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APPLIED MECHANICS
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Propagation of Elastic Waves in a Micropolar Cylinder. II

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1. Introduction

In this paper we are concerned with the problem of propagation of monochromatic torsional waves in an infinite micropolar cylinder made of isotropic, homogeneous, centrosymmetric material. Our considerations, similarly as in the first part of this communication [4] refer to the Cosserat medium, wherein the deformation of the body is defined by two independent vectors: the displacement vector **u** and the rotation vector **ω**. This part is a generalization of the problem of propagation of the torsional wave in an infinite cylinder within the framework of the classical theory of elasticity [5].

2. Basic equations

The starting point for our considerations will be the equations for the micropolar elastic medium [1]—[3]. Equations describing the field of displacements and rotations and the field of stresses are given in the first part of this work [4]. Our considerations are conducted within the system of cylindrical coordinates (r, φ, z).

In the case of propagation of the torsional wave in an infinite cylinder with circular section we assume $u_r = u_z = 0$, $\omega_\varphi = 0$; the third component of the displacement vector u_φ as well as the components of the rotation vector ω_r, ω_z are independent of the angle φ.

Under these assumptions the system of the three following equations is what remains from the system of equations of the problem within the system of cylindrical coordinates (cf. Eqs. (2.8) and (2.9) in [4])

$$(\mu+\alpha)\left(\nabla^2 u_\varphi - \frac{u_\varphi}{r^2}\right) + 2\alpha\left(\frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r}\right) = \varrho \ddot{u}_\varphi,$$

$$(2.1) \qquad (\gamma+\varepsilon)\left(\nabla^2 \omega_r - \frac{\omega_r}{r^2}\right) - 4\alpha\omega_r + (\beta+\gamma-\varepsilon)\frac{\partial \varkappa}{\partial r} - 2\alpha\frac{\partial u_\varphi}{\partial z} = J\ddot{\omega}_r,$$

$$(\gamma+\varepsilon)\nabla^2 \omega_z - 4\alpha\omega_z + (\beta+\gamma-\varepsilon)\frac{\partial \varkappa}{\partial z} + \frac{2\alpha}{r}\frac{\partial}{\partial r}(r u_\varphi) = J\ddot{\omega}_z,$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}, \qquad \varkappa = \frac{1}{r}\frac{\partial}{\partial r}(r\omega_r) + \frac{\partial \omega_z}{\partial z}.$$

The field of displacements $\mathbf{u} = (0, u_\varphi, 0)$ and rotations $\boldsymbol{\omega} = (\omega_r, 0, \omega_z)$ described by Eqs. (2.1) induces the following state of stresses $\boldsymbol{\sigma}$ and couple stresses $\boldsymbol{\mu}$

$$\text{(2.2)} \qquad \boldsymbol{\sigma} = \begin{vmatrix} 0 & \sigma_{r\varphi} & 0 \\ \sigma_{\varphi r} & 0 & \sigma_{\varphi z} \\ 0 & \sigma_{z\varphi} & 0 \end{vmatrix}, \qquad \boldsymbol{\mu} = \begin{vmatrix} \mu_{rr} & 0 & \mu_{rz} \\ 0 & \mu_{\varphi\varphi} & 0 \\ \mu_{zr} & 0 & \mu_{zz} \end{vmatrix},$$

where

$$\sigma_{r\varphi} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) + \frac{\alpha}{r}\left(\frac{\partial}{\partial r}(u_\varphi r) - \frac{\partial u_r}{\partial \varphi}\right) - 2\alpha\omega_z,$$

$$\sigma_{\varphi r} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) - \frac{\alpha}{r}\left(\frac{\partial}{\partial r}(u_\varphi r) - \frac{\partial u_r}{\partial \varphi}\right) + 2\alpha\omega_z,$$

$$\sigma_{\varphi z} = \mu\left(\frac{\partial u_\varphi}{\partial z} + \frac{\partial u_z}{r\partial \varphi}\right) + \frac{\alpha}{r}\left(\frac{\partial u_z}{\partial \varphi} - \frac{\partial}{\partial z}(ru_\varphi)\right) - 2\alpha\omega_r,$$

$$\text{(2.3)} \quad \sigma_{z\varphi} = \mu\left(\frac{\partial u_\varphi}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \varphi}\right) - \frac{\alpha}{r}\left(\frac{\partial u_z}{\partial \varphi} - \frac{\partial}{\partial z}(ru_\varphi)\right) + 2\alpha\omega_r,$$

