VOLUME 4 NUMBER 1 APRIL 1966

International Journal of Engineering Science

Editor-in-Chief A. C. ERINGEN

PERGAMON PRESS
OXFORD . LONDON . NEW YORK . PARIS

Int. J. Engng Sci. Vol. 4, pp. 53–68. Pergamon Press 1966. Printed in Great Britain

SINGULAR INTEGRAL EQUATIONS OF THERMOELASTICITY

J. IGNACZAK and W. NOWACKI

Warsaw, Poland

Abstract—The integral relations analogous to that of Somigliana and Helmholtz in elastokinetics have been introduced by the integration of basic differential equations of coupled thermoelasticity. Thus we have obtained the representation of displacements, temperature and the thermoelastic displacement potential by surface integrals.

The solutions have been utilized to construct the general thermoelastic potentials of single and double layer. By means of these potentials the basic boundary problem of coupled thermoelasticity has been reduced to the solution of a system of singular integral equations.

1. INTRODUCTION

THE purpose of the present paper is to obtain the solutions of the equations of thermoelasticity describing the harmonic vibrations of medium by means of singular integral equations. By the integration of basic differential equations of thermoelasticity one can derive integral relations which constitute a generalization the Somigliana theorem known from the elastokinetics [1]. The integration of the equation of thermoelastic displacement potential leads also to the representation of its solution in the form of surface integrals. In the case when there is no coupling between the temperature and the deformation of a body we obtain the Helmholtz theorem known from the elastokinetics.

The solutions obtained in sections 2 and 3 provide the so called surface potentials of single and double layer. The application of these potentials allows us to reduce the basic boundary problem of thermoelasticity to the solution of a system of singular integral equations. Finally, the theorem on discontinuity of thermoelastic potentials during the passage through the boundary of the equations satisfied by these potentials.

The integral equations here obtained are the Fredholm singular integral equations of the second kind. The integrals encountered here must be understood in the sense of the Cauchy principal values. In the last section of the paper we give the procedure for the construction of the approximate solutions of the equations of thermoelasticity by means of the so called canonical functional integral equations.

This method enables to solve the equations of thermoelasticity in an approximate way for an arbitrary, single connected, three-dimensional body.

2. EQUATIONS OF THERMOELASTICITY

Let us now consider a homogeneous, isotropic, perfectly elastic body occupying the region V and bounded by the surface Σ. For this medium the linearized equations of thermoelasticity [2], [3]

$$\mu u_{i,kk}+(\lambda+\mu)u_{k,ki}+X_i=\gamma\theta_{,i}+\rho\ddot{u}_i, \tag{2.1}$$

$$\theta_{,kk}-\frac{1}{\kappa}\dot{\theta}-\eta\dot{u}_{k,k}=-\frac{Q}{\kappa}, \quad i,\ k=1,\ 2,\ 3, \tag{2.2}$$

hold.

The first equation represents the Lamè equation (the equation of motion) while the second one constitutes the extended equation of thermal conductivity. In these equations $\theta=T-T_0$ denotes the increase of temperature with respect to the natural state T_0 for which the stresses and deformations are equal to zero. u_i are the components of the displacement vector, X_i denote the components of the body force vector. Q is a function describing the intensity of heat sources. The magnitudes μ, λ are the Lamè constants referring to the isothermal state. Next we have $\gamma=(3\lambda+2\mu)\alpha_t$, where α_t is the coefficient of linear thermal expansion, ρ denotes the density, and $\kappa=\lambda_0/\rho c_\varepsilon$ is a coefficient for which λ_0 is the coefficient of thermal conductivity while c_ε is the specific heat for a constant deformation. Finally $Q=W/\rho c_\varepsilon$, where W denotes the amount of heat produced in the unit of time and volume, and $\eta=\gamma T_0/\lambda_0$. The functions u_i, θ, X_i, Q are the functions of place and time. The dot denotes the derivative with respect to time.

Besides equations (2.1) and (2.2) we have the constitutive equations

$$\sigma_{ij}=2\mu\varepsilon_{ij}+(\lambda\varepsilon_{kk}-\gamma\theta)\delta_{ij}, \tag{2.3}$$

and the relations between the deformations and displacements

$$\varepsilon_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i})=u_{(i,j)}\,. \tag{2.4}$$

By the decomposition of the displacement vector and the body forces vector into the potential and solenoidal parts

$$u_i=\phi_{,i}+\epsilon_{ijk}\psi_{k,j}\,, \qquad X_i=\rho(\vartheta_{,i}+\epsilon_{ijk}\chi_{k,j})\,, \tag{2.5}$$

the system of equations (2.1), (2.2) can be reduced to the form

$$\left(\nabla^2-\frac{1}{c_1^2}\partial_t^2\right)\phi=m\theta-\frac{1}{c_1^2}\vartheta\,, \tag{2.6}$$

$$\left(\nabla^2-\frac{1}{c_2^2}\partial_t^2\right)\psi_i=-\frac{1}{c_2^2}\chi_i\,, \tag{2.7}$$

$$\left(\nabla^2-\frac{1}{\kappa}\partial_t\right)\theta-\eta\partial_t\nabla^2\phi=-\frac{Q}{\kappa}\,. \tag{2.8}$$

We have introduced the symbols

$$c_1^2=\frac{\lambda+2\mu}{\rho}\,, \qquad c_2^2=\frac{\mu}{\rho}, \qquad m=\frac{\gamma}{c_1^2\rho}\,, \qquad \partial_t=\frac{\partial}{\partial t}\,.$$

Equation (2.6) represents the longitudinal elastic wave while equation (2.7) the transversal wave. Equation (2.8) is the equation of thermal conductivity. Eliminating from equations (2.6) and (2.8) the temperature we obtain

$$\left(\nabla^2-\frac{1}{c_1^2}\partial_t^2\right)\left(\nabla^2-\frac{1}{\kappa}\partial_t\right)\phi-m\eta\partial_t\nabla^2\phi=-\frac{mQ}{\kappa}-\frac{1}{c_1^2}\left(\nabla^2-\frac{1}{\kappa}\partial_t\right)\vartheta. \tag{2.9}$$

In the sequel we shall consider only the vibrations varying harmonically in time. Since the loading by body forces and heat sources may be reduced to a boundary problem we shall assume that the causes of the vibrations are expressed only by the boundary conditions.

