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(ELASTICITY)

Propagation of Monochromatic Waves in an Infinite Micropolar Elastic Plate

by

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1. Introduction

Recent years witnessed an increasing interest in problems of propagation of monochromatic elastic waves in the general Cosserat medium, wherein the deformation of a body is described by two vectors independent of each other, namely by the displacement vector $\mathbf{u}(\mathbf{x}, t)$ and the rotation vector $\boldsymbol{\omega}(\mathbf{x}, t)$.

The propagation of plane waves in an infinite micropolar medium was discussed by V. A. Palmov [1]; the propagation of rotation waves in an infinite medium was the subject of a paper by one of the present authors (W.N.) [2]. Basic solutions of equations of motion in an infinite medium have been given in [3], where the waves arising under the effect of body couples and body forces were presented. Quite recently, S. Kaliski, J. Kapelewski and C. Rymarz devoted a paper [4] to the problem of propagation of surface waves in a micropolar medium.

In this paper we are concerned with the propagation of monochromatic waves in an infinite elastic plate discussing two characteristic types of propagation of monochromatic waves.

2. Basic equations

To begin with, we shall consider the equations describing an elastic micropolar medium [5], [6]. An elastic, homogeneous, isotropic and centrisymmetric body will be the object of our subsequent remarks. Under the effect of external loadings displacement, $\mathbf{u}(\mathbf{x}, t)$, and rotation $\boldsymbol{\omega}(\mathbf{x}, t)$ field will form in such a body.

The state of strain is described by two asymmetric tensor: The strain tensor γ_{ji} and the curvature-twist tensor $\varkappa_{ji}$. There is

$$\gamma_{ji} = u_{i,j} - \epsilon_{kji}\,\omega_k, \quad \varkappa_{ji} = \omega_{i,j}. \tag{2.1}$$

The state of stress is defined, in turn, by the following two asymmetric tensors: the stress tensor σ_{ij} and the couple-stress tensor μ_{ji}. The relation between the state of strain and that of stress is described by the relations

$$\begin{aligned}\sigma_{ji} &= (\mu+\alpha)\gamma_{ji}+(\mu-\alpha)\gamma_{ij}+\lambda\gamma_{kk}\delta_{ij},\\ \mu_{ji} &= (\gamma+\varepsilon)\varkappa_{ji}+(\gamma-\varepsilon)\varkappa_{ij}+\beta\varkappa_{kk}\delta_{ij}.\end{aligned} \tag{2.2}$$

The quantities $\mu, \lambda, \alpha, \beta, \gamma, \varepsilon$ denote the material constants.

Introducing (2.2) into the equations of motion

$$\sigma_{ji,j}+X_i-\varrho\ddot{u}_i = 0, \quad \epsilon_{ijk}\sigma_{jk}+\mu_{ji,j}+Y_i-J\ddot{\omega}_i = 0. \tag{2.3}$$

and expressing the quantities γ_{ji} and $\varkappa_{ji}$ by the displacements u_i and rotations ω_i — in accordance with the formulae (2.1) — we obtain the system of differential equations. We write them in the vector form

$$\begin{aligned}&(\mu+\alpha)\nabla^2\mathbf{u}+(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}\mathbf{u}+2\alpha\operatorname{rot}\boldsymbol{\omega}+\mathbf{X} = \varrho\ddot{\mathbf{u}},\\ &(\gamma+\varepsilon)\nabla^2\boldsymbol{\omega}+(\gamma+\beta-\varepsilon)\operatorname{grad}\operatorname{div}\boldsymbol{\omega}-4\alpha\boldsymbol{\omega}+2\alpha\operatorname{rot}\mathbf{u}+\mathbf{Y} = J\ddot{\boldsymbol{\omega}}.\end{aligned} \tag{2.4}$$

In the above equations the symbol $\mathbf{X}$ denotes the body forces vector, while $\mathbf{Y}$ stands for the body-couple vector, ϱ for the density and J for the rotational inertia.

