POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNICAL RESEARCH

PROCEEDINGS OF VIBRATION PROBLEMS

QUARTERLY

VOL. 10 WARSAW No. 2
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PROCEEDINGS
OF VIBRATION PROBLEMS
WARSAW, 2, 10 (1969)

THE GENERATION OF WAVES IN AN INFINITE MICROPOLAR ELASTIC SOLID

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1. Introduction

The aim of the present work is the determination of the field of displacement $\mathbf{u}(\mathbf{x}, t)$ and the field of rotations $\boldsymbol{\omega}(\mathbf{x}, t)$ in an infinite micropolar elastic medium, generated as a result of the action of body forces and couples. Such a general approach includes the particular case of the action of concentrated body forces and couples. Displacements and rotations caused by these actions form the set of basic solutions.

We shall consider an elastic, isotropic, homogeneous and circularly symmetrical medium in which the state of strain is determined by two nonsymmetrical tensors, the tensor of deformation γ_{ji} and the torsional-bending tensor $\varkappa_{ji}$, which are defined as follows [1-3]:

$$\gamma_{ji} = u_{i,j} - \epsilon_{kji}\omega_k, \quad \varkappa_{ji} = \omega_{i,j}. \tag{1.1}$$

The state of stresses is determined by two nonsymmetrical tensors, the tensor of stresses $\boldsymbol{\sigma}$ and the tensor of couple stresses $\boldsymbol{\mu}$. The dependence between the state of strain and the state of stress is described by the following relations [1-3]:

$$\begin{aligned} \sigma_{ji} &= (\varkappa+\alpha)\gamma_{ji} + (\varkappa-\alpha)\gamma_{ij} + \lambda\gamma_{kk}\delta_{ji}, \\ \mu_{ji} &= (\gamma+\varepsilon)\varkappa_{ji} + (\gamma-\varepsilon)\varkappa_{ij} + \beta\varkappa_{kk}\delta_{ji}. \end{aligned} \tag{1.2}$$

The equations of motion are

$$\begin{aligned} \sigma_{ji,j} + X_i &= \varrho\ddot{u}_i, \\ \epsilon_{ijk}\sigma_{jk} + \mu_{ji,j} + Y_i &= J\ddot{\omega}_i, \end{aligned} \tag{1.3}$$

where u_i are the coordinates of the vector of displacements; ω_i—the coordinates of the vector of rotations; X_i, Y_i—the coordinates of the vector of external forces and the vector of body couples respectively; ϵ_{ijk}—the unit pseudotensor; α, $\varkappa$, λ, β, γ, ε are material constants, ϱ, J—respectively the density and the rotational inertia. The functions $\mathbf{u}$, $\boldsymbol{\omega}$, $\mathbf{X}$, $\mathbf{Y}$ are functions of position $\mathbf{x}$ and time t.

Expressing the components of the tensor of stresses and the tensor of couples stresses in Eqs. (1.3) from the relations (1.2), and considering (1.1), we obtain the following set of equations for the vector of displacements and the vector of rotation:

$$\begin{aligned} (\varkappa+\alpha)\nabla^2\mathbf{u} + (\lambda+\varkappa-\alpha)\operatorname{grad}\operatorname{div}\mathbf{u} + 2\alpha\operatorname{rot}\boldsymbol{\omega} + \mathbf{X} &= \varrho\ddot{\mathbf{u}}, \\ (\gamma+\varepsilon)\nabla^2\boldsymbol{\omega} + (\gamma+\beta-\varepsilon)\operatorname{grad}\operatorname{div}\boldsymbol{\omega} + 2\alpha\operatorname{rot}\mathbf{u} - 4\alpha\boldsymbol{\omega} + \mathbf{Y} &= J\ddot{\boldsymbol{\omega}}. \end{aligned} \tag{1.4}$$

The set of six differential Eqs. (1.4) may be reduced by way of the decomposition of the vectors of displacement $\mathbf{u}$, rotation $\boldsymbol{\omega}$ and the vector of external body forces $\mathbf{X}$ and the vector of body couples $\mathbf{Y}$ into a potential part and a rotational part to a set of simple wave equations.

Thus by decomposing the vectors $\mathbf{u}$, $\boldsymbol{\omega}$ into:

$$\begin{aligned} \mathbf{u} &= \operatorname{grad}\Phi + \operatorname{rot}\boldsymbol{\Psi}, \quad \operatorname{div}\boldsymbol{\Psi} = 0, \\ \boldsymbol{\omega} &= \operatorname{grad}\varphi + \operatorname{rot}\boldsymbol{\Omega}, \quad \operatorname{div}\boldsymbol{\Omega} = 0, \end{aligned} \tag{1.5}$$

and the vectors

$$\begin{aligned} \mathbf{X} &= \varrho(\operatorname{grad}\vartheta + \operatorname{rot}\boldsymbol{\chi}), \quad \operatorname{div}\boldsymbol{\chi} = 0 \\ \mathbf{Y} &= J(\operatorname{grad}\sigma + \operatorname{rot}\boldsymbol{\eta}), \quad \operatorname{div}\boldsymbol{\eta} = 0, \end{aligned} \tag{1.6}$$

we obtain from the set of Eqs. (1.4) the following equations:

$$\begin{aligned} &\left(\nabla^2 - \frac{1}{c_1^2}\partial_t^2\right)\Phi + \frac{1}{c_1^2}\vartheta = 0, \\ &\left(\nabla^2 - \tau^2 - \frac{1}{c_3^2}\partial_t^2\right)\varphi + \frac{1}{c_3^2}\sigma = 0, \\ &\left[\left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right)\left(\nabla^2 - \nu^2 - \frac{1}{c_4^2}\partial_t^2\right) + \eta_0^2\nabla^2\right]\boldsymbol{\Psi} = \frac{p}{c_4^2}\operatorname{rot}\boldsymbol{\eta} - \frac{1}{c_2^2}\left(\nabla^2 - \nu^2 - \frac{1}{c_4}\partial^2\right)\boldsymbol{\chi}, \\ &\left[\left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right)\left(\nabla^2 - \nu^2 - \frac{1}{c_4^2}\partial_t^2\right) + \eta_0^2\nabla^2\right]\boldsymbol{\Omega} = \frac{s}{c_2^2}\operatorname{rot}\boldsymbol{\chi} - \frac{1}{c_4^2}\left(\nabla^2 - \frac{1}{c_2^2}\partial_t^2\right)\boldsymbol{\eta}, \end{aligned} \tag{1.7}$$

where the following denotations have been introduced:

$$\begin{aligned} &c_1 = \left(\frac{\lambda+2\varkappa}{\varrho}\right)^{1/2}, \quad c_2 = \left(\frac{\varkappa+\alpha}{\varrho}\right)^{1/2}, \quad c_3 = \left(\frac{\beta+2\gamma}{J}\right)^{1/2}, \quad c_4 = \left(\frac{\gamma+\varepsilon}{J}\right)^{1/2}, \\ &\tau^2 = \frac{4\alpha}{\beta+2\gamma}, \quad \nu^2 = \frac{4\alpha}{\gamma+\varepsilon}, \quad \eta_0^2 = \frac{4\alpha^2}{(\gamma+\varepsilon)(\varkappa+\alpha)}, \quad p = \frac{2\alpha}{\varkappa+\alpha}, \quad s = \frac{2\alpha}{\gamma+\varepsilon}. \end{aligned} \tag{1.7a}$$

Equations $(1.7)_1$ and $(1.7)_2$ are decoupled, the first one representing the propagation of longitudinal waves, whereas the second—the propagation of torsional waves. Equations $(1.7)_3$ and $(1.7)_4$ are coupled and present the propagation of the modified transverse waves. The completeness of these potentials has been proved in the work [6].

In the next section, the general solutions of the equations of motion (1.4) will be presented for the case in which the causes bringing about the deformation of the body are body forces and moments. Fourier's quadruple integral exponential transformation will be used for solutions of problems for the case of the plane state of strain and solutions for the case of the static action of concentrated forces and couples will also be included. In Sec. 3, the solutions for the case of harmonic vibrations will be presented. Next in Sec. 4, we shall present solutions for the case of axially symmetrical deformation of the body with the independence of all causes and effects of the angle φ (in the system of cylindrical coordinates r, φ, z). And finally in Sec. 5 the solutions will be presented for the axially symmetrical deformation of the body in the case of the action of body forces and couples harmonically varying in time.

2. General Solutions of the Equations of Motion

To solve the set of Eqs. (1.7), we shall use the quadruple Fourier transformation defined as follows:

$$\tag{2.1}\begin{aligned}\tilde{\Phi}(\xi_1, \xi_2, \xi_3, \mu) &= \frac{1}{4\pi^2}\int\limits_{E_4}\Phi(x_1, x_2, x_3, t)\exp[i(x_k\xi_k+\mu t)]\,dV,\\ \Phi(x_1, x_2, x_3, t) &= \frac{1}{4\pi^2}\int\limits_{W_4}\tilde{\Phi}(\xi_1, \xi_2, \xi_3, \mu)\exp[-i(x_k\xi_k+\mu t)]\,dW,\end{aligned}$$

where $dV = dx_1dx_2dx_3dt$ and E_4 denote the interior of the space x_1, x_2, x_3, t, and $dW = d\xi_1 d\xi_2 d\xi_3 d\mu$, where W_4 is the interior of the space.

From the set of Eqs. (1.7), we obtain, after applying the expressions

$$\tag{2.2}\frac{1}{4\pi}\int\limits_{E_4}\left(\frac{\partial\Phi}{\partial x_j}, \frac{\partial^2\Phi}{\partial t^2}\right)\exp[i(x_k\xi_k+\mu t)]\,dV = -(i\xi_j, \mu^2)\tilde{\Phi},$$

the following transforms:

$$\tag{2.3}\begin{aligned}\tilde{\Phi} &= \frac{1}{c_1^2}\frac{\tilde{\vartheta}}{\boldsymbol{\xi}^2-\sigma_1^2}, \qquad \tilde{\varphi} = \frac{1}{c_3^2}\frac{\tilde{\sigma}}{\boldsymbol{\xi}^2+\tau^2-\sigma_3^2},\\ \tilde{\Psi}_j &= \frac{1}{\Delta}\left[\frac{1}{c_2^2}(\boldsymbol{\xi}^2+\nu^2-\sigma_4^2)\tilde{\chi}_j - \frac{ip\xi_k}{c_4^2}\in_{jkl}\tilde{\eta}_l\right],\\ \tilde{\Omega}_j &= \frac{1}{\Delta}\left[\frac{1}{c_4^2}(\boldsymbol{\xi}^2-\sigma_2^2)\tilde{\eta}_j - \frac{is}{c_2^2}\xi_k\in_{jkl}\tilde{\chi}_l\right],\end{aligned}$$

where the following denotations have been introduced:

$$\tag{2.3a}\begin{aligned}&\sigma_1 = \frac{\mu}{c_1}, \quad \sigma_2 = \frac{\mu}{c_2}, \quad \sigma_3 = \frac{\mu}{c_3}, \quad \sigma_4 = \frac{\mu}{c_4}, \quad \Delta = (\boldsymbol{\xi}^2-\lambda_1^2)(\boldsymbol{\xi}^2-\lambda_2^2),\\ &\lambda_{1,2}^2 = \frac{1}{2}\left[\sigma_2^2+\sigma_4^2+\eta_0^2-\nu^2 \pm \sqrt{[\sigma_4^2-\sigma_2^2-\eta_0^2+\nu^2]^2+4ps\sigma_2^2}\right], \quad \boldsymbol{\xi}^2 = \xi_1^2+\xi_2^2+\xi_3^2.\end{aligned}$$

After carrying out the quadruple Fourier transformation for the expressions (1.5), we shall obtain:

$$\tag{2.4}\begin{aligned}\tilde{u}_j &= -i\xi_j\tilde{\Phi} - i\xi_k\in_{jkl}\tilde{\Psi}_l,\\ \tilde{\omega}_j &= -i\xi_j\tilde{\varphi} - i\xi_k\in_{jkl}\tilde{\Omega}_l.\end{aligned}$$

Inserting into these relations $\tilde{\Phi}, \tilde{\varphi}, \tilde{\Psi}, \tilde{\Omega}_j$ defined by the functions (2.3), and considering that

$$\in_{ljk}\in_{lmn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km},$$

and since $\operatorname{div}\boldsymbol{\chi} = 0$ and $\operatorname{div}\boldsymbol{\eta} = 0$, we obtain the following formulae for the transforms:

$$\tag{2.5}\tilde{u}_j = -\frac{i\xi_j\tilde{\vartheta}}{c_1^2(\boldsymbol{\xi}^2-\sigma_1^2)} + \frac{1}{\Delta}\left[\frac{p}{c_4^2}\boldsymbol{\xi}^2\tilde{\eta}_j - \frac{i}{c_2^2}(\boldsymbol{\xi}^2+\nu^2-\sigma_4^2)\in_{jkl}\xi_k\tilde{\chi}_l\right],$$

$$\tag{2.6}\tilde{\omega}_j = -\frac{i\xi_j\tilde{\sigma}}{c_3^2(\boldsymbol{\xi}^2+\tau^2-\sigma_3^2)} + \frac{1}{\Delta}\left[\frac{1}{c_3^2}\boldsymbol{\xi}^2\tilde{\chi}_j - \frac{i}{c_4^2}(\boldsymbol{\xi}^2-\sigma_2^2)\in_{jkl}\xi_k\tilde{\eta}_l\right].$$

