POLISH ACADEMY OF SCIENCES
INSTITUTE OF BASIC TECHNICAL PROBLEMS

PROCEEDINGS OF VIBRATION PROBLEMS

QUARTERLY

VOL. 2 WARSAW 1961 No 4(9)
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PROCEEDINGS
OF VIBRATION PROBLEMS
WARSAW 1961
VOL. 2, No 4(9)

THE PLANE DYNAMIC PROBLEM OF THERMOELASTICITY

JÓZEF IGNACZAK and WITOLD NOWACKI (WARSAW)

1. Introduction

The object of the present paper is to present several solution methods of plane dynamic problems of thermoelasticity. In the first part, the non-coupled thermoelastic problem is formulated in stresses and in strains. It is shown that the particular integrals differ in the two methods of solution methods by a constant only.

The second part presents two examples of solution of the non-coupled problem by means of the stress function. This is the problem of forced vibration of an infinite rectangular prism and an elastic layer heated at the edge in a manner harmonically variable in time.

The third part contains the solution of the plane dynamic problem of thermoelasticity for coupled temperature and displacement fields, by introducing a stress function and resolving the displacement vector. Such a procedure is illustrated by way of an example of forced vibration of an infinite rectangular prism and an elastic layer due to heat sources harmonic in time and uniformly distributed inside the region of the prism.

2. Plane Non-Coupled Dynamic Problem of Thermoelasticity

Let us consider an elastic body under the action of a temperature field and in a plane state of strain. It is assumed, first, that the temperature field is not coupled with the strain field. It is assumed also that the mass and surface forces are equal to zero.

In plane strain, the stress-strain relations are given by the equations:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma T)\delta_{ij} \qquad (i, j = 1, 2), \tag{2.1}$$

where μ, λ are LAMÉ constants, T — temperature, $\gamma = (3\lambda + 2\mu)\alpha_t$ where α_t is the coefficient of thermal dilatation, δ_{ij} KRONECKER's delta.

Observe that

$$\sigma_{kk} = 2(\lambda+\mu)\varepsilon_{kk}-2\gamma T. \tag{2.2}$$

The strains are connected with the displacements u_i by the relations:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}) \qquad (i,j = 1,2). \tag{2.3}$$

The displacements, strains, stresses and temperature are functions of the time t and the variables x_1, x_2.

The point of departure for our considerations are the equations of motion (the mass forces are disregarded)

$$\sigma_{ij,j} = \varrho \ddot{u}_i \qquad (i,j = 1,2) \tag{2.4}$$

and the heat equation

$$\Box_3^2 T = -Q/\varkappa, \quad \Box_3^2 = \nabla^2-\frac{1}{\varkappa}\partial_t, \tag{2.5}$$

where ϱ is the density per unit volume, Q — a function of heat sources and $\varkappa$ — the coefficient of heat conduction.

Let us differentiate the first of Eqs. (2.4) with respect to x_1, the second with respect to x_2 and add and substract the equations thus obtained, bearing in mind Eqs. (2.1). We obtain

$$\sigma_{11,11}+\sigma_{22,22}+2\sigma_{12,12} = \varrho\ddot{\varepsilon}_{kk} \tag{2.6}$$

$$\left(\partial_1^2-\frac{1}{2c_2^2}\partial_t^2\right)\sigma_{11}-\left(\partial_2^2-\frac{1}{2c_2^2}\partial_t^2\right)\sigma_{22} = 0, \quad c_2^2 = \mu/\varrho. \tag{2.7}$$

Let us differentiate the first equation of the set (2.4) with respect to x_2, the second to x_1, and add. Then:

$$\sigma_{kk,12}+\Box_2^2\sigma_{12} = 0, \quad \Box_2^2 = \nabla^2-\frac{1}{c_2^2}\partial_t^2. \tag{2.8}$$

Substituting the stresses from Eq. (2.1) into (2.6), eliminating the quantity σ_{12}, by means of the compatibility equation

$$\varepsilon_{11,22}+\varepsilon_{22,11} = 2\varepsilon_{12,12}, \tag{2.9}$$

and making use of (2.2), we obtain the equations;

$$\Box_1^2\sigma_{kk}+2\mu\bar{m}\Box_2^2 T = 0, \quad \Box_1^2 = \nabla^2-\frac{1}{c_1^2}\partial_t^2, \quad c_1^2 = \frac{\lambda+2\mu}{\varrho}, \quad \bar{m} = \frac{\gamma}{\lambda+2\mu}. \tag{2.10}$$

Let us express the stresses in terms of the stress function F in the form:

$$\sigma_{ij} = -F_{,ij}+\delta_{ij}\left(\nabla^2-\frac{1}{2c_2^2}\partial_t^2\right)F \qquad (i,j = 1,2). \tag{2.11}$$

It is seen that, by eliminating the inertia terms — that is for the quasi-static problem — this function becomes the AIRY function.

It is also seen that by expressing the stresses in Eqs. (2.7) and (2.8) by means of the function F, these equations are satisfied identically. Substituting in (2.10) the equation

$$\sigma_{kk} = \Box_2^2 F, \tag{2.12}$$

we obtain the following differential equation for the function F

$$\Box_1^2 \Box_2^2 F + 2\mu\bar{m}\Box_2^2 T = 0. \tag{2.13}$$

For the isothermal problem, this equation becomes the equation of J. R. M RADOK, [1]. The solution of (2.13) may be composed of two parts: $F = F_0 + F^*$, where F_0 is a particular integral satisfying the equation

$$\Box_1^2 F_0 + 2\mu\bar{m} T = 0, \tag{2.14}$$

where

$$\sigma_{ij}^{(0)} = -F_{0,ij} + \delta_{ij}\left(\nabla^2 - \frac{1}{2c_2^2}\partial_t^2\right) F_0 \tag{2.15}$$

and F^* is the general integral of the homogeneous equation

$$\Box_1^2 \Box_2^2 F^* = 0. \tag{2.16}$$

Eq. (2.16) may be replaced by a system of two equations, by making use of the theorem of T. BOGGIO, [2]:

$$\Box_1^2 F_1 = 0, \quad \Box_2^2 F_2 = 0, \tag{2.17}$$

where

$$F^* = F_1 + F_2, \quad \sigma_{ij}^* = -F_{ij}^* + \delta_{ij}\left(\nabla^2 - \frac{1}{2c_2^2}\partial_t^2\right) F^*. \tag{2.18}$$

Another way of solving the plane problem consists in assumption solving the displacement vector in the form:

$$\mathbf{u} = \operatorname{grad} \Phi + \operatorname{rot} \boldsymbol{\psi}. \tag{2.19}$$

Substituting (2.19) in the equations of motion in displacements

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \gamma\operatorname{grad} T = \varrho\ddot{\mathbf{u}}, \tag{2.20}$$

we obtain the system of equations:

$$\Box_1^2 \Phi = \bar{m} T, \quad \Box_2^2 \boldsymbol{\psi} = 0. \tag{2.21}$$

The solution of Eqs. (2.21) is composed of a particular integral Φ_0 satisfying the equation

$$\Box_1^2 \Phi_0 = \bar{m} T, \tag{2.22}$$

and of the general integrals of

$$\Box_1^2 \Phi_1 = 0, \quad \Box_2^2 \psi = 0. \tag{2.23}$$

Let us observe that Eq. (2.22) becomes (2.14) if we assume that $F_0 = -2\mu\Phi_0$. It is equally easy to show that the stresses corresponding to the function Φ_0 are given by the equations

$$\sigma_{ij}^{(0)} = 2\mu\left[\Phi_{0,ij} - \delta_{ij}\left(\nabla^2 - \frac{1}{2c_2^2}\partial_t^2\right)\Phi_0\right] \qquad (i,j = 1,2). \tag{2.24}$$

Still another method of solution consists in composing the displacement u_i of the terms $u_i^{(0)} = \Phi_{0,i}$ and of the term $u_i^{(1)}$, satisfying the homogeneous system of displacement equations

$$\mu u_{i,kk}^{(1)} + (\lambda+\mu) u_{k,ki}^{(1)} - \varrho \ddot{u}_i^{(1)} = 0. \tag{2.25}$$

These equations may be expressed in the operational form

$$L_{ij} u_i^{(1)} = 0, \quad L_{ij} = \Box_2^2 \delta_{ij} + a\,\partial_i\,\partial_j, \quad a = \frac{\lambda+\mu}{\mu}. \tag{2.26}$$

Introducing the stress functions χ_1, χ_2, and using them to express the dis placements $u_i^{(1)}$

$$u_1^{(1)} = L_{22}\chi_1 - L_{12}\chi_2, \quad u_2^{(1)} = -L_{21}\chi_1 + L_{11}\chi_2, \tag{2.27}$$

or

$$u_i^{(1)} = [\Box_2^2 \delta_{ij} + a(\nabla^2 \delta_j - \partial_{ii}\partial_j)]\chi_j \qquad (i,j = 1,2), \tag{2.28}$$

we shall obtain the following system of equations

$$\Box_1^2 \Box_2^2 \chi_i = 0 \quad (i = 1,2). \tag{2.29}$$

The functions χ_1 are B. G. Galerkin's functions [3], [4], generalized to the plane dynamic problem. In most cases only one displacement function ($\chi_2 = \chi$, $\chi_1 = 0$, for instance) will suffice to find the stresses.