$$\mu_{rr} = 2\gamma\frac{\partial \omega_r}{\partial r} + \beta\varkappa, \quad \mu_{\varphi\varphi} = 2\gamma\frac{1}{r}\left[\frac{\partial \omega_\varphi}{\partial \varphi} + \omega_r\right] + \beta\varkappa, \quad \mu_{zz} = 2\gamma\frac{\partial \omega_z}{\partial z} + \beta\varkappa,$$

$$\mu_{rz} = \gamma\left(\frac{\partial \omega_r}{\partial z} + \frac{\partial \omega_z}{\partial r}\right) - \varepsilon\left(\frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r}\right),$$

$$\mu_{zr} = \gamma\left(\frac{\partial \omega_r}{\partial z} + \frac{\partial \omega_z}{\partial r}\right) + \varepsilon\left(\frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r}\right).$$

We shall now introduce the potentials φ and ϑ. They are connected with the rotations

$$\text{(2.4)} \qquad \omega_r = \frac{\partial \Phi}{\partial r} - \frac{\partial \vartheta}{\partial z}, \qquad \omega_z = \frac{\partial \Phi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\vartheta).$$

We introduce, moreover, the functions η and χ defined as below

$$\text{(2.5)} \qquad u_\varphi = -\frac{\partial \eta}{\partial r}, \qquad \vartheta = -\frac{\partial \chi}{\partial r}.$$

Substituting (2.4) and (2.5) into (2.1), we obtain the following system of wave equations

$$\text{(2.6)} \qquad \Box^1 \Phi = 0, \quad \Box^2 \chi + 2\alpha\eta = 0, \quad \Box^4 \eta - 2\alpha\nabla^2\chi = 0,$$

where

$$\Box^1 = (2\gamma+\beta)\nabla^2 - 4\alpha - J\partial_t^2, \qquad \Box^2 = (\gamma+\varepsilon)\nabla^2 - 4\alpha - J\partial_t^2,$$

$$\Box^4 = (\mu+\alpha)\nabla^2 - \varrho\partial_t^2.$$

Eq. $(2.6)_1$ represents the torsional wave. Eqs. $(2.6)_2$ and $(2.6)_3$ are mutually coupled. Eliminating from these equations first χ and then η, we obtain the equation

$$(2.7)\qquad (\square^2\,\square^4+4\alpha^2\,\nabla^2)\,(\chi,\eta)=0,$$

describing the propagation of modified transverse and torsional waves.

3. Propagation of torsional wave in an infinite cylinder

Let us assume a torsional wave propagating in an infinite cylinder with circular section along the z-axis with constant phase velocity c. This wave being, by assumption, monochromatic, we may write the functions φ,χ,η, in the form

$$(3.1)\qquad (\Phi,\chi,\eta)=(\Phi^*(r),\chi^*(r),\eta^*(r))\,e^{i(kz-\omega t)},\qquad c=\frac{\omega}{k}.$$

Substituting the presumed form of the solutions of (3.1) into the system of Eqs. (2.1) we obtain

$$(3.2)\qquad \begin{aligned}\Phi&=A\mathfrak{J}_0(\tau r)\,e^{i(kz-\omega t)},\\ \chi&=[B\mathfrak{J}_0(\lambda_3 r)+D\mathfrak{J}_0(\lambda_4 r)]\,e^{i(kz-\omega t)},\\ \eta&=[E\mathfrak{J}_0(\lambda_3 r)+F\mathfrak{J}_0(\lambda_4 r)]\,e^{i(kz-\omega t)},\end{aligned}$$

where the following notations were used

$$\lambda_{3,4}^2=-k^2+\frac{1}{2}\left(\sigma_2^2+\sigma_4^2-\nu_1^2+\eta_1^2\pm\sqrt{(\sigma_2^2+\sigma_4^2-\nu_1^2+\eta_1^2)^2+4\eta_1^2\sigma_2^2}\right),$$

$$\tau=(\sigma_3^2-k^2-\lambda_0^2)^{1/2},\qquad \lambda_0^2=\frac{4\alpha}{2\gamma+\beta},\qquad \sigma_3=\frac{\omega}{c_3},\qquad c_3=\left(\frac{2\gamma+\beta}{J}\right)^{1/2},$$

$$\sigma_2=\frac{\omega}{c_2},\qquad \sigma_4=\frac{\omega}{c_4},\qquad c_2=\left(\frac{\mu+\alpha}{\varrho}\right)^{1/2},\qquad c_4=\left(\frac{\gamma+\varepsilon}{J}\right)^{1/2},$$

$$\nu_1^2=\frac{4\alpha}{\gamma+\varepsilon},\qquad \eta_1^2=\frac{4\alpha^2}{(\mu+\alpha)(\gamma+\varepsilon)}.$$