Substituting the relations

$$\phi(x,t)=\phi^*(x)e^{i\omega t}, \qquad \psi_j(x,t)=\psi_j^*(x)e^{i\omega t},$$

in equations (2.6)–(2.8) and neglecting the heat sources and the body forces on the right hand sides of these equations we obtain the following system of equations [4]

$$\Box_1^2\phi^*-m\theta^*=0, \tag{2.10}$$

$$\Box_2^2\psi_i^*=0, \tag{2.11}$$

$$\Box_3^2\theta^*+\frac{\varepsilon}{m}h_3^2\nabla^2\phi^*=0, \tag{2.12}$$

where we have introduced the symbols

$$\Box_\alpha^2=\nabla^2+h_\alpha^2, \quad \alpha=1,2,3, \quad h_1=\frac{\omega}{c_1}, \quad h_2=\frac{\omega}{c_2}, \quad h_3=\frac{1}{i}\left(\frac{i\omega}{\kappa}\right)^{\frac{1}{2}}, \quad \varepsilon=\eta m\kappa, \quad i=\sqrt{-1}.$$

Eliminating from (2.10) and (2.12) the function θ^* we obtain the equation of longitudinal waves

$$\Box_{k_1}^2\Box_{k_3}^2\phi^*=0, \tag{2.13}$$

where

$$\Box_{k_1}^2=\nabla^2+k_1^2, \qquad \Box_{k_3}^2=\nabla^2+k_3^2.$$

The magnitudes k_1, k_3 are the roots of the equation

$$k^4-k^2[h_1^2+(1+\varepsilon)h_3^2]+h_1^2h_3^2=0,$$

and take the values

$$k_r=\alpha_r-i\beta_r, \qquad r=1,3, \qquad \alpha_r>0, \qquad \beta_r\geq 0.$$

Denoting $k_1=k_1(\varepsilon)$, $k_3=k_3(\varepsilon)$ we have: $k_1(0)=h_1$, $k_3(0)=h_3$. In the subsequent considerations we shall discuss only the vibrations harmonic in time, therefore we shall drop the stars in the symbols of the functions ϕ^*, θ^* etc.

3. INTEGRAL FORM OF THE SOLUTION OF THE THERMOELASTIC POTENTIAL EQUATION

In this section we consider only the longitudinal waves arising in a body V bounded by the surface Σ. Equations (2.13) and (2.10) constitute a point of departure of our considerations. In order to obtain the integral identity determining the function Φ for $x\in V$ by means of surface integrals we make use of the following equality valid for arbitrary two functions $\phi=\phi(x)$, $\bar{\phi}=\bar{\phi}(x)$:

$$\int_V \mathrm{d}V(x)[\bar{\phi}(\xi)\square^2_{k_1}\square^2_{k_3}\phi(\xi)-\phi(\xi)\square^2_{k_1}\square^2_{k_3}\bar{\phi}(\xi)]$$
$$=\int_V \mathrm{d}V(x)[\bar{\phi}(\xi)\nabla^4\phi(\xi)-\phi(\xi)\nabla^4\bar{\phi}(\xi)]+(k_1^2+k_3^2)\int_V \mathrm{d}V(\xi)[\bar{\phi}(\xi)\nabla^2\phi(\xi)-\phi(\xi)\nabla^2\bar{\phi}(\xi)]. \quad (3.1)$$

Making use of the basic formula for bi-Laplacian

$$\int_V \mathrm{d}V(\bar{\phi}\nabla^4\phi-\phi\nabla^4\bar{\phi})=\int_\Sigma \mathrm{d}\Sigma\left(\nabla^2\bar{\phi}\frac{\partial\phi}{\partial n}-\phi\frac{\partial\nabla^2\bar{\phi}}{\partial n}+\bar{\phi}\frac{\partial\nabla^2\phi}{\partial n}-\nabla^2\phi\frac{\partial\bar{\phi}}{\partial n}\right), \quad (3.2)$$

and of the Green transformation

$$\int_V \mathrm{d}V(\bar{\phi}\nabla^2\phi-\phi\nabla^2\bar{\phi})=\int_\Sigma \mathrm{d}\Sigma\left(\bar{\phi}\frac{\partial\phi}{\partial n}-\phi\frac{\partial\bar{\phi}}{\partial n}\right), \quad (3.3)$$

equation (3.1) can be represented in the following form

$$\int_V \mathrm{d}V(\bar{\phi}\square^2_{k_1}\square^2_{k_3}\phi-\phi\square^2_{k_1}\square^2_{k_3}\bar{\phi})$$
$$=\int_\Sigma \mathrm{d}\Sigma\left(\bar{\phi}\frac{\partial}{\partial n}\square^2\phi-\phi\frac{\partial}{\partial n}\square^2\bar{\phi}\right)-\int_\Sigma \mathrm{d}\Sigma\left(\nabla^2\phi\frac{\partial\bar{\phi}}{\partial n}-\nabla^2\bar{\phi}\frac{\partial\phi}{\partial n}\right), \quad (3.4)$$

where $\square^2=\nabla^2+k_1^2+k_3^2$.

Let us assume that the function Φ has no singularity in the region V and satisfies the homogeneous equation

$$\square^2_{k_1}\square^2_{k_3}\phi=0, \qquad x\in V. \quad (3.5)$$

Temperature θ, related with the function $\bar{\Phi}$ is given by the formula

$$\theta=\frac{1}{m}\square^2_1\phi. \quad (3.6)$$

Let the function $\bar{\Phi}$ satisfy the following singular equation in an infinite thermoelastic space

$$\Box_{k_1}^2 \Box_{k_3}^2 \bar{\Phi}(x, \xi) = -m\delta(x-\xi), \tag{3.7}$$

where $\delta(x)$ is the Dirac function.

The function [5]

$$\bar{\Phi}(x, \xi) = -\frac{m}{4\pi r} \cdot \frac{e^{-ik_1 r} - e^{-ik_3 r}}{k_1^2 - k_3^2}, \qquad r^2 = (x_j - \xi_j)(x_j - \xi_j), \qquad j = 1, 2, 3, \tag{3.8}$$

is the solution of this equation.

It can readily be verified (equation 2.9) that the function Φ is the potential of thermoelastic displacement for a concentrated source of heat of the intensity κ applied at a point ξ. For the external regions the function $\bar{\Phi}$ satisfies also the condition of radiation [4] extended on the problem of thermoelasticity.