The simplest of the methods just described is that which involves applying the stress function F. It has the advantage of obtaining the quasi-static problem by rejecting the inertia terms (that is by cancelling the time derivatives of F).

In further considerations, we shall discuss the solution problem of Eq. (2.13).

We shall be concerned first with the particular integral of (2.14). For the infinite plate, this integral constitutes the final solution of the problem. In this case, there are no edge conditions and we are concerned only with a longitudinal thermoelastic wave.

For solving the system of equations

$$\Box_3^2 T = -Q/\varkappa, \quad \Box_1^2 F_0 + 2\mu\overline{m} T = 0, \tag{2.30}$$

we introduce the auxiliary function S, satisfying the equation

$$\Box_1^2 S = -Q/\varkappa, \tag{2.31}$$

with the same boundary conditions as the function T.

Let us perform the operation $\Box_3^2$ on the second of Eqs. (2.30). We obtain:

$$\Box_1^2\Box_3^2 F_0 = \frac{2\mu\bar{m}Q}{\varkappa}. \tag{2.32}$$

Bearing in mind that

$$\Box_1^2\Box_3^2 = \frac{\Box_1^2-\Box_3^2}{(\Box_3^2)^{-1}-(\Box_1^2)^{-1}}, \quad \Box_1^2-\Box_3^2 = \frac{1}{\varkappa}\,\partial_t-\sigma_1^2\partial_t^2, \quad \sigma_1^2 = 1/c_1^2$$

we obtain from Eq. (2.32):

$$\left(\frac{1}{\varkappa}\,\partial_t-\sigma_1^2\partial_t^2\right)F_0 = \frac{2\mu\bar{m}}{\varkappa}\,[(\Box_3^2)^{-1}-(\Box_1^2)^{-1}](Q). \tag{2.33}$$

Since

$$T = -\frac{1}{\varkappa}\,(\Box_3^2)^{-1}(Q), \quad S = -\frac{1}{\varkappa}\,(\Box_1^2)^{-1}(Q),$$

therefore (2.33) takes the form:

$$\left(\frac{1}{\varkappa}\,\partial_t-\sigma_1^2\partial_t^2\right)F_0 = -2\mu\bar{m}[T-S]. \tag{2.34}$$

Let us subject (2.34) to the LAPLACE transformation, assuming that $F_0(x_1,x_2,0) = 0$, $\dot{F}_0(x_1,x_2,0) = 0$. These conditions follow from the assumption that the body is free from stress for $t \leqslant 0$. We have

$$\left\{\begin{aligned} &\bar{F}_0(x_1,x_2,p) = -\frac{2\mu\bar{m}}{p/\varkappa-\sigma_1^2p^2}\,(\bar{T}-\bar{S}),\\ &\bar{F}_0(x_1,x_2,p) = \int_0^\infty e^{-pt}F_0(x_1,x_2,t)dt.\end{aligned}\right. \tag{2.35}$$

The quantities $\bar{T}$ and $\bar{S}$ will be found from the equations:

$$\nabla^2\bar{T}-\frac{p}{\varkappa}\,\bar{T} = -\frac{\bar{Q}}{\varkappa}, \quad \nabla^2\bar{S}-p^2\sigma_1^2\bar{S} = -\frac{\bar{Q}}{\varkappa}. \tag{2.36}$$

Here also, homogeneous initial conditions are assumed for the functions T and S.

The transform of $\bar{F}_0(x_1, x_2, p)$ may be expressed by performing on Eq. (2.35) the double FOURIER transformation

$$(2.37)\qquad \bar{F}_0(x_1, x_2, p) = -\frac{\mu\bar{m}}{\pi p(\varkappa^{-1}-\sigma_1^2 p)} \int_{-\infty}^{\infty}\int (T^*-S^*) \exp[-i(\alpha_1 x_1 + \alpha_2 x_2)] d\alpha_1\, d\alpha_2,$$

where

$$(2.38)\qquad T^* = \frac{Q^*}{\varkappa(\alpha_1^2+\alpha_2^2+p\varkappa^{-1})}, \quad S^* = \frac{Q^*}{\varkappa(\alpha_1^2+\alpha_2^2+p\sigma_1^2)}$$

and

$$(2.39)\qquad Q^*(\alpha_1, \alpha_2, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int \bar{Q}(x_1, x_2, p) \exp[i(\alpha_1 x_1+\alpha_2 x_2)] dx_1\, dx_2.$$

Performing the inverse LAPLACE transformation, we obtain the function sought, F_0, and from Eq. (2.15) the stresses $\sigma_{ij}^{(0)}$. For an infinite plane, they will constitute the final stresses.

In the particular case of axially symmetric temperature field, the function $\bar{F}_0(r, p)$ may be represented by HANKEL transform in the form:

$$(2.40)\qquad \bar{F}_0(r, p) = -\frac{2\mu\bar{m}}{p(\varkappa^{-1}-\sigma_1^2 p)} \int_0^\infty (T^*-S^*)\alpha J_0(\alpha r) d\alpha,$$

where

$$(2.41)\qquad T^*(\alpha, p) = \frac{Q^*}{\varkappa\alpha^2+p}, \quad S^* = \frac{Q^*}{\varkappa(\alpha^{-2}+\sigma_1^2 p^2)}$$

and

$$(2.42)\qquad Q^*(\alpha, p) = \int_0^\infty \bar{Q}(r, p) r J_0(\alpha r) dr.$$

Only a few cases of wave propagation of cylindrical thermoelastic waves in the infinite space or plate have hitherto been solved. Above all should be mentioned the solution of the thermal shock in the region of a cylinder of infinite length, obtained by T. MURA, [5], and that of sudden point heating of the infinite plate by H. PARKUS, [6]. Attention is also directed to the solution by W. DERSKI, [7], and that of the two-dimensional problem by J. IGNACZAK, [8]. The latter concerns a temperature field, discontinuous in time and space.

An example of a plane problem will be given below. A solution of forced vibration of an infinite cylinder will be given with rectangular cross-section heated on the surface in a manner harmonically variable in time. The above stress function method will be used.

3. Forced Thermoelastic Vibration of a Rectangular Prism

Let us consider an infinite rectangular prism with the sides a_1 and a_2, the surface being free from stress and heated to the temperature $T_0 e^{i\omega t}$, where ω is the frequency.

The problem is to solve the heat equation

$$\left(\partial_1^2+\partial_2^2-\frac{1}{\varkappa}\partial_t\right)T=0 \tag{3.1}$$

in the region $|x_1|<\frac{a_1}{2}, \quad |x_2|<\frac{a_2}{2}$

with the boundary conditions

$$\begin{cases} T\left(\pm\frac{a_1}{2}, x_2, t\right)=T_0 e^{i\omega t}=e^{i\omega t}\dfrac{4T_0}{a_2}\displaystyle\sum_{m=1,3,\dots}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{\beta_m}\cos\beta_m x_2; \quad \beta_m=\frac{m\pi}{a_2}, \\ T\left(x_1, \pm\frac{a_2}{2}, t\right)=T_0 e^{i\omega t}=e^{i\omega t}\dfrac{4T_0}{a_1}\displaystyle\sum_{n=1,3,\dots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{\alpha_n}\cos\alpha_n x_1, \quad \alpha_n=\frac{n\pi}{a_1}. \end{cases} \tag{3.2}$$

Next, we solve the equation

$$(\partial_1^2+\partial_2^2-i\eta)U=0, \quad \eta=\omega/\varkappa, \tag{3.3}$$

obtained from (3.1) by inserting $T(x_1, x_2, t)=e^{i\omega t}U(x_1, x_2)$.

