Ultrasonics 122 (2022) 106687

Contents lists available at ScienceDirect

Ultrasonics

journal homepage: www.elsevier.com/locate/ultras

Analytical solution of the nonlinear equations of acoustic in the form of Gaussian beam

Janusz Wójcik

Institute of Fundamental Technological Research, Polish Academy of Sciences, Pawinskiego 5B, 02-106 Warsaw, Poland

ARTICLE INFO

Keywords:
Nonlinear propagation
Analytical solution
Gaussian beam
Nonlinear interactions
Dispersion

ABSTRACT

The nonlinear acoustics equation for a dissipative medium is analytically solved. Continuous wave stimulation and an axisymmetric Gaussian spatial profile of the boundary conditions were assumed. The approximation of the D'Alambert operator by the wave diffusion operator was applied and justified. In this approximation and assuming classical absorption (dispersion), the equation to be solved is presented by the Khokhlov-Zabolotskaya-Kuznetsov model.

A sequence of functions describing the spatial distribution of the harmonic components of the disturbance was determined. They are the form of spatially modulated Gauss functions for harmonic wave numbers (frequencies).

For a lossless medium a universal numerical sequence describing non-linear interactions and harmonic generations was determined. In other cases, the description of the cooperation of dispersion and non-linear interactions in the harmonic generation process is given by a sequence of functions dependent on the dispersion coefficient and with boundary values given by the universal sequence mentioned above.

It was unexpectedly discovered that the influence of geometrical parameters of the beam on nonlinear interactions depends on dispersion, and component of the dispersion, absorption may strengthen harmonic generation. In general, dispersion spatially modulates the amplitude and phase of nonlinear interactions. This is not against the law of conservation of energy. The energy exchange between the fundamental (initiating) component and other harmonics is described.

The analytical solution was compared with the numerical one. The numerical solution was obtained in the scheme implementing the full Helmholtz operator (no axial - wave diffusion- approximation).

1. Introduction

We cannot analytically solve most of the realistic boundary value problems for the nonlinear equations of acoustics, so we use numerical analysis tools. However, the knowledge of solving a suitably complex problem or an accurate concept of the general structure of the solution allows for the improvement of these tools [1], their testing and interpretation of the results. When looking for analytical solutions, we also count on a final or deeper understanding of the phenomenon.

Formulation of mathematical models of nonlinear acoustics and the search for their solutions took place in parallel. An extensive historical outline of the formation of contemporary nonlinear acoustics on the basis of isotropic medium models along with reprinted works by Poisson, Riemann, Stokes and others can be found in books [2–5].

The first work that transformed a nonlinear differential description of sound propagation, or more precisely the motions of a one-dimensional lossless medium, into a functional description was Poisson's work from 1808 [2]. The functional description is obtained as a result of the single integration of the equations of motion and has the form $\mathrm{v}(z,t) = \mathrm{V}(\mathrm{v}(z,t),z,t)$, where v is the solution to be found (acoustical velocity), depending on the z coordinate and time t, $\mathrm{V}()$ is a given function resulting from the boundary conditions.

In 1860, Riemann, adopting more general assumptions and methods than Poisson, transformed the system of differential laws of conservation of matter and the momentum of the one-dimensional Euler fluid (ideal isotropic medium in Euler coordinates) into a functional problem for one dependent variable - velocity [2–4,6]. Earnshaw (1860) obtained an analogous result in the Lagrange coordinates [3,7]. In 1935, Fubini solved the Earnshaw equation for the sinusoidal boundary condition $\mathrm{V}(\mathrm{v}(0,t),0,t) = sin(\omega_0 t)$ and presented v as a Fourier time series with respect to harmonics ω_0 [2,3,5]. The Riemann method, with the given equation of state, resulted in the remaining relationships between the dependent variables (most often an adiabatic transformation is assumed, it may be polytropic). Riemann's solution describes the nonlinear

E-mail address: jwojcik@ippt.pan.pl.

https://doi.org/10.1016/j.ultras.2022.106687
Received 29 September 2021; Received in revised form 10 December 2021; Accepted 13 January 2022
Available online 26 January 2022

evolution of border disturbances moving to the right or left (there is also an isentropic mode). They are commonly called simple waves. The so-called Riemann invariants are related to simple waves - constant quantities in these waves in the whole area of motion [2,8]. The ideas contained in Riemann's solution still play a key role in the theoretical and numerical study of fluid dynamics, including non-linear dynamics of wave motions, the formation, properties and the interactions of shock waves [9,10].

In this paper, we present non-trivially analytical solution of the boundary problem for nonlinear acoustics equations of dissipative medium, corresponding to real situations. The problem is physically three-dimensional but axisymmetric, i.e. mathematically two- dimensional. The solution is a very good approximation of the NAE (Nonlinear Acoustics Equation) solution [5,11,12]. For a classically absorbing medium, the NAE is sometimes called the Westervelt equation. The approximation results from the fact that the basic linear evolutionary operator in this equation is D'Alembercian, i.e. a strictly hyperbolic operator, and the proposed solution is based on Gaussian beam (Lager polynomials of the 0th order).

In a description using variables with retarded times, fast spatial changes in the direction of propagation are compensated by fast temporal changes of the perturbation. The disturbance slowly changes with respect to the coordinate in the direction of propagation and the second derivative in the D'Alambert operator with respect to this coordinate can be omitted. We get the so-called parabolic or paraxial approximation. Gaussian beams satisfy a homogeneous wave equation with a parabolic approximation of this operator. Sometimes, especially in the space-Fourier frequency coordinates, the approximate operator is called the wave diffusion operator. The use of this approximation in the Kuznetsov equation [13] or NAE leads to the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation [5,12,14]. Hence, the presented solution is, in particular, due to the classic absorption model, a strict KZK solution.

An attempt to solve the KZK equation for Gaussian boundary conditions and sinusoidal excitation in time (as in this work) was made in [15]. The method of successive approximations was used. Its computational complexity in nonlinear wave problems grows rapidly. Therefore, it was limited to obtain a description of the generation of the first harmonic. The fact that the fundamental component is a solution of the linearized equation was not used and it can be selected so as to satisfy the boundary condition. The equation was solved "from the beginning" using the Green function method. This leads to a more complex notation of the result, which is to be expected to be a Gaussian beam. The Gaussian form of the first harmonic was obtained from a complex integral expression as a result of approximations supported by numerical calculations.

So far, only in the one-dimensional case, analytical solutions of the NAE boundary problems in the dissipative medium are known (the case of classical absorption). In this case, it convey to the Burgers equation [5,12] and after Cole-Hopf substitution, it becomes linearized to the form of a thermal conductivity equation [2,3,16].

A method for the integration time fractional Burgers-Hopf equation is presented in [17]. The orders of derivatives with respect to time from the interval (0,1] were considered. The obtained solutions are of a highly a periodic kink type. Solutions for the derivatives of 1/2 and 0.95 were demonstrated. The most important case of integer derivatives (the base case) was omitted in the discussion. It should be noted that these solutions do not have a linear asymptotic with respect to the nonlinearity parameter (or Mach number), and unlike conventional solutions, they tend to infinity when this parameter tends to zero. Such a situation may take place in the description of non-linear phenomena, see e.g. [18] where such a non-physical branch of the solutions of the boundary problem was indicated for the Burgers equation. The same work [17] also presents the class of exact solutions of fractional KZK equations, and in particular the "normal" KZK equation. In general, the features of these solutions are identical to those described above for the Burgers equation. Contrary to the authors, it should be stated that the solutions they obtained represent the non-physical branch of solutions of the Burgers and KZK equations and have nothing to do with acoustics. However, they are an interesting and important supplement to the description of the mathematical properties of equations.

Due to the order of non-linearity with respect to the disturbance, we have two branches of solutions (like the roots of a quadratic equation) that are significantly separated from each other due to the acoustic Mach number. One physical one represents the usual solutions of boundary problems with determined parameters, the other one concerns free space with a slightly limited set of dispersion parameters.

Symmetry group of the KZK equation, was determined in the work important from the point of view of investigating the properties of the KZK equation [19]. The set of invariant transformations allows for the determination of successive (infinite number), in other words, solution trajectories from the existing solution. Unfortunately, the paper does not explicitly provide any non-trivial solution. Although for a given boundary condition, a new non-linear system of equations has been proposed. Its solution would constitute the solution of the KZK.

By skillful substitutions and approximations, the cases of spherical and cylindrical symmetry (as well as flat one-dimensional symmetry) were included in one universal propagation model described by the so-called generalized Burgers equation. These are spatial cases, however, mathematically $1D + t$ with coefficients dependent on coordinates. This model, together with approximate solutions, is discussed e.g. in [3–6,20,21].