Requiring the phase velocities to be real and positive, i.e. λ_3^2,λ_4^2 to be positive, we have to satisfy the condition $\omega^2>\dfrac{4\alpha}{J}$. Since λ_3^2,λ_4^2 depend on the frequency ω, the waves described by the functions χ and η undergo dispersion.

In order to get a unique solution of the problem thus formulated we need also the boundary conditions. We assume that the boundary surface of the cylinder is free of stresses, i.e. the following conditions have to be fulfilled

$$(3.3)\qquad \sigma_{r\varphi}=0,\quad \mu_{rz}=0,\quad \mu_{rr}=0\quad \text{for}\quad r=a.$$

These conditions expressed with the help of function (3.1) will take the form

$$\sigma_{r\varphi} = \left| -(\mu+\alpha)\frac{\partial^2 \eta}{\partial r^2} + (\mu-\alpha)\frac{1}{r}\frac{\partial \eta}{\partial r} + 2\alpha\left(\frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r}\frac{\partial \chi}{\partial r} - \frac{\partial \varphi}{\partial z}\right)\right|_{r=a} = 0,$$

$$\text{(3.4)} \quad \mu_{rr} = \left| \beta \nabla^2 \varphi + 2\gamma \frac{\partial^2}{\partial r^2}\left(\varphi + \frac{\partial \chi}{\partial z}\right)\right|_{r=a} = 0,$$

$$\mu_{rz} = \left| \frac{\partial}{\partial r}\left\{\gamma\left[2\left(\frac{\partial \varphi}{\partial z} + \frac{\partial^2 \chi}{\partial z^2}\right) - \nabla^2 \chi\right] + \varepsilon \nabla^2 \chi\right\}\right|_{r=a} = 0.$$

Introducing into the formulae expressing the boundary conditions the function φ, χ, η defined by Eqs. (3.2) we obtain the following system of three homogeneous equations

$$A_1 2\alpha ik \mathfrak{J}_0(\tau a) + B_1\left[2\alpha\lambda_3^2 \mathfrak{J}_0(\lambda_3 a) + b_3\left(-(\mu+\alpha)\mathfrak{J}_0(\lambda_3 a) + \right.\right.$$
$$\left.\left. + 2\mu \frac{\mathfrak{J}_1(\lambda_3 a)}{\lambda_3 a}\right)\lambda_3^2\right] + D_1\left[2\alpha\lambda_4^2 \mathfrak{J}_0(\lambda_4 a) + b_4 \lambda_4^2\left(-(\mu+\alpha)\mathfrak{J}_0(\lambda_4 a) + \right.\right.$$
$$\left.\left. + 2\mu \frac{\mathfrak{J}_1(\lambda_4 a)}{\lambda_4 a}\right)\right] = 0,$$

(3.5)

$$A_1\left\{[\beta(\tau^2+k^2)+2\gamma\tau^2]\mathfrak{J}_0(\tau a) - 2\gamma\tau^2 \frac{\mathfrak{J}_1(\tau a)}{\tau a}\right\} + B_1\left\{2\gamma ik\lambda_3^2\left(\mathfrak{J}_0(\lambda_3 a) - \right.\right.$$
$$\left.\left. - \frac{\mathfrak{J}_1(\lambda_3 a)}{\lambda_3 a}\right)\right\} + D_1\left\{2\gamma ik\lambda_4^2\left(\mathfrak{J}_0(\lambda_4 a) - \frac{\mathfrak{J}_1(\lambda_4 a)}{\lambda_4 a}\right)\right\} = 0,$$

$$A_1\left\{2\gamma ik\tau^2 \frac{\mathfrak{J}_1(\tau a)}{\tau}\right\} + B_1\left\{\lambda_3^2[\gamma(k^2-\lambda_3^2)+\varepsilon(k^2+\lambda_3^2)]\frac{\mathfrak{J}_1(\lambda_3 a)}{\lambda_3 a}\right\} +$$
$$+ D_1\left\{\lambda_4^2[\gamma(k^2-\lambda_4^2)+\varepsilon(k^2+\lambda_4^2)]\frac{\mathfrak{J}_1(\lambda_4 a)}{\lambda_4 a}\right\} = 0,$$

