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APPLIED MECHANICS
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Propagation of Elastic Waves in a Micropolar Cylinder. I

by

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1. Introduction

In this paper we shall be concerned with the problem of propagation of longitudinal monochromatic waves in an infinite cylinder made of an isotropic, homogeneous and centrosymmetric material. Our considerations will be conducted within the framework of the asymmetric theory of elasticity for the Cosserat medium, the deformation of which is described by six independent functions: three components of the displacement vector and three components of the rotation vector.

The problem of propagation of the longitudinal wave in a cylinder was discussed — within the framework of the classical theory of elasticity — by L. Pochhammer [1] and C. Chree [2].

The present paper may be regarded as a generalization of the results obtained by these two authors on the micropolar elastic medium. We derived also transcendental equations permitting to determine the phase velocities of propagating waves. Particular cases of small and large wavelengths with respect to the cylinder radius are considered also. Our considerations are completed with a discussion of the problem of how to pass to the solutions within the framework of classical elastokinetics.

2. Basic equations

Equations of micropolar elastic medium will constitute the starting point of our considerations [3]—[5]. We shall consider an isotropic homogeneous and centrosymmetric body. Under the effect of external loadings a displacement field $\mathbf{u}(\mathbf{x}, t)$ will form in the body and an independent rotation field $\boldsymbol{\omega}(\mathbf{x}, t)$. The state of strain is described by two asymmetric tensors: tensor of strain γ_{ji} and the curvature twist tensor $\varkappa_{ji}$. Here we have

$$\gamma_{ji} = u_{i,j} - \epsilon_{kji}\,\omega_k, \quad \varkappa_{ji} = \omega_{i,j}. \tag{2.1}$$

The state of stress is defined by two asymmetric tensors: tensor of stress σ_{ij} and the couple stress tensor μ_{ij}. The relation between the state of stress and that of strain is described by the following equations:

$$\sigma_{ji} = (\mu+\alpha)\gamma_{ji} + (\mu-\alpha)\gamma_{ij} + \lambda\gamma_{kk}\delta_{ij}, \tag{2.2}$$

$$\mu_{ji} = (\gamma+\varepsilon)\varkappa_{ji} + (\gamma-\varepsilon)\varkappa_{ij} + \beta\varkappa_{kk}\delta_{ij}. \tag{2.3}$$

The symbols $\mu, \lambda, \beta, \gamma, \varepsilon, \alpha$ denote material constants. Substituting Eqs. (2.2) and (2.3) into the equations of motion

$$\sigma_{ji,j} + X_i = \varrho\ddot{u}_i, \tag{2.4}$$

$$\epsilon_{ijk}\sigma_{jk} + \mu_{ji,j} + Y_i - J\ddot{\omega}_i = 0 \tag{2.5}$$

and expressing the quantities γ_{ji} and $\varkappa_{ji}$ by the displacements u_i and rotations ω_i — — according to Eq. (2.1) — we arrive at a system of six differential equations which we write in the vector form

$$(\mu+\alpha)\nabla^2\mathbf{u} + (\lambda+\mu-\alpha)\,\mathrm{grad\,div}\,\mathbf{u} + 2\alpha\,\mathrm{rot}\,\boldsymbol{\omega} + \mathbf{X} = \varrho\ddot{\mathbf{u}}, \tag{2.6}$$

$$(\gamma+\varepsilon)\nabla^2\boldsymbol{\omega} + (\beta+\gamma-\varepsilon)\,\mathrm{grad\,div}\,\boldsymbol{\omega} - 4\alpha\boldsymbol{\omega} + 2\alpha\,\mathrm{rot}\,\mathbf{u} + \mathbf{Y} = J\ddot{\boldsymbol{\omega}}. \tag{2.7}$$

In the above equations $\mathbf{X}$ denotes the vector of body forces, $\mathbf{Y}$ stands for the vector of body couples, ϱ denotes density, J — rotational inertia. The time derivative of the functions $\mathbf{u}, \boldsymbol{\omega}$ is marked by a point.