Next let us perform the quadruple Fourier transform over Eqs. (1.6):

$$\tag{2.7} \begin{aligned} \tilde{X}_j &= -\varrho\,(i\xi_j\tilde{\vartheta}+i\xi_k\epsilon_{jkl}\tilde{\chi}_l),\\ \tilde{Y}_j &= -J(i\xi_j\tilde{\sigma}+i\xi_k\epsilon_{jkl}\tilde{\eta}_l). \end{aligned}$$

From the solution of this set of algebraic equations, we arrive at:

$$\tag{2.8} \begin{aligned} &\tilde{\vartheta} = \frac{i\xi_k\tilde{X}_k}{\varrho\boldsymbol{\xi}^2}, \qquad \tilde{\sigma} = \frac{i\xi_k\tilde{Y}_k}{J\boldsymbol{\xi}^2},\\ &\tilde{\chi}_j = -\frac{i}{\varrho\boldsymbol{\xi}^2}\epsilon_{jkl}\xi_l\tilde{X}_l, \qquad \tilde{\eta}_j = -\frac{i}{J\boldsymbol{\xi}^2}\epsilon_{jkl}\xi_k\tilde{Y}_l. \end{aligned}$$

Substituting (2.8) into the formulae (2.5) and (2.6), and after carrying out the inverse Fourier transformation determined by $(2.1)_2$, we obtain the general solution of the set of Eqs. (1.4) in the form of a quadruple improper integral:

$$\tag{2.9} \begin{aligned} u_j(x_1,x_2,x_3,t) = \frac{1}{4\pi^2}\int\limits_{W_4}\Bigg\{&\frac{\xi_j\xi_k\tilde{X}}{\varrho c_1^2\boldsymbol{\xi}^2(\boldsymbol{\xi}^2-\sigma_1^2)}\\ &-\frac{1}{\Delta}\left[\frac{\boldsymbol{\xi}^2+\nu^2-\sigma_4^2}{c_2^2\varrho\boldsymbol{\xi}^2}(\xi_j\xi_k\tilde{X}_k-\boldsymbol{\xi}^2\tilde{X}_j)+\frac{ip}{Jc_4^2}\epsilon_{jkl}\xi_kY_l\right]\Bigg\}\exp[-i(\xi_kx_k+\mu t)]\alpha W, \end{aligned}$$

$$\tag{2.10} \begin{aligned} \omega_j(x_1,x_2,x_3,t) = \frac{1}{4\pi^2}\int\limits_{W_4}&\frac{\xi_j\xi_k\tilde{Y}_k}{Jc_3^2\boldsymbol{\xi}^2(\boldsymbol{\xi}^2+\tau^2-\sigma_3^2)}\\ &-\frac{1}{\Delta}\left[\frac{\boldsymbol{\xi}^2-\sigma_2^2}{Jc_4^2\boldsymbol{\xi}^2}(\xi_j\xi_k\tilde{Y}_k-\boldsymbol{\xi}^2\tilde{Y}_j)+\frac{is}{\varrho c_2^2}\epsilon_{jkl}\xi_k\tilde{X}_l\right]\Bigg\}\exp[-i(\xi_kx_k+\mu t)]\,dW. \end{aligned}$$

Since the displacements and rotations are known, we can now determine the tensor of deformation γ_{ji} and the tensor of rotations $\varkappa_{ji}$, and on the basis of the formulae (1.2) we can determine the tensor of stresses σ_{ji} and the tensor of couple stresses μ_{ji}.

We shall now consider the particular case in which $\alpha = 0$. Then (1.4) are independent of each other and take the form:

$$\tag{2.11} \begin{aligned} \varkappa\nabla^2\mathbf{u}+(\varkappa+\lambda)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+\mathbf{X} &= \varrho\ddot{\mathbf{u}},\\ (\gamma+\varepsilon)\nabla^2\boldsymbol{\omega}+(\gamma+\beta-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\omega}+\mathbf{Y} &= J\ddot{\boldsymbol{\omega}}. \end{aligned}$$

Equations $(2.11)_1$ are the equations of classical electrokinetics, while Eqs. $(2.11)_2$ describe the motion in an elastic hypothetical medium in which may occur only rotations but no displacements. For $\alpha = 0$, we obtain from (2.9) and (2.10) the following expression for displacements and rotations:

$$\tag{2.12} u_j = \frac{1}{4\pi^2\varkappa}\int\limits_{W_4}\frac{\tilde{X}_j(\delta^2\boldsymbol{\xi}^2-\mu^2/c_2^2)-(\delta^2-1)\xi_j\xi_k\tilde{X}_k}{(\boldsymbol{\xi}^2-\mu^2/c_2^2)(\boldsymbol{\xi}^2\delta^2-\mu^2/c_2^2)}\exp[-i(\xi_kx_k+\mu t)]\,dW.$$

$$\tag{2.13} \omega_j = \frac{1}{4\pi^2(\gamma+\varepsilon)}\int\limits_{W_4}\frac{\tilde{Y}_j(\varrho^2\boldsymbol{\xi}^2-\mu^2/c_4^2)-(\varrho_0^2-1)\xi_j\xi_k\tilde{Y}_k}{(\boldsymbol{\xi}^2-\mu^2/c_4^2)(\boldsymbol{\xi}^2\varrho_0^2-\mu^2/c_4^2)}\exp[-i(\xi_kx_k+\mu t)]\,dW,$$

where

$$\tag{2.13a} \delta^2 = \frac{\lambda+2\varkappa}{\varkappa}, \qquad \varrho_0^2 = \frac{\beta+2\gamma}{\gamma+\varepsilon}.$$

The formula (2.12) has been derived in the work [4].

We shall assume that we are dealing with a plane state of strain in which all causes $(\mathbf{X}, \mathbf{Y})$ and effects $(\mathbf{u}, \boldsymbol{\omega})$ depend only on the variables x_1, x_2, t.

In this case, the set of Eqs. (1.4) disintegrates into two sets of equations independent of each other. In the first set, X_1, X_2, X_3 occur as causes and u_1, u_2, ω_3 as effects, and in the second set Y_1, Y_2, Y_3 are the causes and ω_1, ω_2, u_3 the effects.

Denoting the body forces and couples as functions of x_1, x_2, t by X_j^* and Y_j^*, we shall determine the quantities $\tilde{X}_j$ and $\tilde{Y}_j$ occurring in the formulae (2.9), (2.10) in the following manner:

$$(2.14) \qquad \tilde{X}_j(\xi_1, \xi_2, \xi_3, \mu) = \frac{1}{4\pi^2} \int_{E_3} X_j^*(x_1, x_2, t) \exp[i(\xi_1 x_1 + \xi_2 x_2 + \mu t)] dS \int_{-\infty}^{\infty} e^{i\xi_3 x_3} dx_3.$$

Here, $dS = dx_1 dx_2 dt$, and E_3 is the interior of the space x_1, x_2, t. Since

$$(2.14a) \qquad \int_{-\infty}^{\infty} e^{i\xi_3 x_3} dx_3 = 2\pi\delta(\xi_3),$$

therefore

$$(2.15) \qquad \tilde{X}_j(\xi_1, \xi_2, \xi_3, \mu) = \delta(\xi_3)\sqrt{2\pi}\, \tilde{X}_j^*(\xi_1, \xi_2, \mu),$$

where

$$(2.15a) \qquad \tilde{X}_j^*(\xi_1, \xi_2, \mu) = \frac{1}{(2\pi)^{3/2}} \int_{E_3} X_j^*(x_1, x_2, t) \exp[i(\xi_1 x_1 + \xi_2 x_2 + \mu t)] dS.$$

Substituting (2.15) and the analogous formula for $\tilde{Y}_j(\xi_1, \xi_2, \xi_3, \mu)$ into (2.9) and (2.10), we obtain:

$$(2.16) \qquad u_j = \frac{1}{(2\pi)^{2/3}} \int_{W_3} \left\{ \frac{\xi_j \xi_k \tilde{X}_k^*}{\varrho c_1^2 \boldsymbol{\xi}^2 (\xi_2^2 - \sigma_1^2)} - \frac{1}{\Delta} \left[\frac{\boldsymbol{\xi}^2 + \nu^2 - \sigma_4^2}{\varrho c_2^2 \boldsymbol{\xi}^2} (\xi_j \xi_k \tilde{X}_k^* - \boldsymbol{\xi}^2 \tilde{X}_j^*) + \frac{ip}{J c_4^2} \epsilon_{jkl} \xi_k \tilde{Y}_l^* \right] \right\} \exp[-i(x_k \xi_k + \mu t)] dT,$$

$$(2.17) \qquad \omega_j = \frac{1}{(2\pi)^{3/2}} \int_{W_3} \left\{ \frac{\xi_j \xi_k \tilde{Y}_k^*}{J c_3^2 \boldsymbol{\xi}^2 (\boldsymbol{\xi}^2 + \tau^2 - \sigma_3^2)} - \frac{1}{\Delta} \left[\frac{\boldsymbol{\xi}^2 - \sigma_2^2}{J c_4^2 \boldsymbol{\xi}^2} (\xi_j \xi_k \tilde{Y}_k^* - \boldsymbol{\xi}^2 \tilde{Y}_j^*) + \frac{is}{\varrho c_2^2} \epsilon_{jkl} \xi_k \tilde{X}_l^* \right] \right\} \exp[-i(x_k \xi_k + \mu t)] dT,$$

$$j, k = 1, 2, \quad \boldsymbol{\xi}^2 = \xi_1^2 + \xi_2^2.$$

Here, $dT = dx_1 dx_2 dt$ and W_3 is the interior of the space ξ_1, ξ_2, t.

It can be noticed from these formulae that u_1, u_2, ω_3 might arise from the action of the body forces X_1^*, X_2^* and the body couples Y_3^*. The functions ω_1, ω_2, u_3 are connected with the action of the body forces X_3^* and the body couples Y_1^*, Y_2^*.

We shall presently consider the following particular case-static loads. We assume that in the static problem the body forces are functions of x_i only, viz.:

$$(2.18) \qquad X_j = P_j(x_1, x_2, x_3), \quad Y_j = M_j(x_1, x_2, x_3).$$

Fourier's transform of the component of mass force will be:

$$(2.19) \qquad \tilde{X}_j(\xi_1, \xi_2, \xi_3, \mu) = \sqrt{2\pi}\, \tilde{P}_j(\xi_1, \xi_2, \xi_3)\delta(\mu),$$

where

$$\tilde{P}_j(\xi_1, \xi_2, \xi_3) = \frac{1}{(2\pi)^{3/2}} \int_{B_3} P_j(x_1, x_2, x_3) e^{i\xi_k x_k} dA, \tag{2.19a}$$

where the fact that $\int_{-\infty}^{\infty} e^{i\mu t} dt = 2\pi\delta(\mu)$ has been taken advantage of.

Here $dA = dx_1\, dx_2\, dx_3$, and B_3 is the interior of the space x_1, x_2, x_3.

Analogously, for Fourier's transform of the body couple, we obtain the expression:

$$\tilde{Y}_j(\xi_1, \xi_2, \xi_3, \mu) = \sqrt{2\pi}\, \tilde{M}_j(\xi_1, \xi_2, \xi_3)\, \delta(\mu). \tag{2.20}$$

Substituting (2.19) and (2.20) into (2.9), we obtain the expression for static displacements caused by the action of the body forces $P_j(x_1, x_2, x_3)$ and couples $M_j(x_1, x_2, x_3)$:

$$u_j(x_1, x_2, x_3) = \frac{1}{(2\pi)^3} \int_{D_3} \left\{ \frac{\xi_j \xi_k \tilde{P}_k}{\varrho c_1^2 \boldsymbol{\xi}^4} - \frac{1}{\Delta_0} \left[\frac{ip}{Jc_4^2} \epsilon_{jkl} \xi_k \tilde{M}_l + \frac{\boldsymbol{\xi}^2 + \nu^2}{\varrho c_2^2 \boldsymbol{\xi}^2} (\xi_j \xi_k \tilde{P}_k - \boldsymbol{\xi}^2 \tilde{P}_j) \right] \right\} e^{-i\xi_k x_k} dD, \tag{2.21}$$

and substituting into (2.10), we obtain the following expression for rotations:

$$\omega_j(x_1, x_2, x_3) = \frac{1}{(2\pi)^3} \int_{D_3} \left\{ \frac{\xi_j \xi_k \tilde{M}_k}{Jc_3^2 \boldsymbol{\xi}^2 (\boldsymbol{\xi}^2 + \tau^2)} - \frac{1}{\Delta_0} \left[\frac{1}{Jc_4^2} (\xi_j \xi_k \tilde{M}_k - \boldsymbol{\xi}^2 \tilde{M}_j) + \frac{is}{\varrho c_2^2} \epsilon_{jkl} \xi_k \tilde{P}_l \right] \right\} e^{-i\xi_k x_k} dD, \tag{2.22}$$

where $dD = d\xi_1 d\xi_2 d\xi_3$, and D_3 is the interior of the space ξ_1, ξ_2, ξ_3; and

$$\Delta_0 = (\boldsymbol{\xi}^2 - k_1^2)(\boldsymbol{\xi}^2 - k_2^2), \quad k_1^2 + k_2^2 = \eta_0^2 - \nu^2, \quad k_1^2 k_2^2 = 0. \tag{2.22a}$$

Putting $k_1^2 = 0$, $k_2^2 = \eta_0^2 - \nu^2$, we obtain:

$$\Delta_0 = \boldsymbol{\xi}^2 (\boldsymbol{\xi}^2 - \eta_0^2 + \nu^2). \tag{2.22b}$$

When $\alpha = 0$, we obtain:

$$u_j = \frac{1}{(2\pi)^3 \varkappa} \int_{W_3} \left\{ \frac{\tilde{P}_j}{\boldsymbol{\xi}^2} - \frac{\delta^2 \xi_j \xi_k \tilde{P}_k}{\beta \boldsymbol{\xi}^4} \right\} e^{-i\xi_k x_k} dW_3, \tag{2.23}$$

where $\beta = c_1^2/\hat{c}_2^2$, $\delta^2 = 1 + \lambda/\varkappa$, $\hat{c}_2^2 = \varkappa/\varrho$, which is the expression known in classical elastokinetics [4] and

$$\omega_j = \frac{1}{(2\pi)^3 (\gamma + \varepsilon)} \int_{W_3} \left\{ \frac{\xi_j \xi_k (1 - \beta_1)}{\boldsymbol{\xi}^4 \beta_1} \tilde{M}_k - \frac{\tilde{M}_j}{\boldsymbol{\xi}^2} \right\} e^{-i\xi_k x_k} dW_3, \tag{2.24}$$

where $\beta_1 = c_3^2/c_4^2$, which represents the solution in such a hypothetical medium in which only rotations and couple stresses may occur.

