \\
&\mu_{r\varphi} = \gamma \left(\frac{\partial \omega_\varphi}{\partial r} - \frac{\omega_\varphi}{r} \right) + \varepsilon \left(\frac{\partial \omega_\varphi}{\partial r} + \frac{\omega_\varphi}{r} \right), \\
&\mu_{\varphi r} = \gamma \left(\frac{\partial \omega_\varphi}{\partial r} - \frac{\omega_\varphi}{r} \right) - \varepsilon \left(\frac{\partial \omega_\varphi}{\partial r} + \frac{\omega_\varphi}{r} \right), \\
&\mu_{\varphi z} = (\gamma - \varepsilon) \frac{\partial \omega_\varphi}{\partial z}, \qquad \mu_{z\varphi} = (\gamma + \varepsilon) \frac{\partial \omega_\varphi}{\partial z}.
\end{aligned} \tag{2.10}
$$

Let us now consider a particular case — $a = 0$ —, where the Eqs. (1.1) and (1.2) become independent of each other. From Eqs. (2.7)—(2.9) we have

$$
\begin{aligned}
u_r &= \frac{1}{2\pi\varrho c_1^2} \iint_{-\infty}^{\infty} e^{-i(\xi z + \omega t)} \, d\omega \, d\zeta \int_0^\infty \eta \, \Im_1(\eta r) \times \\
&\quad \times \left[\frac{(\eta^2 + \delta^2 \zeta^2 - \omega^2/c_2^2) \tilde{X}_r + i\zeta\eta (\delta^2 - 1) \tilde{X}_z}{(\alpha^2 - \omega^2/c_1^2)(\alpha^2 - \omega^2/c_2^2)} \right] d\eta, \\
u_z &= \frac{1}{2\pi\varrho c_1^2} \iint_{-\infty}^{\infty} e^{-i(\xi z + \omega t)} \, d\omega \, d\zeta \int_0^\infty \eta \, \Im_0(\eta r) \times \\
&\quad \times \left[\frac{(\zeta^2 + \eta^2 \delta^2 - \omega^2/c_2^2) \tilde{X}_z + i\zeta\eta (\delta^2 - 1) \tilde{X}_r}{(\alpha^2 - \omega^2/c_1^2)(a^2 - \omega^2/c_2^2)} \right] d\eta, \\
\omega_\varphi &= \frac{1}{2\pi J c_4^2} \iint_{-\infty}^{\infty} e^{-i(\xi z + \omega t)} \, d\zeta \, d\omega \int_0^\infty \eta \, \Im_1(\eta r) \frac{\tilde{Y}_\varphi}{\alpha^2 - c_4^2} \, d\eta, \qquad \delta = \frac{c_1}{c_2}.
\end{aligned} \tag{2.11}
$$

The formulae $(2.11)_1$ and $(2.11)_2$ refer to the classical elastic medium [6], while $(2.11)_3$ to a hypothetical medium, wherein only rotations may appear.

Let us now consider the system of wave equations (1.10). Performing on these equations the integral transformation and taking into account that

$$
\begin{aligned}
(\tilde{\varphi}, \tilde{\psi}, \tilde{\Omega}, \tilde{\sigma}) &= \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{i(\zeta z + \omega t)} dz \, dt \int_0^\infty r \, \Im_0(\eta r) (\varphi, \psi, \Omega, \sigma) \, dr, \\
(\tilde{X}_\varphi, \tilde{\eta}_\varphi) &= \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{i(\zeta z + \omega t)} \, dz \, dt \int_0^\infty r \, \Im_1(\eta r) (X_\varphi, \eta_\varphi) \, dr,
\end{aligned} \tag{2.12}
$$

we arrive at the following quantities

$$(2.13)\quad \tilde{\varphi} = \frac{1}{c_3^2} \frac{\tilde{\sigma}}{(\alpha^2+\nu_0^2-\sigma_3^2)}, \qquad \sigma_3 = \frac{\omega}{c_3}, \qquad \tilde{\psi} = \frac{1}{\eta\Delta}\left(\frac{s\tilde{X}_\varphi}{c_2^2\varrho} + \frac{(\alpha^2-\sigma_2^2)}{c_4^2}\tilde{\eta}_\varphi\right),$$

$$\tilde{\Omega} = \frac{1}{\eta\Delta}\left(\frac{\alpha^2+\nu^2-\sigma_4^2}{c_2^2\varrho}\tilde{X}_\varphi - \frac{p\alpha^2}{c_4^2}\tilde{\eta}_\varphi\right).$$