Note that

$$\bar{\theta}(x, \xi) = \frac{1}{m} \Box_1^2 \bar{\Phi} = -\frac{1}{4\pi r(k_1^2 - k_3^2)} [(h_1^2 - k_1^2) e^{-ik_1 r} - (h_1^2 - k_3^2) e^{-ik_3 r}]. \tag{3.9}$$

Substituting (3.7) in equation (3.4) and making use of equation (3.6) we obtain the following basic equations

$$\phi(x) = \frac{1}{m} \int_\Sigma d\Sigma(\xi) \left[\bar{\Phi}(\xi, x) \frac{\partial}{\partial n} \Box^2 \phi(\xi) - \phi(\xi) \frac{\partial}{\partial n} \Box^2 \bar{\Phi}(\xi, x) \right]$$
$$- \frac{1}{m} \int_\Sigma d\Sigma(\xi) \left[\nabla^2 \phi(\xi) \frac{\partial}{\partial n} \bar{\Phi}(\xi, x) - \nabla^2 \bar{\Phi}(\xi, x) \frac{\partial}{\partial n} \phi(\xi) \right], \text{ for } x \in V, \tag{3.10}$$

$$\phi(x) = 0 \text{ for } x \in E - V \tag{3.11}$$

where E denotes the entire space. These equations can be transformed by means of the relation between the functions ϕ and θ in accordance with (3.6). After a little manipulation we find that

$$\phi(x) = \int_\Sigma d\Sigma \left\{ \left[\bar{\Phi} \frac{\partial \theta}{\partial n} - \theta \frac{\partial \bar{\Phi}}{\partial n} \right] + \frac{1}{m} \left[(\Box_k^2 \bar{\Phi}) \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} (\Box_k^2 \bar{\Phi}) \right] \right\}, \qquad x \in V, \tag{3.12}$$

$$\phi(x) = 0 \text{ for } x \in E - V, \tag{3.12'}$$

where $\Box_k^2 = \nabla^2 + k_1^2 + k_3^2 - h_1^2$.

Formula (3.12) expresses the function $\phi(x)$ inside the region V by means of the functions $\phi(\xi)$, $\partial\phi(\xi)/\partial n$, $\theta(\xi)$, $\partial\theta(\xi)/\partial n$ given on the surface Σ.

Performing the operation $1/m\, \Box_1$ on equation (3.12) and making use of the properties of the function $\bar{\Phi}$ and of the relation $-(k_1^2 - h_1^2)(k_3^2 - h_1^2) = \varepsilon h_1^2 h_3^2$ we arrive at the analogous formula for the temperature

$$\theta(x)=\int_{\Sigma} d\Sigma(\xi)\left[\bar{\theta}(\xi,x)\frac{\partial\theta(\xi)}{\partial n}-\theta(\xi)\frac{\partial\bar{\theta}(\xi,x)}{\partial n}\right]+\rho\frac{\omega^2}{\alpha}\int_{\Sigma} d\Sigma(\xi)\left[\bar{\phi}(\xi,x)\frac{\partial\phi(\xi)}{\partial n}-\phi(\xi)\frac{\partial}{\partial n}\bar{\phi}(\xi,x)\right],\qquad x\in V, \tag{3.13}$$

where

$$\alpha=\frac{m\gamma}{\varepsilon h_3^2},\qquad \varepsilon=\eta m\kappa.$$

In the case when there is no coupling ($\varepsilon=0$) we obtain from equation (3.13)

$$\theta_{\varepsilon=0}(x)=\int_{\Sigma} d\Sigma\left(\bar{\theta}\frac{\partial\theta}{\partial n}-\theta\frac{\partial\bar{\theta}}{\partial n}\right)_{\varepsilon=0}. \tag{3.14}$$

Since for the uncoupled problem $k_1(0)=h_1$, $h_3(0)=h_3$, it results from equation (3.9) that

$$\bar{\theta}_{\varepsilon=0}=\frac{1}{4\pi r}e^{-ih_3r}.$$

Formula (3.14) assumes the following form

$$\theta_{\varepsilon=0}(x)=\frac{1}{4\pi}\int_{\Sigma} d\Sigma(\xi)\left[\frac{e^{-ih_3r}}{r}\frac{\partial\theta(\xi)}{\partial n}-\theta(\xi)\frac{\partial}{\partial n}\left(\frac{e^{-ih_3r}}{r}\right)\right],\quad r=r(\xi,x) \tag{3.15}$$

Formula (3.15) is known as the Green formula for uncoupled classical equation of thermal conductivity [6].

Let us return to equation (3.12) and assume that we have to deal with a hypothetical medium for which $\alpha_t=0$.

For such a medium we have $\eta=0$, $m=0$. Taking into consideration that

$$\frac{1}{m}(\Box_k^2\bar{\phi})_{\eta=0}=\frac{1}{4\pi r}e^{-ih_1r},\qquad (\bar{\phi})_{\eta=0}=-\frac{m}{4\pi r}\frac{e^{-ih_1r}-e^{-ih_3r}}{h_1^2-h_3^2}. \tag{3.16}$$

Substituting (3.16) in formula (3.12) we find

$$\phi(x)=\frac{1}{4\pi}\int_{\Sigma} d\Sigma(\xi)\left[\frac{e^{-ih_1r}}{r}\frac{\partial\phi(\xi)}{\partial n}-\phi(\xi)\frac{\partial}{\partial n}\left(\frac{e^{-ih_1r}}{r}\right)\right],\qquad r=r(\xi,x). \tag{3.17}$$

This formula can easily be reduced to the case when the motion in the medium takes place in adiabatic conditions (classical elastokinetics). The isothermic Lamè constants μ_T, λ_T appearing in the magnitude h_1 must then be replaced by the adiabatic constant μ_s, λ_s. Equation (3.17) is the Helmholtz equation [7] known form the elastokinetics.

Formulae (3.12) and (3.13), derived for coupled problem of thermoelasticity, do not constitute the complete functions solving the general boundary problem. Therefore we shall require more general integral representations making use of the direct method of integration of the basic equations of thermoelasticity or applying the reciprocity theorem [8, 9].

4. INTEGRAL FORM OF THE GENERAL SOLUTION OF THE EQUATIONS OF COUPLED THERMOELASTICITY

Our purpose is to represent the vector of the displacement and the temperature at an internal point x of the region V by means of the integrals on the surface Σ bounding the region.

Let us assume that the causes producing the motion of a medium are expressed by the boundary conditions.

We shall construct the solution of the following equations of thermoelasticity

$$\sigma_{ij,j} = -\omega^2 \rho u_i, \tag{4.1}$$

$$\theta_{,jj} + h_3^2 \theta + \frac{\gamma}{\alpha} u_{k,k} = 0, \qquad x \in V, \qquad i, j, k = 1, 2, 3, \tag{4.2}$$

where

$$\alpha = \frac{m\gamma}{\varepsilon h_3^2}, \qquad \sigma_{ij} = \sigma_{ij}(x), \qquad u_i = u_i(x), \qquad \theta = \theta(x)$$

and the constitutive equations

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + (\lambda u_{k,k} - \gamma\theta)\delta_{ij}. \tag{4.3}$$

The differential equations (4.1) and (4.2) and relations (4.3) describe the amplitudes of the motion harmonically varying in time.