Applying the previously described method, we shall pass to the two-dimensional static problem. As a result we obtain

$$u_j = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \left\{ \frac{\xi_j \xi_k \tilde{P}_k}{\varrho c_1^2 \boldsymbol{\xi}^4} - \frac{1}{\Delta_0} \left[\frac{\boldsymbol{\xi}^2 + \nu^2}{\varrho c_2^2 \boldsymbol{\xi}^2} (\xi_j \xi_k \tilde{P}_k - \boldsymbol{\xi}^2 \tilde{P}_j) + \frac{ip}{Jc_4^2} \epsilon_{jkl} \xi_k \tilde{M}_l \right] \right\} \exp[-i(\xi_1 x_1 + \xi_2 x_2)]\, d\xi_1\, d\xi_2, \tag{2.25}$$

(2.26) $$\omega_j = \frac{1}{2\pi}\int\!\!\int_{-\infty}^{\infty}\left\{\frac{\xi_j\xi_k\tilde{M}_k}{Jc_3^2\boldsymbol{\xi}^4} - \frac{1}{\Delta_0}\left[\frac{\boldsymbol{\xi}^2+\nu^2}{Jc_1^2\boldsymbol{\xi}^2}(\xi_j\xi_k\tilde{M}_k-\boldsymbol{\xi}^2\tilde{M}_j) + \frac{is}{\varrho c_2^2}\,\epsilon_{jkl}\xi_k\tilde{P}_l\right]\right\}\exp[-i(\xi_1x_1+\xi_2x_2)]\,d\xi_1\,d\xi_2, \qquad j,k=1,2.$$

Here $\boldsymbol{\xi}^2 = \xi_1^2+\xi_2^2$ and

(2.26a) $$\tilde{M}_j = \frac{1}{2\pi}\int\!\!\int_{-\infty}^{\infty} M_j(x_1,x_2)e^{i(x_1\xi_1+x_2\xi_2)}\,dx_1\,dx_2.$$

Let us consider a particular case. We shall assume that in the origin of the system of coordinates a concentrated force acts along the axis x_1:

(2.27) $$X_k(\mathbf{x},t) = P_0\delta(x_1)\delta(x_2)\delta(x_3)\delta_{1k}, \qquad Y_k = 0.$$

From the formulae (2.21) and (2.22), we obtain:

(2.28)
$$u_j^{(1)} = \frac{P_0}{(2\pi)^3}\int\!\!\int\!\!\int_{-\infty}^{\infty}\left\{\frac{\xi_j\xi_1}{\varrho c_1^2\boldsymbol{\xi}^4} - \frac{\boldsymbol{\xi}^2+\nu^2}{\boldsymbol{\xi}^4(\boldsymbol{\xi}^2-\eta_0^2+\nu^2)\varrho c_2^2}(\xi_j\xi_1-\boldsymbol{\xi}^2)\right\}e^{-ix_k\xi_k}\,d\xi_1\,d\xi_2\,d\xi_3,$$
$$\omega_j^{(1)} = -\frac{P_0}{(2\pi)^3}\int\!\!\int\!\!\int_{-\infty}^{\infty}\frac{si}{\boldsymbol{\xi}^2(\boldsymbol{\xi}^2-\eta_0^2+\nu^2)\varrho c_2^2}\,\epsilon_{jkl}\xi_1\,e^{-i\xi_kx_k}\,d\xi_1\,d\xi_2\,d\xi_3.$$

After performing the integration and bearing in mind that

(2.28a) $$\int\!\!\int\!\!\int_{-\infty}^{\infty}\frac{e^{-ix_k\xi_k}}{\boldsymbol{\xi}^2-k_2^2}\,d\xi_1\,d\xi_2\,d\xi_3 = 2\pi^2\frac{e^{ik_2R}}{R}; \qquad \int\!\!\int\!\!\int_{-\infty}^{\infty}\frac{e^{-ix_k\xi_k}}{\boldsymbol{\xi}^4}\,d\xi_1\,d\xi_2\,d\xi_3 = -\pi^2R,$$

we obtain the following expressions for displacements and rotations caused by the action of the concentrated force (2.27):

(2.29)
$$u_j^{(1)} = -\frac{P_0}{8\pi\mu}\left\{\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_1}\left[\frac{\lambda+\mu}{\lambda+2\mu}R - \frac{\gamma+\varepsilon}{2\mu}\left(\frac{e^{ik_2R}}{R}-\frac{1}{R}\right)\right] + \left(\frac{2\alpha}{\mu+\alpha}\frac{e^{ik_2R}}{R}-\frac{2}{R}\right)\delta_{1j}\right\},$$
$$\omega_j^{(1)} = -\frac{P_0}{8\mu\pi}\epsilon_{1jk}\frac{\partial}{\partial x_k}\left(\frac{e^{ik_2R}}{R}-\frac{1}{R}\right),$$

where

$$k_2 = \left[\frac{4\alpha\varkappa}{(\gamma+\varepsilon)(\alpha+\varkappa)}\right]^{1/2}.$$

Those results are consistent with the solutions of N. SANDRU [5].

In the case in which in the origin of the system there acts a static concentrated couple with the vector oriented in the positive direction of the axis x_1:

(2.30) $$Y_k(\mathbf{x},t) = M_0\delta(x_1)\delta(x_2)\delta(x_3)\delta_{1k}, \qquad X_k = 0,$$

then for the displacements and rotations, we obtain from (2.21) and (2.22) the following expressions:

$$(2.31)\quad \begin{aligned} u_j^{(1)} &= -\frac{M_0}{(2\pi)^3}\iiint\limits_{-\infty}^{\infty} \epsilon_{jkl}\xi_k \frac{ip}{Jc_4^2} e^{-i\xi_k x_k} d\xi_1 d\xi_2 d\xi_3, \\ \omega_j^{(1)} &= \frac{M_0}{(2\pi)^3}\iiint\limits_{-\infty}^{\infty} \frac{\xi_j\xi_1}{Jc_3^2\xi^2(\xi^2+\tau^2)} e^{-i\xi_k x_k} d\xi_1 d\xi_2 d\xi_3, \end{aligned}$$

from which we obtain upon integration:

$$(2.32)\quad \begin{aligned} u_j^{(1)} &= -\frac{M_0}{8\pi\mu}\epsilon_{1jk}\frac{\partial}{\partial x_k}\left(\frac{e^{ik_2R}}{R}-\frac{1}{R}\right), \\ \omega_j^{(1)} &= \frac{M_0}{16\pi\alpha}\left\{\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_1}\left[\left(\frac{e^{i\tau R}}{R}-\frac{1}{R}\right)-\left(1+\frac{\alpha}{\mu}\right)\left(\frac{e^{ik_2R}}{R}-\frac{1}{R}\right)\right]+\frac{4\alpha e^{ik_2R}}{R(\gamma+\varepsilon)}\delta_{1j}\right\}. \end{aligned}$$

When $\alpha \to 0$, then from (2.29), we obtain the solution for the classical theory of elasticity for the action of a static force concentrated in the origin of the system of coordinates in the direction of the x_1 axis:

$$(2.33)\quad u_j^{(1)} = -\frac{P_0(\lambda+\varkappa)}{8\pi\mu(\lambda+2\varkappa)}\left(\frac{1}{R}-\frac{x_jx_1}{R^3}\right)-\frac{2}{R}\delta_{1j}, \quad \omega_j^{(1)} = 0.$$

3. Vibration Harmonically Varying in Time

Let us consider vibration harmonically varying in time caused by the action of body forces and couples:

$$(3.1)\quad X_i(\mathbf{x}, t) = X_i^*(\mathbf{x},)e^{-i\omega t}; \quad Y_i(\mathbf{x}, t) = Y_i^*(\mathbf{x})e^{-i\omega t}.$$

The formulae for the displacements $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^*(\mathbf{x})e^{-i\omega t}$ and rotations $\boldsymbol{\omega}(\mathbf{x}, t) = \boldsymbol{\omega}^*(\mathbf{x})e^{-i\omega t}$ are obtained from the transformations of the formulae (2.9) and (2.10) for forces variable arbitrarily in time. The transforms occurring in those formulae are expressed in the following manner:

$$(3.2)\quad \tilde{X}_i(\boldsymbol{\xi}, \mu) = \frac{1}{4\pi^2}\int\limits_{W_3} X_i^*(\mathbf{x})e^{ix_k\xi_k}dV_3 \int\limits_{-\infty}^{\infty} e^{it(\mu-\omega)}dt.$$

Since

$$\int\limits_{-\infty}^{\infty} e^{it(\mu-\omega)}dt = 2\pi\delta(\mu-\omega),$$

we have:

$$(3.3)\quad \tilde{X}_i(\boldsymbol{\xi}, \mu) = \sqrt{2\pi}\,\tilde{X}_i^*(\boldsymbol{\xi})(\mu-\omega),$$

where

$$(3.4)\quad \tilde{X}_i^*(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{3/2}}\int\limits_{W_3} X_i^*(\mathbf{x})e^{ix_k\xi_k}dV_3; \quad dV_3 = dx_1dx_2dx_3.$$

Substituting (3.4) into the formulae (2.9), (2.10), we obtain the following expressions for the displacements $\mathbf{u}(\mathbf{x}, t)$ and rotations $\boldsymbol{\omega}(\mathbf{x}, t)$

$$(3.5) \quad u_j = \frac{e^{-i\omega t}}{(2\pi)^{3/2}} \int\limits_{W_3} \left\{ \frac{\xi_j \xi_k \tilde{X}_k^*}{\varrho c_1^2 \boldsymbol{\xi}^2(\boldsymbol{\xi}^2 - \sigma_1^2)} - \frac{1}{\Delta} \left[\frac{\boldsymbol{\xi}^2 + \nu^2 - \sigma_4^2}{c_2^2 \varrho \boldsymbol{\xi}^2} (\xi_j \xi_k \tilde{X}_k^* - \boldsymbol{\xi}^2 \tilde{X}_j^*) + \frac{ip}{Jc_4^2} \epsilon_{jkl} \xi_k \tilde{Y}_l^* \right] \right\}_{\mu=\omega} \exp(-ix_k \xi_k) dW_3, \quad dW_3 = d\xi_1 d\xi_2 d\xi_3,$$

$$(3.6) \quad \omega_j = \frac{e^{-i\omega t}}{(2\pi)^{3/2}} \int\limits_{W_3} \left\{ \frac{\xi_j \xi_k \tilde{Y}_k^*}{Jc_3^2 \boldsymbol{\xi}^2(\boldsymbol{\xi}^2 - \sigma_3^2 + \tau^2)} - \frac{1}{\Delta} \left[\frac{\boldsymbol{\xi}^2 - \sigma_2^2}{Jc_4^2 \boldsymbol{\xi}^2} (\xi_j \xi_k \tilde{Y}_k^* - \boldsymbol{\xi}^2 \tilde{Y}_j^*) + \frac{is}{\varrho c_2^2} \epsilon_{jkl} \xi_k \tilde{X}_l^* \right] \right\}_{\omega=\omega} \exp(-ix_k \xi_k) dW_3.$$

We shall consider the particular case of the concentrated body force acting in the origin of the system of coordinates and oriented along the x_1-axis:

$$(3.7) \qquad X_k(\mathbf{x}, t) = P_0 \delta(x_1) \delta(x_2) \delta(x_3) \delta_{1k} e^{-i\omega t}, \quad Y_k = 0.$$