Let us perform also the Fourier–Hankel transformation on the relations (1.8) and (1.9)

$$(2.14)\qquad \tilde{\omega}_r = -\eta\tilde{\varphi}+i\zeta\eta\tilde{\psi}, \quad \tilde{\omega}_z = -i\zeta\tilde{\varphi}+\eta^2\tilde{\psi}, \quad u_\varphi = \eta\tilde{\Omega},$$

$$(2.15)\qquad \tilde{Y}_r = -J(\eta\tilde{\sigma}-i\zeta\tilde{\eta}_\varphi), \quad \tilde{Y}_z = -J(i\zeta\tilde{\sigma}-\eta\tilde{\eta}_\varphi).$$

It results from the relations (2.15) that

$$(2.16)\qquad \tilde{\sigma} = \frac{1}{J\alpha^2}(i\zeta\tilde{Y}_z-\eta\tilde{Y}_r), \qquad \tilde{\eta}_\varphi = \frac{1}{J\alpha^2}(\eta\tilde{Y}_z-i\zeta\tilde{Y}_r).$$

Substituting (2.2) into (2.14) and bearing in mind (2.16), we obtain the transforms $\tilde{\omega}_r, \tilde{\omega}_z, \tilde{u}_\varphi$. Performing now the inverse Fourier–Hankel transformation, we get

$$(2.17)\quad \omega_r = -\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\omega\,d\zeta\int_0^\infty \eta\,\mathfrak{I}_1(\eta r)\left\{\frac{\eta(i\zeta\tilde{Y}_z-\eta\tilde{Y}_r)}{c_3^2J\alpha^2(\alpha^2+\nu_0^2-\sigma_3^2)} - \frac{i\zeta}{\Delta}\left[\frac{\alpha^2-\sigma_2^2}{Jc_4^2\alpha^2}(\eta\tilde{Y}_z-i\zeta\tilde{Y}_r)+\frac{s\tilde{X}_\varphi}{c_2^2\varrho}\right]\right\}d\eta,$$

$$(2.18)\quad \omega_z = -\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\zeta z-\omega t)}\,d\omega\,d\zeta\int_0^\infty \eta\,\mathfrak{I}_0(\eta r)\left\{\frac{i\zeta(i\varrho\tilde{Y}_z-\eta\tilde{Y}_r)}{c_3^2J\alpha^2(\alpha^2+\nu_0^2-\sigma_3^2)} - \frac{\eta}{\Delta}\left[\frac{\alpha^2-\sigma_2^2}{Jc_4^2\alpha^2}(\eta\tilde{Y}_z-i\zeta\tilde{Y}_r)+\frac{s\tilde{X}_\varphi}{c_2^2\varrho}\right]\right\}d\eta,$$

$$(2.19)\quad u_\varphi = \frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\omega\,d\zeta\int_0^\infty \frac{\eta\mathfrak{I}_1(\eta r)}{\Delta}\left\{\frac{\alpha^2+\nu_0^2-\sigma_4^2}{c_2^2\varrho}\tilde{X}_\varphi - \frac{p}{Jc_4^2}(\eta\tilde{Y}_z-i\zeta\tilde{Y}_r)\right\}d\eta.$$