Let us now consider a particular case, where the vectors $\mathbf{u}$ and $\boldsymbol{\omega}$ are functions only of the variable x_1, x_2 and time t. In this case we can derive from (2.4) two systems of equations independent of each other

$$\begin{aligned}&(\mu+\alpha)\nabla_1^2 u_1+(\mu+\lambda-\alpha)\partial_1 e+2\alpha\partial_2\omega_3 = \varrho\ddot{u}_1,\\ &(\mu+\alpha)\nabla_1^2 u_2+(\mu+\lambda-\alpha)\partial_2 e-2\alpha\partial_1\omega_3 = \varrho\ddot{u}_2,\\ &(\gamma+\varepsilon)\nabla_1^2\omega_3-4\alpha\omega_3+2\alpha(\partial_1 u_2-\partial_2 u_1) = J\ddot{\omega}_3,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&(\gamma+\varepsilon)\nabla_1^2\omega_1+(\gamma+\beta-\varepsilon)\partial_1\varkappa-4\alpha\omega_1+2\alpha\partial_2 u_3 = J\ddot{\omega}_1,\\ &(\gamma+\varepsilon)\nabla_1^2\omega_2+(\gamma+\beta-\varepsilon)\partial_2\varkappa-4\alpha\omega_2-2\alpha\partial_1 u_3 = J\ddot{\omega}_2,\\ &(\mu+\alpha)\nabla_1^2 u_3+2\alpha(\partial_1\omega_2-\partial_2\omega_1) = \varrho\ddot{u}_3.\end{aligned} \tag{2.6}$$

Here we have: $\nabla_1^2 = \partial_1^2+\partial_2^2$; $e = \partial_1 u_1+\partial_2 u_2$; $\varkappa = \partial_1\omega_1+\partial_2\omega_2$.

The displacement and rotation field $\mathbf{u} = (u_1, u_2, 0)$, $\boldsymbol{\omega} = (0, 0, \omega_3)$ described by Eqs. (2.5) induces the following stress $\boldsymbol{\sigma}$ and couple-stress $\boldsymbol{\mu}$ state

$$\boldsymbol{\sigma} = \begin{vmatrix}\sigma_{11} & \sigma_{12} & 0\\ \sigma_{21} & \sigma_{22} & 0\\ 0 & 0 & \sigma_{33}\end{vmatrix}, \quad \boldsymbol{\mu} = \begin{vmatrix}0 & 0 & \mu_{13}\\ 0 & 0 & \mu_{23}\\ \mu_{31} & \mu_{32} & 0\end{vmatrix}, \tag{2.7}$$

where — in virtue of (2.2) — there is

$$\begin{aligned}&\sigma_{11} = 2\mu\partial_1 u_1+\lambda e, \quad \sigma_{22} = 2\mu\partial_2 u_2+\lambda e, \quad \sigma_{33} = \lambda e,\\ &\sigma_{12} = \mu(\partial_1 u_2+\partial_2 u_1)+\alpha(\partial_1 u_2-\partial_2 u_1)-2\alpha\omega_3,\\ &\sigma_{21} = \mu(\partial_1 u_2+\partial_2 u_1)-\alpha(\partial_1 u_2-\partial_2 u_1)+2\alpha\omega_3,\\ &\mu_{13} = (\gamma+\varepsilon)\partial_1\omega_3, \quad \mu_{31} = (\gamma-\varepsilon)\partial_1\omega_3,\\ &\mu_{23} = (\gamma+\varepsilon)\partial_2\omega_3, \quad \mu_{32} = (\gamma-\varepsilon)\partial_2\omega_3.\end{aligned} \tag{2.8}$$