From (3.4) we obtain

$$(3.8) \qquad \tilde{X}_l^* = \frac{P_0}{(2\pi)^{3/2}} \int\!\!\int\!\!\int_{-\infty}^{\infty} \delta(x_1)\delta(x_2)\delta(x_3)\delta_{1k} dV_3 = \frac{P_0}{(2\pi)^{3/2}} \delta_{1k}, \quad \tilde{Y}_l^* = 0.$$

Substituting the above expressions into the formulae (3.5), (3.6), we obtain the formulae for the displacements and rotations caused by the concentrated force (3.7):

$$(3.9) \quad u_j = \frac{P_0 e^{-i\omega t}}{(2\pi)^3} \int\limits_{W_3} \left\{ \frac{\xi_j \xi_1}{\varrho c_1^2 \boldsymbol{\xi}^2(\boldsymbol{\xi}^2 - \sigma_1^2)} - \frac{1}{\Delta} \left[\frac{\boldsymbol{\xi}^2 + \nu^2 - \sigma_4^2}{\varrho c_2^2 \boldsymbol{\xi}^2} (\xi_j \xi_1 - \boldsymbol{\xi}^2 \delta_{1j}) \right] \right\}_{\mu=\omega} \exp(-ix_k \xi_k) dW_3,$$

$$(3.10) \qquad \omega_j = \frac{P_0 e^{-i\omega t}}{(2\pi)^3 \varrho c_2^2} s \epsilon_{jk1} \int\limits_{W_3} \frac{i\xi_k \exp(-ix_k \xi_k)}{\Delta} dW_3.$$

However, if in the origin of the system there acts a concentrated couple oriented in the direction of the x_1-axis

$$(3.11) \qquad Y_i(\mathbf{x}, t) = M_0 \delta(x_1) \delta(x_2) \delta(x_3) \delta_{1i} e^{-i\omega t}, \quad X_k = 0,$$

then, from the expression identical with (3.4), we obtain for the couple transform:

$$(3.12) \qquad Y_i^*(\boldsymbol{\xi}) = \frac{M_0}{(2\pi)^{3/2}} \delta_{1i},$$

and next substituting the above values into the formulae (3.5), (3.6), we obtain:

$$(3.13) \quad u_j = \frac{M_0 e^{-i\omega t}}{(2\pi)^3 Jc_4^2} \epsilon_{jk1} ip \int\limits_{W_3} \frac{\xi_k \exp(-ix_k \xi_k)}{\Delta} dW_3,$$

$$(3.14) \quad \omega_j = \frac{M_0 e^{-i\omega t}}{(2\pi)^3} \int\limits_{W_3} \left\{ \frac{\xi_j \xi_1}{Jc_3^2 \boldsymbol{\xi}^2(\boldsymbol{\xi}^2 + \tau^2 - \sigma_3^2)} - \frac{1}{\Delta} \left[\frac{\boldsymbol{\xi}^2 - \sigma_2^2}{Jc_4^2 \boldsymbol{\xi}^2} (\xi_j \xi_1 - \boldsymbol{\xi}^2 \delta_{1j}) \right] \right\} \exp(-ix_k \xi_k) dW_3.$$

The integrals occurring in the formulae (3.9), (3.10) and (3.13) (3.14) can be directly determined as:

$$I_1 = \frac{1}{c_1^2 \varrho} \int_{W_3} \frac{\xi_j \xi_1 \exp(-ix_k \xi_k)}{\xi^2(\xi^2-\sigma_1^2)} dW_3 = -\frac{2\pi^2}{\varrho\omega^2} \partial_1 \partial_j \left(\frac{e^{i\sigma_1 R}}{R} - \frac{1}{R} \right),$$

$$I_2 = \frac{1}{c_2^2 \varrho} \int_{W_3} \frac{(\xi^2+\nu^2-\sigma_4^2)\xi_j \xi_1}{\xi^2(\xi^2-\lambda_1^2)(\xi^2-\lambda_2^2)} \exp(-ix_k \xi_k) dW_3$$

(3.14a)
$$= \frac{2\pi^2}{\varrho\omega^2} \partial_1 \partial_j \left[A_1 \left(\frac{e^{i\lambda_1 R}}{R} - \frac{1}{R} \right) + A_2 \left(\frac{e^{i\lambda_2 R}}{R} - \frac{1}{R} \right) \right],$$

$$I_3 = \frac{1}{c_2^2 \varrho} \int_{W_3} \frac{\xi^2+\nu^2-\sigma_4^2}{(\xi^2-\lambda_1^2)(\xi^2-\lambda_2^2)} \exp(-ix_k \xi_k) dW_3$$

$$= \frac{2\pi^2}{\varrho\omega^2} \left(A_1 \lambda_1^2 \frac{e^{i\lambda_1 R}}{R} + A_2 \lambda_2^2 \frac{e^{i\lambda_2 R}}{R} \right),$$

where

(3.14b)
$$A_1 = \frac{\sigma_2^2-\lambda_2^2}{\lambda_1^2-\lambda_2^2}, \quad A_2 = \frac{\sigma_2^2-\lambda_1^2}{\lambda_2^2-\lambda_1^2}, \quad R = (x_1^2+x_2^2+x_3^2)^{1/2}.$$

Therefore for the displacement $u_j^{(1)}$ and the rotation $\omega_j^{(1)}$ caused by the action of the concentrated force located in the origin of the system of coordinates and oriented in the direction of x_1 axis, we obtain:

(3.15)
$$u_j^{(1)} = \frac{P_0 e^{-i\omega t}}{4\pi\varrho\omega^2} \left(A_1 \lambda_1^2 \frac{e^{i\lambda_1 R}}{R} + A_2 \lambda_2^2 \frac{e^{i\lambda_2 R}}{R} \right) \delta_{1j}$$
$$+ \frac{P_0 e^{-i\omega t}}{4\pi\varrho\omega^2} \frac{\partial}{\partial x_1 \partial x_j} \left(A_1 \frac{e^{i\lambda_1 R}}{R} + A_2 \frac{e^{i\lambda_2 R}}{R} - \frac{e^{i\sigma_1 R}}{R} \right), \quad j = 1, 2, 3,$$

(3.16)
$$\omega_j^{(1)} = \frac{p P_0 e^{-i\omega t}}{4\pi\varrho c_2^2 (\lambda_1^2-\lambda_2^2)} \epsilon_{1jk} \frac{\partial}{\partial x_k} \left(\frac{e^{i\lambda_1 R} - e^{i\lambda_2 R}}{R} \right), \quad j = 1, 2, 3.$$

We shall now shift the concentrated force to the point $\boldsymbol{\eta}$ and orient it in the direction of the x_l axis. Then for $P_0 = 1$, we obtain:

(3.17)
$$u_j = U_j^{(l)}(\mathbf{x}, \boldsymbol{\eta}, t) = \frac{e^{-i\omega t}}{4\pi\varrho\omega^2} \left(A_1 \lambda_1^2 \frac{e^{i\lambda_1 R}}{R} + A_2 \lambda_2^2 \frac{e^{i\lambda_2 R}}{R} \right) \delta_{jl}$$
$$+ \frac{e^{-i\omega t}}{4\pi\varrho\omega^2} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} \left(A_1 \frac{e^{i\lambda_1 R}}{R} + A_2 \frac{e^{i\lambda_2 R}}{R} - \frac{e^{i\sigma_1 R}}{R} \right),$$

(3.18)
$$\omega_j = \Omega_j^{(l)}(\mathbf{x}, \boldsymbol{\eta}, t) = \frac{e^{-i\omega t} s}{4\pi\varrho c_2^2 (\lambda_1^2-\lambda_2^2)} \epsilon_{ljk} \frac{\partial}{\partial x_k} \left(\frac{e^{i\lambda_1 R}}{R} - \frac{e^{i\lambda_2 R}}{R} \right),$$
$$j, l = 1, 2, 3.$$

In these equations, we have $R = [(x_i-\eta_i)(x_i-\eta_i)]^{1/2}$. In this way, we have obtained the displacement tensor $U_j^{(l)}(\mathbf{x}, \boldsymbol{\eta}, t)$ and the tensor of rotations $\Omega_j^{(l)}(\mathbf{x}, \boldsymbol{\eta}, t)$. These tensors form symmetrical matrices.

Substituting into the formulae (3.17), (3.18) $\alpha = 0$, we obtain the transition to classical elastokinetics [4]:

(3.19)
$$U_j^{(l)}(\mathbf{x}, \boldsymbol{\eta}, t) = \frac{e^{-i\omega t}}{4\pi\varrho\omega^2}\left[\tau_0^2 \frac{e^{i\tau_0 R}}{R}\delta_{jl} - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}\left(\frac{e^{i\sigma_1 R} - e^{i\tau_0 R}}{R}\right)\right],$$
$$\Omega_j^{(l)}(\mathbf{x}, \boldsymbol{\eta}, t) = 0.$$

Here,

(3.19a)
$$\tau_0 = \frac{\omega}{\hat{c}_2}, \quad \hat{c}_2 = \left(\frac{\varkappa}{\varrho}\right)^{1/2}, \quad \sigma_1 = \frac{\omega}{c_1}, \quad c_1 = \left(\frac{\lambda + 2\varkappa}{\varrho}\right)^{1/2}.$$

The formulae (3.17) and (3.18) have been derived in a different way in the work [5].

Let us now pass to the action of the concentrated body couple located in the origin of the system of coordinates with the vector oriented in the direction of the x_1 axis and described by (3.11). Integrating analogously the formulae (3.13), (3.14), we obtain the following expressions:

(3.20)
$$u_j^{(1)} = \frac{pM_0 e^{-i\omega t}}{4\pi J c_4^2(\lambda_1^2 - \lambda_2^2)} \epsilon_{1jk} \frac{\partial}{\partial x_k}\left(\frac{e^{i\lambda_1 R}}{R} - \frac{e^{i\lambda_2 R}}{R}\right),$$

(3.21)
$$\omega_j^{(1)} = \frac{M_0 e^{-i\omega t}}{4\pi J c_4^2}\left\{\left(\lambda_1^2 C_1 \frac{e^{i\lambda_1 R}}{R} + \lambda_2^2 C_2 \frac{e^{i\lambda_2 R}}{R}\right)\delta_{1j} + \partial_1 \partial_j\left(C_1 \frac{e^{i\lambda_1 R}}{R} + C_2 \frac{e^{i\lambda_2 R}}{R} + C_3 \frac{e^{i\lambda_3 R}}{R}\right)\right\}, \quad j = 1, 2, 3,$$

where

(3.21a)
$$C_1 = \frac{\lambda_1^2 - \sigma_2^2}{\lambda_1^2(\lambda_1^2 - \lambda_2^2)}, \quad C_2 = \frac{\lambda_2^2 - \sigma_2^2}{\lambda_2^2(\lambda_2^2 - \lambda_1^2)}, \quad C_3 = -\frac{\sigma_2^2}{\lambda_1^2 \lambda_2^2}, \quad \lambda_3 = \left(\frac{\omega^2 - \omega_0^2}{c_3^2}\right)^{1/2},$$
$$\omega_0^2 = \frac{4\alpha}{J}.$$

It is noticed that the action of the concentrated couple $Y_j^* = \delta_{1j}\delta(\mathbf{x})$ incurs a zero value of the displacement in the direction of the x_1 axis ($u_1^{(1)} = 0$) which in turn causes that the component of deformation $\gamma_{11} = 0$.

Moving the concentrated couple to the point $\boldsymbol{\eta}$ and orienting the vector of couple parallel to the x_l axis, we obtain the displacement tensor $V_j^{(l)}(\mathbf{x}, \boldsymbol{\eta}, t)$ and the tensor of rotation $W_j^{(l)}(\mathbf{x}, \boldsymbol{\eta}, t)$. For $M_0 = 1$, we have

(3.22)
$$V_j^{(l)}(\mathbf{x}, \boldsymbol{\eta}, t) = \frac{e^{-i\omega t} p}{4\pi J c_4^2(\lambda_1^2 - \lambda_2^2)} \epsilon_{ljk} \frac{\partial}{\partial x_k}\left(\frac{e^{i\lambda_1 R}}{R} - \frac{e^{i\lambda_2 R}}{R}\right),$$

(3.23)
$$W_j^{(l)}(\mathbf{x}, \boldsymbol{\eta}, t) = \frac{e^{-i\omega t}}{4\pi J c_4^2}\left\{\left(\lambda_1^2 C_1 \frac{e^{i\lambda_1 R}}{R} + \lambda_2^2 C_2 \frac{e^{i\lambda_2 R}}{R}\right)\delta_{lj} + \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_j}\left(C_1 \frac{e^{i\lambda_1 R}}{R} + C_2 \frac{e^{i\lambda_2 R}}{R} + C_3 \frac{e^{i\lambda_3 R}}{R}\right)\right\}, \quad j, l = 1, 2, 3.$$

The matrix of the tensors $V_j^{(l)}$ and $W_j^{(l)}$ is symmetrical. In the case in which $\alpha = 0$, we arrive at:

(3.24)
$$V_j^{(l)} = 0,$$
$$W_j^{(l)}(\mathbf{x}, \boldsymbol{\eta}, t) = \frac{e^{-i\omega t}}{4\pi J \omega^2}\left[\tau_1^2 \frac{e^{i\tau_1 R}}{R}\delta_{lj} - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}\left(\frac{e^{i\tau_1 R}}{R} - \frac{e^{i\tau_2 R}}{R}\right)\right],$$

where

$$\tau_1 = \frac{\omega}{c_4}, \quad \tau_2 = \frac{\omega}{c_3}.$$

The expression $(2.4)_2$ applies to the hypothetical medium in which there may occur only rotations and couple stresses.