A number of works (e.g. [22–25]) announce solutions of the KZK equation in the title or in the introduction. At the same time, reference is made to source works, publications [14] or other publications (e.g. [2,5]) which duplicate the result [14] which is the source of the equation form. It should be noted that the KZK equation from the source work and the aforementioned work differ significantly primarily in the differentiation operation of the non-linear term. In the KZK equation, it is differentiation with respect to retarded time and not with respect to the coordinate of slow changes in the direction of propagation. In the KZK equation, time is defined as retarded time in the sense described above. Moreover, the equations from these works cannot be transformed by coordinate transformations in KZK equation. In work [23] there is not even the "parabolic approximation" characteristic for KZK equation. The equations in these works correspond to the Kadomtsev-Petviashvili (KP) model equation.

The role of absorption, or more generally dispersion, in the description of nonlinear interactions and generation of harmonic beams is surprising. Then in Chapter 2 we recalled the basic facts for describing this phenomenon. This may be useful to the reader less familiar with the problem or in further research. Furthermore, we note that the classical absorption model is not analytic (fully dispersive) even though it was derived from the analytic dispersion model for the Navier-Stokes equations. These issues are developed in Appendix A using examples of a Maxwell-type medium (remembers dispersive stresses) and a medium with absorption characteristic of biological substances (e.g., blood).

Then we introduce a normalized system of variables so that the Mach number appears explicitly in the equations. We present a solution in the form of a Fourier series. We transform NAE into a system of nonlinear equations on the series coefficients. We analyze the order of equations and the Fourier components. We show that in each equation (for the disturbance component) there are two types of nonlinear terms. They differ by two orders respect the Mach number. They are components of two vectors whose sum constitutes a complete non-linear description. One of them, for a complete disturbance, is of the first order respect to the Mach number. The second is of the second order. NAE was derived with precision of the first order terms. The above observations resulting from the analysis of orders in chapter 2 constitute the basis of the adopted method of solving the NAE and the KZK equation. In chapter 3 we solve a nonlinear system of equations and obtain an analytical and explicit description of nonlinear interactions. In chapter 4 we present visualizations of the obtained analytical solution and compare them with numerical solutions. Chapter 5 discusses the results.

In contrast to optics, Gaussian boundary conditions and Gaussian beams are rare in acoustics. However, they have sufficiently realistic features, which also characterize the most common limited sources, that the presented results can be considered as common to a wide class of boundary problems of practical importance.

2. Basic equations and relations

2.1. Solution form and order analysis

In the normalized system of variables the NAE describing the propagation of an acoustical disturbance in a lossy and nonlinear medium has the form [1]

$$2\partial_{\tau z}P - \Delta P + 2\partial_{\tau}AP = q\partial_{\tau\tau}P^2 + o(q^2) \tag{1}$$

where Δ is the Laplace operator; $\Delta := \Delta_{\perp} + \partial_{zz}$, where $\Delta_{\perp}$ is the transverse component of the Laplacian. In this work only axisymmetric case of propagation in the cylindrical coordinates is considered then, $\Delta_{\perp} = \partial_{rr} + (1/r)\partial_r$; (r, z) denotes the normalized space coordinates; $\tau := (t - z)$ is normalized retarded time; t is normalized time; $P = P(r, z, \tau)$ is the normalized acoustic pressure; q is the coefficient of nonlinearity.

Generally A is the convolution-type operator with kernel $A(\mathbf{x}, t)$ describing an dispersion $A^{x,t}P := A \underset{\mathbf{x},t}{\otimes} P$. Generalized Fourier transform of the kernel A, $\widehat{a}^{\varsigma,\omega}(\varsigma, \widehat{\omega}) = \mathrm{F}^{x,t}[A]$ is a function of dispersion, where ς is the complex wave number, $\widehat{\omega}$ is the complex pulsation. That is $\widehat{a}^{\varsigma,\omega}$ is the eigenvalue of A corresponding to the eigenfunction $f = exp(-i\varsigma \mathbf{e_K}\cdot\mathbf{x} + i\widehat{\omega}t)$, $Af = \widehat{a}^{\varsigma,\omega}f$; $\mathbf{K} = \varsigma\mathbf{e_K}$ is the complex wave vector; i is the imaginary unit. Using [26] we obtain the analytical form of the dispersion coefficient $\widehat{a}^{\omega}(\widehat{\omega}) = \widehat{a}^{\varsigma,\omega}(\varsigma(\widehat{\omega}), \widehat{\omega})$, where $\varsigma(\widehat{\omega})$ is the solution to the dispersion equation. On the real axis $\omega = \mathrm{Re}(\widehat{\omega})$ we get $\widehat{a}(\omega) = a(\omega) + ih(\omega)$, where $a(\omega)$ is the weak-signal absorption coefficient, $\mathrm{n}_I = (c_0/\omega)a$, $1 - \mathrm{n}_r = (c_0/\omega)h$; n_I, $\mathrm{n}_r := c_0/c_f(\omega)$ imaginary and real part of the refraction coefficient, $c_f(\omega)$ is the phase velocity. As is known a i h related by Kramers-Kronig relations (Hilbert transforms). We note that the procedure presented here in brief allows us to replace the time-dependent (mixed) model of dispersion by a time-dependent (homogeneous) model since the transition from $\widehat{a}^{\varsigma,\omega} \rightarrow \widehat{a}^{\omega}$ corresponds to $A^{x,t} \rightarrow A^{t}$ (details in [26]). We also have $A^{t}f(\widehat{\omega}, t) = \widehat{a}^{\omega}f(\widehat{\omega}, t)$, $f = exp(i\widehat{\omega}t)$. Moreover, the integro-differential operations with respect to time and retarded time are identical. Because of the relatively easy measurement of the absorption coefficient, it is usually the basis for the determination of the full dispersion, i.e. h. However, analytical determination of using the Kramers-Kronig formulae is difficult or impossible. Similarly, in the case of experimental data because they are available only from a limited frequency band and not in the required range $\omega \in [0, \infty)$. In [27], a very good approximate solution to this problem is presented. In Appendix A, two dispersion models, specific to acoustics, are presented. The first represents the Maxwell medium. In the zero relaxation time limit, the dispersion operator of this medium corresponds to the Navier-Stokes viscous stress model which is the source of the classical absorption model. In the second model we follow the opposite direction. For the absorption coefficient characteristic of many organic media, we determine the full dispersion. Eq.(1) was normalized according to

$$P := \frac{P'}{P'_0}\tau := \Omega'_0\tau' = \Omega'_0(t' - z'/c'_0), (r, z) := K'_0\cdot(r', z') \tag{2}$$

$$\partial_{\tau} := \frac{1}{\Omega'_0}\partial_{\tau'}, \nabla := \frac{1}{K'_0}\nabla', \omega := \frac{\omega'}{\Omega'_0}, k := \frac{k'}{K'_0}, K'_0c'_0 = \Omega'_0$$

Where P'_0 is the characteristic pressure (here the peak of the absolute pressure value at the source surface);ω' is the angular frequency; c'_0 is the equilibrium sound velocity; g'_0 is the equilibrium density; $\widehat{q} := P'_0/g'_0c_0^2$ is the acoustical Mach number, q is the nonlinearity coefficient; γ is the exponent of the adiabate, for the empirical state equation $\gamma := (B/A) + 1$ where (B/A) is the nonlinearity parameter of the medium. The normalization of small signal coefficient of absorption and dispersion coefficient $\widehat{a}(\omega)$ was performed as follows $a(\omega) = a'(\omega'/\Omega'_0)/K'_0$. Usually coefficient of the absorption is given in the form $a'(\omega') = \alpha'_{\chi}\cdot(\omega'/2\pi)^{\chi}$ where α'_{χ} is the parameter of the absorption $(\mathrm{Np}/(\mathrm{m}\cdot\mathrm{Hz}^{\chi}))$; χ is the power of the growth of the absorption, $\chi = 2$ for classical absorbing media (water, glycerine), $\chi \simeq 1$ for the blood and many other organic media. Then in this cases $a(\omega) = \alpha_{\chi}|\omega|^{\chi}$ $\alpha_{\chi} := \alpha'_{\chi}\cdot(\Omega'_0/2\pi)^{\chi}/K'_0$. The imposition of the relation $K'_0c'_0 = \Omega'_0$ for the values K'_0 and Ω'_0 causes that the fundamental dispersion relation $\omega'(k') = \mp k'c'_0$ for lossless linear and unbounded medium (no dispersion) takes the form $\omega(k) = k$, $k(\omega) = \pm\omega$, where $k' = \pm|k'|$ $k'(\omega')$ is the wave number. Thus, we introduced a common measure of time and distance in space in which the equilibrium speed of sound $c_0 = 1$. For Fourier series representations of the disturbances ω and k are discrete variables which numerate the components of the series for $k = \omega = 1$, $\omega' = \Omega'_0, k' = K'_0$. In not normalized units $k' = k\cdot K'_0$, $\omega' = \omega\cdot\Omega'_0$, $k, \omega = 1, 2, \ldots$. For continuous sinusoidal stimulations of the medium, it is natural to adopt the normalization, in which Ω'_0 is the pulsation of the excitation and $K'_0 = \Omega'_0/c'_0$ is the wave number corresponding to it.