As concerns the displacement and rotation field $\mathbf{u} = (0, 0, u_3)$ and $\boldsymbol{\omega} = (\omega_1, \omega_2, 0)$ described by Eqs. (2.6), it induces the following stresses $\boldsymbol{\sigma}$ and couple stresses $\boldsymbol{\mu}$

$$\boldsymbol{\sigma} = \begin{vmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{vmatrix}, \qquad \boldsymbol{\mu} = \begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{21} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}, \tag{2.9}$$

where

$$\begin{aligned} &\sigma_{13} = (\mu+a)\,\partial_1 u_3 + 2a\omega_2, \quad \sigma_{31} = (\mu-a)\,\partial_1 u_3 - 2a\omega_2, \\ &\sigma_{23} = (\mu+a)\,\partial_2 u_3 - 2a\omega_1, \quad \sigma_{32} = (\mu-a)\,\partial_2 u_3 + 2a\omega_1, \\ &\mu_{11} = 2\gamma\partial_1\omega_1 + \beta\varkappa, \quad \mu_{22} = 2\gamma\partial_2\omega_2 + \beta\varkappa, \quad \mu_{33} = \beta\varkappa, \\ &\mu_{12} = \gamma(\partial_1\omega_2 + \partial_2\omega_1) + \varepsilon(\partial_1\omega_2 - \partial_2\omega_1), \\ &\mu_{21} = \gamma(\partial_1\omega_2 + \partial_2\omega_1) - \varepsilon(\partial_1\omega_2 - \partial_2\omega_1). \end{aligned} \tag{2.10}$$

We shall show that the system of Eqs. (2.5) leads to monochromatic waves known in the classical elastokinetics as Lamb's waves Eqs. (2.5) lead to the waves of Love's type.

3. Modified Lamb's waves

Let us now consider an elastic plate — we assume its thickness to be $2h$ — wherein a monochromatic wave propagates along the x_2-axis. We assume that the edges of the layer $x_1 = \pm h$ are free of stresses. The following conditions should be satisfied on these edges

$$\sigma_{11} = 0, \quad \sigma_{12} = 0, \quad \mu_{13} = 0, \quad \text{for} \quad x_1 = \pm h. \tag{3.1}$$

Expressing the displacements by the potentials Φ, Ψ

$$u_1 = \partial_1\Phi - \partial_2\Psi, \quad u_2 = \partial_2\Phi + \partial_1\Psi, \tag{3.2}$$

we can derive (putting $\mathbf{X} = \mathbf{Y} = 0$) from the system of Eqs. (2.5) the following equations

$$\begin{gathered} (\lambda+2\mu)\,\nabla_1^2\Phi - \varrho\ddot{\Phi} = 0, \quad (\mu+a)\,\nabla_1^2\Psi - \varrho\ddot{\Psi} - 2a\omega_3 = 0, \\ [(\gamma+\varepsilon)\,\nabla_1^2 - 4a - J\partial_t^2]\,\omega_3 + 2a\nabla_1^2\Psi = 0. \end{gathered} \tag{3.3}$$

Eliminating from the last two equations first the quantity Ψ and then ω_3 we have

$$\{[(\mu+a)\,\nabla_1^2 - \varrho\partial_t^2]\,[(\gamma+\varepsilon)\,\nabla_1^2 - 4a - J\partial_t^2] + 4a^2\,\nabla_1^2\}\,(\Psi, \omega_3) = 0. \tag{3.4}$$

Eq. $(3.3)_1$ describes the longitudinal wave, while Eq. (3.4) the modified transverse wave.

The solutions of Eqs. $(3.3)_1$ and (3.4) will be sought for in the form

$$(\Phi, \Psi, \omega_3) = (\Phi^*(x_1), \Psi^*(x_1), \omega^*(x_1)\,e^{i(kx_2-\omega t)}. \tag{3.5}$$

These solutions are as follows

$$\begin{aligned}\Phi^* &= A \operatorname{sh} \delta x_1 + B \operatorname{ch} \delta x_1, \quad \delta = (k^2-\sigma_1^2)^{1/2},\\ \Psi^* &= C \operatorname{sh} \lambda_1 x_1 + D \operatorname{ch} \lambda_1 x_1 + E \operatorname{sh} \lambda_2 x_1 + F \operatorname{ch} \lambda_2 x_1,\\ \omega^* &= C' \operatorname{sh} \lambda_1 x_1 + D' \operatorname{ch} \lambda_1 x_1 + E' \operatorname{sh} \lambda_2 x_1 + F' \operatorname{ch} \lambda_2 x_1.\end{aligned} \tag{3.6}$$