Let us assume another set of equations corresponding to thermoelastic infinite medium in which there act a concentrated source of heat of intensity κ harmonically varying in time. All the functions appearing in these equations will be denoted by horizontal bars

$$\bar{\sigma}_{ij,j} = -\omega^2 \rho \bar{u}_i \tag{4.4}$$

$$\bar{\theta}_{,jj} + h_3^2 \bar{\theta} + \frac{\gamma}{\alpha} \bar{u}_{k,k} = -\delta(x - \xi),$$

$$\bar{\sigma}_{ij} = \bar{\sigma}_{ij}(x, \xi), \qquad \bar{u}_i = \bar{u}_i(x, \xi), \qquad \bar{\theta} = \bar{\theta}(x, \xi). \tag{4.5}$$

Besides we have to add the Duhamel–Neumann equations

$$\bar{\sigma}_{ij} = 2\mu \bar{\varepsilon}_{ij} + (\lambda \bar{u}_{k,k} - \gamma\bar{\theta})\delta_{ij}. \tag{4.6}$$

Combining the sets of equations (4.1), (4.2), (4.3), and (4.4), (4.5), (4.6) integrating them correspondingly over the region V and making use of the Green transformation we obtain the following equation

$$\theta(x) = \int_\Sigma d\Sigma(\xi) \left[\bar{\theta}(\xi, x) \frac{\partial \theta(\xi)}{\partial n} - \theta(\xi) \frac{\partial \bar{\theta}(\xi, x)}{\partial n} \right] + \frac{1}{\alpha} \int_\Sigma d\Sigma(\xi) [\bar{u}_i(\xi, x) p_i(\xi) - u_i(\xi) \bar{p}_i(\xi, x)], \qquad x \in V. \tag{4.7}$$

Here we have

$$p_i(\xi)=\sigma_{ij}(\xi)n_j(\xi), \qquad \bar{p}_i(\xi, x)=\bar{\sigma}_{ij}(\xi, x)n_j(\xi),$$

while the differentiation under the sign of surface integrals is performed with respect to the variables ξ. Formula (4.7) represents a relation between the temperature θ at a point $x\in V$ and the functions θ, $\partial\theta/\partial n$, u_i, p_i on the surface. The comparison of relations (4.7) and (3.13) gives the relations on the surface Σ between the values of the potential ϕ and $\partial\phi/\partial n$ and the loadings p_i and the surface displacements u_i. In order to obtain an integral representation for the vector $u_i(x)$ for $x\in V$ we must assume two other systems of relations, namely the set of equations (4.1), (4.2), (4.3) and the set of equations

$$\sigma^s_{ij,j}=-\omega^2\rho u^s_i-\delta(x-\xi)\delta_{is}, \tag{4.8}$$

$$\theta^s_{,kk}+h_3^2\theta^s+\frac{\gamma}{\alpha}u^s_{k,k}=0, \tag{4.9}$$

$$\sigma^s_{ij}=2\mu\varepsilon^s_{ij}+(\lambda u^s_{k,k}-\gamma\theta^s)\delta_{ij}, \qquad i, j, k, s=1, 2, 3,$$

$$\sigma^s_{ij}=\sigma^s_{ij}(x, \xi), \qquad u^s_i=u^s_i(x, \xi), \qquad \theta^s=\theta^s(x, \xi). \tag{4.10}$$

This set of equations refers to a concentrated force, harmonically varying in time, applied at a point ξ, and directed along the x_s axis.

Combining correspondingly the sets of equations (4.1), (4.2), (4.3) and (4.8), (4.9), (4.10) we obtain the following formula

$$u_s(x)=\int_\Sigma d\Sigma(\xi)[u^s_k(\xi, x)p_k(\xi)-u_k(\xi)p^s_k(\xi,x)] \\ +\alpha\int_\Sigma d\Sigma(\xi)\left[\theta^s(\xi, x)\frac{\partial\theta(\xi)}{\partial n}-\theta(\xi)\frac{\partial\theta^s(\xi, x)}{\partial n}\right], \tag{4.11}$$

where

$$u^s_k(x, \xi)=u^k_s(x, \xi)=u^s_k(\xi, x), \qquad p^s_k(\xi, x)=\sigma^s_{ij}(\xi, x)n_j(\xi).$$

The functions u^s_k, θ^s as well as $\bar{u}_i$, $\bar{\theta}$ are the Green functions for thermoelastic medium [5] and are the known magnitudes.

Formulae (4.7) and (4.11) constitute the coupled integral representation of the general solution of thermoelasticity. These formulae constitute the generalization of the Somigliana formulae known in elastostatics on the case of thermoelasticity [1]. Observe that the functions θ^s and $\bar{u}_s$ are not arbitrary and satisfy the following relations

$$\bar{u}_s(x, \xi)=\alpha\theta^s(\xi, x), \qquad \theta^s(x, \xi)=-\theta^s(\xi, x). \tag{4.12}$$

This relation can be obtained either by the appropriate combination of the sets of equations (4.4–4.6) and (4.8–4.10) or directly from the reciprocity theorem.

If the Green functions $\bar{u}_i$, $\bar{\theta}$ and u_i^s, θ^s are chosen in such a way that they refer to a body occupying the region V and bounded by a surface Σ and if the boundary conditions

$$\bar{u}_i=0\,, \qquad \bar{\theta}=0\,, \qquad u_i^s=0\,, \qquad \theta^s=0$$

hold on the surface Σ, then equations (4.7) and (4.11) can be reduced to the form

$$\theta(x)=-\int_{\Sigma}\mathrm{d}\Sigma(\xi)\left[\theta(\xi)\frac{\partial\bar{\theta}(\xi,\,x)}{\partial n}+\frac{1}{\alpha}u_i(\xi)\bar{p}_i(\xi,\,x)\right], \tag{4.13}$$

$$u_s(x)=-\int_{\Sigma}\mathrm{d}\Sigma(\xi)\left[u_k(\xi)p_k^s(\xi,\,x)+\alpha\theta(\xi)\frac{\partial\theta^s(\xi,\,x)}{\partial n}\right]. \tag{4.14}$$

Formulae (4.13), (4.14) constitute the solution of the first boundary problem for which the displacements u_i and the temperature θ are given on Σ.