The solution of (3.3) with the boundary conditions (3.2) takes the form:

$$U(x_1, x_2)=4T_0\left\{\frac{1}{a_1}\sum_{n=1,3,\dots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{\alpha_n}\frac{\operatorname{ch}\gamma_n x_2}{\operatorname{ch}\frac{\gamma_n a_2}{2}}\cos\alpha_n x_1+\right. \tag{3.4}$$

$$\left.+\frac{1}{a_2}\sum_{m=1,3,\dots}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{\beta_m}\frac{\operatorname{ch}\delta_m x_1}{\operatorname{ch}\frac{\delta_m a_1}{2}}\cos\beta_m x_2\right\}$$

$$\gamma_n=\sqrt{\alpha_n^2+i\eta}, \quad \delta_m=\sqrt{\beta_m^2+i\eta},$$

Next, making use of (2.34), we obtain the function F_0, if the function $S(x_1, x_2, t)=e^{i\omega t}V(x_1, x_2)$ is known, satisfying the equation

$$(\partial_1^2+\partial_2^2+k_1^2)V=0, \quad k_1^2=\frac{\omega^2}{c_1^2}, \tag{3.5}$$

and the same boundary conditions as the function U. Thus, a solution is obtained analogous to (3.4):

$$V(x_1, x_2) = 4T_0\left\{\frac{1}{a_1}\sum_{n=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{\alpha_n} \frac{\operatorname{ch}\xi_n x_2}{\operatorname{ch}\xi_n \frac{a_2}{2}}\cos\alpha_n x_1 + \right.$$
$$\left. + \frac{1}{a_2}\sum_{m=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{m-1}{2}}}{\beta_m} \frac{\operatorname{ch}\eta_m x_1}{\operatorname{ch}\frac{\eta_m a_1}{2}}\cos\beta_m x_2\right\} \tag{3.6}$$

$$\xi_n = \sqrt{\alpha_n^2 - k_1^2}, \qquad \eta_m = \sqrt{\beta_m^2 - k_1^2}.$$

It is easy now to find F_0, from the Eq. (2.34), and the stresses $\sigma_{ij}^{(0)}$ from (2.15).

Substituting $F_0 = e^{i\omega t}G_0$ in (2.34), we obtain:

$$G_0 = -\frac{2\mu\overline{m}}{i\eta + k_1^2}(U - V). \tag{3.7}$$

The stresses $\sigma_{ij}^{(0)}$ attached to the function F_0 are given by the equations

$$\sigma_{ij}^{(0)} = e^{i\omega t}\left[-G_{0,ij} + \delta_{ij}\left(\nabla^2 + \frac{k_1^2}{2}\right)G_0\right] \qquad (i, j = 1, 2). \tag{3.8}$$

It is easy to find from (3.7) and (3.8) that $G_0 = 0$ on the boundary of the prism and that $\sigma_{11}^{(0)} = 0$ on the edge $x_1 = \pm a_1/2$, and $\sigma_{22}^{(0)} = 0$ on the edge $x_2 = \pm a_2/2$. The shear stresses σ_{12}^{0} are different from zero. These stresses assume the form:

$$\sigma_{12}^{(0)}\left(\frac{a_1}{2}, x_2, t\right) = -\frac{8\mu\overline{m}T_0 e^{i\omega t}}{i\eta + k_1^2}\left\{\frac{1}{a_1}\sum_{n=1,3,\ldots}^{\infty}\left(\frac{\gamma_n \operatorname{sh}\gamma_n x_2}{\operatorname{ch}\gamma_n \frac{a_2}{2}} - \frac{\xi_n \operatorname{sh}\xi_n x_2}{\operatorname{ch}\xi_n \frac{a_2}{2}}\right) + \right.$$
$$\left. + \frac{1}{a_2}\sum_{m=1,3,\ldots}^{\infty}(-1)^{\frac{m-1}{2}}\left(\delta_m \operatorname{th}\frac{\delta_m a_1}{2} - \eta_m \operatorname{th}\frac{\eta_m a_1}{2}\right)\sin\beta_m x_2\right\}, \tag{3.9}$$

$$\sigma_{12}^{(0)}\left(x_1, \frac{a_2}{2}, t\right) = -\frac{8\mu\overline{m}T_0 e^{i\omega t}}{i\eta + k_1^2}\left\{\frac{1}{a_1}\sum_{n=1,3,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\left(\gamma_m \operatorname{th}\frac{\gamma_n a_2}{2} - \right.\right.$$
$$\left.\left. -\xi \operatorname{th}\frac{\xi_n a_2}{2}\right)\sin\alpha_n x_1 + \frac{1}{a_2}\sum_{m=1,3,\ldots}^{\infty}\left(\frac{\delta_m \operatorname{sh}\delta_m x_1}{\operatorname{ch}\delta_m \frac{a_1}{2}} - \frac{\eta_m \operatorname{sh}\eta_m x_1}{\operatorname{ch}\eta_m \frac{a_1}{2}}\right)\right\}. \tag{3.10}$$

To cancel the stresses $\sigma_{12}^{(0)}$ on the boundaıy of the prism an appropriate state of stress should be superposed, σ_{ij}^{*} expressed in terms of the functions F_1 and F_2 satisfying Eqs. (2.17). Introducing the notations

$$F^{*} = F_1+F_2 = e^{i\omega t}(\Phi+\Psi), \quad F^{*} = G_1 e^{i\omega t}, \tag{3.11}$$

we reduce Eqs. (2.17) to the form

$$(\nabla^2+k_1^2)\Phi = 0, \quad (\nabla^2+k_2^2)\Psi = 0, \quad k_2^2 = \frac{\omega^2}{c_2^2}. \tag{3.12}$$

The solutions of these equations, taking into consideration the symmetry in relation to the x_1 and x_2-axis, are the functions

$$\Phi = \sum_{n=1,3,\dots}^{\infty} C_n \operatorname{ch} \xi_n x_2 \cos \alpha_n x_1 + \sum_{m=1,3,\dots}^{\infty} A_m \operatorname{ch} \eta_m x_1 \cos \beta_m x_2, \tag{3.13}$$

$$\Psi = \sum_{n=1,3,\dots}^{\infty} D_n \operatorname{ch} \bar{\xi}_n x_2 \cos \alpha_n x_1 + \sum_{m=1,3,\dots}^{\infty} B_m \operatorname{ch} \bar{\eta}_m x_1 \cos \beta_m x_2, \tag{3.14}$$

where

$$\bar{\xi}_n = \sqrt{\alpha_n^2-k_2^2}, \quad \bar{\eta}_m = \sqrt{\beta_m^2-k_2^2}.$$

The additional stresses σ_{ij}^{*} are expressed by Eq. (2.18), where the function F^{*} is expressed in terms of Φ and Ψ by Eq. (3.11).

In order to cancel the shear stresses $\sigma_{12}^{(0)}$ on the boundary, the following boundary conditions should be satisfied:

$$\text{for} \quad x_1 = \pm \frac{a_1}{2}$$

$$\sigma_{11}^{(0)}+\sigma_{11}^{*} = 0, \quad \sigma_{12}^{(0)}+\sigma_{12}^{*} = 0; \tag{3.15}$$

or

$$G_{1,22}+\frac{k_1^2}{2} G_1 = 0, \quad -G_{1,12}+\tau_{12}^{(0)} = 0, \quad \sigma_{12}^{(0)} = e^{i\omega t}\tau_{12}^{(0)}; \tag{3.15'}$$

and for $\quad x_2 = \pm \dfrac{a_2}{2}$

$$\sigma_{22}^{(0)}+\sigma_{22}^{*} = 0, \quad \sigma_{12}^{(0)}+\sigma_{12}^{*} = 0; \tag{3.16}$$

or

$$G_{1,11}+\frac{k_1^2}{2} G_1 = 0, \quad -G_{1,12}+\tau_{12}^{(0)} = 0. \tag{3.16'}$$