We look for the solutions of Eq. (1) in the half-space $z \geq 0$ in the form of a Fourier series

$$P(r, z, \tau) = \sum_{k=1}^{K} P_k = \frac{1}{2}\sum_{k=1}^{K} F_k(r, z)e^{-ik\tau} + c.c \tag{3}$$

In the coordinates with retarded time, the functions $F_k(r, z)$ are slowly varying spatial envelopes of the quickly varying factor $exp(ikz)$ (in non-normalized units $exp(ik'z') = exp(i(\omega'/c'_0)z'))$, $c.c$ denotes complex conjugation.

After substituting Eq. (3) into Eq. (1) and using the orthogonality of the set of functions $exp(\mp ik\tau)$, $l = 1, 2, \ldots, k, \ldots$ we obtain the system of equations

$$(2ik\partial_z + 2ik\widehat{a}(k) + \Delta_{\perp} + \partial_{zz})F_k = \frac{1}{2}qk^2\left(\sum_{l=1}^{k-1}F_{k-l}F_l + 2\sum_{l=k+1}^{K}F^{*}_{l-k}F_l\right)k = 1, 2, 3, \ldots, K \tag{4}$$

Theoretically $K \rightarrow \infty$. In numerical calculations, the effective dimension of the Fourier representation is used according to the increasing or decreasing intensity of nonlinear interactions $K = K(z)$ (see [1,11,28,29]). In the case of an infinite dimensional representation of F_k, the non-linear term in Eq.(4) is a correct form of the reduction of the self-convolution F_k to a given dimension K. Therefore, K can be arbitrarily enlarged without changing the structure of the description of nonlinear interactions. Fundamental component $F_1(r, z)$ satisfies the boundary condition, $F_1(r, z = 0) = F_1(r)$, $F_1(r = 0) = 1$ such that $P(r, z = 0, \tau) = F_1(r)exp(-i\tau) + c.c.$ So the remained components must meet the conditions $F_l(r, 0) = 0$, $l \geq 2$. So $F_1(r, z)$ is the field that initiates the generation of the other components.

$$\begin{pmatrix} (2i\partial_z + 2i\widehat{a}(1) + \Delta_\perp + \partial_{zz})F_1 \\ (4i\partial_z + 4i\widehat{a}(2) + \Delta_\perp + \partial_{zz})F_2 \\ \vdots \\ \vdots \\ (2ik\partial_z + 2ik\widehat{a}(k) + \Delta_\perp + \partial_{zz})F_k \\ \vdots \end{pmatrix} = \frac{q}{2}k^2 \begin{pmatrix} 0 & 2F_1^* & 2F_2^* & 2F_3^* & \cdots & 2F_{k-1}^* & \cdots & & & 2F_{K-1}^* \\ F_1 & 0 & 2F_1^* & 2F_2^* & & \vdots & & & & \vdots \\ F_2 & F_1 & 0 & 2F_1^* & & & & & & \\ F_3 & F_2 & F_1 & 0 & & & & & & \\ \vdots & & & & \ddots & & & & & \\ F_{k-1} & F_{k-2} & \cdots & F_2 & F_1 & 0_{(k,k)} & 2F_1^* & 2F_2^* & \cdots & 2F_{K-k}^* \\ \vdots & & & & & \ddots & & & & \vdots \\ & & & & & \vdots & & & 0 & 2F_1^* \\ F_{K-1} & F_{K-2} & & & \cdots & F_{K-k} & \cdots & F_2 & F_1 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_k \\ \vdots \\ F_K \end{pmatrix} \tag{5}$$

To better illustrate the generation cascade, we have presented Eq. (4) in the matrix form Eq. (5). The first summations in the system of equations in Eq.(4) are created by rows in triangle matrices below diagonal and in sequence $l = 1,2,...,k-1$, $k = 2,..,K-1$. By $o(f)$ we denote the order of magnitude of f with respect to the nonlinearity coefficient (Mach number). Especially $o(q^0) \sim 1$, $o(q^k) \sim q^k$, $o(q^{l_1} + ... + q^{l_s} + ... + q^{l_{mx}}) \sim q^{min l_s}$. Because $o(F_1) \sim q^0 = 1$ then nonlinear term in Eqs.(4),(5) of the lowest order with respect to q, determining evolution F_2 is $o(F_2) = o(qF_1F_1) \sim q$. Hence, based on the terms on the right side of the diagonal in Eq.(5), or on the second sum in Eq.(4), we still have $o(F_1) \sim 1 + o(qF_1^*F_2) \sim 1 + o(q^2)$. Then, the lowest-order "source" term for F_3 is $o(F_3) = o(qF_2F_1) \sim q^2$. Wherein, determined on this basis the correction to the terms on the right-hand side of the diagonal in Eq. (5) in the lowest row is $o(qF_1^*F_3) \sim q^3$. Moreover, the initial evolution (initialization) of the mode F_K (as well as the previous F_k, $k = 2,3,...$) depends solely on all existing $K-1$ ($k-1$) and does not depend on F_K, (F_k). Generally, denoting the first term by $NL_k^{(1)}$ and the second term with $NL_k^{(2)}$ on the right hand side of Eq.(4) and assuming that $o(F_l) = q^{l-1}$, we get

$$o\left(NL_k^{(1)}\right) = o\left(q\sum_{l=1}^{k-1} F_{k-l}F_l\right) \sim q^{k-1} \tag{6}$$

$$o\left(NL_k^{(2)}\right) = o\left(2q\sum_{l=k+1}^{K} F_{l-k}^*F_l\right) = o\left(2q\sum_{l=1}^{K-k} F_{k+l}^*F_l\right) = o\left(q^{k+1}\sum_{l=1}^{K-k} q^{2(l-1)}\right) \sim q^{k+1} \tag{7}$$

It means $o(F_k) = (q^{k-1})_{NL^{(1)}} + (q^{k+1})_{NL^{(2)}}$, which is consistent with the assumption and the previous specific considerations. Thus $o\left(NL_k^{(2)}\right) = q^2 o\left(NL_k^{(1)}\right)$. It should be stressed that from the analysis of orders F_k with respect to q does not automatically result $|F_l| >> |F_k|$ for $l < k$ and $q << 1$. This is shown, for example, by the numerical solutions of the system Eq.(4), where for close l and k we observe $|F_l| > |F_k|$ rather than the strong inequality and $F_k \sim F_l$ for $q \sim 0.0001$, and also $F_k \sim 1$ for small k. However, we assume that q is small enough so that for the above relations $|F_l| >> q|F_k|$. The term $NL_k^{(2)}$ is two orders of magnitude with respect to q smaller than $NL_k^{(1)}$. From a formal point of view it could be omitted in the system Eq.(4) (triangular matrix over the diagonal in Eq. (5)). The first term is the source term for the equation on F_k and does not depend on F_k. If $NL_k^{(2)}$ is omitted, all previous modes provide energy to F_k but they do not change themselves. For example, F_1 generates F_2 (without losing energy), further and similarly F_1 and F_2 generate F_3 without losing energy. The change in the amplitude of the F_k mode due to the loss of energy on the generation of the remaining modes is described by $NL_k^{(2)}$. Thus, omitting $NL_k^{(2)}$ means breaking the law of energy conservation.

Summarizing Eq.(1) is a nonlinear equation of order q with respect to the full perturbation P (vector $F := [.., F_k, ..]$). Nonlinear "source" vector $NL^{(1)} := \left[.., NL_k^{(1)}, ...\right]$ with respect to the full disturbance F is of the same order. The non-linear "source" vector $NL^{(2)} := \left[.., NL_k^{(2)}, ...\right]$ is of the order of q^2 with respect to the full perturbation. That is, $o(NL^{(2)}) = q \cdot o(NL^{(1)})$ and for components of the same order k $o\left(NL_k^{(2)}\right) = q^2 o\left(NL_k^{(1)}\right)$. A schematic picture of the relationship between the orders of the nonlinear interaction vectors and between the components of these vectors is shown in Fig. 1.