If the functions $\bar{u}_i$, $\bar{\theta}$ and u_i^s, θ^s referred to a body occupying a bounded region V and free from stresses and temperature on the surface then the magnitudes

$$\bar{p}_i=0\,, \qquad \bar{\theta}=0\,, \qquad p_i^s=0\,, \qquad \theta^s=0 \text{ on } \Sigma\,,$$

would have to be substituted in equation (4.7) and (4.11). Then formulae (4.7) and (4.11) take the form

$$\theta(x)=-\int_{\Sigma}\mathrm{d}\Sigma(\xi)\left[\theta(\xi)\frac{\partial\bar{\theta}(\xi,\,x)}{\partial n}-\frac{1}{\alpha}u_i(\xi,\,x)p_i(\xi)\right], \tag{4.15}$$

$$u_s(x)=\int_{\Sigma}\mathrm{d}\Sigma(\xi)\left[u_k^s(\xi,\,x)p_k(\xi)-\theta(\xi)\frac{\partial\theta^s(\xi,\,x)}{\partial n}\right], \tag{4.16}$$

and constitute the solution of the second boundary problem for which the loadings p_k and the temperature θ are given on Σ.

But the application of formulae (4.13–4.16) is limited on account of the difficulties arising in course of determination of the Green functions $\bar{u}_i$, $\bar{\theta}$, u_k^s, θ^s satisfying the prescribed boundary conditions.

5. THERMOELASTIC POTENTIALS AND INTEGRAL EQUATIONS FOR BOUNDARY VALUE PROBLEMS

Let us introduce the thermoelastic surface potentials similarly as for the potentials of elastokinetics [10]. The system of relations

$$V_s(x)=2\int_{\Sigma}\mathrm{d}\Sigma(\xi)\varphi_k(\xi)u_s^k(\xi,\,x)+2\alpha\int_{\Sigma}\mathrm{d}\Sigma(\xi)\psi(\xi)\theta^s(\xi,\,x)\,,$$

$$V(x)=2\int_{\Sigma}\mathrm{d}\Sigma(\xi)\psi(\xi)\bar{\theta}(\xi,\,x)+\frac{2}{\alpha}\int_{\Sigma}\mathrm{d}\Sigma(\xi)\varphi_k(\xi)\bar{u}_k(\xi,\,x)\,, \tag{5.1}$$

will be called the thermoelastic potential of single layer. Here $\varphi_k=\varphi_k(\xi)$, $\psi=\psi(\xi)$ are unknown surface densities of an appropriate regularity. The functions $\bar{u}_k$, $\bar{\theta}$, u_s^k, θ^s are the Green functions given by equations (4.4–4.6) and (4.8–4.10) and they refer to the thermoelastic infinite space. The system

$$W_s(x)=2\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)p_k^s(\xi, x)+2\alpha\int_\Sigma d\Sigma(\xi)\psi(\xi)\frac{\partial\theta^s(\xi, x)}{\partial n},$$

$$W(x)=2\int_\Sigma d\Sigma(\xi)\psi(\xi)\frac{\partial\bar{\theta}(\xi, x)}{\partial n}+\frac{2}{\alpha}\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)\bar{p}_k(\xi, x), \tag{5.2}$$

will be called the thermoelastic potential of double layer. The following symbols have been introduced in formulae (5.2)

$$p_k^s(\xi, x)=[2\mu\varepsilon_{kj}^s+(\lambda u_{p,p}^s-\gamma\theta^s)\delta_{kj}]n_j,$$

$$\bar{p}_k(\xi, x)=[2\mu\bar{\varepsilon}_{kj}+(\lambda\bar{u}_{p,p}-\gamma\bar{\theta})\delta_{kj}]n_j. \tag{5.3}$$

Now, let us define a thermoelastic potential which is a combination of the potentials of single and double layer.

$$M_s(x)=2\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)p_k^s(\xi, x)+2\alpha\int_\Sigma d\Sigma(\xi)\psi(\xi)\theta^s(\xi, x),$$

$$M(x)=2\int_\Sigma d\Sigma(\xi)\psi(\xi)\bar{\theta}(\xi, x)+\frac{2}{\alpha}\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)\bar{p}_k(\xi, x). \tag{5.4}$$

Investigating the jumps of the potentials (5.1), (5.2) and (5.4) at the passage through the surface Σ we introduce the following notation. Let $W_s(\xi_0)$, $W_s^{(i)}(\xi_0)$, and $W_s^{(e)}(\xi_0)$ denote the limits of the vector $W_s(\xi)$ for $\xi\to\xi_0\in\Sigma$ over the surface Σ, $W_s(\xi)$ for $\xi\to\xi_0\in V$ inside the region V, and $W_s(\xi)$ for $\xi\to\xi_0\in\Sigma$ for $\xi\in E-V$, respectively. It can be proved that the potentials $V_s(x)$, $V(x)$ are the continuous functions of points $x\in\Sigma$. We show, however, that the potential of double layer $W_s(x)$, $W(x)$ is discontinuous on this surface. We have

$$\begin{aligned} W_s^{(i)}(\xi_0)&=-\varphi_s(\xi_0)+W_s(\xi_0), \qquad & W^{(i)}(\xi_0)&=-\psi(\xi_0)+W(\xi_0),\\ W_s^{(e)}(\xi_0)&=\varphi_s(\xi_0)+W_s(\xi_0), & W^{(e)}(\xi_0)&=\psi(\xi_0)+W(\xi_0). \end{aligned} \tag{5.5}$$

These relations are analogous to those for jumps of the harmonic potential of double layer. We prove that the first surface integral in formulae (5.2) is a discontinuous function while the second one represents a continuous function.

From the first equation of the set (4.8) the following relation results

$$\varphi_k(x)\sigma_{kj,j}^s(x, \xi)=-\delta(x-\xi)\varphi_s(\xi)-\omega^2\rho u_k^s(x, \xi)\varphi_k(x) \tag{5.6}$$

Observing that

$$\int_V \mathrm{d}V(x)\delta(x-\xi)=h(\xi)=\begin{cases}1 \text{ for } \xi\in V\,,\\ \frac{1}{2} \text{ for } \xi\in\Sigma\,,\\ 0 \text{ for } \xi\in E-V\,,\end{cases} \tag{5.7}$$

and integrating relation (5.6) with respect to $x\in V$ and changing the variables we get

$$2\int_\Sigma \mathrm{d}\Sigma(\xi)\varphi_k(\xi)p_k^s(\xi,\,x)=-2h(x)\varphi_s(x)+g_s(x) \tag{5.8}$$

where

$$g_s(x)=2\int_V \mathrm{d}V(\xi)\varphi_{i,j}(\xi)\sigma_{ij}^s(\xi,\,x)-2\rho\omega^2\int_V \mathrm{d}V(\xi)u_k^s(x,\,\xi)\varphi_k(\xi)\,.$$

It can be proved that $g_s(x)$ is a combination of volume integrals is continuous for $x\in\Sigma$.