Use was made here of the relations

$$\text{(3.6)} \qquad E = b_3 B \quad F = b_4 D,$$

where

$$b_k = p\frac{k^2+\lambda_k^2}{k^2+\lambda_k^2-\sigma_2^2}, \quad p = \frac{2\alpha}{\mu+\alpha}, \quad k = 3, 4.$$

The relations (3.6) are due to the fact that the functions χ and η are coupled by Eqs. $(2.6)_{2,3}$.

Putting equal to zero the determinant of the system of Eqs. (3.5), we obtain the characteristic transcendental equation permitting to determine the phase velocity $c = \frac{\omega}{k}$ of the propagation of waves in the cylinder.

$$
\begin{aligned}
(3.7)\quad & -2\alpha k \Im_0(\tau a)\left\{2\gamma k\lambda_3^2\left(\Im_0(\lambda_3 a)-\frac{\Im_1(\lambda_3 a)}{\lambda_3 a}\right)\lambda_4^2\frac{\Im_1(\lambda_4 a)}{\lambda_4 a}g_4-2\gamma k\lambda_4^2\times\right.\\
&\times\left.\left(\Im_0(\lambda_4 a)-\frac{\Im_1(\lambda_4 a)}{\lambda_4 a}\right)\lambda_3^2\frac{\Im_1(\lambda_3 a)}{\lambda_3 a}g_3\right\}-\left\{[\beta(\tau^2+k^2)+2\gamma\tau^2]\Im_0(\tau a)-\right.\\
&\left.-2\gamma\tau^2\frac{\Im_1(\tau a)}{\tau a}\right\}\left[2\alpha\lambda_3^2\Im_0(\lambda_3 a)-b_3\left((\alpha+\mu)\Im_0(\lambda_3 a)-2\mu\frac{\Im_1(\lambda_3 a)}{\lambda_3 a}\right)\lambda_3^2\right]\times\\
&\times\lambda_4^2\frac{\Im_1(\lambda_4 a)}{\lambda_4 a}g_4-\left[2\alpha\lambda_4^2\Im_0(\lambda_4 a)-b_4\left((\alpha+\mu)\Im_0(\lambda_4 a)-2\mu\frac{\Im_1(\lambda_4 a)}{a\lambda_4}\right)\lambda_4^2\right]\times\\
&\times\lambda_3^2\frac{\Im_1(\lambda_3 a)}{\lambda_3 a}g_3\Big\}-4\gamma^2k^2\tau^2\frac{\Im_1(\tau a)}{\tau a}\left\{\left[2\alpha\lambda_3^2\Im_0(\lambda_3 a)-b_3\left((\alpha+\mu)\Im_0(\lambda_3 a)-\right.\right.\right.\\
&\left.\left.-2\mu\frac{\Im_1(\lambda_3 a)}{a\lambda_3}\right)\lambda_3^2\right]\lambda_4^2\left(\Im_0(\lambda_4 a)-\frac{\Im_1(\lambda_4 a)}{\lambda_4 a}\right)-\lambda_3^2\left(\Im_0(\lambda_3 a)-\right.\\
&\left.\left.-\frac{\Im_1(\lambda_3 a)}{\lambda_3 a}\right)\left[2\alpha\lambda_4^2\Im_0(\lambda_4 a)-g_4\lambda_4^2\left((\alpha+\mu)\Im_0(\lambda_4 a)-\frac{\Im_1(\lambda_4 a)}{\lambda_4 a}\right)\right]\right\}=0,
\end{aligned}
$$

where

$$g_r=\gamma(k^2-\lambda_r^2)+\varepsilon(k^2+\lambda_r^2),\qquad r=3,4.$$

Eq. (3.7) is most complicated and in its general form unsuitable for discussion.

In a particular case when the wave is long as compared with the radius of the cylinder we get — expanding the Bessel functions into a series and retaining but first two terms of the expanded expression — the first approximation in the form

$$
(3.8)\qquad c=\left\{\frac{4\gamma\omega^2(2\gamma-\varepsilon)\,c_3^2\,c_4^2}{2\omega^2\{[2\gamma^2+\varepsilon(2\gamma+\beta)]\,c_4^2-\gamma(\gamma-\varepsilon)\,c_3^2\}-\left\{4\alpha\gamma\frac{\gamma-\varepsilon}{\gamma+\varepsilon}+\lambda_0^2\,[6\gamma+\beta+(\gamma-\varepsilon)(2\gamma+\beta)]\right\}c_4^2\,c_3^2}\right\}^{1/2}.
$$

The torsional wave undergoes dispersion which is due — as seen from (3.8) — to the fact that the phase velocity depends on the frequency.