We introduced here the notation specified below

$$\begin{aligned}&\sigma_1 = \frac{\omega}{c_1}, \quad c_1 = \left(\frac{\lambda+2\mu}{\varrho}\right)^{1/2}, \quad \sigma_2 = \frac{\omega}{c_2}, \quad c_2 = \left(\frac{\mu+\alpha}{\varrho}\right)^{1/2},\\ &\sigma_4 = \frac{\omega}{c_4}, \quad c_4 = \left(\frac{\gamma+\varepsilon}{J}\right)^{1/2}, \quad \nu^2 = \frac{4\alpha}{\gamma+\varepsilon}, \quad \eta^2 = \frac{4\alpha^2}{(\gamma+\varepsilon)(\mu+\alpha)},\\ &\lambda_{1,2}^2 = k^2 + \tfrac{1}{2}\left(\nu^2-\eta^2-\sigma_2^2-\sigma_4^2 \pm \sqrt{(\sigma_2^2+\sigma_4^2+\eta^2-\nu^2)^2 - 4\sigma_2^2(\sigma_4^2-\nu^2)}\right).\end{aligned} \tag{3.7}$$

Since the quantities λ_1^2 and λ_2^2 have to be positive (this follows from the postulate that the phase velocities be real), we have $\omega^2 > 4\alpha/J$. Eqs. $(3.6)_2$ and $(3.6)_3$ are connected through Eqs. $(3.3)_2$ and $(3.3)_3$, respectively.

Similarly as in classical elastokinetics, the general problem of propagation of waves may be reduced to the solution of two simple problems, i.e. to the consideration of the symmetric and antisymmetric vibrations.

a. Symmetric vibrations are characterized by the symmetry of displacements u_2 and stresses σ_{11}, σ_{22} and μ_{13} with respect to the plane $x_1 = 0$. In this case we have to put in the expressions (3.6): $A = D = F = D' = F' = 0$. In view of the coupling of Eqs. $(3.3)_2$ and $(3.3)_3$, we have

$$C' = \varkappa_1 C, \quad E' = \varkappa_2 E, \tag{3.8}$$

where

$$\varkappa_r = \frac{1}{p}(\sigma_2^2+k^2-\lambda_r^2), \quad r = 1, 2, \quad p = \frac{2\alpha}{\mu+\alpha}.$$

Expressing the boundary conditions (3.1) by the functions Φ^*, Ψ^* and ω^*, we obtain a system of three homogeneous equations. Making equal to zero the determinant of this system, we arrive at the following characteristic equation

$$\frac{\operatorname{tgh}(\delta h)}{\operatorname{tgh}(\lambda_1 h)} = \frac{(2\mu+\lambda)\delta^2 - k^2\lambda}{4\mu^2 k^2 \lambda_1 \delta(\varkappa_2-\varkappa_1)}\left(a_1\varkappa_2 - a_2\varkappa_1 \frac{\lambda_1}{\lambda_2}\frac{\operatorname{tgh}(\lambda_2 h)}{\operatorname{tgh}(\lambda_1 h)}\right) \tag{3.9}$$

where $a_r = \mu(k^2+\lambda_r^2) + \alpha(\lambda_r^2-k^2) - 2\alpha\varkappa_r$, $r = 1, 2$. The quantity c is the phase velocity sought for. From the transcendental Eq. (3.9) we obtain an infinite number of roots k. To each of these roots there corresponds a definite form of vibrations.