If we now assume that at the point $\mathbf{x}'$ there acts a concentrated force $P_0 = 1$ in the direction of the x_l-axis, then from (3.18) the rotation will be expressed as:

$$\Omega_j^{(l)}(\mathbf{x}, \mathbf{x}', t) = \frac{se^{-i\omega t}}{4\pi\varrho c_2^2(\lambda_1^2-\lambda_2^2)} \in_{ljk} \frac{\partial}{\partial x_k}\left(\frac{e^{i\lambda_1 R}}{R} - \frac{e^{i\lambda_2 R}}{R}\right). \tag{3.25}$$

If however, we assume that at the point $\mathbf{x}'$ acts a concentrated couple $M_0 = 1$ in the direction of the x_l axis, then from (3.22), we obtain:

$$V_j^{(l)}(\mathbf{x}, \mathbf{x}', t) = \frac{pe^{-i\omega t}}{4\pi\varrho c_4^2(\lambda_1^2-\lambda_2^2)} \in_{ljk} \frac{\partial}{\partial x_k}\left(\frac{e^{i\lambda_1 R}}{R} - \frac{e^{i\lambda_2 R}}{R}\right). \tag{3.26}$$

Therefore, by comparing these formulae, we

$$\Omega_j^{(l)}(\mathbf{x}, \mathbf{x}', t) = V_j^{(l)}(\mathbf{x}', \mathbf{x}, t), \tag{3.27}$$

since there occurs the equality $s/c^2 = p/c_4^2$. This conclusion might be also taken from the theorem of reciprocity [6].

In the case in which the displacements $\mathbf{u}$ and rotations $\boldsymbol{\omega}$ are independent of one space variable—viz., we are dealing with two-dimensional problems—we can obtain the solution of those problems by performing transitions for the known solutions of (3.15), (3.16) and (3.20), (3.21). The above transitions to two-dimensional problems for the action of body forces $X_j = \delta(x_1)\delta(x_2)\delta_{1j}e^{-i\omega t}$ and body couples $Y_j = \delta(x_1)\delta(x_2)\delta_{1j}e^{-i\omega t}$ have been performed in the work [5].

4. Axially Symmetrical Deformation of a Body

In this section we shall consider the case of the axially symmetrical deformation of a body. The field of displacements $\mathbf{u}$ and rotations $\boldsymbol{\omega}$ is characterized by an axial symmetry with respect to the z axis.

In cylindrical coordinates (r, φ, z), and assuming independence of all causes and effects of the angle φ, we obtain from (1.4) two sets of equations independent of each other:

$$\begin{aligned}
&(\varkappa+\alpha)\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) + (\lambda+\varkappa-\alpha)\frac{\partial e}{\partial r} - 2\alpha\frac{\partial \omega_\varphi}{\partial z} + X_r = \varrho \ddot{u}_r,\\
&(\varkappa+\alpha)\nabla^2 u_z + (\lambda+\varkappa-\alpha)\frac{\partial e}{\partial z} + \frac{2\alpha}{r}\frac{\partial}{\partial r}(r\omega_\varphi) + X_z = \varrho \ddot{u}_z,\\
&(\gamma+\varepsilon)\left(\nabla^2 \omega_\varphi - \frac{\omega_\varphi}{r^2}\right) - 4\alpha\omega_\varphi + 2\alpha\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + Y_\varphi = J\ddot{\omega}_\varphi,
\end{aligned} \tag{4.1}$$

and

$$(\gamma+\varepsilon)\left(\nabla^2 \omega_r - \frac{\omega_r}{r^2}\right) - 4\alpha\omega_r + (\beta+\gamma-\varepsilon)\frac{\partial \varkappa}{\partial r} - 2\alpha\frac{\partial u_\varphi}{\partial z} + Y_r = J\ddot{\omega}_r, \tag{4.2}$$

(4.2) (cont.)
$$(\gamma+\varepsilon)\nabla^2\omega_z-4\alpha\omega_z+(\beta+\gamma-\varepsilon)\frac{\partial\varkappa}{\partial z}+\frac{2\alpha}{r}\frac{\partial}{\partial r}(ru_\varphi)+Y_z=J\ddot{\omega}_z,$$
$$(\varkappa+\alpha)\left(\nabla^2u_\varphi-\frac{u_\varphi}{r^2}\right)+2\alpha\left(\frac{\partial\omega_r}{\partial z}-\frac{\partial\omega_z}{\partial r}\right)+X_\varphi=\varrho\ddot{u}_\varphi,$$

where the following denotations have been introduced:

$$\mathbf{u}\equiv(u_r,u_\varphi,u_z),\quad \boldsymbol{\omega}=(\omega_r,\omega_\varphi,\omega_z),\quad \mathbf{X}\equiv(X_r,X_\varphi,X_z),$$

(4.2a)
$$\mathbf{Y}\equiv(Y_r,Y_\varphi,Y_z),\quad e=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z},\quad \varkappa=\frac{1}{r}\frac{\partial}{\partial r}(r\omega_r)+\frac{\partial\omega_z}{\partial z},$$
$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

Let us express the displacements and rotations in Eqs. (4.1) by the potentials Φ, Ψ, Γ:

(4.3)
$$u_r=\frac{\partial\Phi}{\partial r}+\frac{\partial\Psi}{\partial r\partial z},\quad u_z=\frac{\partial\Phi}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right),\quad \omega_\varphi=-\frac{\partial\Gamma}{\partial r},$$

and decompose the body forces and couples into the potential part and the rotational part:

(4.4)
$$X_r=\varrho\left(\frac{\partial\vartheta}{\partial r}-\frac{\partial\chi_\varphi}{\partial z}\right),\quad X_z=\varrho\left[\frac{\partial\vartheta}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(r\chi_\varphi)\right].$$

Substituting (4.3) and (4.4) into Eqs. (4.1), we obtain the following set of wave equations:

(4.5)
$$\left(\nabla^2-\frac{1}{c_1^2}\partial_t^2\right)\Phi+\frac{1}{c_1^2}\vartheta=0,$$
$$-\frac{\partial}{\partial r}\left[\left(\nabla^2-\frac{1}{c_2^2}\partial_t^2\right)\Psi+p\Gamma\right]+\frac{1}{c_2^2}\chi_\varphi=0,$$
$$-\frac{\partial}{\partial r}\left[\left(\nabla^2-\nu^2-\frac{1}{c_4^2}\partial_t^2\right)\Gamma-s\nabla^2\Psi\right]+\frac{Y_\varphi}{Jc_4^2}=0.$$

Let us now express the displacements and rotations in Eqs. (4.2) by the potentials:

(4.6)
$$\omega_r=\frac{\partial\varphi}{\partial r}+\frac{\partial^2\psi}{\partial r\partial z},\quad \omega_z=\frac{\partial\varphi}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right),\quad u_\varphi=-\frac{\partial\Omega}{\partial r},$$

and decompose the mass moments into the potential part and rotational part:

(4.7)
$$Y_r=J\left(\frac{\partial\sigma}{\partial r}-\frac{\partial\eta_\varphi}{\partial z}\right),\quad Y_z=J\left[\frac{\partial\sigma}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(r\eta_\varphi)\right].$$

Substituting (4.6) and (4.7) into Eqs. (4.2), we obtain the following set of wave equations:

(4.8)
$$\left(\nabla^2-\tau^2-\frac{1}{c_3^2}\partial_t^2\right)\varphi+\frac{1}{c_3^2}\sigma=0,$$
$$-\frac{\partial}{\partial r}\left[\left(\nabla^2-\nu^2-\frac{1}{c_4^2}\partial_t^2\right)\psi+s\Omega\right]+\frac{1}{c_4^2}\eta_\varphi=0,$$
$$-\frac{\partial}{\partial r}\left[\left(\nabla^2-\frac{1}{c_2^2}\partial_t^2\right)\Omega-p\nabla^2\psi\right]+\frac{1}{\varrho c_2^2}X_\varphi=0.$$

We shall find the general solution of the wave Eqs. (4.5) and (4.8) by applying the Fourier-Hankel integral transformation. The Fourier-Hankel integral transformation, applied to the set of Eqs. (4.5) has the form [7]:

$$\tag{4.9} \begin{aligned} \tilde{\Phi}(\eta,\zeta,\mu) &= \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} e^{-i(\xi z+\mu t)}\,dz\,dt\int\limits_0^{\infty} r\mathscr{J}_0(r\eta)\,\Phi(r,z,t)\,dr,\\ \Phi(r,z,t) &= \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} e^{-i(\xi z+\mu t)}\,d\zeta\,d\mu\int\limits_0^{\infty} \eta\mathscr{J}_0(r\eta)\,\tilde{\Phi}(\eta,\zeta,\mu)\,d\eta. \end{aligned}$$

Analogous expressions are obtained for the functions Ψ, Ω. Performing the integral transformations of Eqs. (4.5), we obtain a set of algebraic equations whose solutions are given by the transforms:

$$\tag{4.10} \begin{aligned} &\tilde{\Phi} = \frac{1}{c_1^2}\frac{\tilde{\vartheta}}{\alpha^2-\sigma_1^2},\\ &\tilde{\Psi} = \frac{1}{\eta\Delta}\left[\frac{(\alpha^2+\nu^2-\sigma_4^2)}{c_2^2}\,\tilde{\chi}_\varphi+\frac{p\tilde{Y}_\varphi}{Jc_4^2}\right],\qquad \tilde{\Omega} = \frac{1}{\eta\Delta}\left[\frac{\alpha^2 s\tilde{\chi}_\varphi}{c_2^2}+\frac{(\alpha^2-\sigma_2^2)}{Jc_4^2}\,\tilde{Y}_\varphi\right]. \end{aligned}$$

Here

$$\tag{4.10a} \Delta = (\alpha^2-\lambda_1^2)(\alpha^2-\lambda_2^2),\qquad \alpha^2 = \zeta^2+\eta^2,$$

where

$$\tag{4.10b} \lambda_{1,2}^2 = \frac{1}{2}\left(\sigma_2^2+\sigma_4^2+\eta_0^2-\nu^2\mp\sqrt{(\sigma_2^2-\sigma_4^2-\eta_0^2+\nu^2)^2+4\sigma_2^2\eta_0^2}\right).$$

Let us conduct the Fourier-Hankel transformation of the relations (4.3) and (4.4). Assuming that

$$\tag{4.11} \begin{aligned} (\tilde{u}_r,\tilde{X}_r,\chi_\varphi) &= \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} e^{i(\zeta z+\mu t)}\,dz\,dt\int\limits_0^{\infty} r\mathscr{J}_1(r\eta)\,(u_r,X_r,\chi_\varphi)\,dr,\\ (\tilde{\vartheta},\tilde{X}_z) &= \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} e^{i(\zeta z+\mu t)}\,dz\,dt\int\limits_0^{\infty} r\mathscr{J}_0(r\eta)\,(\vartheta,X_z)\,dr, \end{aligned}$$

we obtain:

$$\tag{4.12} \tilde{u}_r = -\eta\tilde{\Phi}+i\zeta\eta\tilde{\Psi},\quad \tilde{u}_z = -i\zeta\tilde{\Phi}+\eta^2\tilde{\Psi},\quad \tilde{\omega}_\varphi = \eta\tilde{\Gamma},$$

$$\tag{4.13} \tilde{X}_r = -\varrho\eta\tilde{\vartheta}+\varrho i\zeta\tilde{\chi}_\varphi,\qquad \tilde{X}_z = -\varrho i\zeta\tilde{\vartheta}+\varrho\eta\tilde{\chi}_\varphi.$$

From the relations (5.13), we obtain:

$$\tag{4.14} \tilde{\vartheta} = \frac{1}{\varrho\alpha^2}(i\zeta\tilde{X}_z-\eta\tilde{X}_r),\qquad \tilde{\chi}_\varphi = \frac{1}{\varrho\alpha}(\eta\tilde{X}_z-i\zeta\tilde{X}_r).$$

Substituting (4.10) into (4.12), and taking into account (4.14), we obtain the transforms of the quantities $\tilde{u}_r$, $\tilde{u}_z$, $\tilde{\omega}_\varphi$ expressed by the transforms $\tilde{X}_r$, $\tilde{X}_z$, $\tilde{Y}_\varphi$.