Equation (1) was derived from the general equations of the isotropic medium, omitting the terms of the order q^2 with respect to the full disturbance (of the q order with respect to the nonlinear term in Eq. (1)). Formally, this means that $NL^{(2)}$ is out of the range of precision for which Eq. (1) was derived and $o(NL^{(2)}) = q \cdot o(q\partial_{\tau\tau}P^2)$. According to the above considerations, the solution of the system Eq.(4) is presented in the form

$$F_k = e^{-\widehat{a}(k)z}(C_k + D_k) \tag{8}$$

where $o(D_k) = q^2 o(C_k)$. Substituting Eq. (8) into Eq. (4) and dividing by $2ik$ we get

$$\left(\partial_z + \frac{1}{2ik}\Delta_\perp + \frac{1}{2ik}\partial_{zz}\right)C_k = q\frac{k}{4i}\sum_{l=1}^{k-1} C_{k-l}C_l e^{\widehat{a}_g(k,l)z} \tag{9}$$

$$\left(\partial_z + \frac{1}{2ik}\Delta_\perp + \frac{1}{2ik}\partial_{zz}\right)D_k = q\frac{k}{2i}\left(\sum_{l=1}^{K-k} C_{k+l}C_l^* e^{-\widehat{a}_d(k,l)z} + \sum_{l=1}^{k-1} D_l C_{k-l} e^{\widehat{a}_g(k,l)z}\right) \tag{10}$$

where $\widehat{a}_g(k,l) := \widehat{a}(k) - \widehat{a}(k-l) - \widehat{a}(l)$, $\mathrm{Re}(\widehat{a}_g) \geq 0$, $\widehat{a}_d(k,l) := \widehat{a}(k+l) + \widehat{a}(l) - \widehat{a}(k)\mathrm{Re}(\widehat{a}_d) \geq 0$. For the non dissipative media $\widehat{a}_g(k,l) = \widehat{a}_d(k,l) \equiv 0$. Examples $\widehat{a}_g$ and $\widehat{a}_d$ for classical absorption and some biological

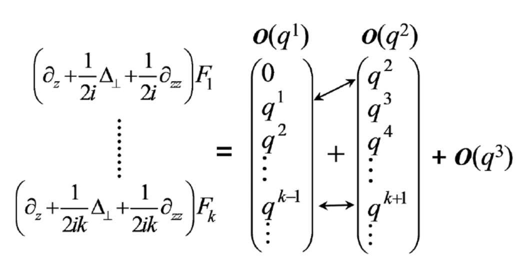

Fig. 1. Relations between nonlinear "source" vectors extracted on the basis of Eq. (5) and between components of these vectors (horizontal arrow).

substances see Appendix A. On the basis of Eq.(9,10), it can be seen that the dispersion modulates spatially, amplitude and phase non-linear interactions. The influence of $\widehat{a}_g$ is stronger because it concerns the equations Eqs. (9) of lower order respect *q*.

The right side of Eq. (10) is of the order q^{k+1}. Terms of the order q^{k+3} and higher are omitted. However, for $l \geq 2$ the first sum in Eq.(10) generates terms of the order $q^{k+2 \cdot l-1}$ and as a consequence only the $C_{k+1}C_1^*$ term should be kept.

In the case that there were $o(D_k) = q \cdot o(C_k)$(not $o(D_k) = q^2 \cdot o(C_k)$) and we limited ourselves only to determine C_k, we could talk only about an asymptotically strict solution $C_k \rightarrow F_k$, $q \rightarrow 0, q \neq 0$. Relation $o(D_k) = q^2 \cdot o(C_k)$ is much stronger. Therefore, according to the above order analysis, it can be concluded that the exact solution of the system Eq. (5) is also one that is based only on C_k satisfying Eq. (9), it means F_k tends to C_k very quickly when $q \rightarrow 0$, $q \neq 0$. We will call the C_k coefficients the "core" of the solution, while coefficients D_k "supplement". Nevertheless, C_1 should be considered an exception as it is a solution of a linear problem. The harmonics are generated at a cost of the energy of the fundamental component. It is therefore necessary to add an explicit form to the description of the solution, at least D_1 so to get F_1 and a description of the basic process of energy transfer from the fundamental mode to the remaining ones. Next we present the solutions for D_k in particular, in the explicit form for D_1 and D_2.

2.2. *The basic properties of Gaussian beams*

The Lager polynomials of the order 0 (popular Gauss) are of the form

$$\Psi_k(z,r) = \frac{z_N}{\varsigma(z)} exp\left(-\frac{kr^2}{2\varsigma(z)}\right), \varsigma(z) := z_0 + i(z - z_e) \tag{11}$$

$$= \frac{z_N}{\varsigma(z)} exp\left(-\frac{r^2}{d_k^2(z)}\right) exp\left(\frac{-ikr^2}{2R_k(z)}\right) \tag{12}$$

where, $z_N := |\varsigma(0)| = \sqrt{z_0^2 + z_e^2}$ is the normalization constant that $|\Psi_k(0,0)| = 1$, z_0 is the Rayleigh length which characterize convergence-divergence of the beam, z_e is the position of the maximum and the waist of the beam $\Psi_k(z_e,0) = z_N/z_0$, $\varsigma(z)$ is the Kogelnik parameter [30], $d_k^2(z) := |\varsigma(z)|^2 2/z_0 k = d_{0k}^2 |\varsigma(z)|^2 / z_0^2$, $R(z) := |\varsigma(z)|^2/(z_e - z)$. The waist of the beam is $d_{0k} := d_k(z_e) = \sqrt{2z_0/k} = d_0/\sqrt{k}$, $d_0 = \sqrt{2z_0}$. In dimensional units $d_{0k}' = d_0'/\sqrt{k}$, $d_0' = \sqrt{2z_0'/K_0'}$. Although z_0 is the basic parameter, we can set it for all *k* through task d_0, which is intuitively more palpable and it makes easier to make the beam similar to other types of beam. The Gaussian beam satisfies the equation (wave diffusion)

$$\left(\partial_z + \frac{1}{2ik}\Delta_\perp\right)\Psi_k = 0 \tag{13}$$

in hemi space $z \geq 0$, assuming a boundary value $\Psi_k(0,r)$, $|\Psi_k(0,0)| = 1$. For the same boundary conditions (Gaussian) the function

$$\Psi_k^H := -\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \Psi_k(0,s)\frac{z}{R}\partial_R \frac{e^{ik(R-z)}}{R} d\phi s ds \tag{14}$$

where $R := \sqrt{z^2 + r^2 + s^2 - 2rscos(\phi)}$, is the Rayleigh-Sommerfeld solution of the Helmholtz Eq.(15) for Dirichlet boundary conditions,

$$\left(\partial_z + \frac{1}{2ik}\Delta_\perp + \frac{1}{2ik}\partial_{zz}\right)\Psi_k^H = 0 \tag{15}$$

It means that $\Psi_k(z,r) \neq \Psi_k^H(z,r)$ and Ψ_k don't satisfy Eq.(15). However if $kz_0 = (d_0k)^2/2 \gg 1$ then $\Psi_k \cong \Psi_k^H$ and Gaussian beams are very good approximations of the solution of the Eq.(15) see Appendix B. For $k = 1$, this condition occurs even for highly focused (converging - diverging) beam. In non-normalized units, for $k = 1, k' = K_0'$, supposing $d_0' = 2\pi/K_0'$ we obtain $z_0 = K_0' z_0' = (d_0' K_0')^2/2 = 2\pi^2 \gg 1$. A large relative phase error occurs in an area where the Gaussian beam amplitude is negligibly small.

Even more so, under the described conditions, for k > 1 the approximation of Eq.(13) of the Helmholtz operator is justified. Typically, replacing the Helmholtz operator Eq. (15) with the wave diffusion operator Eq. (13) is called a paraxial or parabolic approximation. The omission ∂_{zz} averages the spatial distributions of the disturbance along the axis *z* (ie "elimination" of the Fresnel zones between the boundary condition surface and the far zone). Hence, for smooth boundary conditions, in Helmholtz's description, these zones are poorly manifested, and in the case of Gaussian conditions not at all, and the approximation describes well the convergent-divergent beams.