Similarly, taking into account equation (4.9) we verify that the second term of the potential W_s is a continuous function. Formulae (5.8) and (5.2) furnish the first group of relations (5.5). The discontinuity of the function $W=W(x)$ can be derived in the same way by the integration of equations (4.4) and (4.5). The second term in formula for $W(x)$ is continuous on the surface Σ.

We introduce the notation

$$\hat{p}_i(x)=[2\mu V_{(i,\,j)}+\lambda V_{k,\,k}\delta_{ij}-\gamma V\delta_{ij}]n_j(x)\,,$$
$$\hat{\theta}(x)=V_{,k}n_k(x)\,. \tag{5.9}$$

where the functions V_s, V are given by formula (5.1). It can be shown that

$$\hat{p}_k^{(i)}(\xi_0)=\varphi_k(\xi_0)+\hat{p}_k(\xi_0)\,,\qquad \hat{\theta}^{(i)}(\xi_0)=\psi(\xi_0)+\hat{\theta}(\xi_0)\,,$$
$$\hat{p}_k^{(e)}(\xi_0)=-\varphi_k(\xi_0)+\hat{p}_k(\xi_0)\,,\qquad \hat{\theta}^{(e)}(\xi_0)=-\psi(\xi_0)+\hat{\theta}(\xi_0)\,. \tag{5.10}$$

The thermoelastic potentials (5.1–5.4) and the jump relations for these potentials allows us to reduce the basic boundary problems of thermoelasticity to the solution of singular integral equations.

Let us confine our considerations only to certain typical problems. We consider the case when the displacements are given on the boundary and simultaneously the temperature or flux of temperature are prescribed on the surface Σ.

Let us assume that the displacements $u_s(\xi_0)=f_s(\xi_0)$ and the temperature $\theta(\xi_0)=g(\xi_0)$ are given on the boundary Σ.

The solution to the problem is required in the form of the potential of double layer (5.2); we assume

$$U_s(x)=W_s(x),\qquad \theta(x)=W(x)\,. \tag{5.11}$$

We can verify easily that the functions $U_s(x)$, $\theta(x)$ satisfy the equation

$$L_{sk}U_k-\gamma\partial_s\theta=0\,,\qquad \Box_3^2\theta+\frac{\gamma}{\alpha}\partial_kU_k=0\,,\qquad x\in V\,, \tag{5.12}$$

where

$$L_{sk}=(\mu\partial_p\partial_p+\omega^2\rho)\delta_{sk}+(\lambda+\mu)\partial_s\partial_k\,.$$

Taking into account relations (5.5) for the functions $\varphi_k(\xi)$, $\psi(\xi)$ we obtain the system of coupled integral equations

$$\begin{aligned}
&\varphi_s(\xi_0)-2\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)p_k^s(\xi,\,\xi_0)-2\alpha\int_\Sigma d\Sigma(\xi)\psi(\xi)\frac{\partial\theta^s(\xi,\,\xi_0)}{\partial n}=-f_s(\xi_0)\,,\\
&\psi(\xi_0)-2\int_\Sigma d\Sigma(\xi)\psi(\xi)\frac{\partial\bar\theta(\xi,\,\xi_0)}{\partial n}-\frac{2}{\alpha}\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)\bar p_k(\xi,\,\xi_0)=-g(\xi_0)\,.
\end{aligned} \tag{5.13}$$

Equations (5.13) have the form of Fredholm singular integral equations of the second kind; the integrals appearing there should be understood in the sense of Cauchy principal values.

For the uncoupled problem we obtain from equations (5.11) and (5.13)

$$\begin{aligned}
U_s\big|_{\varepsilon=0}&=2\int_\Sigma d\Sigma\varphi_k(p_k^s)_{\varepsilon=0}+2\int_\Sigma d\Sigma\psi\left(\frac{\partial\bar u_s}{\partial n}\right)_{\varepsilon=0},\\
\theta\big|_{\varepsilon=0}&=2\int_\Sigma d\Sigma\psi\left(\frac{\partial\bar\theta}{\partial n}\right)_{\varepsilon=0},
\end{aligned} \tag{5.14}$$

respectively. Here the functions φ_k and ψ satisfy the uncoupled integral equations

$$\begin{aligned}
&\varphi_s(\xi_0)-2\int_\Sigma d\Sigma\varphi_k(p_k^s)_{\varepsilon=0}=-f_s(\xi_0)+2\int_\Sigma d\Sigma\psi\left(\frac{\partial\bar u_s}{\partial n}\right)_{\varepsilon=0},\\
&\psi(\xi_0)-2\int_\Sigma d\Sigma\psi\left(\frac{\partial\bar\theta}{\partial n}\right)_{\varepsilon=0}=-g(\xi_0)\,.
\end{aligned} \tag{5.15}$$

Let us assume that the displacements $u_i(\xi_0)=f_i(\xi_0)$ and the flux of heat $\partial\theta/\partial n|_{\xi=\xi_0}=S(\xi_0)$ are given on the boundary Σ. The solution is required in the form

$$\hat U_s(x)=M_s(x)\,,\qquad \tilde\theta(x)=M(x)\,,\qquad x\in V\,, \tag{5.16}$$

where the functions M_s, M are given by formulae (5.4). We easily verify that inside the region V equations (5.12) are satisfied and that the unknown densities $\varphi_k(\xi)$, $\psi(\xi)$ satisfy the following system of singular integral equations

$$\begin{aligned}
&\varphi_s(\xi_0)-2\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)p_k^s(\xi,\,\xi_0)-2\alpha\int_\Sigma d\Sigma(\xi)\theta^s(\xi,\,\xi_0)=-f_s(\xi_0)\,,\\
&\psi(\xi_0)+2\int_\Sigma d\Sigma(\xi)\psi(\xi)\frac{\partial\bar\theta(\xi,\,\xi_0)}{\partial n_0}+\frac{2}{\alpha}\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)\frac{\partial\bar p_k(\xi,\,\xi_0)}{\partial n_0}=S(\xi_0)\,,
\end{aligned} \tag{5.17}$$

where

$$\frac{\partial\bar{\theta}(\xi,\xi_0)}{\partial n_0}=\lim_{x\to\xi_0}\frac{\partial}{\partial n_x}\bar{\theta}(\xi,x)\text{ for }x\in\Sigma,$$

and in the same way we define $\partial p_k(\xi,\xi_0)/\partial n_0$.