In the sequel it will be more convenient to present the system of Eqs. (2.6) and (2.7) in the framework of cylinder coordinates (r, φ, z). Thus we have:

$$\begin{aligned}
&(\mu+\alpha)\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial\varphi}\right) + (\lambda+\mu-\alpha)\frac{\partial e}{\partial r} + \\
&\qquad + 2\alpha\left(\frac{1}{r}\frac{\partial\omega_z}{\partial\varphi} - \frac{\partial}{\partial z}(r\omega_\varphi)\right) + X_r = \varrho\ddot{u}_r, \\
&(\mu+\alpha)\left((\nabla^2 u_\varphi - \frac{u_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\varphi}\right) + (\lambda+\mu-\alpha)\frac{1}{r}\frac{\partial e}{\partial\varphi} + \\
&\qquad + 2\alpha\left(\frac{\partial\omega_r}{\partial z} - \frac{\partial\omega_z}{\partial r}\right) + X_\varphi = \varrho\ddot{u}_\varphi, \\
&(\mu+\alpha)\nabla^2 u_z + (\lambda+\mu-\alpha)\frac{\partial e}{\partial z} + 2\alpha\frac{1}{r}\left[\frac{\partial}{\partial r}(r\omega_\varphi) - \frac{\partial\omega_r}{\partial\varphi}\right] + X_z = \varrho\ddot{u}_z,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&(\gamma+\varepsilon)\left(\nabla^2\omega_r - \frac{\omega_r}{r^2} - \frac{2}{r^2}\frac{\partial\omega_\varphi}{\partial\varphi}\right) - 4\alpha\omega_r + (\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial r} + \\
&\qquad + 2\alpha\left(\frac{1}{r}\frac{\partial u_z}{\partial\varphi} - \frac{\partial u_\varphi}{\partial z}\right) + Y_r = J\ddot{\omega}_r, \\
&(\gamma+\varepsilon)\left(\nabla^2\omega_\varphi - \frac{\omega_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial\omega_r}{\partial\varphi}\right) - 4\alpha\omega_\varphi + (\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{r\partial\varphi} + \\
&\qquad + 2\alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + Y_\varphi = J\ddot{\omega}_\varphi, \\
&(\gamma+\varepsilon)\nabla^2\omega_z - 4\alpha\omega_z + (\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial z} + 2\alpha\frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\varphi) - \frac{\partial u_r}{\partial\varphi}\right) + Y_z = J\ddot{\omega}_z.
\end{aligned} \tag{2.9}$$

The following notations have been introduced

$$e = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z}, \qquad \varkappa = \frac{1}{r}\frac{\partial}{\partial r}(r\omega_r) + \frac{1}{r}\frac{\partial \omega_\varphi}{\partial \varphi} + \frac{\partial \omega_z}{\partial z}.$$

In the case of propagation of longitudinal wave in an infinite cylinder with circular section we put $u_\varphi = 0$, $\omega_r = \omega_z = 0$. We assume, moreover, that the quantities u_r, u_z, ω_φ do not depend on the angle φ. Thus, from the system of Eqs. (2.8), (2.9) it remains only the system of three equations

$$(2.10)\qquad \begin{aligned} &(\mu+\alpha)\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) + (\lambda+\mu-\alpha)\frac{\partial e}{\partial r} - 2\alpha\frac{\partial \omega_\varphi}{\partial z} = \varrho \ddot{u}_r,\\ &(\mu+\alpha)\nabla^2 u_z + (\lambda+\mu-\alpha)\frac{\partial e}{\partial z} + 2\alpha\frac{1}{r}\frac{\partial}{\partial r}(r\omega_\varphi) = \varrho \ddot{u}_z,\\ &(\gamma+\varepsilon)\left(\nabla^2 \omega_\varphi - \frac{\omega_\varphi}{r^2}\right) - 4\alpha\omega_\varphi + 2\alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) = J\ddot{\omega}_\varphi, \end{aligned}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \qquad e = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}.$$

To the displacement $\mathbf{u} = (u_r, 0, u_z)$ and rotation field $\boldsymbol{\omega} = (0, \omega_\varphi, 0)$ are subordinated the state of stress $\boldsymbol{\sigma}$ and the state of couple stress $\boldsymbol{\mu}$.