Now, the rotations ω_r, ω_z and the displacement u_φ being known we may determine the stresses σ_{ji} and the couple stresses μ_{ji} from the formulae

$$(2.20)\quad \begin{aligned} &\mu_{rr} = 2\gamma\frac{\partial\omega_r}{\partial r}+\beta\varkappa, \quad \mu_{\varphi\varphi} = 2\gamma\frac{\omega_r}{r}+\beta\varkappa, \quad \mu_{zz} = 2\gamma\frac{\partial\omega_z}{\partial z}+\beta\varkappa, \\ &\mu_{rz} = \gamma\left(\frac{\partial\omega_z}{\partial r}+\frac{\partial\omega_r}{\partial z}\right)+\varepsilon\left(\frac{\partial\omega_r}{\partial z}-\frac{\partial\omega_z}{\partial r}\right), \end{aligned}$$

$$
\begin{aligned}
\mu_{zr} &= \gamma\left(\frac{\partial\omega_z}{\partial r}+\frac{\partial\omega_r}{\partial z}\right)-\varepsilon\left(\frac{\partial\omega_r}{\partial z}-\frac{\partial\omega_z}{\partial r}\right),\\
\sigma_{r\varphi} &= \mu\left(\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}\right)+\frac{\alpha}{r}\frac{\partial}{\partial r}(u_\varphi r)-2\alpha\omega_z,\\
\sigma_{\varphi r} &= \mu\left(\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}\right)-\frac{\alpha}{r}\frac{\partial}{\partial r}(u_\varphi r)+2\alpha\omega_z,\\
\sigma_{\varphi z} &= \mu\frac{\partial u_\varphi}{\partial z}-\frac{\alpha}{r}\frac{\partial}{\partial z}(ru_\varphi)-2\alpha\omega_r,\\
\sigma_{z\varphi} &= \mu\frac{\partial u_\varphi}{\partial z}+\frac{\alpha}{r}\frac{\partial}{\partial z}(ru_\varphi)+2\alpha\omega_r.
\end{aligned}
\tag{2.20}
$$

In the particular case of classical elastokinetics, i.e. for $\alpha \to 0$ we obtain from (2.17)—(2.19)

$$
\begin{aligned}
\omega_r &= -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\zeta\,d\omega\int\limits_0^\infty \eta\,\mathfrak{J}_1(\eta r)\left[\frac{\eta(i\zeta\tilde{Y}_z-\eta\tilde{Y}_r)}{Jc_3^2\alpha^2(\alpha^2-\sigma_3^2)}-\right.\\
&\qquad\left.-\frac{i\zeta(\eta\tilde{Y}_z-i\zeta\tilde{Y}_r)}{Jc_4^2\alpha^2(\alpha^2-\sigma_4^2)}\right]d\eta,\\
\omega_z &= -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\zeta\,d\omega\int\limits_0^\infty \eta\,\mathfrak{J}_0(\eta r)\left[\frac{i\zeta(i\zeta\tilde{Y}_z-\eta\tilde{Y}_r)}{Jc_3^2\alpha^2(\alpha^2-\sigma_3^2)}-\right.\\
&\qquad\left.-\frac{\eta(\eta\tilde{Y}_z-i\zeta\tilde{Y}_r)}{Jc_4^2\alpha^2(\alpha^2-\sigma_4^2)}\right]d\eta,\\
u_\varphi &= \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}\,d\zeta\,d\omega\int\limits_0^\infty \eta\,\mathfrak{J}_1(\eta r)\frac{\tilde{X}_\varphi}{c_2^2\varrho(\alpha^2-\hat{\sigma}_2^2)}\,d\eta,\\
\hat{\sigma}_2 &= \left(\frac{\mu}{\varrho}\right)^{1/2}.
\end{aligned}
\tag{2.21}
$$

Formula $(2.21)_3$ refers to the classical elastic medium, while the formulae $(2.21)_1$ and $(2.21)_2$ to a hypothetical medium, wherein, only the rotations may appear.

Similarly, as it was shown in [6], one may derive the formulae describing the displacements and rotations in the static problem as well as for vibrations changing harmonically in time.

A more ample discussion of the problems briefly presented in this paper will be published in Proceedings of Vibration Problems.

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В. НОВАЦКИЙ и В. К. НОВАЦКИЙ, **ГЕНЕРИРОВАНИЕ ВОЛН В БЕСКОНЕЧНОЙ МИКРОПОЛЯРНОЙ УПРУГОЙ СРЕДЕ.** II.

В работе дается решение проблемы пропагации волн в бесконечной микрополярной упругой среде, возникших под воздействием массовых сил и моментов. Определено поле перемещений, оборотов и поле напряжений и моментовых напряжений для случая осе-симметрической деформации тела.