For waves short as compared with the radius of the cylinder the characteristic equation — after asymptotic transition — reduces to the equation characteristic of the surface waves of Love's type in an elastic half-space [6]:

$$
\begin{aligned}
(3.9)\qquad 4k^2\gamma\tau(\lambda_1 d_3-\lambda_2 d_4)&=4\gamma\alpha k^2(e_4 d_3-e_3 d_4)+\\
&+[(2\gamma+\beta)\,\tau^2-\beta^2k^2]\,(e_3 d_4-e_4 d_3),
\end{aligned}
$$

where

$$e_r=\gamma(k^2+\lambda_r^2)-\varepsilon(k^2-\lambda_r^2),\qquad d_r=(\mu+\alpha)\,\lambda_r\,b_k+2\alpha\lambda_r,\qquad r=3,4.$$

Putting in the characteristic equation (3.7) $\alpha=0$ we obtain two following equations:

$$\mathfrak{J}_0(\mathring{\lambda}_4 a) = 2\frac{\mathfrak{J}_1(\mathring{\lambda}_4 a)}{\mathring{\lambda}_4 a},$$

(3.10)

$$\left\{\beta\sigma_3^2 \mathfrak{J}_0(\mathring{\tau}a) + 2\gamma(\sigma_3^2 - k^2)\left(\mathfrak{J}_0(\mathring{\tau}a) - \frac{\mathfrak{J}_1(\mathring{\tau}a)}{\mathring{\tau}a}\right)\right\}\frac{\mathfrak{J}_1(\mathring{\lambda}_4 a)}{\mathring{\lambda}_4 a}[2k^2\gamma + (\varepsilon-\gamma)\sigma_4^2] +$$
$$+4\gamma^2 k^2(\sigma_3^2 - k^2)\frac{\mathfrak{J}_1(\mathring{\tau}a)}{\mathring{\tau}a}\left(\mathfrak{J}_0(\mathring{\lambda}_4 a) - \frac{\mathfrak{J}_1(\mathring{\lambda}_4 a)}{\mathring{\lambda}_4 a}\right) = 0,$$

The marks zero over the quantities λ_3, λ_4 and τ denote their values for $\alpha = 0$. Eq. $(3.10)_1$ is known from the classical theory of elasticity [5]; itis the solution of

$$\mu\left(\nabla^2 u_\varphi - \frac{u_\varphi}{r^2}\right) = \varrho \ddot{u}_\varphi, \tag{3.11}$$

derived from $(2.1)_1$ for $\alpha = 0$. Assuming that the wave propagates along the z-axis with constant phase velocity, assuming further the solution of Eq. (3.11) in the form

$$u_\varphi = -\frac{\partial \eta}{\partial r} = -\frac{\partial}{\partial r}\eta^*(r)\, e^{i(kz-\omega t)} \tag{3.12}$$

and making use of the boundary condition $\sigma_{r\varphi} = 0$ for $r = a$ we obtain the characteristic Eq. $(3.10)_1$.

Solving the system of equations

(3.13)

$$(\gamma+\varepsilon)\left(\nabla^2 \omega_r - \frac{\omega_r}{r^2}\right) + (\beta+\gamma-\varepsilon)\frac{\partial \varkappa}{\partial r} = J\ddot{\omega}_r,$$
$$(\gamma+\varepsilon)\nabla^2 \omega_z + (\beta+\gamma-\varepsilon)\frac{\partial \varkappa}{\partial z} = J\ddot{\omega}_z,$$

under the boundary conditions $\mu_{rr} = 0$, $\mu_{rz} = 0$ for $r = a$ we obtain the transcendental equation $(3.10)_2$.

From Eq. $(3.10)_2$ we may determine the phase velocities of torsional waves in a hypothetical medium, wherein only rotations may appear.