$$(2.11)\qquad \boldsymbol{\sigma} = \begin{vmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\varphi\varphi} & 0 \\ \sigma_{zr} & 0 & \sigma_{zz} \end{vmatrix}, \qquad \boldsymbol{\mu} = \begin{vmatrix} 0 & \mu_{r\varphi} & 0 \\ \mu_{\varphi r} & 0 & \mu_{\varphi z} \\ 0 & \mu_{z\varphi} & 0 \end{vmatrix},$$

where

$$(2.12)\qquad \begin{aligned} &\sigma_{rr} = 2\mu\frac{\partial u_r}{\partial r} + \lambda e, \qquad \sigma_{\varphi\varphi} = 2\mu\frac{u_r}{r} + \lambda e, \qquad \sigma_{zz} = 2\mu\frac{\partial u_z}{\partial z} + \lambda e,\\ &\sigma_{rz} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) + \alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + 2\alpha\omega_\varphi,\\ &\sigma_{zr} = \mu\left(\frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z}\right) - \alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) - 2\alpha\omega_\varphi,\\ &\mu_{r\varphi} = \gamma\left(\frac{\partial \omega_\varphi}{\partial r} - \frac{\omega_\varphi}{r}\right) + \varepsilon\left(\frac{\partial \omega_\varphi}{\partial r} + \frac{\omega_\varphi}{r}\right),\\ &\mu_{\varphi r} = \gamma\left(\frac{\partial \omega_\varphi}{\partial r} - \frac{\omega_\varphi}{r}\right) - \varepsilon\left(\frac{\partial \omega_\varphi}{\partial r} + \frac{\omega_\varphi}{r}\right),\\ &\mu_{\varphi z} = (\gamma-\varepsilon)\frac{\partial \omega_\varphi}{\partial z}, \qquad \mu_{z\varphi} = (\gamma+\varepsilon)\frac{\partial \omega_\varphi}{\partial z}. \end{aligned}$$

We introduce the potentials Φ, W connected with the displacements u_r, u_z by the relations

$$u_r = \frac{\partial \Phi}{\partial r} - \frac{\partial W}{\partial z}, \quad u_z = \frac{\partial \Phi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rW). \tag{2.13}$$

Introducing the above expressions into Eqs. (2.10), we obtain the following system of wave equations

$$\Box_1 \Phi = 0, \tag{2.14}$$

$$\overset{0}{\Box}_2 W + 2\alpha\omega_\varphi, \quad \overset{0}{\Box}_4 \omega_\varphi - 2\alpha\left(\nabla^2 W - \frac{W}{r^2}\right) = 0, \tag{2.15}$$

where

$$\Box_1 = (\lambda+2\mu)\nabla^2 - \varrho\partial_t^2, \quad \overset{0}{\Box}_2 = (\mu+\alpha)\left(\nabla^2 - \frac{1}{r^2}\right) - \varrho\partial_t^2,$$

$$\overset{0}{\Box}_4 = (\gamma+\varepsilon)\left(\nabla^2 - \frac{1}{r^2}\right) - 4\alpha - J\partial_t^2.$$

Introducing the functions Ψ, Γ such that

$$W = -\frac{\partial \Psi}{\partial r}, \quad \omega_\varphi = -\frac{\partial \Gamma}{\partial \varphi}, \tag{2.16}$$

we reduce Eqs. (2.14) and (2.15) to the simple form

$$\Box_1 \Phi = 0, \tag{2.17}$$

$$\Box_2 \Psi + 2\alpha\Gamma = 0 \quad \Box_4 \Gamma - 2\alpha\nabla^2\Psi = 0, \tag{2.18}$$

where

$$\Box_2 = (\mu+\alpha)\nabla^2 - \varrho\partial_t^2, \quad \Box_4 = (\gamma+\varepsilon)\nabla^2 - 4\alpha - J\partial_t^2.$$

Eqs. $(2.18)_1$ and $(2.18)_2$ are mutually coupled. Eliminating from these equations first Γ and then Ψ we obtain

$$(\Box_2\Box_4 + 4\alpha^2\nabla^2)(\Psi, \Gamma) = 0. \tag{2.19}$$

Eq. (2.17) describes longitudinal waves, while Eq. (2.19) the modified transverse and twist waves.