For classically absorbing media in approximation $\Delta \cong \Delta_\perp$, from Eqs. (1), (4) we get the so-called KZK model [5,12,14]. Here we assume that, $1/kz_0$ is a small parameter and $z_0 = K_0' z_0' \geq \pi^2$ holds. We will use the following properties of Gaussian beams

$$\Psi_{k-l}\Psi_l = \frac{z_N}{\varsigma(z)}\Psi_k \tag{16}$$

$$\Psi_{k+l}\Psi_l^* = \frac{\varsigma(z)}{z_N}\Psi_k|\Psi_l|^2 = \frac{z_N}{\varsigma(z)^*}\Psi_k exp\left(-\frac{2r^2}{d_l^2(z)}\right)$$

$$= \frac{z_N^2}{|\varsigma(z)|^2} exp\left(-r^2 \cdot \frac{k\varsigma(z)^* + 2z_0 l}{2|\varsigma(z)|^2}\right) \tag{17}$$

3. Solutions

3.1. *Solution for C_k*

We search the solutions of the system Eq. (9) in the form

$$C_l(z,r) = W_l(z)\Psi_l(z,r) W_1(z) = 1 W_l(0) = 0 \tag{18}$$

After substituting Eq.(18) into Eq.(9), using of Eq.(16) we obtain

$$\partial_z\left(1 + \frac{1}{2ik}\partial_z\right)W_k = S_k := \frac{-iqz_N}{4}\frac{k}{\varsigma(z)}\sum_{l=1}^{k-1} W_{k-l}W_l e^{\widehat{a}_g(k,l)z} \tag{19}$$

The solution of this equation is

$$W_k = 2ik\int_0^z e^{-2ik(z-z')}\widetilde{W}_k dz' \widetilde{W}_k(z) := \int_0^z S_k(z')dz' \tag{20}$$

$$W_k = \widetilde{W}_k - \int_0^z e^{-2ik(z-z')}\partial_{z'}\widetilde{W}_k dz' = \widetilde{W}_k - \frac{\partial_{z'}\widetilde{W}_k}{2ik} + \ldots \tag{21}$$

Due to the form of W_1 and S_2, it is reasonable to assume that all $\partial_{z'}\widetilde{W}_k = S_k$ are slowly varying functions with respect to the quickly varying exponential factor. Then we will show that $|\partial_z W_k/2k| \sim o(1/z_0)|W_k|$ or $|\partial_{zz} W_k/2k| \sim o(1/z_0)|\partial_z W_k|$. So we have $W_k = \widetilde{W}_k + o(1/z_0)$ and

$$W_k = \frac{-iqz_N k}{4}\int_0^z \sum_{l=1}^{k-1} W_{k-l}W_l e^{\widehat{a}_g(k,l)z'}\frac{dz'}{\varsigma(z')} k = 2,3,\ldots, \tag{22}$$

$$W_k = \frac{-qz_N k}{4}\int_0^z \left(\sum_{l=1}^{k-1} W_{k-l}W_l e^{\widehat{a}_g(k,l)z'}\right)\partial_{z'} ln\left(\frac{\varsigma(z'))}{\varsigma(0)}\right)dz' \tag{23}$$

The system of integral equations Eq.(22), (23) is also a recursive solution of the differential system formed from Eq.(9). Knowing the previous W_l, $l < k$ allows to calculate W_k. Equations Eq.(20), (21) are exact solutions in the NAE model. For a medium with classical absorption, Eq.(22), (23) are strict solutions of the parabolic approximation of

NAE, in particular of the KZK model. Below we present examples of unraveling this recursion, that is, determining the W_k without referring to the previous ones. Strictly speaking, we reduce functional recursion to a series of recursively determined constants.

Eq.(23) suggests changing the measure from dz' to $dw = dln[\cdot]$, $w = ln[\varsigma(z)/\varsigma(0)]$. In the case of a medium that is lossless or has an absorption coefficient $a(k) = \alpha_1 k$, and $\mathrm{Im}(\widehat{a}_g(k,l))z_{pr} \ll \pi$, where z_{pr} is the propagation range along z axis. Eq.(23) takes the form

$$W_k = \frac{-qz_N k}{4}\int_0^w \sum_{l=1}^{k-1} W_{k-l}(w')W_l(w')dw' k = 2,3,.... \quad (24)$$

It is easy to see that the system Eq.(24) has a solution in the domain of monomials with respect to w. Omitting details, substitution into Eq.(24) expressions

$$W_l = b_l(qz_N w)^{l-1} l = 2,3,...,k,...W_1 = 1 \quad (25)$$

shows that they are solutions of the system Eqs. (23), (24) if

$$b_k = \frac{-k}{4(k-1)}\sum_{l=1}^{k-1} b_{k-l}b_l b_1 = 1 \quad (26)$$

Where b_k, $k = 1,2,...,$ is an alternating sequence of numbers 1, $-1/2, 3/8, -1/3, 125/384,...$ with initially decreasing and then for $k > 5$ increasing absolute values. For $k \to \infty$, $|b_k/b_{k-1}| < 3/2$. In fact, it is Eq.(26) that is the basic element of the description of the nonlinear generation of harmonics (in the case of excitation with a continuous wave).

$$W_k(z) = b_k \cdot (qz_N \cdot w(z))^{k-1} k = 1,2,3,.... \quad (27)$$

$$w(z) = ln\left[\frac{\varsigma(z)}{\varsigma(0)}\right] = ln\left[1 + \frac{i\cdot z}{\varsigma(0)}\right] \quad (28)$$

In Appendix C, the range of the applied approximations allowing the omission of $\partial_{zz}W_k$ was determined. In order to solve the recursion of Eq.(23) in the general case of the loss medium we proceed as above. We use substitutions

$$W_l(z) = \widetilde{b}_l(qz_N w(z))^{l-1} \quad (29)$$

where now $\widetilde{b}_l = \widetilde{b}_l(z)$. From Eq. (23), we get a recursive solution on $\widetilde{b}_k$

$$\widetilde{b}_k w(z)^{k-1} = \frac{-k}{4(k-1)}\int_0^z \partial_{z'}\left(w(z')^{k-1}\right)\left(\sum_{l=1}^{k-1}\widetilde{b}_{k-l}\widetilde{b}_l e^{\widehat{a}_g(k,l)z'}\right)dz' \quad (30)$$

Integrating by parts we get $\widetilde{b}_k$ in the form corresponding to Eq. (26),

$$\widetilde{b}_k + \frac{k}{4(k-1)}\sum_{l=1}^{k-1}\widetilde{b}_{k-l}\widetilde{b}_l e^{\widehat{a}_g(k,l)z} = \frac{k}{4(k-1)}\int_0^z \left(\frac{w(z')}{w(z)}\right)^{k-1}\partial_{z'}\sum_{l=1}^{k-1}\widetilde{b}_{k-l}\widetilde{b}_l e^{\widehat{a}_g(k,l)z'}dz' \quad (31)$$

$\widetilde{b}_1(z) = b_1 = 1$. Now, dispersion has a significant influence on the description of non-linear interactions and the generation of harmonics, and $\widetilde{b}_k(z)$ is a sequence of complex functions with real boundary values. For $z = 0$, the term on the right of Eq.(31) is zero, hence $\widetilde{b}_k(0) = b_k + 0\cdot i$. Further discussion of Eq. (31) is provided in the conclusions. For the above-discussed case $a_g(k,l) = 0$, Eqs.(30), (31) change into Eq.(26).

The functions $C_k(z,r)$ have the form,

$$C_k(z,r) = W_k(z)\Psi_k(z,r) = (qz_N w(z))^{k-1}\cdot\widetilde{b}_k(z)\cdot\Psi_k(z,r)$$

$$= (q\cdot w(z)\cdot\varsigma(z))^{k-1}\cdot\widetilde{b}_k(z)\cdot\Psi_1(z,r)^k$$

$$= \left(q\cdot z_N\cdot ln\left[\frac{\varsigma(z)}{\varsigma(0)}\right]\right)^{k-1}\cdot\widetilde{b}_k(z)\cdot\frac{z_N}{\varsigma(z)}exp\left(-\frac{kr^2}{2\varsigma(z)}\right)k = 1,2,3,... \quad (32)$$

The C_k functions satisfy the required boundary conditions.