From the first condition (3.15′) and from the first condition (3.16′), we obtain:

$$B_m = -A_m t\,(\eta_m^{*}, \bar{\eta}_m^{*}), \quad D_n = -C_n t(\xi_n^{*}, \bar{\xi}_n^{*}), \tag{3.17}$$

with the following notations

$$t(\eta_m^*, \overline{\eta}_m^*) = \frac{\operatorname{ch} \eta_m \frac{a_1}{2}}{\operatorname{ch} \overline{\eta}_m \frac{a_1}{2}}, \qquad t(\xi_n^*, \overline{\xi}_n^*) = \frac{\operatorname{ch} \xi_n \frac{a_2}{2}}{\operatorname{ch} \overline{\xi}_n \frac{a_2}{2}}.$$

From the second condition (3.15′), we obtain the equation:

$$(3.18) \quad \sum_{n=1,3,\dots}^{\infty} C_n \left[\xi_n \operatorname{sh} \xi_n x_2 - t(\xi_n^*, \overline{\xi}_n^*)\overline{\xi}_n \operatorname{sh} \overline{\xi}_n x_2\right](-1)^{\frac{n-1}{2}} \alpha_n +$$

$$+ \sum_{m=1,3,\dots}^{\infty} A_m \left[\left[\eta_m \operatorname{sh} \eta_m \frac{a_1}{2} - t(\eta_m^*, \overline{\eta}_m^*)\overline{\eta}_m \operatorname{sh} \overline{\eta}_m \frac{a_1}{2}\right] \beta_m \sin \beta_m x_2 - \right.$$

$$- \frac{8\mu \overline{m} T_0}{i\eta + k_1^2} \left\{ \frac{1}{a_1} \sum_{n=1,3,\dots}^{\infty} \left[\gamma_n \frac{\operatorname{sh} \gamma_n x_2}{\operatorname{ch} \gamma_n \frac{a_2}{2}} - \xi_n \frac{\operatorname{sh} \xi_n x_2}{\operatorname{ch} \xi_n \frac{a_2}{2}} \right] + \right.$$

$$\left. + \frac{1}{a_2} \sum_{m=3,1,\dots}^{\infty} (-1)^{\frac{m-1}{2}} \left(\delta_m \operatorname{th} \frac{\delta_m a_1}{2} - \eta_m \operatorname{th} \frac{\eta_m a_1}{2} \right) \sin \beta_m x_2 \right\} = 0.$$

Expressing the function $\operatorname{sh} \xi_n x_2, \operatorname{sh} \overline{\xi}_n x_2, \operatorname{sh} \gamma_n x_2$ in the form of the infinite series

$$(3.19) \quad \begin{cases} \operatorname{sh} \xi_m x_2 = \sum_{m=1,3,\dots}^{\infty} E_{nm} \sin \beta_m x_2, \quad |x_2| < a_2/2, \\ E_{nm} = \frac{4\xi_n}{a_2} \frac{(-1)^{\frac{m-1}{2}}}{\xi_n^2 + \beta_m^2} \operatorname{ch} \frac{\xi_n a_2}{2} \quad \text{and so on,} \end{cases}$$

and introducing the notations

$$(3.20) \begin{cases} \nabla_{nm} = \alpha_n^2 + \beta_m^2, \\ b_m = \frac{1}{\operatorname{ch} \overline{\eta}_m \frac{a_1}{2}} \left(\eta_m \operatorname{sh} \eta_m \frac{a_1}{2} \operatorname{ch} \overline{\eta}_m \frac{a_1}{2} - \overline{\eta}_m \operatorname{sh} \overline{\eta}_m \frac{a_1}{2} \operatorname{ch} \eta_m \frac{a_1}{2} \right), \end{cases}$$

we reduce Eq. (3.18) to the form:

$$(3.21) \quad A_m \beta_m b_m - \frac{4}{a^2} (k_1^2 - k_2^2)(-1)^{\frac{m-1}{2}} \beta_m^2 \sum_{n=1,3,\dots}^{\infty} \frac{C_n (-1)^{\frac{n-1}{2}} \alpha_n \operatorname{ch} \xi_n \frac{a_2}{2}}{(\Delta_{nm} - k_1^2)(\Delta_{nm} - k_2^2)} -$$

$$-\frac{8\mu\bar{m}T_0}{i\eta+k_1^2}\,\frac{(-1)^{\frac{m-1}{2}}}{a_2}\left(\frac{k_1^2}{\eta_m}\operatorname{th}\eta_m\frac{a_1}{2}+\frac{i\eta}{\delta_m}\operatorname{th}\frac{\delta_m a_1}{2}\right)=0\quad(m=1,3,5,\dots,\infty).$$

From a boundary condition of the group (3.16), we obtain a system of equations analogous to (3.21):

$$(3.22)\qquad C_n\alpha_n c_n-\frac{4}{a_1}(k_1^2-k_2^2)(-1)^{\frac{n-1}{2}}\alpha_n^2\sum_{m=1,3,\dots}^{\infty}\frac{A_m(-1)^{\frac{m-1}{2}}\beta_m\operatorname{ch}\eta_m\frac{a_1}{2}}{(\Delta_{nm}-k_1^2)(\Delta_{nm}-k_2^2)}-$$

$$-\frac{8\mu\bar{m}T_0}{i\eta+k_1^2}\,\frac{(-1)^{\frac{n-1}{2}}}{a_1}\left(\frac{k_1^2}{\xi_n}\operatorname{th}\xi_n\frac{a_2}{2}+\frac{i\eta}{\gamma_n}\operatorname{th}\gamma_n\frac{a_2}{2}\right)=0$$

$$(n=1,3,5,\dots,\infty),$$

where

$$c_n=\frac{1}{\operatorname{ch}\bar{\xi}_n\frac{a_2}{2}}\left(\xi_n\operatorname{sh}\frac{\xi_n a_2}{2}\operatorname{ch}\frac{\bar{\xi}_n a_2}{2}-\bar{\xi}_n\operatorname{sh}\bar{\xi}_n\frac{a_2}{2}\operatorname{ch}\xi_n\frac{a_2}{2}\right).$$

We have obtained an infinite system of non-homogeneous equations of which for given frequencies of temperature changes ω, we can determine the constants A_m, B_m, C_n, D_n. Let us observe that for a fixed and finite value ω it is that $\alpha_n^2-k_1^2<0$ or that for a certain n the quantities ξ_n become imaginary.

The same applies to $\bar{\xi}_n$, η_m, $\bar{\eta}_m$.

The system of equations (3.21) and (3.22) can be written in the form:

$$(3.23)\qquad \begin{cases} A_m e_m+\sum\limits_{n=1}^{\infty}C_n f_{nm}=d_m,\\ C_n g_n+\sum\limits_{m=1}^{\infty}A_m h_{nm}=k_n,\end{cases}\qquad (n,\ m=1,\ 3,\ \dots,\ \infty).$$

Since the quantities A_m, C_n, e_m, ... etc. are complex $A_m=A_m^0+i\bar{A}_m$, ..., therefore the system of equations should be split up into two systems:

$$(3.24)\qquad \begin{cases} A_m^0 e_m^0-\bar{A}_m\bar{e}_m+\sum\limits_{n=1}^{\infty}(C_n^0 f_{nm}^0-\bar{C}_n\bar{f}_{nm})=d_m^0,\\ C_n^0 g_n^0-\bar{C}_n\bar{g}_n+\sum\limits_{m=1}^{\infty}(A_m^0 h_{nm}^0-\bar{A}_m\bar{h}_{nm})=k_n^0,\end{cases}$$

and

$$(3.25)\qquad \begin{cases} \bar{A}_m e_m^0 + \bar{e}_m A_m^0 + \sum_{n=1}^{\infty} (\bar{C}_n f_{nm}^0 + \bar{f}_{nm}^0 C_n^0) = \bar{d}_m, \\ \bar{C}_n g_n^0 + \bar{g}_n C_n^0 + \sum_{m=1}^{\infty} (\bar{A}_m h_{nm}^0 + A_m^0 \bar{h}_{nm}) = \bar{k}_n. \end{cases}$$

In the case of a quadratic prism, considerable simplification can be achieved because $C_n^0 = A_n^0$, $\bar{C}_n = \bar{A}_n$. The following system of equations is to be solved:

$$(3.26)\qquad \begin{cases} A_m^0 e_m^0 - \bar{A}_m \bar{e}_m + \sum_{n=1}^{\infty} (A_n^0 f_{nm}^0 - \bar{A}_n \bar{f}_{nm}) = d_m^0, \\ \bar{A}_m e_m^0 + A_m^0 \bar{e}_m + \sum_{n=1}^{\infty} (\bar{A}_n f_{nm}^0 + A_n^0 \bar{f}_{nm}) = \bar{d}_m. \end{cases}$$