In the case of the uncoupled problem equations (5.17) can be considerably simplified

$$\varphi_s(\xi_0)-2\int_\Sigma d\Sigma(\xi)\varphi_k(\xi)(p_k^s(\xi,\xi_0))_{\varepsilon=0}-2\int_\Sigma d\Sigma(\xi)\psi(\xi)(\bar{u}_s(\xi_0,\xi))_{\varepsilon=0}=-f_s(\xi_0),$$

$$\psi(\xi_0)+2\int_\Sigma d\Sigma(\xi)\psi(\xi)\left(\frac{\partial\bar{\theta}(\xi,\xi_0)}{\partial n}\right)_{\varepsilon=0}=S(\xi_0). \tag{5.18}$$

We are not going to rewrite the integral equations for a boundary problem for which loadings are given on the boundary. We observe, however, that if a loading $p_i=p_i(\xi_0)$ and a flux $S=S(\xi_0)$ are given on the boundary Σ then the required solution is given in the form of the potentials of single layer $V_s(x)$, $V(x)$. By means of relations (5.9) and (5.10) the corresponding singular integral equations can be written down in the explicit form.

The investigation of the existence and the uniqueness of the obtained integral equations can be proved in a similar way as it was done for the elastic potentials [10].

6. CANONICAL FUNCTIONAL INTEGRAL EQUATIONS AND APPROXIMATE SOLUTIONS

Let us assume that the normal derivative of the thermoelastic displacement potential $\partial\phi/\partial n|_\Sigma=f(\xi)$ and the temperature $\theta(\xi)=g(\xi)$ are given on the boundary Σ of a body. Taking into account relations (3.12), (3.13) we arrive at the following functional equations for $x\in E-V$:

$$0=\int_\Sigma d\Sigma(\xi)\left\{\left[\bar{\Phi}(\xi,x)\frac{\partial\theta(\xi)}{\partial n}-g(\xi)\frac{\partial\bar{\Phi}(\xi,x)}{\partial n}\right]+\frac{1}{m}\left[(\Box_k^2\bar{\Phi}(\xi,x)f(\xi)-\phi(\xi)\frac{\partial}{\partial n}(\Box_k^2\bar{\Phi}(\xi,x))\right]\right\}, \tag{6.1}$$

$$0=\int_\Sigma d\Sigma(\xi)\left[\bar{\theta}(\xi,x)\frac{\partial\theta(\xi)}{\partial n}-g(\xi)\frac{\partial\bar{\theta}(\xi,x)}{\partial n}\right]+\frac{\rho\omega^2}{\alpha}\int_\Sigma d\Sigma(\xi)\left[\bar{\Phi}(\xi,x)f(\xi)-\phi(\xi)\frac{\partial\bar{\Phi}(\xi,x)}{\partial n}\right]. \tag{6.2}$$

In equations (6.1) and (6.2) the unknown functions are $\partial\theta/\partial n$ and ϕ on the surface Σ.

If these functions are known then also θ and ϕ for $x\in V$ are known in accordance with formulae (3.12) and (3.13). Equations (6.1) and (6.2) for which the functions ϕ, $\partial\theta/\partial n$ appear only under the sign of integral, and for which the regions of variability of the points x and ξ do not coincide, are called the canonical functional equations. It can be proved that equations (6.1) and (6.2) have only one solution for the functions ϕ and $\partial\theta/\partial n$ for $\xi\in\Sigma$.

Let us introduce the symbols

$$\frac{\partial\theta(\xi)}{\partial n}=X(\xi), \qquad \phi(\xi)=Y(\xi). \tag{6.3}$$

Equations (6.1) and (6.2) can be solved in an approximate way replacing them by linear algebraic equations by means of the mechanical quadrature.

We select N points $x_j (j=1, 2, \ldots, N)$ on a certain surface Σ' containing the entire region V. The points x_j may be taken the points of intersection of the N normals to the surface Σ with the surface Σ'. As a rule, these normals are uniformly distributed on Σ. In this case the kernels of the integrands in equations (6.1) and (6.2) are bounded functions for $x_j \in \Sigma'$ and $\xi \in \Sigma$. Thus we can apply the mechanical quadrature.

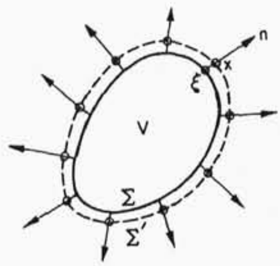

FIG. 1.

Equations (6.1), (6.2) lead to the approximate relations

$$\sum_{i=1}^{N} A_i^{(N)}\bar{\Phi}(\xi_i, x_j)X(\xi_i)-\frac{1}{m}\sum_{i=1}^{N} A_i^{(N)}\frac{\partial}{\partial n}(\Box_k^2\bar{\Phi}(\xi_i, x_j))Y(\xi_i)=a_j, \tag{6.4}$$

$$\sum_{i=1}^{N} A_i^{(N)}\bar{\theta}(\xi_i, x_j)X(\xi_i)-\frac{\rho\omega^2}{\alpha}\sum_{i=1}^{N} A_i^{(N)}\frac{\partial}{\partial n}\bar{\Phi}(\xi_i, x_j)Y(\xi_i)=b_j, \tag{6.5}$$

where

$$a_j=\int_\Sigma \mathrm{d}\Sigma(\xi)\left[g(\xi)\frac{\partial}{\partial n}\bar{\Phi}(\xi, x_j)-\frac{1}{m}f(\xi)\Box_k^2(\bar{\Phi}(\xi, x_j))\right],$$

$$b_j=\int_\Sigma \mathrm{d}\Sigma(\xi)\left[g(\xi)\frac{\partial}{\partial n}\bar{\theta}(\xi, x_j)-\frac{\rho\omega^2}{\alpha}f(\xi)\bar{\Phi}(\xi, x_j)\right].$$

Here $A_i^{(N)}$ are the coefficients of the given mechanical quadrature. The set of equations (6.4), (6.5) has $2N$ unknowns $X(\xi_i)$, $Y(\xi_i)$, $i=1, 2, \ldots N$, and can be solved when its fundamental determinant is not equal to zero, this, as a rule, can be achieved by an appropriate selection of the points x_j on the surface Σ'.