3. Propagation of longitudinal wave in a cylinder with circular section

Let us assume that in an infinite cylinder with circular section the longitudinal wave propagates with constant phase velocity c along the cylinder axis, i.e. along the z-axis .We assume, further, that it is a monochromatic wave. We write the functions Φ, Ψ, Γ in the form

$$(\Phi, \Psi, \Gamma) = \left(\Phi^*(r), \Psi^*(r), \Gamma^*(r)\right)e^{i(kz-\omega t)}, \quad c = \frac{\omega}{k}. \tag{3.1}$$

Substituting (3.1) into Eq. (2.17) and (2.18) we get the following solutions of those equations

(3.2) $$\Phi = A\mathfrak{J}_0(\sigma r)\, e^{i(kz-\omega t)},$$

(3.3) $$\Psi = [B\mathfrak{J}_0(\lambda_1 r)+C\mathfrak{J}_0(\lambda_2 r)]\, e^{i(kz-\omega t)},$$

(3.4) $$\Gamma = [E\mathfrak{J}_0(\lambda_1 r)+F\mathfrak{J}_0(\lambda_2 r)]\, e^{i(kz-\omega t)}.$$

The following notations have been introduced

$$\lambda_{1,2}^2 = -k^2+\frac{1}{2}\left(\sigma_2^2+\sigma_4^2-\nu^2+\eta^2\pm\sqrt{(\sigma_4^2-\sigma_2^2-\nu^2+\eta^2)^2+4\eta^2\sigma_2^2}\right),$$

$$\sigma = (\sigma_1^2-k^2)^{1/2},\quad \sigma_1=\frac{\omega}{c_1},\quad \sigma_2=\frac{\omega}{c_2},\quad \sigma_4=\frac{\omega}{c_4},$$

$$c_1=\left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2},\quad c_2=\left(\frac{\mu+\alpha}{\varrho}\right)^{1/2},\quad c_4=\left(\frac{\gamma+\varepsilon}{J}\right)^{1/2},$$

$$\nu^2=\frac{4\alpha}{\gamma+\varepsilon},\quad \eta^2=4\alpha^2/(\gamma+\varepsilon)(\alpha+\mu).$$

As the quantities λ_1^2, λ_2^2 have to be positive (what results from the requirement that the phase velocities be real and positive), there is $\omega^2>\frac{4\alpha}{J}$. The expressions (3.3) and (3.4) represent dispersion waves, λ_1, λ_2 being dependent on ω.

The constants B an E as well as C and F are coupled in view of the coupling of the quantities Ψ and Γ in Eqs. $(2.18)_1$ and $(2.18)_2$.

We assume the surface of the cylinder to be free of stresses. In this way we have the following three boundary conditions

(3.5) $$\sigma_{rr}=0,\quad \sigma_{rz}=0,\quad \mu_{r\varphi}=0\quad \text{for}\quad r=a,$$

where a denotes the radius of the cylinder. The conditions (3.5) will be expressed by the functions Φ, Ψ, Γ. We get

(3.6) $$\begin{aligned}\sigma_{rr}\big|_{r=a} &= \left|2\mu\frac{\partial^2}{\partial r^2}\left(\Phi+\frac{\partial\Psi}{\partial z}\right)+\lambda\nabla^2\Phi\right|_{r=a}=0,\\ \sigma_{rz}\big|_{r=a} &= \left|\frac{\partial}{\partial r}\left[2\mu\left(\frac{\partial^2\Psi}{\partial z^2}-\frac{\partial\Phi}{\partial z}\right)-(\mu+\alpha)\nabla^2\Psi\right]-2\alpha\frac{\partial\Gamma}{\partial r}\right|_{r=a}=0,\\ \mu_{r\varphi}\big|_{r=a} &= \left|(\gamma-\varepsilon)\frac{1}{r}\frac{\partial\Gamma}{\partial r}-(\gamma+\varepsilon)\frac{\partial^2\Gamma}{\partial r^2}\right|_{r=a}=0.\end{aligned}$$

Introducing the functions Ψ, Γ into Eq. $(2.18)_1$ we obtain the relations

(3.7) $$E=b_1 B,\quad F=b_2 D,$$

where

$$b_k=\frac{1}{p}(\lambda_k^2+k^2-\sigma_2^2),\quad p=\frac{2\alpha}{\mu+\alpha},\quad k=1,2.$$