The frequency of forced vibration ω must be chosen so that it does not coincide with the natural frequency of the prism. The natural frequency of the prism will be obtained from the homogeneous system of equations (3.21), (3.22) by setting the determinant of this system equal to zero.
The case of an elastic layer ($a_2 \to \infty$) is therefore particulary simple in the case where $T\left(\pm \frac{a_1}{2}, t\right) = T_0 e^{i\omega t}$, if the temperature and the stresses depend on x and t only.

$$(3.27)\qquad U(x_1) = T_0 \frac{\operatorname{ch} \varepsilon x_1}{\operatorname{ch} \varepsilon \frac{a_1}{2}}, \qquad V(x_1) = T_0 \frac{\cos k_1 x_1}{\cos k_1 \frac{a_1}{2}}, \qquad \varepsilon = \sqrt{i\eta},$$

and

$$(3.28)\qquad F_0 = -\frac{2\mu\bar{m}}{i\eta + k_1^2} (U - V) e^{i\omega t}.$$

To determine the stresses, it suffices to know the function F_0 because the boundary conditions $\sigma_{11} = \sigma_{12} = 0$ on the boundaries $x_1 = \pm a_1/2$ are satisfied.

The stresses will be obtained from the equations:

$$(3.29)\qquad \sigma_{11} = \frac{k_1^2}{2} e^{i\omega t} G_0, \qquad \sigma_{22} = e^{i\omega t}\left(G_{0,11} + \frac{k_1^2}{2} G_0\right), \qquad \sigma_{12} = 0.$$

In particular, for σ_{11} we obtain the equation:

$$(3.30)\qquad \sigma_{11} = -\frac{\mu\bar{m}k_1^2 T_0 e^{i\omega t}}{(i\eta + k_1^2)} \left(\frac{\operatorname{ch}\sqrt{i\eta}\, x_1}{\operatorname{ch}\sqrt{i\eta}\, \frac{a_1}{2}} - \frac{\cos k_1 x}{\cos k_1 \frac{a_1}{2}} \right).$$

If the temperature $T_0 \cos \omega t$ is prescribed on the boundary the stresses σ_{11} are given by the real part of the equation (3.29) if $T_0 \sin \omega t$ on the boundary then — by the imaginary part of (3.29).

Let us observe that the stresses increase indefinitely if $\cos k_1 a \to 0$. The case of $\cos k_1 a = 0$ determines the natural frequencies $\omega_0 = (\pi/a_1)(2n-1)c_1$ of an elastic layer.

4. The Plane Dynamic Coupled Problem of Thermoelasticity

Let us consider an elastic body acted on by a non-steady-state temperature field in a plane state of stress. The coupling between the temperature field and the strain field will, however, be taken into account. In this case we have the generalized heat equation

$$\Box_3^2 T - \partial_t \chi \varepsilon_{kk} = -\frac{Q}{\varkappa}, \tag{4.1}$$

where $\chi = \gamma T_0/\varrho c$ and T_0+T is the absolute temperature and the state $T=0$ is identical with the state where the stresses and displacements are zero; c — is the specific heat.

For the coupled problem, Eqs. (2.1)-(2.3) are valid. If we confine ourselves to the temperature field only the mass forces being disregarded 1, equations (2.11), (2.12) and (2.13) remain valid.

Expressing ε_{kk} in (4.1) in terms of σ_{kk} by means of (2.2) and in terms of the stress function F by means of (2.12) we obtain finally the system of two equations

$$\Box_4^2 T - \varepsilon \partial_t \Box_2^2 F = -Q/\varkappa, \quad \Box_4^2 = \nabla^2 - \frac{1}{\varkappa_0}\partial_t, \quad \frac{1}{\varkappa_0} = \frac{1}{\varkappa} + 2\varepsilon\gamma, \tag{4.2}$$

$$\varepsilon = \frac{\chi}{2(\lambda+\mu)}$$

$$\Box_1^2 \Box_2^2 F + 2\mu\bar{m}\Box_2^2 T = 0. \tag{4.3}$$

Eliminating from these equations the temperature, we obtain the differential equation:

$$\Box_2^2 \left\{ [\Box_1^2 \Box_3^2 - \bar{m}\chi\partial_t \nabla^2] F - \frac{2\mu\bar{m}Q}{\varkappa} \right\} = 0. \tag{4.4}$$

It is seen that if the coupling is disregarded ($\chi \to 0$) Eq. (4.4) becomes (2.13) subject to the operation $\Box_3^2$.

The particular integral will be obtained from the equation:

$$(\Box_1^2 \Box_3^2 - \bar{m}\chi\partial_t \nabla^2) F_0 = \frac{2\mu\bar{m}Q}{\varkappa}, \tag{4.5}$$

and the stress corresponding to the function F_0 — from Eqs. (2.15). For the infinite region, F_0, constitutes the solution of the problem.

For a bounded region, the general integral F^* of the equation

$$\square_2^2(\square_1^2\square_3^2-\bar{m}\chi\partial_t\nabla^2)F^*=0 \tag{4.6}$$

should be added to the particular integral F_0, where $F^*=F_1+F_2$ and the functions F_1, F_2 should satisfy the equations:

$$(\square_1^2\square_3^2-\bar{m}\chi\partial_t\nabla^2)F_1=0, \quad \square_2^2F_2=0. \tag{4.7}$$

The stresses connected with the function F^* will be found from Eqs. (2.18).

The second method, also very convenient, consists in reassumption of the displacement vector in the form:

$$\mathbf{u}=\operatorname{grad}\Phi+\operatorname{rot}\boldsymbol{\psi}. \tag{4.8}$$

Substituting (4.8) in the displacement equations (2.20) and in the heat equation (4.1), we obtain a system of three equations:

$$\square_3^2T-\chi\partial_t\nabla^2\Phi=-Q/\varkappa, \tag{4.9}$$

$$\square_1^2\Phi=\bar{m}T, \quad \square_2^2\boldsymbol{\psi}=0. \tag{4.10}$$

Eliminating T we shall obtain a system of equations [10]:

$$(\square_1^2\square_3^2-\bar{m}\chi\partial_t\nabla^2)\Phi_0=-\frac{\bar{m}Q}{\varkappa}, \quad \square_2^2\boldsymbol{\psi}=0. \tag{4.11}$$

Finding the function Φ from Eq. (4.11), we shall obtain the temperature T from the first of Eqs. (4.10).

If we are concerned with an infinite body, the function Φ_0 which is a particular integral of the equation

$$(\square_1^2\square_3^2-\chi\bar{m}\partial_t\nabla^2)\Phi_0=-\frac{\bar{m}Q}{\varkappa} \tag{4.12}$$

is the solution of the problem. The stresses σ_{ij}^0 connected with the function Φ_0 are given by the equations

$$\sigma_{ij}^{(0)}=2\mu\left\{\Phi_{0,ij}-\delta_{ij}\left[\nabla^2-\frac{1}{2c_2^2}\partial_t^2\right]\Phi_0\right\} \qquad (i,j=1,2). \tag{4.13}$$

For a bounded region, we should solve also the system of equations

$$(\square_1^2\square_3^2-\bar{m}\chi\partial_t\nabla^2)\Phi^*=0, \quad \square_2^2\psi=0. \tag{4.14}$$

[1] The consideration of the influence of the mass forces requires the introduction of three stress functions. Cf. the Ref. [9].

The additional stresses σ_{ij}^* are given by the equations

$$(4.15)\qquad \begin{cases} \sigma_{11}^* = 2\mu\left(-\Phi^*_{,22}+\dfrac{1}{2c_2^2}\partial_t^2\Phi^*\right)+2\mu\psi_{,12}, \\ \sigma_{22}^* = 2\mu\left(-\Phi^*_{,11}+\dfrac{1}{2c_2^2}\partial_t^2\Phi^*\right)-2\mu\psi_{,12}, \\ \sigma_{12}^* = 2\mu\Phi^*_{,12}+\mu(\psi_{,22}-\psi_{,11}). \end{cases}$$

The final expressions for stresses are obtained by superposition

$$(4.16)\qquad \sigma_{ij} = \sigma_{ij}^{(0)}+\sigma_{ij}^*.$$

5. Forced Coupled Thermoelastic Vibration of a Rectangular Prism

Let us consider an infinite rectangular cylinder inside which uniformly distributed heat sources act in a harmonic manner $Q = Q_0\, e^{i\omega t}$.