For $\alpha \to 0$ (what corresponds to the classical theory of elasticity) Eq. (3.9) reduces to the known transcendental equation for Lamb's waves [7]:

$$\frac{\operatorname{tgh}(kh\sqrt{1-c^2/c_1^2})}{\operatorname{tgh}(kh\sqrt{1-c^2/\hat{c}_2^2})} = \frac{\left(2-\frac{c^2}{\hat{c}_2^2}\right)^2}{4\sqrt{\left(1-\frac{c^2}{c_1^2}\right)\left(1-\frac{c^2}{\hat{c}_2^2}\right)}}, \quad \hat{c}_2 = \left(\frac{\mu}{\varrho}\right)^{1/2}. \tag{3.10}$$

Let us now consider two particular cases. We assume first that the wavelength is small as compared with the thickness of the plate $2h$. Then the quantities δh, $\lambda_1 h$ and $\lambda_2 h$ are large such that it is plausible to assume the relation of hyperbolic tangents as equal to one. Then

$$\text{(3.11)} \qquad \frac{\varkappa_2 a_1}{\varkappa_2 - \varkappa_1} - \frac{a_2 \lambda_1}{\lambda_2} \frac{\varkappa_1}{\varkappa_2 - \varkappa_1} = \frac{4\mu^2 k^2 \lambda_1 \delta}{(2\mu+\lambda)\delta^2 - k^2 \lambda}.$$

The above equation coincides with the dispersional equation for the surface wave in a micropolar medium [4]. For $\alpha \to 0$ we obtain from (3.11) the equation characteristic for Rayleigh waves [8]

$$\text{(3.12)} \qquad \left(2 - \frac{c^2}{\hat{c}_2^2}\right)^2 = 4\sqrt{\left(1 - \frac{c^2}{c_1^2}\right)\left(1 - \frac{c^2}{\hat{c}_2^2}\right)}.$$

For long waves, as compared with the thickness $2h$, the quantities δh, $\lambda_1 h$, $\lambda_2 h$ are small and the hyperbolic tangents in (3.9) may be replaced by their arguments. We have

$$\text{(3.13)} \qquad 4\mu^2 k^2 \delta^2 (\varkappa_2 - \varkappa_1) = [(2\mu+\lambda)\delta^2 - k^2 \lambda](a_1 \varkappa_2 - a_2 \varkappa_1).$$

In the particular case $\alpha \to 0$ there is

$$c = \frac{2\hat{c}_2}{c_1}(c_1^2 - \hat{c}_2^2)^{1/2}.$$

b. Antisymmetric vibrations. Let us now consider the particular case where the displacement u_2 and the stresses σ_{11}, σ_{22} and μ_{13} are antisymmetric with respect to the plane $x_1 = 0$. Then we have to put in the expressions (3.6) $B = C = = E = C' = E' = 0$ and $D' = \varkappa_1 D_1$ $F' = \varkappa_2 F$.

Making use of the boundary conditions (3.1) we arrive at the transcendental equation

$$\text{(3.14)} \qquad \left(\frac{a_1 \varkappa_2 \lambda_2}{\operatorname{tgh}(\lambda_1 h)} - \frac{a_2 \varkappa_1 \lambda_1}{\operatorname{tgh}(\lambda_2 h)}\right) \operatorname{tgh}(\delta h) = \frac{4\mu^2 k^2 \delta^2 \lambda_1 \lambda_2 (\varkappa_2 - \varkappa_1)}{(2\mu+\lambda)\delta^2 - k^2 \lambda}$$

which permits to determine the successive values of the parameter k.

For $\alpha \to 0$ we obtain from Eq. (3.14) the transcendental equation of classical elastokinetics [8]

$$\text{(3.15)} \qquad \frac{\operatorname{tgh}(kh\sqrt{1 - c^2/c_1^2})}{\operatorname{tgh}(kh\sqrt{1 - c^2/\hat{c}_2^2})} = \frac{4\sqrt{\left(1 - \frac{c^2}{c_1^2}\right)\left(1 - \frac{c^2}{\hat{c}_2^2}\right)}}{\left(2 - \frac{c^2}{\hat{c}_2^2}\right)^2}.$$