Performing the inverse Fourier-Hankel transformation, we obtain finally:

$$(4.15)\qquad u_r = -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)}\,d\zeta\,d\mu \int\limits_0^{\infty} \eta_1(\mathscr{I}\eta r)\Bigg\{\frac{\eta\,(i\zeta\tilde{X}_z-\eta\tilde{X}_r)}{\varrho c_1^2(\boldsymbol{\alpha}^2-\sigma_1^2)\boldsymbol{\alpha}^2} - \frac{i\zeta}{\Delta}\left[\frac{(\boldsymbol{\alpha}^2+\nu^2-\sigma_4^2)}{\varrho c_2^2\boldsymbol{\alpha}^2}(\eta\tilde{X}_z-i\zeta\tilde{X}_r)+\frac{p}{Jc_4^2}\tilde{Y}_\varphi\right]\Bigg\}d\eta,$$

$$(4.16)\qquad u_z = -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)}\,d\zeta\,d\mu \int\limits_0^{\infty} \eta\mathscr{I}_0(\eta r)\Bigg\{\frac{i\zeta\,(i\zeta\tilde{X}_z-\eta\tilde{X}_r)}{\varrho c_1^2\boldsymbol{\alpha}^2(\boldsymbol{\alpha}^2-\sigma_1^2)} - \frac{\eta}{\Delta}\left[\frac{(\boldsymbol{\alpha}^2+\nu^2-\sigma_4^2)}{\varrho\boldsymbol{\alpha}^2 c_2^2}(\eta\tilde{X}_z-i\zeta\tilde{X}_r)+\frac{p}{Jc_4^2}\tilde{Y}_\varphi\right]\Bigg\}d\eta,$$

$$(4.17)\qquad \omega_\varphi = \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)}\,d\zeta\,d\mu \int\limits_0^{\infty} \frac{\eta\mathscr{I}_1(\eta r)}{\Delta}\left[\frac{s(\eta\tilde{X}_z-i\zeta\tilde{X}_r)}{\varrho c_2^2}+\frac{(\boldsymbol{\alpha}^2-\sigma_2^2)\tilde{Y}_\varphi}{Jc_4^2}\right]d\eta.$$

Knowing the displacements and rotations, we can determine the components of the tensor of stresses σ_{ji} and the tensor of couple stresses μ_{ji} from the formulae:

$$(4.18)\qquad \begin{aligned}
&\sigma_{rr} = 2\varkappa\frac{\partial u_r}{\partial r}+\lambda e,\quad \sigma_{\varphi\varphi} = 2\varkappa\frac{u_r}{r}+\lambda e,\quad \sigma_{zz} = 2\varkappa\frac{\partial u_z}{\partial z}+\lambda e,\\
&\sigma_{rz} = \varkappa\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)-\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)+2\alpha\omega_\varphi,\\
&\sigma_{zr} = \varkappa\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)+\alpha\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)-2\alpha\omega_\varphi,\\
&\mu_{r\varphi} = \gamma\left(\frac{\partial\omega_\varphi}{\partial r}-\frac{\omega_\varphi}{r}\right)+\varepsilon\left(\frac{\partial\omega_\varphi}{\partial r}+\frac{\omega_\varphi}{r}\right),\\
&\mu_{\varphi r} = \gamma\left(\frac{\partial\omega_\varphi}{\partial r}-\frac{\omega_\varphi}{r}\right)-\varepsilon\left(\frac{\partial\omega_\varphi}{\partial r}+\frac{\omega_\varphi}{r}\right),\\
&\mu_{\varphi z} = (\gamma-\varepsilon)\frac{\partial\omega_\varphi}{\partial z},\quad \mu_{z\varphi} = (\gamma+\varepsilon)\frac{\partial\omega_\varphi}{\partial z}.
\end{aligned}$$

Let us consider the particular case $\alpha = 0$, for which Eqs. (1.4) become independent of each other. From the formulae (4.15) to (4.17), we obtain:

$$(4.19)\qquad \begin{aligned}
u_r &= \frac{1}{2\pi\varrho c_1^2}\iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)}\,d\zeta\,d\mu \int\limits_0^{\infty} \eta\mathscr{I}_1(\eta r)\left[\frac{(\eta^2-\delta^2\zeta^2-\mu^2/c_2^2)\tilde{X}_r+i\zeta\eta\,(\delta^2-1)\tilde{X}_z}{(\boldsymbol{\alpha}^2-\mu^2/c_1^2)(\boldsymbol{\alpha}^2-\mu^2/\hat{c}_2^2)}\right]d\eta,\\
u_z &= \frac{1}{2\pi\varrho c_1^2}\iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)}\,d\zeta\,d\mu \int\limits_0^{\infty} \eta\mathscr{I}_0(\eta r)\left[\frac{(\zeta^2+\eta^2\delta^2-\mu^2/c_2^2)\tilde{X}_z+i\zeta\eta\,(\delta^2-1)\tilde{X}_r}{(\boldsymbol{\alpha}^2-\mu^2/c_1^2)(\boldsymbol{\alpha}^2-\mu^2/\hat{c}_2^2)}\right]d\eta,\\
\omega_\varphi &= \frac{1}{2\pi Jc_4^2}\iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)}\,d\zeta\,d\mu \int\limits_0^{\infty} \eta\mathscr{I}_1(\eta r)\frac{\tilde{Y}_\varphi}{\boldsymbol{\alpha}^2-c_4^2}\,d\eta,\qquad \delta = \frac{c_1}{\hat{c}_2}.
\end{aligned}$$

The formulae $(4.19)_{1,2}$ apply to the classical elastic medium [4], whereas the formula $(4.19)_3$ applies to the elastic medium in which only rotations may occur.

We shall now pass to the set of wave Eqs. (4.8). Performing the integral transformation of them and bearing in mind that

$$
\begin{aligned}
(\tilde{\varphi}, \tilde{\psi}, \tilde{\Omega}, \tilde{\sigma}) &= \frac{1}{2\pi} \iint\limits_{-\infty}^{\infty} e^{i(\zeta z+\mu t)} dz\, dt \int\limits_0^{\infty} r \mathscr{J}_0(\eta r)(\varphi, \psi, \Omega, \sigma)\, dr, \\
(\tilde{X}_\varphi, \tilde{\eta}_\varphi) &= \frac{1}{2\pi} \iint\limits_{-\infty}^{\infty} e^{i(\zeta z+\omega t)} dz\, dt \int\limits_0^{\infty} r \mathscr{J}_1(\eta r)(X_\varphi, \eta_\varphi)\, dr,
\end{aligned} \tag{4.20}
$$

we arrive at the following quantities:

$$
\begin{aligned}
\tilde{\varphi} &= \frac{1}{c_3^2} \frac{\tilde{\sigma}}{(\alpha^2+\tau^2-\sigma_3^2)}, \\
\tilde{\psi} &= \frac{1}{\eta \Delta} \left(\frac{s\tilde{X}_\varphi}{c_2^2 \varrho} + \frac{(\alpha^2-\sigma_2^2)}{c_4^2} \tilde{\eta}_\varphi \right), \\
\tilde{\Omega} &= \frac{1}{\eta \Delta} \left(\frac{\alpha^2+\nu^2-\sigma_4^2}{\varrho c_2^2} \tilde{X}_\varphi - \frac{p\alpha^2}{c_4^2} \tilde{\eta}_\varphi \right).
\end{aligned} \tag{4.21}
$$

Let us perform once again the Fourier-Hankel transformation of the relations (4.6) and (4.7):

$$
\tilde{\omega}_r = -\eta\tilde{\varphi} + i\zeta\eta\tilde{\psi}, \qquad \tilde{\omega}_z = -i\zeta\tilde{\varphi} + \eta^2\tilde{\psi}, \qquad u_\varphi = \eta\tilde{\Omega}, \tag{4.22}
$$

$$
\tilde{Y}_r = -J(\eta\tilde{\sigma} - i\zeta\tilde{\eta}_\varphi), \qquad \tilde{Y}_z = -J(i\zeta\tilde{\sigma} - \eta\tilde{\eta}_\varphi). \tag{4.23}
$$

From the formulae (4.23), it results that

$$
\tilde{\sigma} = \frac{1}{J\alpha^2}(i\zeta\tilde{Y}_z - \eta\tilde{Y}_r), \qquad \tilde{\eta}_\varphi = \frac{1}{J\alpha^2}(\eta\tilde{Y}_z - i\zeta\tilde{Y}_r). \tag{4.24}
$$

Substituting (4.10) into (4.22) and considering (4.24), we obtain the transforms $\tilde{\omega}_r$, $\tilde{\omega}_z$, $\tilde{u}_\varphi$. Performing the inverse Fourier-Hankel transformation of them, we obtain:

$$
\begin{aligned}
\omega_r = -\frac{1}{2\pi} \iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)} d\zeta\, d\mu \int\limits_0^{\infty} \eta \mathscr{J}_1(\eta r) \Bigg\{ & \frac{\eta(i\zeta\tilde{Y}_z - \eta\tilde{Y}_r)}{c_3^2 J\alpha^2(\alpha^2+\nu_0^2-\sigma_3^2)} \\
& - \frac{i\zeta}{\Delta} \left[\frac{\alpha^2-\sigma_2^2}{Jc_4^2\alpha^2}(\eta\tilde{Y}_z - i\zeta\tilde{Y}_r) + \frac{s\tilde{X}_\varphi}{\varrho c_2^2} \right] \Bigg\} d\eta,
\end{aligned} \tag{4.25}
$$

$$
\begin{aligned}
\omega_z = -\frac{1}{2\pi} \iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)} d\mu\, d\zeta \int\limits_0^{\infty} \eta \mathscr{J}_0(\eta r) \Bigg\{ & \frac{i\zeta(i\zeta\tilde{Y}_z - \eta\tilde{Y}_r)}{c_3^2 J\alpha^2(\alpha^2+\nu_0^2-\sigma_3^2)} \\
& - \frac{\eta}{\Delta} \left[\frac{\alpha^2-\sigma_2^2}{Jc_4^2\alpha^2}(\eta\tilde{Y}_z - i\zeta\tilde{Y}_r) + \frac{s\tilde{X}_\varphi}{\varrho c_2^2} \right] \Bigg\} d\eta,
\end{aligned} \tag{4.26}
$$

$$
u_\varphi = \frac{1}{2\pi} \iint\limits_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)} d\mu\, d\zeta \int\limits_0^{\infty} \frac{\eta \mathscr{J}_1(\eta r)}{\Delta} \left\{ \frac{\alpha^2+\nu_0^2-\sigma_4^2}{\varrho c_2^2} \tilde{X}_\varphi - \frac{p}{Jc_4^2}(\eta\tilde{Y}_z - i\zeta\tilde{Y}_r) \right\} d\eta. \tag{4.27}
$$

Knowing beforehand the rotations ω_r, ω_z and the displacement u_φ, we can determine the stresses σ_{ji} and the couple stresses μ_{ji} from the formulae:

$$\mu_{rr} = 2\gamma \frac{\partial \omega_r}{\partial r} + \beta\varkappa, \quad \mu_{\varphi\varphi} = 2\gamma \frac{\omega_r}{r} + \beta\varkappa, \quad \mu_{zz} = 2\gamma \frac{\partial \omega_z}{\partial z} + \beta\varkappa,$$

$$\mu_{rz} = \gamma\left(\frac{\partial \omega_z}{\partial r} + \frac{\partial \omega_r}{\partial z}\right) + \varepsilon\left(\frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r}\right),$$

$$\mu_{zr} = \gamma\left(\frac{\partial \omega_z}{\partial r} + \frac{\partial \omega_r}{\partial z}\right) - \varepsilon\left(\frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r}\right),$$

$$\sigma_{r\varphi} = \varkappa\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) + \frac{\alpha}{r}\frac{\partial}{\partial r}(u_\varphi r) - 2\alpha\omega_z, \tag{4.28}$$

$$\sigma_{\varphi r} = \varkappa\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) - \frac{\alpha}{r}\frac{\partial}{\partial r}(u_\varphi r) + 2\alpha\omega_z,$$

$$\sigma_{\varphi z} = \varkappa \frac{\partial u_\varphi}{\partial z} - \frac{\alpha}{r}\frac{\partial}{\partial z}(r u_\varphi) - 2\alpha\omega_r,$$

$$\sigma_{z\varphi} = \varkappa \frac{\partial u_\varphi}{\partial z} + \frac{\alpha}{r}\frac{\partial}{\partial z}(r u_\varphi) + 2\alpha\omega_r.$$

In the particular case of classical elastokinetics ($\alpha = 0$), we obtain from (4.25)–(4.27):

$$\omega_r = -\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)} d\mu\, d\zeta \int_0^\infty \eta \mathscr{I}_1(\eta r)\left[\frac{\eta(i\zeta\tilde{Y}_z - \eta\tilde{Y}_r)}{Jc_3^2\alpha^2(\alpha^2-\sigma_3^2)} - \frac{i\zeta(\eta\tilde{Y}_z - i\zeta\tilde{Y}_r)}{Jc_4^2\alpha^2(\alpha^2-\sigma_4^2)}\right] d\eta,$$

$$\omega_z = -\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)} d\mu\, d\zeta \int_0^\infty \eta \mathscr{I}_0(\eta r)\left[\frac{i\zeta(i\zeta\tilde{Y}_z - \eta\tilde{Y}_r)}{Jc_3^2\alpha^2(\alpha^2-\sigma_3^2)} - \frac{\eta(\eta\tilde{Y}_z - i\zeta\tilde{Y}_r)}{Jc_4^2\alpha^2(\alpha^2-\sigma_4^2)}\right] d\eta, \tag{4.29}$$

$$u_\varphi = \frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\zeta z+\mu t)} d\mu\, d\zeta \int_0^\infty \eta \mathscr{I}_1(\eta r) \frac{\tilde{X}_\varphi}{\hat{c}_2^2 \varrho(\alpha^2 - \hat{\sigma}_2^2)} d\eta, \quad \hat{c}_2^2 = \frac{\mu}{\varrho}.$$

The first of these formulae refers to the classical elastic medium, while the two remaining formulae refer to the hypothetical medium in which only rotations may occur.