Let $T = 0$ on the lateral surface of the prism. Substituting in (4.9) and (4.10)

$$(5.1)\qquad T = Ue^{i\omega t}, \quad \Phi = \theta e^{i\omega t}, \quad \psi = \Psi e^{i\omega t},$$

we obtain the system of equations:

$$(5.2)\qquad \nabla^2 U - i\eta U - \chi i\omega\nabla^2\theta = -\frac{Q_0}{\varkappa} \quad \eta = \frac{\omega}{\varkappa}.$$

$$(5.3)\quad (\nabla^2+\sigma^2)\theta = \overline{m}U, \quad (\nabla^2+\tau^2)\Psi = 0, \quad \sigma^2 = \frac{\omega^2}{c_1^2}, \quad \tau^2 = \frac{\omega^2}{c_2^2}.$$

Eliminating from these equations first U and then θ, we obtain:

$$(5.4)\qquad (\nabla^2+\varkappa_1^2)(\nabla^2+\varkappa_2^2)U = -\frac{1}{\varkappa}(\nabla^2+\sigma^2)Q_0,$$

$$(5.5)\qquad (\nabla^2+\varkappa_1^2)(\nabla^2+\varkappa_2^2)\theta = -\frac{\overline{m}}{\varkappa}Q_0, \quad (\nabla^2+\tau^2)\psi = 0,$$

where

$$\varkappa_1^2+\varkappa_2^2 = -q(1+\varepsilon)+\sigma^2, \quad \varkappa_1^2\varkappa_2^2 = -q\sigma^2, \quad q = \frac{i\omega}{\varkappa}, \quad \varepsilon = \chi\varkappa\overline{m}.$$

The solution of (5.4) is the double series:

$$(5.6)\quad U(x_1, x_2) = \frac{16Q_0}{a_1a_2\varkappa}\sum_{n,m=1,3,\dots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}(-1)^{\frac{m-1}{2}}(\Delta_{nm}-\sigma^2)\cos\alpha_n x_1}{\alpha_n\beta_m(\Delta_{nm}-\varkappa_1^2)(\Delta_{nm}-\varkappa_1^2)}\cos\beta_m x_2,$$

where

$$\Delta_{nm} = \alpha_n^2+\beta_m^2, \quad \alpha_n = \frac{n\pi}{a_1}, \quad \beta_m = \frac{m\pi}{a_2}.$$

The boundary conditions $T = 0$ are satisfied for $x_1 = \pm a_1/2$, $x_2 = \pm a_2/2$. Next, we determine a particular integral of Eq. (5.5). It has the form:

$$(5.7) \quad \theta_0 = -\frac{16Q_0\overline{m}}{a_1a_2\varkappa} \sum_{n,m=1,3,\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}(-1)^{\frac{m-1}{2}}}{\alpha_n\beta_m(\Delta_{nm}-\varkappa_1^2)(\Delta_{nm}-\varkappa_2^2)} \cos\alpha_n x_1 \cos\beta_n x_2,$$

or

$$\theta_0 = -\frac{8Q_0\overline{m}}{a_1\varkappa(\varkappa_1^2-\varkappa_2^2)} \sum_{n=1,3,\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}\cos\alpha_n x_1}{\alpha_n} \left[\frac{1}{\lambda_n^2}\left(1-\frac{\operatorname{ch}\lambda_n x_2}{\operatorname{ch}\frac{\lambda_n a_2}{2}}\right) - \frac{1}{\bar{\lambda}_n^2}\left(1-\frac{\operatorname{ch}\bar{\lambda}_n x_2}{\operatorname{ch}\frac{\bar{\lambda}_n a_2}{2}}\right)\right],$$

$$\lambda_n = \sqrt{\alpha_n^2-\varkappa_1^2}, \quad \bar{\lambda}_n = \sqrt{\alpha_u^2-\varkappa_2^2}.$$

It can easily be verified that the normal stresses vanish on the boundary of prism. The stress σ_{12}^0 remain different from zero

$$(5.8) \quad \sigma_{12}^{(0)} = -\frac{32\overline{m}\mu Q_0}{a_1a_2\varkappa} \sum_{n,m=1,3,\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}(-1)^{\frac{m-1}{2}}}{(\Delta_{nm}-\varkappa_1^2)(\Delta_{nm}-\varkappa_2^2)} \sin\alpha_n x_1 \sin\beta_m x_2.$$

The stresses $\sigma_{ij}^{(0)}$ at the edge will be suppressed by adding to the state $\sigma_{12}^{(0)}$ the state of stress σ_{ij}^*, expressed in terms of the functions Φ^* and Ψ^*. Thus, Eqs. (4.14) should be solved with the following boundary conditions

$$(5.9) \quad \begin{cases} \sigma_{11}^{(1)}+\sigma_{11}^* = 0, \quad \sigma_{12}^{(0)}+\sigma_{12}^* = 0, \quad T^* = 0 \quad \text{for} \quad x_1 = a_1/2, \\ \sigma_{22}^{(0)}+\sigma_{22}^* = 0, \quad \sigma_{12}^{(0)}+\sigma_{12}^* = 0, \quad T^* = 0 \quad \text{for} \quad x_2 = a_2/2, \end{cases}$$

where the function T^* is determined by the relation

$$(5.10) \quad T^* = \frac{1}{\overline{m}}\square_1^2\Phi^*.$$

Introducing the notations

$$(5.11) \quad \Phi^*(x_1, x_2, t) = e^{i\omega t}\theta^*(x_1, x_2), \quad \Psi = e^{i\omega t}\Psi^*, \quad T^* = e^{i\omega t}U^*,$$

Eqs. (4.14) are reduced to the form:

$$(5.12) \quad (\nabla^2+\varkappa_1^2)(\nabla^2+\varkappa_2^2)\theta^* = 0, \quad (\nabla^2+\tau^2)\Psi^* = 0$$

$$(5.13) \quad U^* = \frac{1}{\overline{m}}(\nabla^2+\sigma^2)\theta^*.$$

The functions θ^* and Ψ^* will be assumed in the form:

$$(5.14)\qquad \theta^* = \sum_{n=1,3,\dots}^{\infty} (E_n \operatorname{ch} \lambda_n x_2 + F_n \operatorname{ch} \bar{\lambda}_n x_2) \cos \alpha_n x_1 + \sum_{m=1,3,\dots}^{\infty} (G_m \operatorname{ch} \xi_m x_1 + H_m \operatorname{ch} \bar{\xi}_m x_1) \cos \beta_m x_2,$$

where

$$(5.15)\qquad \begin{cases} \xi_m = \sqrt{\beta_m^2 - \varkappa_1^2}, \quad \bar{\xi}_m = \sqrt{\beta_m^2 - \varkappa_2^2}. \\ \Psi^* = \sum_{n=1,3,\dots}^{\infty} A_n \operatorname{sh} \gamma_n x_2 \sin \alpha_n x_1 + \sum_{m=1,3,\dots}^{\infty} B_m \operatorname{sh} \eta_m x_1 \sin \beta_m x_2, \end{cases}$$

and

$$\gamma_n = \sqrt{\alpha_n^2 - \tau^2}, \quad \eta_m = \sqrt{\beta_m^2 - \tau^2}.$$

From (5.13), we obtain

$$(5.16)\quad U^* = \frac{1}{\bar{m}} \left\{ \sum_{n=1,3,\dots}^{\infty} [E_n(\sigma^2 - \varkappa_1^2) \operatorname{ch} \lambda_n x_2 + F_n(\sigma^2 - \varkappa_2) \operatorname{ch} \bar{\lambda}_n x_2] \cos \alpha_n x_1 + \sum_{m=1,3,\dots}^{\infty} [G_m(\sigma^2 - \varkappa_1^2) \operatorname{ch} \xi_m x_1 + H_m(\sigma^2 - \varkappa_2^2) \operatorname{ch} \bar{\xi}_m x_1] \cos \beta_m x_2 \right\}.$$