If the wavelength is very small as compared with the thickness of the plate $2h$, Eq. (3.14) goes into (3.11). If, on the contrary, the length of the wave is large as compared with the thickness of the plate, then expanding the hyperbolic tangents into a series and retaining but two terms of the expanded form we obtain the equation

$$(3.16)\quad \left(1-\frac{\delta h^2}{3}\right)\left[\frac{a_1\varkappa_2}{\lambda_1^2\left(1-\frac{\lambda_2^2h^2}{3}\right)}-\frac{a_2\varkappa_1}{\lambda_2^2\left(1-\frac{\lambda_2^2h^2}{3}\right)}\right]=\frac{4\mu^2k^2(\varkappa_2-\varkappa_1)}{(2\mu+\lambda)\,\delta^2-k^2\lambda}.$$

Therefrom we are able to determine the phase velocity $c=\frac{\omega}{k}$ of the flexural wave. For $\alpha\to 0$ we obtain an expression known from the classical elastokinetics [8]

$$c^2=\frac{4}{3}(kh)^2\,\hat{c}_2^2\left(1-\frac{\hat{c}_2^2}{c_1^2}\right).$$

4. The modified Love's waves

Let us now consider an elastic plate $2h$ thick; the propagation of the monochromatic wave in such a medium is described by the system of Eqs. (2.6). We assume that the waves propagate with constant velocity along the x_2-axis. Then there is

$$(4.1)\qquad (\omega_1,\omega_2,u_3)=\left(\omega_1^*(x_1),\omega_2^*(x_1),u^*(x_1)\right)e^{i(kx_2-\omega t)}.$$

Introducing into Eqs. (2.6) the potentials φ and ψ connected with the rotations ω_1,ω_2 by the relations

$$(4.2)\qquad \omega_1=\partial_1\varphi-\partial_2\psi,\quad \omega_2=\partial_2\varphi+\partial_1\psi,$$

we separate these equations obtaining the following system of equations

$$(4.3)\qquad \begin{aligned}&[(\beta+2\gamma)\nabla_1^2-4\alpha-J\partial_t^2]\varphi=0,\\ &[(\gamma+\varepsilon)\nabla_1^2-4\alpha-J\partial_t^2]\psi-2\alpha u_3=0,\\ &[(\mu+\alpha)\nabla_1^2-\varrho\partial_t^2]u_3+2\alpha\nabla_1^2\psi=0.\end{aligned}$$

Eliminating from the two last equations first the quantity ψ and then u_3 we get an equation identical as to its structure with Eq. (3.4).

$$(4.4)\qquad \{[(\gamma+\varepsilon)\nabla_1^2-4\alpha-J\partial_t^2][(\mu+\alpha)\nabla_1^2-\varrho\partial_t^2]+4\alpha^2\nabla_1^2\}(\psi,u_3)=0,$$

Now, requiring the boundary of the plate to be free of stresses, we have the following boundary conditions

$$(4.5)\qquad \mu_{11}=0,\quad \mu_{12}=0,\quad \sigma_{13}=0\quad \text{for}\quad x_1=\pm h.$$

Eq. $(4.3)_1$ represent the rotational wave while Eq. (4.4) describes the transverse and twist wave.

The solutions of Eqs. $(4.3)_1$ and (4.4) will be sought for in the form

$$(4.6)\qquad \begin{aligned}&\varphi^*=A\,\mathrm{sh}\,\sigma x_1+B\,\mathrm{ch}\,\sigma x_1,\\ &\psi^*=C\,\mathrm{sh}\,\lambda_1x_1+D\,\mathrm{ch}\,\lambda_1x_1+E\,\mathrm{sh}\,\lambda_2x_1+F\,\mathrm{ch}\,\lambda_2x_1,\\ &u^*=C'\,\mathrm{sh}\,\lambda_1x_1+D'\,\mathrm{ch}\,\lambda_1x_1+E'\,\mathrm{sh}\,\lambda_2x_1+F'\,\mathrm{ch}\,\lambda_2x_1.\end{aligned}$$

The following notations have been introduced into the above formulae

$$\sigma = (k^2+\nu_0^2-\sigma_3^2), \quad \sigma_3 = \frac{\omega}{c_3}, \quad c_3 = \left(\frac{\beta+2\gamma}{J}\right)^{1/2}, \quad \nu_0^2 = \frac{4\alpha}{\beta+2\gamma}.$$