5. Harmonic Vibration in the Case of Axially Symmetrical Deformation of a Body

We shall consider vibration harmonically varying in time caused by the action of body forces and couples:

$$X_i(r, z, t) = X_i^*(r, z)e^{-i\omega t}, \quad Y_i(r, z, t) = Y_i^*(r, z)e^{-i\omega t}. \tag{5.1}$$

For the displacements u_r, u_z and rotation ω_φ, we have the formulae (4.15)–(4.17). The transforms occurring in them will be expressed as follows:

$$\tilde{X}_j(\eta, \zeta, \mu) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{i(\zeta z+\mu t)} dz\, dt \int_0^\infty r_1 \mathscr{I}_1(\eta r)\tilde{X}_j^*(r, z)e^{-i\omega t} dr = \sqrt{2\pi}\,\delta(\mu-\omega)\,\tilde{X}_j^*(\eta, \zeta),$$

(5.2)

where

$$\tilde{X}_j^*(\eta,\zeta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\zeta z}dz\int_0^{\infty}X_j^*(r,z)rJ_1(\eta r)\,dr. \tag{5.3}$$

Analogously, we shall obtain the expression for the transform Y:

$$\tilde{Y}_j(\eta,\zeta,\mu)=\sqrt{2\pi}\,\delta(\mu-\omega)\tilde{Y}_j^*(\eta,\zeta), \tag{5.4}$$

where

$$\tilde{Y}_j^*(\eta,\zeta)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{i\zeta z}dz\int_0^{\infty}Y_j^*(r,z)r\mathscr{J}_1(\eta r)\,dr. \tag{5.5}$$

Therefore, we obtain from (4.15)–(4.17):

$$\mu_r=-\frac{1}{\sqrt{2\pi}}e^{-i\omega t}\int_{-\infty}^{\infty}e^{-i\zeta z}d\zeta\int_0^{\infty}\eta\mathscr{J}_1(\eta r)\left\{\frac{\eta(i\zeta\tilde{X}_z^*-\eta\tilde{X}_r^*)}{\varrho c_1^2(\boldsymbol{\alpha}^2-\sigma_1^2)\boldsymbol{\alpha}^2}-\frac{i\zeta}{\Delta}\left[\frac{\boldsymbol{\alpha}^2+\nu^2-\sigma_4^2}{\varrho c_2^2\boldsymbol{\alpha}^2}(\eta\tilde{X}_z^*-i\zeta\tilde{X}_z^*)+\frac{p}{Jc_4^2}\tilde{Y}_\varphi^*\right]\right\}_{\mu=\omega}d\eta, \tag{5.6}$$

$$\mu_z=-\frac{1}{\sqrt{2\pi}}e^{-i\omega t}\int_{-\infty}^{\infty}e^{-i\zeta z}d\zeta\int_0^{\infty}\eta\mathscr{J}_0(\eta r)\left\{\frac{i\zeta(i\zeta\tilde{X}_z^*-\eta\tilde{X}_r^*)}{\varrho c_1^2\boldsymbol{\alpha}^2(\boldsymbol{\alpha}^2-\sigma_1^2)}-\frac{\eta}{\Delta}\left[\frac{\boldsymbol{\alpha}^2+\nu^2-\sigma_4^2}{\varrho\boldsymbol{\alpha}^2c_2^2}\times(\eta\tilde{X}_z^*-i\zeta\tilde{X}_r^*)+\frac{p}{Jc_4^2}\tilde{Y}_\varphi^*\right]\right\}_{\mu=\omega}d\eta, \tag{5.7}$$

$$\omega_\varphi=\frac{1}{\sqrt{2\pi}}e^{-i\omega t}\int_{-\infty}^{\infty}e^{-i\zeta z}d\zeta\int_0^{\infty}\frac{\eta}{\Delta}\mathscr{J}_1(\eta r)\left[\frac{s(\eta\tilde{X}_z^*-i\zeta\tilde{X}_r^*)}{\varrho c_2^2}+\frac{(\boldsymbol{\alpha}^2-\sigma_2^2)\tilde{Y}_\varphi^*}{Jc_4^2}\right]_{\mu=\omega}d\eta. \tag{5.8}$$

From the solution of the set of Eqs. (4.2)—namely, from Eqs. (5.25) to (5.27), we obtain the expressions for the displacements u_φ and rotations ω_r, ω_z:

$$u_\varphi=\frac{1}{\sqrt{2\pi}}e^{-i\omega t}\int_{-\infty}^{\infty}e^{-i\zeta z}d\zeta\int_0^{\infty}\frac{\eta}{\Delta}\mathscr{J}(\eta r)\left\{\frac{\boldsymbol{\alpha}^2+\tau^2-\sigma_4^2}{c_2^2\varrho}\tilde{X}_\varphi^*-\frac{p}{Jc_4^2}(\eta\tilde{Y}_z^*-i\zeta\tilde{Y}_r^*)\right\}_{\mu=\omega}d\eta, \tag{5.9}$$

$$\omega_r=-\frac{1}{\sqrt{2\pi}}e^{-i\omega t}\int_{-\infty}^{\infty}e^{-i\zeta z}d\zeta\int_0^{\infty}\eta\mathscr{J}_1(\eta r)\left\{\frac{\eta(i\zeta\tilde{Y}_z^*-\eta\tilde{Y}_r^*)}{c_3^2J\boldsymbol{\alpha}^2(\boldsymbol{\alpha}^2+\tau^2-\sigma_3^2)}-\frac{i\zeta}{\Delta}\left[\frac{(\boldsymbol{\alpha}^2-\sigma_2^2)}{Jc_4^2\boldsymbol{\alpha}^2}(\eta\tilde{Y}_z^*-i\zeta\tilde{X}_r^*)+\frac{s\tilde{X}_\varphi^*}{\varrho c_2^2}\right]\right\}_{\mu=\omega}d\eta, \tag{5.10}$$

$$\omega_z=-\frac{1}{\sqrt{2\pi}}e^{-i\omega t}\int_{-\infty}^{\infty}e^{-i\zeta z}d\zeta\int_0^{\infty}\eta\mathscr{J}_0(\eta r)\left\{\frac{i\zeta(i\zeta\tilde{Y}_z^*-\eta\tilde{Y}_r^*)}{c_3^2J\boldsymbol{\alpha}^2(\boldsymbol{\alpha}^2+\tau^2-\sigma_3^2)}-\frac{\eta}{\Delta}\left[\frac{\boldsymbol{\alpha}^2-\sigma_2^2}{Jc_4^2\boldsymbol{\alpha}^2}(\eta\tilde{Y}_z^*-i\zeta\tilde{Y}_r^*)+\frac{s\tilde{X}_\varphi^*}{c_2^2\varrho}\right]\right\}_{\mu=\omega}d\eta. \tag{5.11}$$

We shall now consider two particular cases: (1) the body force concentrated in the origin of the system of coordinates and oriented along the z-axis; and (2) the couple concentrated in the origin of the system of coordinates with the vector oriented in the direction of the positive z-axis.

1. The action of the concentrated force:

$$X_z(r, z, t) = \frac{P_0}{2\pi r}\delta(r)\delta(z)e^{-i\omega t}, \quad Y_\varphi = Y_r = 0, \tag{5.12}$$

from which we obtain:

$$\tilde{X}_z^*(\eta, \zeta) = \frac{P_0}{(2\pi)^{3/2}}, \quad \tilde{Y}_\varphi^* = \tilde{X}_r^* = 0, \tag{5.13}$$

and therefore from the formulae (5.6)–(5.8) we shall have:

$$u_r = -\frac{P_0}{4\pi^2}e^{-i\omega t}\int_{-\infty}^{\infty} e^{-i\zeta z}d\zeta \int_0^{\infty} \eta^2 \mathscr{J}_1(\eta r)\left\{\frac{i\zeta}{\varrho c_1^2(\alpha^2-\sigma_1^2)\alpha^2} - \frac{i\zeta}{\Delta}\frac{\alpha^2+\nu^2-\sigma_4^2}{\varrho c_2^2\alpha^2}\right\}_{\mu=\omega} d\eta, \tag{5.14}$$

$$u_z = \frac{P_0}{4\pi^2}e^{-i\omega t}\int_{-\infty}^{\infty} e^{-i\zeta z}d\zeta \int_0^{\infty} \eta \mathscr{J}_0(\eta r)\left\{\frac{\zeta^2}{\varrho c_1^2(\alpha^2-\sigma_1^2)\alpha^2} + \frac{\eta^2}{\Delta}\frac{\alpha^2+\nu^2-\sigma_4^2}{\varrho \alpha^2 c_2^2}\right\}_{\mu=\omega} d\eta, \tag{5.15}$$

$$\omega_\varphi = \frac{P_0}{4\pi^2}e^{-i\omega t}\int_{-\infty}^{\infty} e^{-i\zeta z}d\zeta \int_0^{\infty} \left\{\frac{s\eta^2}{\varrho c_2^2 \Delta}\right\}_{\mu=\omega} \mathscr{J}_1(\eta r)\, d\eta. \tag{5.16}$$

2. The action of the concentrated couple:

$$Y_z(r, z, t) = \frac{M_0}{2\pi r}\delta(r)\delta(z)e^{-i\omega t}, \quad X_\varphi = Y_r = 0, \tag{5.17}$$

hence

$$\tilde{Y}_z^*(\eta, \zeta) = \frac{M_0}{(2\pi)^{3/2}}, \quad \tilde{X}_\varphi^* = \tilde{Y}_r^* = 0, \tag{5.18}$$

and therefore from the formulae (5.9)–(5.11), we obtain:

$$u_\varphi = -\frac{M_0}{4\pi^2}e^{-i\omega t}\int_{-\infty}^{\infty} e^{-i\zeta z}d\zeta \int_0^{\infty} \left\{\frac{p\eta^2}{J c_4^2 \Delta}\right\}_{\mu=\omega} \mathscr{J}_1(\eta r)\, d\eta, \tag{5.19}$$

$$\omega_r = -\frac{M_0}{4\pi^2}e^{-i\omega t}\int_{-\infty}^{\infty} e^{-i\zeta z}d\zeta \int_0^{\infty} \eta^2 \mathscr{J}_1(\eta r)\left\{\frac{i\zeta}{J c_3^2\alpha^2(\alpha^2+\tau^2-\sigma_3^2)} - \frac{i\zeta(\alpha^2-\sigma_2^2)}{J c_4^2 \alpha^2 \Delta}\right\}_{\mu=\omega} d\eta, \tag{5.20}$$

$$\omega_z = \frac{M_0}{4\pi^2}e^{-i\omega t}\int_{-\infty}^{\infty} e^{-i\zeta z}d\zeta \int_0^{\infty} \eta \mathscr{J}_0(\eta r)\left\{\frac{\zeta^2}{J c_3^2\alpha^2(\alpha^2+\tau^2-\sigma_3^2)} + \frac{\eta^2(\alpha^2-\sigma_2^2)}{\Delta J c_4^2 \alpha^2}\right\}_{\mu=\omega} d\eta. \tag{5.21}$$

It is still necessary to calculate the integrals occurring in the above formulae for the displacements and rotations. Performing the integration, and knowing that

$$\int_{-\infty}^{\infty} \frac{\cos\zeta z}{\zeta^2+a^2}\, d\zeta = \frac{\pi}{a}e^{-za}, \quad \int_0^{\infty} \frac{e^{-z\sqrt{\eta^2-a^2}}}{\sqrt{\eta^2-a^2}}\eta \mathscr{J}_0(nr)\, d\eta = \frac{1}{R}e^{-iaR}, \tag{5.21a}$$

where $R = (r^2+z^2)^{1/2}$, and applying Bessel's equation

$$\frac{d^2\mathscr{J}_0(\eta r)}{dr} + \frac{1}{r}\frac{d\mathscr{J}_0(\eta r)}{dr} + \eta^2\mathscr{J}_0(\eta r) = 0, \tag{5.21b}$$

after numerous transformations, we obtain:

1. In the case of the action of a concentrated force in the origin of the system of coordinates oriented in the direction of the z-axis:

$$u_r = \frac{P_0 e^{-i\omega t}}{4\pi\varrho\omega^2}\frac{\partial^2}{\partial r\partial z}\left(A_1\frac{e^{i\lambda_1 R}}{R}+A_2\frac{e^{i\lambda_2 R}}{R}-\frac{e^{i\sigma_1 R}}{R}\right),$$

(5.22)
$$u_z = \frac{P_0 e^{-i\omega t}}{4\pi\varrho\omega^2}\left[\frac{\omega^2}{c_1^2}\frac{e^{i\sigma_1 R}}{R}-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(A_1\frac{e^{i\lambda_1 R}}{R}+A_2\frac{e^{i\lambda_2 R}}{R}-\frac{e^{i\sigma_1 R}}{R}\right)-\frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)\right],$$

$$\omega_\varphi = -\frac{P_0 s e^{-i\omega t}}{4\pi\varrho c_2^2(\lambda_1^2-\lambda_2^2)}\frac{\partial^2}{\partial r\partial z}\left(\frac{e^{i\lambda_1 R}}{R}-\frac{e^{i\lambda_2 R}}{R}\right),\quad R=(r^2+z^2)^{1/2}.$$