The constants $A_n, B_n, \dots, H_m$ will be determined from the boundary conditions (5.9) which, according to Eqs. (5.11), take the form:

$$(5.17)\quad \begin{cases} -2\mu\left(\partial_2^2 + \frac{\tau^2}{2}\right)\theta^* + 2\mu\Psi^*_{,12} = 0, \quad \sigma_{12}^0 + 2\mu\theta^*_{,12} + \mu(\partial_2^2 - \partial_1^2)\Psi^* = 0, \\ \qquad U^* = 0 \quad \text{for } x_1 = a_1/2, \\ -2\mu\left(\partial_1^2 + \frac{\tau^2}{2}\right)\theta^* - 2\mu\Psi^*_{,12} = 0, \quad \sigma_{12}^0 + 2\mu\theta^*_{,12} + \mu(\partial_2^2 - \partial_1^2)\Psi^* = 0, \\ \qquad U^* = 0 \quad \text{for } x_2 = a_2/2. \end{cases}$$

Let us eliminate the quantities F_n and H_m by means of the conditions of zero temperature on the boundary of the prism. Then,

$$(5.18)\quad \theta^* = \sum_{n=1,3,\dots}^{\infty} E_n \left(\operatorname{ch} \lambda_n x_2 - \varkappa_{12} \frac{\operatorname{ch} \lambda_n \frac{a_2}{2}}{\operatorname{ch} \bar{\lambda}_n \frac{a_2}{2}} \operatorname{ch} \bar{\lambda}_n x_2 \right) \cos \alpha_n x_1 + \sum_{m=1,3,\dots}^{\infty} G_m \left(\operatorname{ch} \xi_m x_1 - \varkappa_{12} \frac{\operatorname{ch} \xi_m \frac{a_1}{2}}{\operatorname{ch} \bar{\xi}_m \frac{a_1}{2}} \operatorname{ch} \bar{\xi}_m x_1 \right) \cos \beta_m x_2,$$

where

$$\varkappa_{12} = \frac{\sigma^2 - \varkappa_1^2}{\sigma^2 - \varkappa_2^2}$$

$$(5.19) \quad U^* = \frac{\sigma^2 - \varkappa_1^2}{m} \left[\sum_{n=1,3,\ldots}^{\infty} E_n \left(\operatorname{ch} \lambda_n x_2 - \frac{\operatorname{ch} \lambda_n \frac{a_2}{2}}{\operatorname{ch} \bar{\lambda}_n \frac{a_2}{2}} \operatorname{ch} \bar{\lambda}_n x_2 \right) \cos \alpha_n x_1 + \right.$$

$$\left. + \sum_{m=1,3,\ldots}^{\infty} G_m \left(\operatorname{ch} \xi_m x_1 - \frac{\operatorname{ch} \xi_m \frac{a_1}{2}}{\operatorname{ch} \bar{\xi}_m \frac{a_1}{2}} \operatorname{ch} \bar{\xi}_m x_1 \right) \cos \beta_m x_2 \right].$$

By means of the conditions of zero normal stress on the edge of the cylinder, we eliminate the constants E_n, G_m. The following expression is obtained for the function θ^*,

$$(5.20) \quad \theta^* = \frac{2\mu}{1 - \varkappa_{12}} \left[\sum_{n=1,3,\ldots}^{\infty} A_n a_n^0 \left(\operatorname{ch} \lambda_n x_2 - \varkappa_{12} \frac{\operatorname{ch} \lambda_n \frac{a_2}{2}}{\operatorname{ch} \bar{\lambda}_n \frac{a_2}{2}} \operatorname{ch} \bar{\lambda}_n x_2 \right) \cos \alpha_n x_1 - \right.$$

$$\left. - \sum_{m=1,3,\ldots}^{\infty} B_m b_m^0 \left(\operatorname{ch} \xi_m x_1 - \varkappa_{12} \frac{\operatorname{ch} \xi_m \frac{a_1}{2}}{\operatorname{ch} \bar{\xi}_m \frac{a_1}{2}} \operatorname{ch} \bar{\xi}_n x_1 \right) \cos \beta_m x_2, \right],$$

where

$$(5.21) \quad \begin{cases} a_n^0 = \dfrac{\alpha_n \gamma_n}{2\mu\alpha_n^2 - \varrho\omega^2} \dfrac{\operatorname{ch} \gamma_n \frac{a_2}{2}}{\operatorname{ch} \lambda_n \frac{a_2}{2}}, \\[2ex] b_m^0 = \dfrac{\beta_m \eta_m}{2\mu\beta_m^2 - \varrho\omega^2} \dfrac{\operatorname{ch} \eta_m \frac{a_1}{2}}{\operatorname{ch} \xi_m \frac{a_1}{2}}, \end{cases}$$

Finally, from the condition of zero shear stress on the boundary we obtain a system of linear equations with an infinite number of coefficients A_n, B_m

where the hyperbolic functions have been expanded in series according to Eqs. (3.19):

(5.22)

$$\frac{4}{a_2}\sum_{n=1,3,\ldots}^{\infty}(-1)^{\frac{n+m-2}{2}}A_m\left\{\frac{(2\alpha_n^2-\tau^2)\gamma_n\operatorname{ch}\gamma_n\frac{a_2}{2}}{\Delta_{nm}-\tau^2}-\frac{4\mu}{1-\varkappa_{12}}\alpha_n a_n^0\operatorname{ch}\lambda_n\frac{a_2}{2}\left[1-\varkappa_{12}-\right.\right.$$

$$\left.\left.-\beta_m^2\left(\frac{1}{\Delta_{nm}-\varkappa_1^2}-\varkappa_{12}\frac{1}{\Delta_{nm}-\varkappa_2^2}\right)\right]\right\}+B_m\left[\frac{4\mu}{1-\varkappa_{12}}b_m^0\beta_m i_m-(2\beta_m^2-\tau^2)\operatorname{sh}\eta_m\frac{a_1}{2}\right]-$$

$$-\frac{Ma_1}{4}\frac{(-1)^{\frac{m-1}{2}}}{\varkappa_1^2-\varkappa_2^2}\left(\frac{1}{\xi_m}\operatorname{th}\xi_m\frac{a_1}{2}-\frac{1}{\bar{\xi}_m}\operatorname{th}\bar{\xi}_m\frac{a_1}{2}\right)=0.$$

(5.23)

$$\frac{4}{a_1}\sum_{m=1,3,\ldots}^{\infty}(-1)^{\frac{n+m-2}{2}}B_m\left\{\frac{(2\beta_m^2-\tau^2)\eta_m\operatorname{ch}\eta_m\frac{a_1}{2}}{\Delta_{nm}-\tau^2}-\frac{4\mu}{1-\varkappa_{12}}\beta_m b_m^0\operatorname{ch}\xi_m\frac{a_1}{2}\left[1-\varkappa_{12}-\right.\right.$$

$$\left.\left.-\alpha_n^2\left(\frac{1}{\Delta_{nm}-\varkappa_1^2}-\varkappa_{12}\frac{1}{\Delta_{nm}-\varkappa_2^2}\right)\right]\right\}+A_n\left[\frac{4\mu}{1-\varkappa_{12}}a_n^0\alpha_n j_n-(2\alpha_n^2-\tau^2)\operatorname{sh}\gamma_n\frac{a_2}{2}\right]+$$

$$+\frac{Ma_2}{4}\frac{(-1)^{\frac{n-1}{2}}}{\varkappa_1^2-\varkappa_2^2}\left(\frac{1}{\lambda_n}\operatorname{th}\frac{\lambda_n a_2}{2}-\frac{1}{\bar{\lambda}_n}\operatorname{th}\frac{\bar{\lambda}_n a_2}{2}\right)=0$$

$$(n,\ m,=1,3,5,\ldots,\infty),$$

where

$$(5.24)\quad\begin{cases} i_m=\left(\xi_m\operatorname{sh}\xi_m\frac{a_1}{2}-\varkappa_{12}\dfrac{\operatorname{ch}\xi_m\frac{a_1}{2}}{\operatorname{ch}\bar{\xi}_m\frac{a_1}{2}}\bar{\xi}_m\operatorname{sh}\bar{\xi}_m\frac{a_1}{2}\right), \\ j_n=\left(\lambda_n\operatorname{sh}\lambda_n\frac{a_2}{2}-\varkappa_{12}\dfrac{\operatorname{ch}\lambda_n\frac{a_2}{2}}{\operatorname{ch}\bar{\lambda}_n\frac{a_2}{2}}\bar{\lambda}_n\operatorname{sh}\bar{\lambda}_n\frac{a_2}{2}\right), \qquad M=\dfrac{32Q_0\bar{m}}{a_1a_2\varkappa}. \end{cases}$$

The system of equations (5.22), (5.23) may be reduced to one infinite system of equations for the sequence $\{A_n\}$ or $\{B_m\}$.