Substituting the functions Φ, Ψ, Γ from Eqs. (3.2)—(3.4) into the formula for boundary conditions (3.6) and taking into account the relations (3.7) we obtain the following system of three homogeneous equations

$$\begin{aligned}(3.8)\quad &\left[2\mu\sigma^2\left(\mathfrak{J}_0(\sigma a)-\frac{\mathfrak{J}_1(\sigma a)}{\sigma a}\right)+\lambda(k^2+\sigma^2)\,\mathfrak{J}_0(\sigma a)\right]A+\left[2\mu ik\lambda_1^2\left(\mathfrak{J}_0(\lambda_1 a)-\right.\right.\\ &\left.\left.-\frac{\mathfrak{J}_1(\lambda_1 a)}{\lambda_1 a}\right)\right]B+\left[2\mu ik\lambda_2^2\left(\mathfrak{J}_0(\lambda_2 a)-\frac{\mathfrak{J}_1(\lambda_2 a)}{\lambda_2 a}\right)\right]D=0,\\ &-2\mu\sigma ik\,\mathfrak{J}_1(\sigma a)\,A+[2\mu k^2-(\mu+\alpha)(k^2+\lambda_1^2)+2\alpha b_1]\,\lambda_1\,\mathfrak{J}_1(\lambda_1 a)\,B+\\ &+[2\mu k^2-(\mu+\alpha)(k^2+\lambda_2^2)+2\alpha\, b_2]\,\lambda_2\,\mathfrak{J}_1(\lambda_2 a)=0,\\ &\left[2\gamma\frac{\mathfrak{J}_1(\lambda_1 a)}{\lambda_1 a}-(\gamma+\varepsilon)\,\mathfrak{J}_0(\lambda_1 a)\right]\lambda_1^2 B+\left[2\gamma\frac{\mathfrak{J}_1(\lambda_2 a)}{\lambda_2 a}-\right.\\ &\left.-(\gamma+\varepsilon)\,\mathfrak{J}_0(\lambda_2 a)\right]\frac{b_2}{b_1}\lambda_2^2 D=0.\end{aligned}$$

Making equal to zero the determinant of the above system of equations we obtain a transcendental equation permitting to determine the phase velocity $c=\dfrac{\omega}{k}$ of the propagation of the wave

$$\begin{aligned}(3.9)\quad &\left[2\mu\sigma^2\left(\mathfrak{J}_0(\sigma a)-\frac{\mathfrak{J}_1(\sigma a)}{\sigma a}\right)+\lambda(k^2+\sigma^2)\,\mathfrak{J}_0(\sigma a)\right]\left\{\frac{\mathfrak{J}_1(\lambda_1 a)}{\lambda_1 a}\,\frac{b_2\,d_1}{b_1}\times\right.\\ &\times\left[2\gamma\frac{\mathfrak{J}_1(\lambda_2 a)}{\lambda_2 a}-(\gamma+\varepsilon)\mathfrak{J}_0(\lambda_2 a)\right]-\frac{\mathfrak{J}_1(\lambda_2 a)}{\lambda_2 a}\,d_2\left[2\gamma\frac{\mathfrak{J}_1(\lambda_1 a)}{\lambda_1 a}-\right.\\ &\left.\left.-(\gamma+\varepsilon)\,\mathfrak{J}_0(\lambda_1 a)\right]\right\}-4\mu^2\sigma^2k^2\frac{\mathfrak{J}_1(\sigma a)}{\sigma a}\left\{\left[\mathfrak{J}_0(\lambda_1 a)-\frac{\mathfrak{J}_1(\lambda_1 a)}{\lambda_1 a}\right]\times\right.\\ &\times\left[2\gamma\frac{\mathfrak{J}_1(\lambda_2 a)}{\lambda_2 a}-(\gamma+\varepsilon)\,\mathfrak{J}_0(\lambda_2 a)\right]\frac{b_2}{b_1}+\left[2\gamma\frac{\mathfrak{J}_1(\lambda_1 a)}{\lambda_1 a}-\right.\\ &\left.\left.-(\gamma+\varepsilon)\,\mathfrak{J}_0(\lambda_1 a)\right]\left[\mathfrak{J}_0(\lambda_2 a)-\frac{\mathfrak{J}_1(\lambda_2 a)}{\lambda_2 a}\right]\right\}=0.\end{aligned}$$

The following notations have been introduced. The transcendental Eq. (3.9) is most complicated and hardly suitable for discussion in such a general form. This is why we shall consider two particular cases.