3.2. Solution for D_k

After substituting Eq.(32) into Eq.(10), we obtain a recursive system of equations on D_k. In the wave diffusion equation approximation, the solution is of the form $D_k = D_k^C + D_k^D$, where $D_1 = D_1^C$. The explicit parts of D_k, are given below

$$D_k^C = \frac{(qz_N)^{k+1}z_N k}{2i}\int_0^z \frac{w(s)^k}{|\varsigma(s)|^2}\sum_{l=1}^{K-k}(qz_N|w(s)|)^{2(l-1)}\widetilde{b}_{k+l}(s)\widetilde{b}_l^*(s)e^{-\widehat{a}_d(k,l)s}G_{k,l}(z,s,r)ds \quad (33)$$

$$G_{k,l}(z,s,r) := \frac{|\varsigma(s)|^2}{\xi_{k,l}(s)Z_{k,l}(z,s)}exp\left(\frac{-r^2k}{Z_{k,l}(z,s)}\right) \quad (34)$$

$$\xi_{k,l}(s) = (\varsigma(s)^* + 2z_0 l/k), Z_{k,l}(z,s) = \frac{|\varsigma(s)|^2}{\xi_{k,l}(s)} + i(z-s) \quad (35)$$

while D_k^D is a solution of the following recursive system for zero boundary conditions

$$\left(\partial_z + \frac{1}{2ik}\Delta_\perp\right)D_k^D = q\frac{k}{2i}\sum_{l=1}^{k-1}D_l C_{k-l}e^{\widehat{a}_g(k,l)z}k = 2,3,...,K,... \quad (36)$$

$$D_k^D(z,r) = q\frac{k}{2i}\int_0^z\int_0^\infty \mathrm{G}_k(z-s,r,\rho)\sum_{l=1}^{k-1}D_l(s,\rho)C_{k-l}(s,\rho)e^{\widehat{a}_g(k,l)s}\rho d\rho ds \quad (37)$$

$$\mathrm{G}_k(z,r,\rho) := \frac{k}{iz}exp\left(\frac{ik}{2z}\left(r^2+\rho^2\right)\right)\mathrm{J}_0\left(\frac{kr\rho}{z}\right) \quad (38)$$

As one can easily see, sequentially for each k, the integration with respect to ρ can be done based on equation 11.4.29 in [31]. From the above, it follows that all D_k have a Gaussian transverse profile and are k-fold integrals with respect to the coordinate z.

Since the dependence of D_l^C on r is explicitly defined by Eqs.(33),(34) then the part of the integral Eq.(37) with respect to ρ can be calculated. Particularly, the explicit form of D_2^D is obtained. The result is shown in Appendix D. Thus we have $F_2 = C_2 + D_2 = C_2 + D_2^C + D_2^D$.

The quantity $D_1 = D_1^C$ is important for the description of nonlinear interactions because it determines the change in amplitude of the fundamental component resulting from the generation and transfer of energy to the harmonics. In general, the energy exchange between the fundamental and the harmonics is proportional to the difference between the power intensity of the solution of a linear and a nonlinear problem, i.e.

$$\delta I(z,r) = |C_1|^2 - |F_1|^2 = -2\mathrm{Re}(C_1 D_1) - |D_1|^2 \quad (39)$$

The importance of D_1 for the solution F_1 and the energy flow direction map (fundamental component $\leftrightarrow$ harmonics) are illustrated in the next section. According to the remarks after Eq.(10), for consistency, only the $l = 1$ component should be preserved in Eq.(33).

3.3. Convergence of the solution

Based on the considerations in Chapter 2, we have

$$|F_k| = exp(-a(k)\cdot z)|C_k + D_k|$$

$$\leq exp(-a(k)\cdot z)\cdot(|C_k| + |D_k|) \quad (40)$$

$$= exp(-a(k)\cdot z)\cdot|C_k| + o\left(q^2\right)$$

Mathematically rigorous demonstration that $|F_k| \leq |C_k|$ if possible is very difficult or tedious. From the considerations in Chapter 2, the "supplement" D_k, $o(D_k) = q^2 \cdot o(C_k)$ describes the transfer of energy from F_k to the others, especially the newly generated ones. So, based on physical considerations it should be $|F_k| \le |C_k|$. Also, the results of numerical calculations, presented below, confirm this inequality. That is, the terms of the series Eq. (3) are majorized by $|C_k| exp(-a(k) \cdot z)$ and the convergence of the series Eq.(3) is characterized by the quotient ε_k of the sequence $C_k \cdot exp(-a(k) \cdot z)$ in Eq.(8) for $k \to \infty$

$$\varepsilon_k(z) = q \cdot z_N \cdot |w(z)| \cdot \left| \frac{\widetilde{b}_k exp(-a(k) \cdot z)}{\widetilde{b}_{k-1} exp(-a(k-1) \cdot z)} \right| \tag{41}$$

For $\varepsilon_k(z) < 1$, $(k \to \infty)$ the series Eq.(3) converges to a regular function. At points z where $\varepsilon_k(z) \ge 1$, the solution on P"blows up" that is, it takes a distributional character. The form ε_k based only on C_k is particularly simple and overtly shows the factors that determine the nature of convergence of a strict series based on F_k. Defining $\varepsilon_k^F := |F_{k+1}/F_k|$ with overtly typed D_k does not mean that we obtain an expression that is more quantitatively accurate or that brings new qualities to the analysis. Rather, we obtain an expression that is not clear with large sizes. It can be shown that $\varepsilon_k^F = \varepsilon_k(1 + o(q^2))$. The core C_k of the solution is therefore the carrier of the analytical properties of solution F_k.

4. Numerical calculations and solutions comparison.

In the following, we present the numerical solutions of Eqs.(1),(4),(5) and compare them with the obtained analytical solution. The numerical algorithm to solve the above equations is based on the full Helmholtz operator [1,28,29]. We recall that the analytical solution on F_k of the form Eq.(8), Eq.(32) was obtained by approximating the Helmholtz operator in Eqs.(4,5,9,10) with the wave diffusion operator Eq.(13) (quasi parabolic).

A medium with material parameters similar to water, speed of sound $c_0^{'} = 1500$ m/s, density $\rho_0^{'} = 1000$ kg/m^3, nonlinearity parameter$\gamma - 1 = B/A = 5$, absorption parameter $\alpha_2^{'} \cong 2.8 \cdot 10^{-14}$ Np/m was assumed. Based on the data in Appendix A, dispersions of the speed of sound were omitted. The following boundary conditions corresponding to a Gaussian beam Eq.(12) were used for $k = 1$, $z_e^{'} = 80$ mm, $z_0^{'} = 12.47$ mm which gives a beam width at the boundary $d_1^{'}(0) \cong 9.147$ mm and at the waist $d_1^{'}(z_e^{'}) \cong 1.41$ mm, $z_N^{'} = |\varsigma^{'}(0)| = 80.97$ mm. The excitation frequency was equal to $\omega^{'}/2\pi = \Omega_0^{'}/2\pi = 3$ MHz, $K_0^{'} = 12.57$ 1/mm. After normalization $\alpha_2 \cong 2.01$–10^{-5}, $z_e = K_0^{'} z_e^{'}$=1005.31, $z_0 = K_0^{'} z_0^{'}$=156.7, $z_N = z_N^{'} K_0^{'} \cong 1017$. Calculations were performed for two cases of boundary pressures $P_0^{'} = 0.18$ MPa and $P_0^{'} = 0.20067$ MPa, corresponding to $q = 2.8 \cdot 10^{-4}$ and $q = 3.122 \cdot 10^{-4}$, and in a limited range for $P_0^{'} = 0.22$ MPa, $q = 3.422 \cdot 10^{-4}$.

The plots of the function $w(z) := ln[\varsigma(z)/\varsigma(0)]$ and for example w^2, w^5, w^{10} are shown in Fig. 2

It should be noted that all $\widetilde{b}_k(z) \cdot exp(-a(k)z)$ are decreasing. It can be seen that with increasing z the index of the minimal function $\left|\widetilde{b}_k\right|$ in the

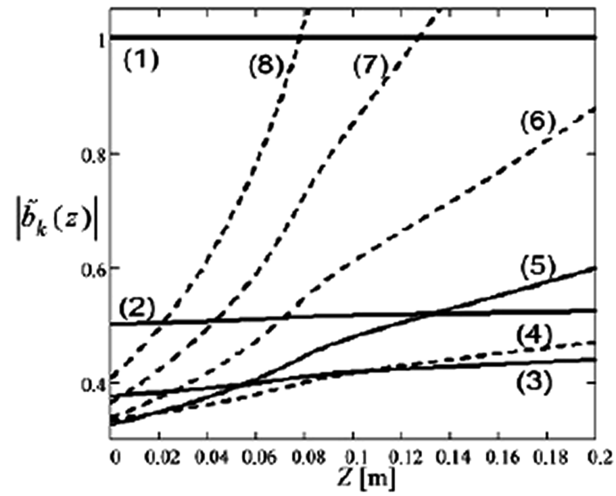

Fig. 3. Harmonic's generation. Solutions of Eq.(31) describing harmonic generation in the case of classical absorption. Subsequent numbers (1)…(8) denote the moduli of the functions $\widetilde{b}_k(z)$, $b_k = \widetilde{b}_k(0)$, $k = 1, ..., 8$, of the solution in the case of either an ideal medium or one with absorption linearly dependent on frequency.