Substituting $\alpha = 0$ into the formulae (5.22), we obtain the transitions to classical elastokinetics:

$$u_r = -\frac{P_0 e^{-i\omega t}}{4\pi\varrho\omega^2}\frac{\partial^2}{\partial r\partial z}\left(\frac{e^{i\sigma_1 R}}{R}-\frac{e^{i\sigma_2 R}}{R}\right),$$

(5.23)
$$u_z = \frac{P_0 e^{-i\omega t}}{4\pi\varrho\omega^2}\left\{\frac{p^2}{c_1^2}\frac{e^{i\sigma_1 R}}{R}-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{e^{i\sigma_2 R}}{R}-\frac{e^{i\sigma_1 R}}{R}\right)-\frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)\right\}.$$

$$\omega_\varphi = 0.$$

2. In the case of the action of a body couple concentrated in the origin of the system of coordinates with the vector oriented in the direction of the positive z-axis:

$$u_\varphi = \frac{M_0 e^{-i\omega t}}{4\pi J(\lambda_1^2-\lambda_2^2)c_4^2}\frac{\partial}{\partial r}\left(\frac{e^{i\lambda_1 R}}{R}-\frac{e^{i\lambda_2 R}}{R}\right),$$

(5.24)
$$\omega_r = \frac{M_0 e^{-i\omega t}}{4\pi J\omega^2}\frac{\partial}{\partial r\partial z}\left(-\frac{e^{i\lambda_3 R}}{R}+A_1\frac{e^{i\lambda_1 R}}{R}+A_2\frac{e^{i\lambda_2 R}}{R}\right),\quad \lambda_3=(\sigma_3^2-\tau^2)^{1/2},$$

$$\omega_z = \frac{M_0 e^{-i\omega t}}{4\pi J\omega^2}\left\{\frac{\omega^2}{c_3^2}\frac{e^{i\lambda_3 R}}{R}-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(C_1\frac{e^{i\lambda_1 R}}{R}+C_2\frac{e^{i\lambda_2 R}}{R}-\frac{e^{i\lambda_3 R}}{R}\right)-\frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)\right\}.$$

For $\alpha = 0$, we obtain from (5.24):

$$u_\varphi = 0,$$

(5.25)
$$\omega_r = \frac{M_0 e^{-i\omega t}}{4\pi J\omega^2}\frac{\partial^2}{\partial r\partial z}\left(\frac{e^{i\tau_2 R}}{R}-\frac{e^{i\tau_1 R}}{R}\right),$$

$$\omega_z = \frac{M_0 e^{-i\omega t}}{4\pi J\omega^2}\left\{\frac{\omega^2}{c_3^2}\left(\frac{e^{i\tau_2 R}}{R}\right)-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{e^{i\tau_1 R}}{R}-\frac{e^{i\tau_2 R}}{R}\right)-\frac{1}{2}\frac{\partial}{\partial z^2}\left(\frac{1}{R}\right)\right\}.$$

We shall pass to static loads. We assume that in the problem of static loads the body forces and couples are functions of r and z only—viz.:

(5.26)
$$X_i = P_i(r,z),\qquad Y_i = M_i(r,z).$$

The Fourier-Hankel transforms of the component of body force and body couple will have the form:

$$\tilde{X}_i = \sqrt{2\pi}\,\tilde{P}_i(\eta,\zeta)\delta(\mu), \quad \tilde{Y}_i = \sqrt{2\pi}\,\tilde{M}_i(\eta,\zeta)\delta(\mu), \tag{5.27}$$

where

$$(\tilde{P}_i, \tilde{M}_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\zeta z} d\zeta \int_0^{\infty} r\mathcal{J}_1(\eta r)(P_i, M_i)\,dr. \tag{5.27a}$$

Therefore, the expressions for static displacements and rotations caused by the concentrated force P_z, P_r and the couple M_φ will be given by:

$$u_r = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta z} d\zeta \int_0^{\infty} \eta\mathcal{J}_1(\eta r)\left\{\frac{\eta(i\zeta\tilde{P}_z - \eta\tilde{P}_r)}{\varrho\alpha^4 c_1^2} - \frac{i\zeta}{\Delta_0}\left[\frac{\alpha^2+\nu_0^2}{\varrho c_2^2 \alpha_2}(\eta\tilde{P}_z - i\zeta\tilde{P}_r) + \frac{p}{Jc_4^2}\tilde{M}_\varphi\right]\right\} d\eta,$$

$$u_z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta z} d\zeta \int_0^{\infty} \eta\mathcal{J}_0(\eta r)\left\{\frac{i\zeta(i\zeta\tilde{P}_z - \eta\tilde{P}_r)}{\varrho c_1^2 \alpha^4} - \frac{\eta}{\Delta_0}\left[\frac{\alpha^2+\nu_0^2}{\varrho c_2^2 \alpha^2}(\eta\tilde{P}_z - i\zeta\tilde{P}_r) + \frac{p}{Jc_4^2}\tilde{M}_\varphi\right]\right\} d\eta, \tag{5.28}$$

$$\omega_\varphi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta z} d\zeta \int_0^{\infty} \frac{\eta\mathcal{J}_1(\eta r)}{\Delta_0}\left[\frac{s(\eta\tilde{P}_z - i\zeta\tilde{P}_r)}{\varrho c_2^2} + \frac{\alpha^2}{Jc_4^2}\tilde{M}_\varphi\right] d\eta.$$

The displacements caused by the action of the force P_φ and the couples M_r and M_z will be given by:

$$\omega_r = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta z} d\zeta \int_0^{\infty} \eta\mathcal{J}_1(\eta r)\left\{\frac{\eta(i\zeta\tilde{M}_z - \eta\tilde{M}_r)}{Jc_3^2\alpha^2(\alpha^2+\nu_0^2)} - \frac{i\zeta}{\Delta_0}\left[\frac{\eta\tilde{M}_z - i\zeta\tilde{M}_r}{Jc_4^2} + \frac{sP_\varphi}{c_2^2\varrho}\right]\right\} d\eta,$$

$$\omega_z = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta z} d\zeta \int_0^{\infty} \eta\mathcal{J}_0(\eta r)\left\{\frac{i\zeta(i\zeta\tilde{M}_z - \eta\tilde{M}_r)}{Jc_3^2\alpha^2(\alpha^2+\nu_0^2)} - \frac{\eta}{\Delta_0}\left[\frac{\eta\tilde{M}_z - i\zeta\tilde{M}_r}{Jc_4^2} + \frac{sP_\varphi}{c_2^2\varrho}\right]\right\} d\eta, \tag{5.29}$$

$$u_\varphi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\zeta z} d\zeta \int_0^{\infty} \frac{\eta\mathcal{J}_1(\eta r)}{\Delta_0}\left\{\frac{\alpha^2+\nu_0^2}{\varrho c_2^2}\tilde{P}_\varphi - \frac{p}{Jc_4^2}(\eta\tilde{M}_z - i\zeta\tilde{M}_r)\right\} d\eta,$$

where

$$\Delta_0 = \alpha^2(\alpha^2 - k_1^2), \quad \alpha^2 = \zeta^2 + \eta^2, \quad k_1^2 = \eta_0^2 - \nu^2. \tag{5.29a}$$

Let us consider here two particular cases: (1) the action of a concentrated force in the origin of the system of coordinates in the direction of the positive z-axis:

$$P_z(r,z) = \frac{P_0}{2\pi r}\delta(r)\delta(z), \quad P_\varphi = P_r = 0; \tag{5.30}$$

and (2) the action of a couple concentrated in the origin of the system of coordinates, whose vector coincides with the positive z-axis:

$$M_z(r,z) = \frac{M_0}{2\pi r}\,\delta(r)\delta(z), \quad P_\varphi = P_r = 0. \tag{5.31}$$

After performing the appropriate integrations, we obtain from (5.28) the expressions for the static displacements u_r, u_z and rotation ω_φ in the case of the action of the concentrated force (5.30):

$$\begin{aligned} u_r &= \frac{P_0}{8\pi\varkappa}\left[\frac{\gamma+\varepsilon}{2\varkappa}\frac{\partial^2}{\partial r\partial z}\left(\frac{e^{ik_1R}}{R}-\frac{1}{R}\right)+\frac{\varkappa+\lambda}{\lambda+2\varkappa}\frac{zr}{R^3}\right], \\ u_z &= \frac{P_0}{8\pi\varkappa}\left(\frac{\lambda+3\varkappa}{\lambda+2\varkappa}\frac{1}{R}+\frac{\varkappa+\lambda}{\lambda+2\varkappa}\frac{z^2}{R}\right)+\frac{P_0(\gamma+\varepsilon)}{16\varkappa^2\pi}\left[\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{e^{ik_1R}}{R}\right)\right. \\ &\qquad \left.+\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)\right], \\ \omega_\varphi &= \frac{P_0}{16\varkappa\pi}\left(\frac{e^{ik_1R}}{R}-\frac{1}{R}\right). \end{aligned} \tag{5.32}$$

For the displacement u_φ and rotations ω_r, ω_z caused by the action of the concentrated couple (5.31), we obtain from (5.29) the following expressions:

$$\begin{aligned} u_\varphi &= \frac{P_0}{16\pi\mu}\left(\frac{e^{ik_1R}}{R}-\frac{1}{R}\right), \\ \omega_r &= \frac{M_0}{4\pi(\gamma+\varepsilon)}\frac{\partial^2}{\partial r\partial z}\left[\frac{\gamma+\varepsilon}{4\alpha}\left(\frac{e^{i\tau R}}{R}-\frac{1}{R}\right)+\frac{1}{k_1^2}\left(\frac{e^{ik_1R}}{R}-\frac{1}{R}\right)\right], \\ \omega_z &= -\frac{M_0}{4\pi(\gamma+\varepsilon)k_1^2}\left[\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{e^{ik_1R}}{R}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)\right] \\ &\quad +\frac{M_0}{4\pi(\beta+2\gamma)}\left\{\frac{e^{i\tau R}}{R}-\frac{\beta+2\gamma}{4\alpha}\left[\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{e^{i\tau R}}{R}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)\right]\right\}. \end{aligned} \tag{5.33}$$

References

1. A. C. Eringen, E. S. Suhubi, *Nonlinear theory of micro-elastic solids* (I), Int. J. Engin. Sci., **2** (1964), 189.
2. A. C. Eringen, E. S. Suhubi, *Nonlinear theory of micro-elastic solids* (II), Int. J. Engin. Sci., **2** (1964), 389.
3. N. A. Palmov, *Podstawowe równania niesymetrycznej sprężystości* [w jęz. rosyjskim], Prik. Mat. Mech., **28** (1964).
4. G. Eason, J. Fulton, I. N. Sneddon, *The generation of waves in an infinite elastic solid by veriable body forces*, Phil. Trans. R. S. A 955, **284** (1956), 575.
5. N. Sandru, *i i some problems of the linear theory of the asymmetric elasticity*, Int. J. Eng. Sci, 1, **4** (1966).
6. W. Nowacki, *Green functions for micropolar elasticity*, Proc. Vibr. 1, **10** (1969).
7. I. N. Sneddon, *Fourier Transforms*, New-York-Toronto-London, 1951.

Streszczenie

ROZPRZESTRZENIANIE SIĘ FAL W NIEOGRANICZONYM MIKROPOLARNYM OŚRODKU SPRĘŻYSTYM

W pracy podano rozwiązanie ogólne równań ruchu dla nieskończonego ośrodka sprężystego, mikropolarnego. Przyczynami powodującymi deformację ciała są momenty i siły masowe. Rozwiązań dokonano przy użyciu transformacji całkowej wykładniczej Fouriera. Przedstawiono ogólne rozwiązania dla sił i momentów masowych zmieniających się dowolnie w czasie; zmieniających się harmonicznie w czasie oraz niezależnych od zmiennej czasowej. Rozważono zagadnienia trójwymiarowe i dwuwymiarowe. Określono również przy użyciu transformacji całkowej Fouriera-Hankela pole przemieszczeń, obrotów i pole naprężeń w przypadku osiowo symetrycznej deformacji ciała.

Резюме

РАСПРОСТРАНЕНИЕ ВОЛН В НЕОГРАНИЧЕННОЙ МИКРОПОЛЯРНОЙ УПРУГОЙ СРЕДЕ

В работе дается общее решение уравнений движения для бесконечной, микрополярной, упругой среды. Причинами, вызывающими деформацию тела, являются массовые моменты и силы. Решение произведено при использовании интегрального экспоненциального преобразования Фурье. Представлены общие решения для массовых сил и моментов, изменяющихся произвольно во времени, меняющихся гармонически во времени, а также независящих от временной переменной. Рассмотрены трехмерные и двумерные проблемы. Определены тоже, при использовании интегрального преобразования Фурье-Ханкеля, поле перемещений, вращений и поле напряжений в случае осесимметричной деформации тела.

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Received December 13, 1968