It is seen that the coupled problem can be represented by the simple series $\theta^* + \theta_0 = \theta$, of which the coefficients can be determined from the system of equations (5.22), (5.23). Assuming $\chi = 0$ in (5.2) and the subsequent equations, the calculations are simplified considerably for the uncoupled problem.

Let us consider in addition the case of $a_2 \to \infty$ that is the case of an elastic layer with the heat sources $Q(x_1. t) = Q_0 e^{i\omega t}$ and $T = 0$ on the edge.

Then

$$\theta(x_1) = -\frac{4Q_0\overline{m}}{a_1\varkappa} \sum_{n=1,3,\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}} \cos \alpha_n x_1}{\alpha_n(\alpha_n^2-\varkappa_1^2)(\alpha_n^2-\varkappa_2^2)}, \tag{5.25}$$

and the stresses are obtained from the equations:

$$\sigma_{11} = -\varrho\omega^2\theta(x_1)e^{i\omega t}, \qquad \sigma_{22} = -2\mu e^{i\omega t}\left(\theta_{,11} + \frac{\varrho\omega^2}{2\mu}\theta\right), \qquad \sigma_{12} = 0. \tag{5.26}$$

These are the final results, all the boundary conditions being satisfied. The temperature field is:

$$T = Ue^{i\omega t} = \frac{4Q_0 e^{i\omega t}}{a_1\varkappa} \sum_{n=1,3,\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}} (\alpha_n^2-\sigma^2)}{\alpha_n(\alpha_n^2-\varkappa_1^2)(\alpha_n^2-\varkappa_2^2)} \cos \alpha_n x_1. \tag{5.27}$$

Let us consider the particular case where the heat sources are $Q(x_1, t) = Q_0 \cos \omega t$. Then, taking the real part of Eq. (5.27), we obtain:

$$\begin{aligned} T = \frac{4Q_0}{a_1\varkappa} \sum_{n=1,3,\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{\alpha_n} & \\ \frac{(\alpha_n^2-\sigma^2)\{\alpha_n^2(\alpha_n^2-\sigma^2)\cos\omega t + \eta[\alpha_n^2(1+\varepsilon)-\sigma^2]\sin\omega t\}}{\alpha_n^4(\alpha_n^2-\sigma^2)^2+\eta^2[\alpha_n^2(1+\varepsilon)-\sigma^2]^2} & \cos\alpha_n x_1, \\ \eta = \frac{\omega}{\varkappa}. & \end{aligned} \tag{5.28}$$

The stress σ_{11} is given by the equation:

$$\sigma_{11} = \frac{4Q_0\overline{m}\varrho\omega^2}{a_1\varkappa} \sum_{n=1,3,\dots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{\alpha_n} \cdot \frac{\alpha_n^2(\alpha_n^2-\sigma^2)\cos\omega t + \eta[\alpha_n^2(1+\varepsilon)-\sigma^2]\sin\omega t}{\alpha_n^4(\alpha_n^2-\sigma^2)^2+\eta^2[\alpha_n^2(1+\varepsilon)-\sigma^2]^2} \cos\alpha x_1. \tag{5.29}$$

The forced vibration has the character of a damped vibration. There is no indefinite increase in the stresses. For the non-coupled problem ($\varepsilon = 0$), the stresses increase indefinitely if $\sigma_n \to \alpha_n$ ($n = 1, 3, \dots, \infty$).

For the coupled quasi-static problem, we have ($\sigma = 0$) from (5.29):

$$T = \frac{4Q_0}{a_1 \varkappa} \sum_{n=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{\alpha_n} \frac{\alpha_n^2 \cos \omega t + \eta(1+\varepsilon) \sin \omega t}{\alpha_n^4 + \eta^2(1+\varepsilon)^2} \cos \alpha_n x_1 . \tag{5.30}$$

The stresses σ_{11} and σ_{12} will be zero. The stresses σ_{22} and σ_{33} will be different from zero, and $\sigma_{22} = \sigma_{33}$.

References

[1] J. R. M. Radok, *On the solution of problems of dynamic plane elasticity*, Quart. Appl. Math., **14** (1956).

[2] T. Boggio, *Sull integrazione di alcune equazioni alle derivati parziali*, Annali di matem. p.ed.appl., s. III, VIII, 181 (1903).

[3] S. Kaliski, *Some boundary-value problems of the theory of elastic and anelastic bodies*, (in Polish), Warszawa 1957.

[4] W. Nowacki, *Problems of thermoelasticity*, (in Polish), Warszawa 1960.

[5] T. Mura, *Dynamical thermal stresses due to thermal shocks*, Res. Rep. Fac. of Engng. Meiji Univ, 8, 1956.

[6] H. Parkus, *Stresses in a centrally heated disc*, Proc. Second U.S.A. Nat. Congr. Appl. Mechan., 307, 1954.

[7] W. Derski, *On transient thermal stresses in an infinite thin plate*, Bull. Acad. Polon. Sci., Serie Sa. Techn., 9, **8** (1960).

[8] J. Ignaczak, *A plane problem of dynamic thermal distortions in thermoelasticity*, Arch. Mech. Stos., 5—6, **12** (1960).

[9] W. Nowacki, *On the treatment of the two-dimensional coupled thermoelastic problems in terms of stresses*, Bull. Acad. Polon. Sci., Ser. Sci. Tech., 3, **9** (1961).

[10] W. Nowacki, *Some dynamic problems of thermoelasticity*, Arch. Mech. Stos., 2, **11** (1959).

Streszczenie

PŁASKIE DYNAMICZNE ZAGADNIENIE TERMOSPRĘŻYSTOŚCI

Przedmiotem pracy jest przedstawienie kilku dróg rozwiązania płaskich zagadnień dynamicznych termosprężystości. W pierwszej części pracy sformułowano niesprzężone zagadnienie termosprężystości w naprężeniach oraz w przemieszczeniach, przy czym wykazano, że całki szczególne w obu sposobach rozwiązania różnią się jedynie stałym współczynnikiem.

W drugiej części pracy przedstawiono dwa przykłady rozwiązania zagadnienia niesprzężonego przy użyciu funkcji naprężenia, mianowicie drgania wymuszone walca nieograniczonego o przekroju prostokątnym i warstwy sprężystej, ogrzanych na brzegu w sposób harmoniczny w czasie.

W trzeciej części podano rozwiązanie zagadnienia płaskiego dynamicznego termosprężystości przy uwzględnieniu sprzężenia pola temperatury i pola deformacji i to na drodze wprowadzenia funkcji naprężenia oraz przez dekompozycję wektora przemieszczenia. Tok postępowania objaśniono przykładem drgań wymuszonych walca nieograniczonego o przekroju prostokątnym i warstwy sprężystej, wywołanych działaniem harmonicznie w czasie zmiennych źródeł ciepła i rozłożonych równomiernie w objętości walca.

Резюме

ПЛОСКАЯ ДИНАМИЧЕСКАЯ ЗАДАЧА УПРУГОСТИ

Рассматривается несколько способов решения плоских динамических задач термоупругости. В первой части работы формулируется несопряженная задача термоупругости в напряжениях и в перемещениях, причем доказывается, что частные интегралы в обоих способах решения разнятся единственно постоянным коэффициентом.

Во второй части приводятся два примера решения несопряженной задачи при использовании функции напряжения, а именно вынужденного колебания бесконечного цилиндра прямоугольного сечения и упругого слоя, нагретых на краю гармонически во времени.

В третьей части дается решение плоской динамической задачи термоупругости при учете сопряжения температурного поля и поля деформации и то путем введения функции напряжения и путем разложения вектора перемещения на две части: потенциальную и ротационную. Ход проведения операции объясняется на примере вынужденных колебаний бесконечного цилиндра прямоугольного сечения и упругого слоя, вызванных действием, гармонически во времени переменных источников тепла и распределенных равномерно в объеме цилиндра.

DEPARTMENT OF MECHANICS OF CONTINUOUS MEDIA
IBTP POLISH ACADEMY OF SCIENCES

Received April 10, 1961.