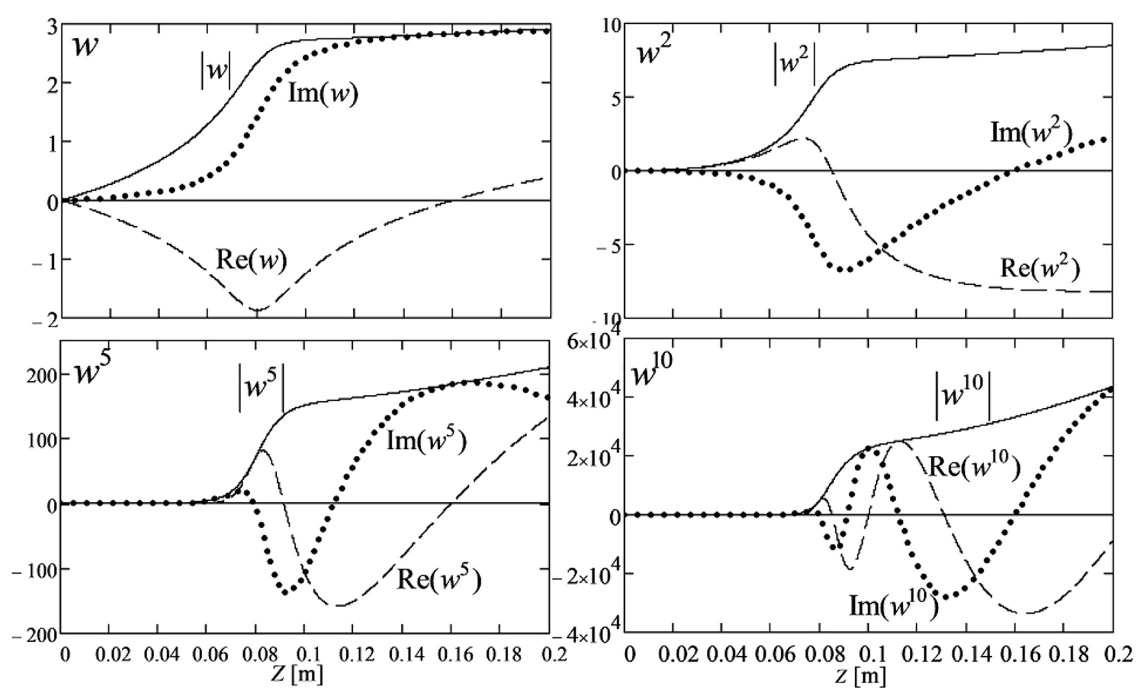

Fig. 2. Harmonic's generation. Example solution factors of Eq.(19) forming the harmonic form of Eqs.(29,30,31,32) along the z axis $w(z) = ln(\varsigma(z)/\varsigma(0))$

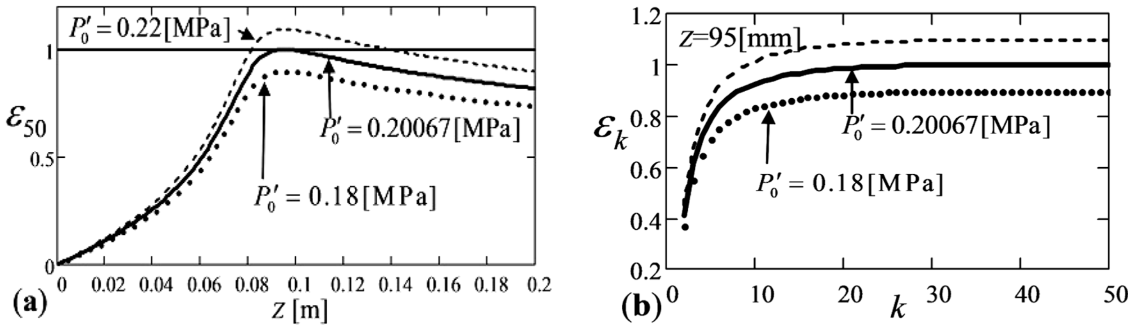

Fig. 4. Convergence conditions. Convergence of series Eq.(32) and convergence characteristics of series Eq.(3). (a) - Dependence of the series quotient on pressure and coordinate. (b) - Stabilization of the quotient of the series ε_k for increasing wave number k as a function of pressure and at a distance for which the limit of classical convergence $\varepsilon_k = 1$ occurs.

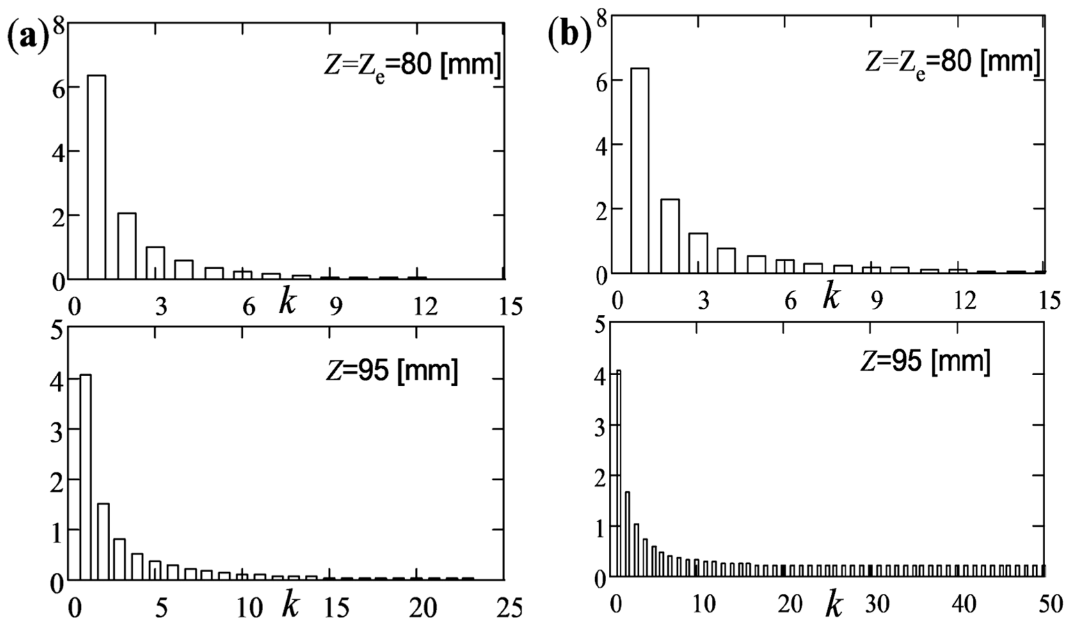

Fig. 5. Fourier spectra C_k in focus z_e and in maximum of the function $\varepsilon(\mathrm{z})_{k=50}$(Fig. 4.a.), for pressure (**a**) -$P_0' = 0.18$ MPa and (**b**) -$P_0' = 0.20067$ MPa. On axis, normalized amplitudes versus normalized frequencies.

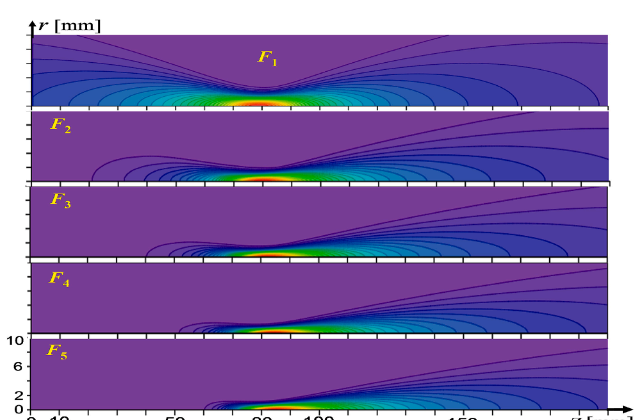

Fig. 6. Spatial distributions (axial z- radial r) of numerically calculated of the fundamental and the harmonics components of the Gaussian beam for the pressure $P_0' = 0.20067$ MPa.

set $\tilde{b}_k$, decreases. In the range z in Fig. 3, from $k = 5$ to $k = 3$.

Plots of the function $\varepsilon(\mathrm{z}) = \varepsilon_\infty(z) \cong \varepsilon(\mathrm{z})_{k=50}$ Eq.(41) for the applied pressures are shown in Fig. 4.a. The approximation follows from the number of $\tilde{b}_k(z)$ calculated and the numerically observed rapidly decreasing changes in $\varepsilon(\mathrm{z})_k$ for $k > 30$ Fig. 4.b. Of course, such approximations are reasonable if the last factor in Eq.(41) is bounded for $k \rightarrow \infty$.