The quantities λ_1, λ_2 are given by the formulae (3.7).

a. Symmetric vibrations. We require the rotation ω_2 and the stresses μ_{11}, μ_{22}, σ_{13} to be symmetric with respect to the plane $x_1 = 0$. This postulate will be satisfied if we assume $A = D = F = D' = F' = 0$ and $C' = \varrho_1 C$, $E' = \varrho_2 E$, where the quantities ϱ_1 and ϱ_2 may be determined from Eq. $(4.3)_3$. Thus we have

$$\varrho_r = \frac{p(k^2-\lambda_r^2)}{\lambda_r^2-k^2+\sigma_2^2}, \quad p = \frac{2\alpha}{\mu+\alpha}, \quad r = 1, 2.$$

Taking into account boundary conditions expressed by (4.5) we obtain the system of three homogeneous equations. Making equal to zero the determinant of this system we get a transcendental equation as below

$$\text{(4.7)} \quad \frac{\operatorname{tgh}(\sigma h)}{\operatorname{tgh}(\lambda_1 h)} = \frac{4\gamma\alpha k^2\left(c_2\lambda_1\dfrac{\operatorname{tgh}(\lambda_2 h)}{\operatorname{tgh}(\lambda_1 h)} - c_1\lambda_2\right) + [(2\gamma+\beta)\sigma^2-\beta k^2]\left(c_1 d_2 - c_2 d_1\dfrac{\operatorname{tgh}(\lambda_2 h)}{\operatorname{tgh}(\lambda_1 h)}\right)}{4k^2\gamma^2\sigma(\lambda_1 d_2-\lambda_2 d_1)},$$

where

$$c_r = \gamma(\lambda_r^2+k^2)+\varepsilon(\lambda_r^2-k^2), \quad d_r = (\mu+\alpha)\lambda_r\varrho_r+2\alpha\lambda_r, \quad r = 1, 2.$$

The successive values of the parameter $k = \dfrac{\omega}{c}$ and the corresponding phase velocities c and forms of waves may be determined from Eq. (4.7).

For small lengths of waves as compared with the thickness of the plate we obtain the equation

$$\text{(4.8)} \quad 4k^2\gamma\sigma(\lambda_1 d_1-\lambda_2 d_2) = 4\gamma\alpha k^2(c_2 d_1-c_1 d_2) + [(2\gamma+\beta)\sigma^2-\beta^2 k^2](c_1 d_2-c_2 d_1).$$

wherefrom we may determine the phase velocity of the surface wave in an elastic half-space. Thus it appears that in a micropolar elastic medium we have not only the waves of Rayleigh type but also the waves of Love type $u_3(x_1, x_2, t) = u_3^*(x_1)\,e^{i(kx_2-\omega t)}$ accompanied by the modified twist waves ω_1, ω_2. In the classical medium the appearance of Love waves was possible only in a layered half-space provided certain definite inequalities concerning the material constants were satisfied [9].

Let us return once more to Eq. (4.7), assuming $\alpha \to 0$. We obtain

$$\text{(4.9)} \quad \frac{\operatorname{tgh}(kh\sqrt{1-c^2/c_3^2})}{\operatorname{tgh}(kh\sqrt{1-c^2/c_4^2})} = \frac{\left(2-\dfrac{c^2}{c_0^2}\right)^2}{4\sqrt{1-c^2/c_3^2}\,(1-c^2/c_4^2)}, \quad c_0 = \left(\frac{\gamma}{J}\right)^{1/2}.$$

We shall consider a particular case. For $\alpha \to 0$ Eqs. (4.3) take the form

$$
\begin{aligned}
&[(\beta+2\gamma)\, \nabla_1^2 - J\partial_t^2]\, \varphi = 0,\\
(4.10)\qquad &[(\gamma+\varepsilon)\, \nabla_1^2 - J\partial_t^2]\, \psi = 0,\\
&[\mu\, \nabla_1^2 - \varrho\partial_t^2]\, u_3 = 0.
\end{aligned}
$$