We assume that the length of the wave is large with respect to the radius of the cylinder. In such a case we can expand the Bessel functions into an infinite series, taking into consideration only two first terms of this expanded form. In this way we obtain — as the first approximation — the following value for the phase velocity

$$(3.10)\quad c=\left\{\frac{4\left(\dfrac{1}{c_2^2}-\dfrac{1}{c_4^2}\right)-2\beta_3\dfrac{1}{c_1^2}-\beta_3\left(2\beta_1\dfrac{1}{c_1^2}+\beta_2\dfrac{1}{c_2^2}\right)+4\dfrac{\eta^2}{c_2^2\,\beta_3\,k^2}}{4\dfrac{1}{c_1^2}\left(\dfrac{1}{c_2^2}-\dfrac{1}{c_4^2}\right)+\left(2\beta_1\dfrac{1}{c_1^2}+\beta_2\dfrac{1}{c_2^2}\right)\left[2\left(\dfrac{1}{c_2^2}-\dfrac{1}{c_4^2}\right)+4\dfrac{1}{\beta_3\,c_2^2}\right]-2\dfrac{\beta_1\,\beta_2\,\beta_3}{c_1^2\,c_2^2}}\right\}^{1/2},$$

where

$$\beta_1 = 1 + \frac{\lambda}{\mu}, \quad \beta_2 = 1 + \frac{\alpha}{\mu}, \quad \beta_3 = \frac{\nu^2 - \eta^2}{k^2}.$$

For $\alpha \to 0$, i.e. for the classical elastokinetics, we obtain from (3.10) the known Pochhammer's formula [1] $c = (E/\varrho)^{1/2}$.

If the length of the wave is small as compared with the radius of the cylinder, then Eq. (3.9), after asymptotic transition, reduces to the equation characteristic for the surface waves in an elastic half-space [6]

$$[(2\mu+\lambda)\,\sigma^2 - k^2\,\lambda]\left(\frac{b_2\,a_1}{b_2 - b_1} - a_2\,\frac{\lambda_1\,b_1}{\lambda_2\,(b_2 - b_1)}\right) - 4\mu^2\,k^2\,\lambda_1\,\sigma = 0, \tag{3.11}$$

where

$$a_r = \mu\,(k^2 - \lambda_r^2) + \alpha\,(\lambda_r^2 - k^2) - 2\alpha\,b_r, \quad r = 1, 2.$$

Let us return once more to the transcendental Eq. (3.9). Putting in this equation $\alpha = 0$ we obtain the following dispersion equations

$$\left[2\mu\sigma^2\left(\mathfrak{J}_0(\sigma a) - \frac{\mathfrak{J}_1(\sigma a)}{\sigma a}\right) + \lambda\sigma_1^2\,\mathfrak{J}_0(\sigma a)\right](2k^2 - \sigma_2^2)\,\frac{\mathfrak{J}_1(\lambda_2^0\,a)}{\lambda_2^0\,a} + 4\mu k^2\,\sigma^2\,\frac{\mathfrak{J}_1(\sigma a)}{\sigma a}\left(\mathfrak{J}_0(\lambda_2^0 a) - \frac{\mathfrak{J}_1(\lambda_2^0\,a)}{\lambda_2^0\,a}\right) = 0, \tag{3.12}$$

$$\frac{\mathfrak{J}_1(\lambda_1^0\,a)}{\lambda_1^0\,a} - \frac{\gamma+\varepsilon}{2\gamma}\,\mathfrak{J}_0(\lambda_1^0\,a) = 0, \tag{3.13}$$

where

$$\sigma_1 = \frac{\omega}{c_1}, \quad \sigma_2 = \frac{\omega}{\hat{c}_2}, \quad \lambda_2^0 = (\sigma_2^2 - k^2)^{1/2}, \quad \lambda_1^0 = (\sigma_3^2 - k^2)^{1/2},$$

$$c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}, \quad \hat{c}_2 = \left(\frac{\mu}{\varrho}\right)^{1/2}.$$

Eq. (3.12) is a transcendental equation derived by Pochhammer [1] within the framework of classical theory of elasticity, Eq. (3.13) refers to the propagation of the wave described by the equation

$$(\gamma+\varepsilon)\left(\nabla^2\,\omega_\varphi - \frac{\omega_\varphi}{r^2}\right) = J\ddot{\omega}_\varphi, \tag{3.14}$$

derived from $(2.10)_3$ putting $\alpha = 0$.