The approximate form of the potential $\phi(x)$, $\theta(x)$ for the points situated inside the region V is given by the formulae

$$\phi(x)=\sum_{i=1}^{N} A_i^{(N)}\left[\bar{\Phi}(\xi_i, x)X(\xi_i)-\frac{1}{m}\frac{\partial}{\partial n}(\Box_k^2\bar{\Phi}(\xi_i, x))Y(\xi_i)\right]$$
$$-\int_\Sigma \mathrm{d}\Sigma(\xi)\left[g(\xi)\frac{\partial\bar{\Phi}(\xi, x)}{\partial n}-\frac{1}{m}f(\xi)\Box_k^2\bar{\Phi}(\xi, x)\right], \tag{6.6}$$

$$\theta(x)=\sum_{i=1}^{N} A_i^{(N)}\left[\bar{\theta}(\xi_i, x)X(\xi_i)-\frac{\rho\omega^2}{\alpha}\frac{\partial}{\partial n}\bar{\phi}(\xi_i, x)Y(\xi_i)\right] \\ -\int_\Sigma d\Sigma(\xi)\left[g(\xi)\frac{\partial\bar{\theta}(\xi, x)}{\partial n}-\frac{\rho\omega^2}{\alpha}f(\xi)\bar{\phi}(\xi, x)\right]. \tag{6.7}$$

Let us consider an arbitrary, bounded, thermoelastic body with the displacements $f_i(\xi)$ and the temperature $g(\xi)$ prescribed on Σ. Making use of relations (4.7) and (4.11) and applying the analogous procedure of the approximate solution we obtain for the displacements and the temperature inside the body the following relations

$$u_s(x)=\sum_{i=1}^{N} A_i^{(N)}[\varphi_k(\xi_i)u_k^s(x, \xi_i)+\alpha\psi(\xi_i)\theta^s(\xi_i, x)] \\ -\int_\Sigma d\Sigma(\xi)\left[f_i(\xi)p_i^s(\xi, x)+\alpha g(\xi)\frac{\partial\theta^s(\xi, x)}{\partial n}\right], \tag{6.8}$$

$$\theta(x)=\sum_{i=1}^{N} A_i^{(N)}\left[\frac{1}{\alpha}\varphi_k(\xi_i)\bar{u}_k(\xi_i, x)+\psi(\xi_i)\bar{\theta}(\xi_i, x)\right] \\ -\int_\Sigma d\Sigma(\xi)\left[\frac{1}{\alpha}f_k(\xi)\bar{p}_k(\xi, x)+g(\xi)\frac{\partial\bar{\theta}(\xi, x)}{\partial n}\right], \tag{6.9}$$

where $3N+N$ unknown values $\varphi_k(\xi_i)$, $\psi(\xi_i)$ $k=1, 2, 3$, $i=1, 2, \ldots, N$ constitute the solution of the following linear system of algebraic equations

$$u_s(x_j)=0, \qquad \theta(x_j)=0, \qquad x_j\in\Sigma', \qquad V'\supset V, \qquad j=1, 2, \ldots, N. \tag{6.10}$$

REFERENCES

[1] E. Trefftz, *Mathematische Elastizitätslehre*, Vol. 6, Chapter 2. Springer (1926).
[2] M. A. Biot, *J. Appl. Phys.* **27**, 240 (1956).
[3] P. Chadwick, *Progress in Solid Mechanics*, Vol. 1, Chapter 6. North Holland (1960).
[4] J. Ignaczak and W. Nowacki, *Arch. Mech. Stos.* **14**, 3 (1962).
[5] W. Nowacki, *Bull. L'Acad. Polon. Sci. Sèr. sci. techn.* **12**, 315 (1964).
[6] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Vol. 1, Chapter 7. McGraw-Hill (1953).
[7] B. B. Baker and E. T. Copson, *The Mathematical Theory of Huygens' Principle.* Clarendon Press, Oxford (1953).
[8] V. Ionescu-Cazimir, *Bull. L'Acad. Polon. Sci. Sèr. sci. techn.* **12**, 473 (1964).
[9] W. Nowacki, *Bull. L'Acad. Polon. Sci. Ser. sci. techn.* **12**, 514 (1964).
[10] V. D. Kupradze, *Progress in Solid Mechanics*, Vol. 3, Chapter 6. North Holland (1963).

(*Received* 10 *July* 1965)

Résumé—Les relations intégrales analogues à celles de Somigliana et Helmoltz en élastocinétique ont été utilisées pour l'intégration des équations différentielles de base de thermoélasticité couplée. On obtient ainsi la représentation des déplacements, de la température et du potentiel thermoélastique du déplacement sur les intégrales de surface.

On a utilisé les solutions pour déterminer les potentiels thermoélastiques généraux de simple ou de double couche. Au moyen de ces potentiels le problème de la base des limites de thermoélasticité couplée sont ramenés à la solution d'un systéme d'équations intégrales simples.

Zusammenfassung—Durch das Integrieren der Grunddifferentialgleichungen war es möglich integrale Beziehungen einzuführen, welche den von Somigliana und Helmholtz gefundenen elastokinetischen Beziehungen analog sind. Auf diese Weise konnten die Verschiebungen, die Temperatur und das thermoelastische Verschiebepotential in der Form von Oberflächenintegralen erhalten werden.

Die Lösungen wurden dazu herangezogen, die allgemeinen thermoelastischen Potentiale von Einzel- und Doppelschichten zu konstruieren, Mit Hilfe dieser Potentiale konnte die Lösung des grundlegenden Grenzwertproblems der gekoppelten Thermoelastizität auf die Lösung eines Systems singulärer Integralgleichungen reduziert werden.

Sommario—I rapporti integrali analoghi a quello del Somigliane e dell'Helmholtz nell'elastocinetica sono stati introdotti con l'integrazione di equazioni basiche differenziali di termoelasticità accoppiata. In tal modo abbiamo ottenuto la rappresentazione degli spostamenti, della temperatura e del potenziale di spostamento termoelastico mediante integrali di superficie.

Le soluzioni sono state utilizzate per costruire i potenziali termoelastici generali di strato singolo e doppio. Tramite tali potenziali, il problema fondamentale limite della termoelasticità accoppiata è stato ridotto alla soluzione di un sistema di equazioni integrali singolari.

Абстракт—С помощью интегрирования основных дифференциальных уравнений термоупругости с взаимодействием вволятся интегральные соотношения аналогичные с соотношениами Сомиляна и Гельмгольтца в теории динамической упругости. Таким образом получается представление перемещений, температуры и потенциала термоупругого перемещения в виде поверхностных интегралов.

Решения используются для построения общих термоупругих потенциалов простого и двойного слоя. С помощью этих потенциалов основная краевая задача термоупругости с взаимодействием сводится к решению системы сингулярных интегральных уравнений.