From the two first boundary conditions (4.5) we have the system of equations

$$
\begin{aligned}
&B\,[(2\gamma+\beta)\, \sigma_0^2 - \beta k^2]\, \mathrm{ch}\,(\sigma_0 h) - 2C\gamma ik\eta\, \mathrm{ch}\,(\eta h) = 0,\\
&B2\gamma ik\sigma_0\, \mathrm{sh}\, \sigma_0 h + [(\gamma+\varepsilon)\, \eta^2 + (\gamma+\varepsilon)\, k^2]\, \mathrm{sh}\, \eta h = 0,
\end{aligned}
$$

where

$$\sigma_0 = (k^2 - \sigma_3^2)^{1/2}, \quad \eta = (k^2 - \sigma_4^2)^{1/2}.$$

Making equal to zero the determinant of this system of equations we obtain, after some simple transformations, the transcendental Eq. (4.9). This equation refers to a hypothetical medium, wherein only rotations may occur, but no displacements.

b. Antisymmetric vibrations. Let us consider the case, where the rotation ω_2 and the stresses μ_{11}, μ_{22}, σ_{13} are antisymmetric with respect to the plane $x_1 = 0$. Assuming in (4.6) $B = C = E = C' = E' = 0$ and $D' = \varrho_1 D$, $F' = \varrho_2 F$ we obtain — taking into account the boundary conditions (4.5) — the following transcendental equation

$$
(4.11)\qquad \frac{\mathrm{tgh}\,(\sigma h)}{\mathrm{tgh}\,(\lambda_1 h)} =
$$

$$
= \frac{4k^2\gamma^2\,\sigma(\lambda_1 d_2 - \lambda_2 d_1)}{[(2\gamma+\beta)\,\sigma^2 - \beta k^2]\left[c_1 d_2 - d_1 c_2 \dfrac{\mathrm{tgh}\,(\lambda_1 h)}{\mathrm{tgh}\,(\lambda_2 h)}\right] + 4\gamma\alpha^2 h^2 \left[\lambda_1 c_2 \dfrac{\mathrm{tgh}\,(\lambda_1 h)}{\mathrm{tgh}\,(\lambda_2 h)} - \lambda_2 c_1\right]}.
$$

For very small lengths of waves as compared with the thickness of the plate we get from Eq. (4.12) again Eq. (4.8). In the particular case $\alpha \to 0$ we obtain the equation

$$
(4.12)\qquad \frac{\mathrm{tgh}\,(kh\sqrt{1 - c^2/c_3^2})}{\mathrm{tgh}\,(kh\sqrt{1 - c^2/c_4^2})} = \frac{4\,(1 - c^2/c_3^2)^{1/2}\,(1 - c^2/c_4^2)^{1/2}}{\left(2 - \dfrac{c^2}{c_0^2}\right)^2}.
$$

If in all transcendental equations referring to the modified Lamb's and Love's problem we put $k = 0$, these equations will refer to the free vibrations of the elastic layer which depend solely on x_1 and t. It means that they are monochromatic one dimensional vibrations.

A more ample discussion of problems of propagation of waves considered in this paper will be published before long in Proceedings of Vibration Problems.

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В. НОВАЦКИЙ и В. К. НОВАЦКИЙ, **РАСПРОСТРАНЕНИЕ МОНОХРОМАТИЧЕСКИХ ВОЛН В БЕСКОНЕЧНОЙ МИКРОПОЛЯРНОЙ УПРУГОЙ ПЛАСТИНЕ**

В работе представлены два типа распространения монохроматических волн в бесконечной микрополярной пластине. Первый тип, охарактеризованный векторами $\mathrm{u} = (u_1, u_2, 0)$ и $\omega = (0, 0, \omega_3)$, является распространением на микрополярную среду проблемы волн Лямба, второй тип, охарактеризованный векторами $\mathrm{u} = (0, 0, u_3)$ и $\omega = (\omega_1, \omega_2, 0)$, соответствует волнам Лова. Показано, что в однородной, микрополярной пластине возможно распространение волн Лова.