Assuming that this wave propagates along the z-axis with constant phase velocity $c = \dfrac{\omega}{k}$, i.e. assuming $\omega_\varphi = \omega^*(r)\,e^{i(kz-\omega t)}$, and making use of the boundary condition $\mu_{r\varphi} = 0$ for $r = a$, we obtain the transcendental Eq. (3.13)

4. Radial vibrations

To conclude, let us consider the one-dimensional problem, where the functions u_r, u_z, ω_φ do not depend neither on φ nor on z. In this case the system of Eqs. (2.10) simplifies to the form

$$
\begin{gathered}
(\mu+\alpha)\left(\nabla_r^2 u_r-\frac{u_r}{r^2}\right)+(\lambda+\mu-\alpha)\frac{\partial e}{\partial r}=\varrho\ddot{u}_r,\\
(\mu+\alpha)\nabla_r^2 u_z+2\alpha\frac{1}{r}\frac{\partial}{\partial r}(r\omega_\varphi)=\varrho\ddot{u}_z,\\
(\gamma+\varepsilon)\left(\nabla_r^2\omega_\varphi-\frac{\omega_\varphi}{r^2}\right)-4\alpha\omega_\varphi-2\alpha\frac{\partial u_z}{\partial r}=J\ddot{\omega}_\varphi,
\end{gathered}
\tag{4.1}
$$

where

$$\nabla_r^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r},\qquad e=\frac{1}{r}\frac{\partial}{\partial r}(ru_r).$$

Assuming that we have to deal with monochromatic vibrations

$$(u_r, u_z, \omega_\varphi)=\left(u_r^*(r), u_z^*(r), \omega_\varphi^*(r)\right)e^{-i\omega t}, \tag{4.2}$$

we obtain from (4.1) the appropriate transcendental equation. We obtain it immediately from Eq. (3.9) putting therein $k=0$. Thus we have

$$\mathfrak{J}_0(\sigma_1 a)=\frac{2\mu}{\lambda+2\mu}\frac{\mathfrak{J}_1(\sigma_1 a)}{\sigma_1 a},\qquad \sigma_1=\frac{\omega}{c_1}, \tag{4.3}$$

$$\frac{\dfrac{\mathfrak{J}_1(\eta_2 a)}{\eta_2 a}}{\dfrac{\mathfrak{J}_1(\eta_1 a)}{\eta_1 a}}=\frac{\left[2\gamma\dfrac{\mathfrak{J}_1(\eta_2 a)}{\eta_2 a}-(\gamma+\varepsilon)\mathfrak{J}_0(\eta_2 a)\right](\eta_2^2-\sigma_2^2)}{\left[2\gamma\dfrac{\mathfrak{J}_1(\eta_1 a)}{\eta_1 a}-(\gamma+\varepsilon)\mathfrak{J}_0(\eta_1 a)\right](\eta_1^2-\sigma_2^2)}, \tag{4.4}$$

where

$$\eta_{1,2}^2=\tfrac{1}{2}\left(\nu^2-\eta^2-\sigma_2^2-\sigma_4^2\pm\sqrt{(\sigma_2^2+\sigma_4^2+\eta^2-\nu^2)^2-4\sigma_2^2(\sigma_4^2-\nu^2)}\right).$$

Making use of Eq. (4.3) we obtain successive values for free radial vibrations. The form of these equations is identical with that known from classical elastikinetics.

Eq. (4.4) is a transcendental equation for modified transverse free vibrations, depending solely on the radius r. For $\alpha\to 0$, i.e. for an elastic Hooke's body, only radial vibrations may appear.

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В. НОВАЦКИЙ и В. К. НОВАЦКИЙ, **ПРОПАГАЦИЯ УПРУГИХ ВОЛН В МИКРОПОЛЯРНОМ ЦИЛИНДРЕ.** I.

В настоящем сообщении рассматривается проблема пропагации монохроматических волн в бесконечном цилиндре, изготовленном из микрополярного упругого материала. Предметом рассуждений авторов являются продольные волны, распространяющиеся вдоль оси цилиндра. В результате получается трансцендентное уравнение, благодаря которому можно определить фазовые скорости.