4. One-dimensional torsional vibrations

We shall now consider the one-dimensional problem, wherein the functions $u_\varphi, \omega_r, \omega_z$ do not depend on φ and on z. Under these assumptions the system of Eqs. (2.1) will take the following form

(4.1)

$$(\mu+\alpha)\left(\nabla_r^2 u_\varphi - \frac{u_\varphi}{r^2}\right) - 2\alpha\frac{\partial \omega_z}{\partial r} = \varrho\ddot{u}_\varphi,$$
$$(\gamma+\varepsilon)\left(\nabla_r^2 \omega_r - \frac{\omega_r}{r^2}\right) - 4\alpha\omega_r + (\beta+\gamma-\varepsilon)\frac{\partial \varkappa}{\partial r} = J\ddot{\omega}_r,$$
$$(\gamma+\varepsilon)\nabla_r^2 \omega_z - 4\alpha\omega_z + \frac{2\alpha}{r}\frac{\partial}{\partial r}(ru_\varphi) = J\ddot{\omega}_z,$$

where

$$\nabla_r^2 = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \qquad \varkappa = \frac{1}{r}\frac{\partial}{\partial r}(r\omega_r).$$

Assuming the existence of monochromatic waves we have

$$(4.2) \qquad (u_\varphi, \omega_r, \omega_z) = \left(u_\varphi^*(r), \omega_r^*(r), \omega_z^*(r)\right) e^{-i\omega t},$$

Substituting (4.2) into (4.1) we obtain the transcendental equation. We can obtain it also directly from (3.7) putting therein $k = 0$. Thus we obtain

$$(4.3) \qquad \mathfrak{J}_0(\tau_1 a) = \frac{2\gamma}{\beta+2\gamma}\,\frac{\mathfrak{J}_1(\tau_1 a)}{\tau_1 a},$$

$$(4.4) \qquad \frac{\dfrac{\mathfrak{J}_1(\eta_4 a)}{\eta_4 a}\eta_4^2}{\dfrac{\mathfrak{J}_1(\eta_3 a)}{\eta_3 a}\eta_3^2} = \frac{2\alpha^2\,\mathfrak{J}_0(\eta_4 a) - (\eta_4^2+\nu_1^2-\sigma_4^2)(\gamma+\varepsilon)\times\left[(\mu+\alpha)\,\mathfrak{J}_0(\eta_4 a) - 2\mu\dfrac{\mathfrak{J}_1(\eta_4 a)}{\eta_4 a}\right]}{2\alpha^2\,\mathfrak{J}_0(\eta_3 a) - \dfrac{1}{2}(\gamma+\varepsilon)(\eta_3^2+\nu_1^2-\sigma_4^2)\times\left[(\mu+\alpha)\,\mathfrak{J}_0(\eta_3 a) - 2\mu\dfrac{\mathfrak{J}_1(\eta_3 a)}{\eta_3 a}\right]},$$

where

$$\tau_1 = (\sigma_3^2-\lambda_0^2)^{1/2}; \qquad \eta_{3,4}^2 = \frac{1}{2}\left(\sigma_2^2+\sigma_4^2-\nu_1^2+\eta_1^2 \pm \sqrt{(\sigma_2^2-\sigma_4^2+\nu_1^2-\eta_1^2)^2+4\eta_1^2\sigma_2^2}\right).$$

Solving Eq. (4.3) we obtain the successive values of free torsional vibrations of the cylinder. The form of these vibrations is identical with that known from the classical theory of elasticity. Eq. (4.4) describes the modified free transverse vibrations which are the function of the radius r only. Performing the transition $\alpha \to 0$, i.e. the transition to the classical theory of elasticity, we are satisfied that within the framework of this theory only the torsional waves described by Eq. (4.3) are admissible. From Eq. (4.4) we obtain $J_1(\sigma_2 a) = 0$ or $J_1(\sigma_4 a) = 0$. These equations refer to a hypothetical medium wherein only rotations appear.

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В. НОВАЦКИЙ и В. К. НОВАЦКИЙ, ПРОПАГАЦИЯ УПРУГИХ ВОЛН В МИКРОПОЛЯРНОМ ЦИЛИНДРЕ. II.

В работе обсуждается проблема пропагации монохроматических волн в бесконечном упругом цилиндре. Рассуждения проведены в рамках несимметрической теории упругости для среды Коссэра. Рассматривается случай пропагации крутильных волн, распространяющихся вдоль оси цилиндра.

Получено характеристическое уравнение, позволяющее определить фазовые скорости. Работу следует считать расширением результатов, полученных в [4], на случай крутильных колебаний цилиндра.