As can be seen in Fig. 5.(b) in the limit of classical convergence, the Fourier spectrum does not go to zero. For this case Fig. 6 showes the spatial distribution of the first few Fourier components of pressure.

A comparison of axial and radial distributions of Fourier coefficients of the analytical and numerical solution is shown in Fig. 7. and Fig. 8, for pressures $P_0' = 0.18$ MPa and $P_0' = 0.20067$ MPa.

We take the numerically obtained solution F_1 as reference. We find from Fig. 7,8. that the supplement $D_1 = D_1^C$ of Eq.(33) leads to an analytical solution $F_1 = C_1 + D_1^C$ consistent with the numerical one. This confirms the correctness of Eq.(33). For $k \geq 2$ in the range z to $\sim z_e$, D_k^C although not a complete addition to the solution improves it bringing $F_k = C_k + D_k^C$ closer to the numerical Fig. 7,8. For $P_0' = 0.18$ MPa $k \geq 3$ and $z > 0.1$ m considering only D_k^C even worsens the solution relative to the "core"C_k. This is all the more evident for $P_0' = 0.20067$ MPa.

The comparison of pressure time profiles on the propagation axis at different distances from the source at $z = 0$, for pressures $P_0' = 0.18$ MPa,

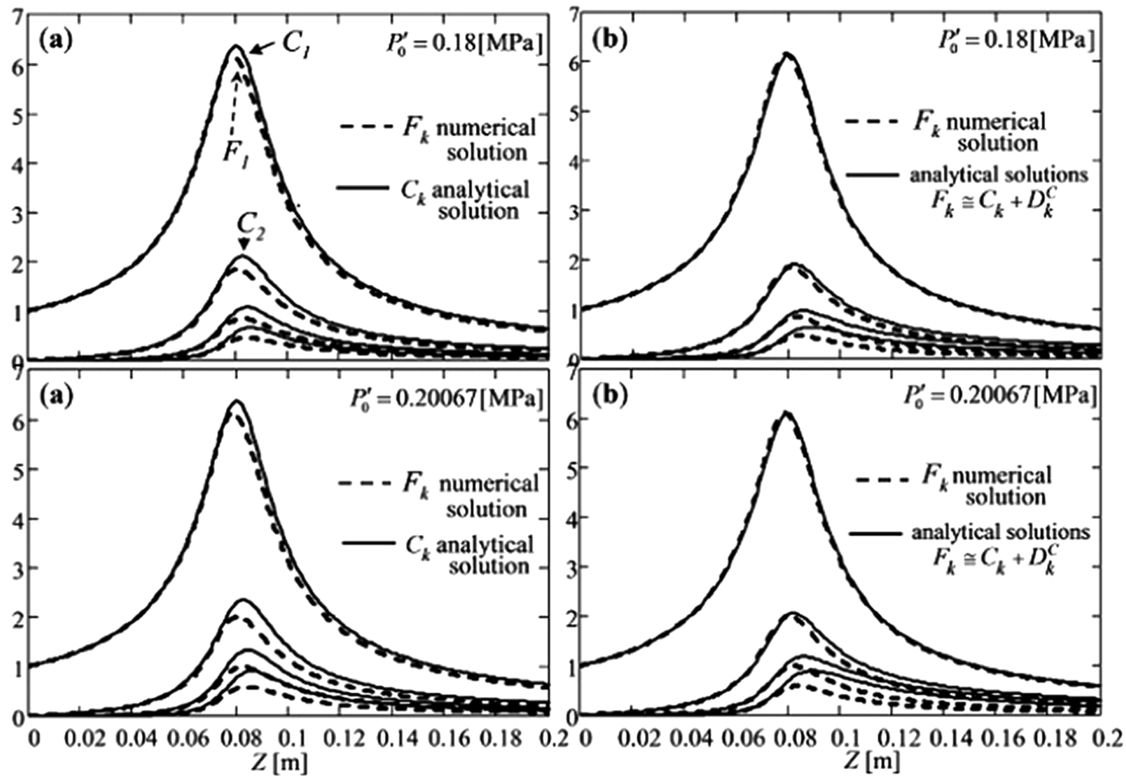

Fig. 7. Comparison on-axis Fourier components of the normalized pressure distributions without correction (**a**) and with correction (**b**), for boundary pressures $P_0' =$ 0.18 MPa on the top figer and $P_0' = 0.20067$ MPa on the down. The boundary values for $z = 0$ corresponds to 1

and $P_0' = 0.20067$ MPa is shown in the Fig. 9. and Fig. 10.

In Fig. 9, we observe very good agreement between the analytical and numerical results, even though $\varepsilon_{50} \cong 0.9$ and no D_k^D applications.

The Fig. 10. corresponds to the situation where the limit of classical convergence $\varepsilon_{50} = 1$ is reached. We observe a very good agreement of the wave profiles in the range $z \le z_e$. In the $z \in (90, 100)$mm range ε_{50} is very close to or equal to 1 what is marked by characteristic peaks visible on the wave profiles. The solution not including D_k^D, significantly deviates from the numerical one in terms of the mapping of the rising edge. These edges are formed by the high frequency components of the spectrum. Although the occurrence of Blow up in Fig. 10 for the analytical solution is due to omission of D_k^D, this effect is present in numerical solutions and is observed experimentally, however, for higher pressures. Fourier spectra of such time profiles are of the type presented in Fig. 5.b. down.

The sign map of Eq.(39) is shown in the Fig. 11. Although it is not visualized, the intense energy transfer to the harmonics takes place in a narrow region along the propagation axis and in the focal area.

5. Conclusions and discussion

A exact solution of the parabolic approximation of NAE equation for any dispersive (dissipative) medium was determined. The approximations used were justified. In particular, for classical absorption it is a exact solution of the KZK equation. Gaussian boundary conditions and time excitation of the medium with a sinusoidal continuous wave form were assumed.

The solution has the form of a Fourier series Eq.(3) with coefficients $F_k = exp(-\widehat{a}(k)z) \cdot (C_k + D_k)$ Eq.(8). Where the dominant "core" of the solution, the components of C_k, has been determined in explicit form. It was shown that for the "supplement" D_k of the solution gives $o(D_k) = q^2 \cdot o(C_k)$. That is, the properties of the "supplement" D_k (e.g., amplitude) decay q^2 times faster than the core C_k when $q \rightarrow 0$. As stated at the end of chapter 2.1, F_k tends to C_k very quickly when $q \rightarrow 0$, $q \neq 0$. Formally, the forms of $D_k = D_k^C + D_k^D$ including "explicitly" all D_k^C have been determined (by which we mean that the integral with respect to ρ in the formal solutions has been calculated). In the important case $k = 1$ this gives $D_1 = D_1^C$, $D_1^D \equiv 0$. On this basis, the energy flow between the fundamental and harmonic components is described - see Eq.(39) and Fig. 11. Until now, this effect could only be analyzed numerically. It has also been explicitly determined D_2^D that is $D_2 = D_2^C + D_2^D$. However we obtained factorization for D_2^D we did not use it to demonstrate the effect on the solution. It does not have the same important descriptive and quantitative meaning as D_1^C(or D_2^C).

As shown by the relation between ε_k^F and ε_k in section 3.3, and also due to the asymptoptic properties of the solution ($F \Rightarrow C$ for $q \rightarrow 0$, $q \neq 0$) the analytic and essential properties of the presented solution F_k are concentrated in the "core" C_k and in $D_1 = D_1^C$ describing the total energy transfer from F_1 to the remainder and the marginal off-axis energy return transfer as seen in Fig. 11.

Of course, if we are interested in a more precise description of the energy exchange between F_k and the others (not only the explicitly available description D_k^C) then a recursive complement D_k^D for $k \ge 2$ Eq.(37), Eq.(D1) is necessary. In principle, however, a deeper analysis of $D_k^D k \ge 2$, due to the high complexity of Eq.(D1), requires a separate work with a large numerical contribution. Moreover, the visualization of the influence D_k^C on the solution was intended to check the correctness of the derivation of these complex analytical formulas since only in the small range k the numerical solution can be considered as a reference. If for $k = 1$ no consistency was obtained it means that an error was made for all k.

From the numerical calculation and also from the effect of the correction on the total sum it follows that $|F_k(z, 0)| \le |C_k(z, 0)|$ and the convergence characteristic determined on the basis of the series C_k also applies to the full solution of $F_k(z, 0)$.

In the compared numerical and analytical solutions, we applied

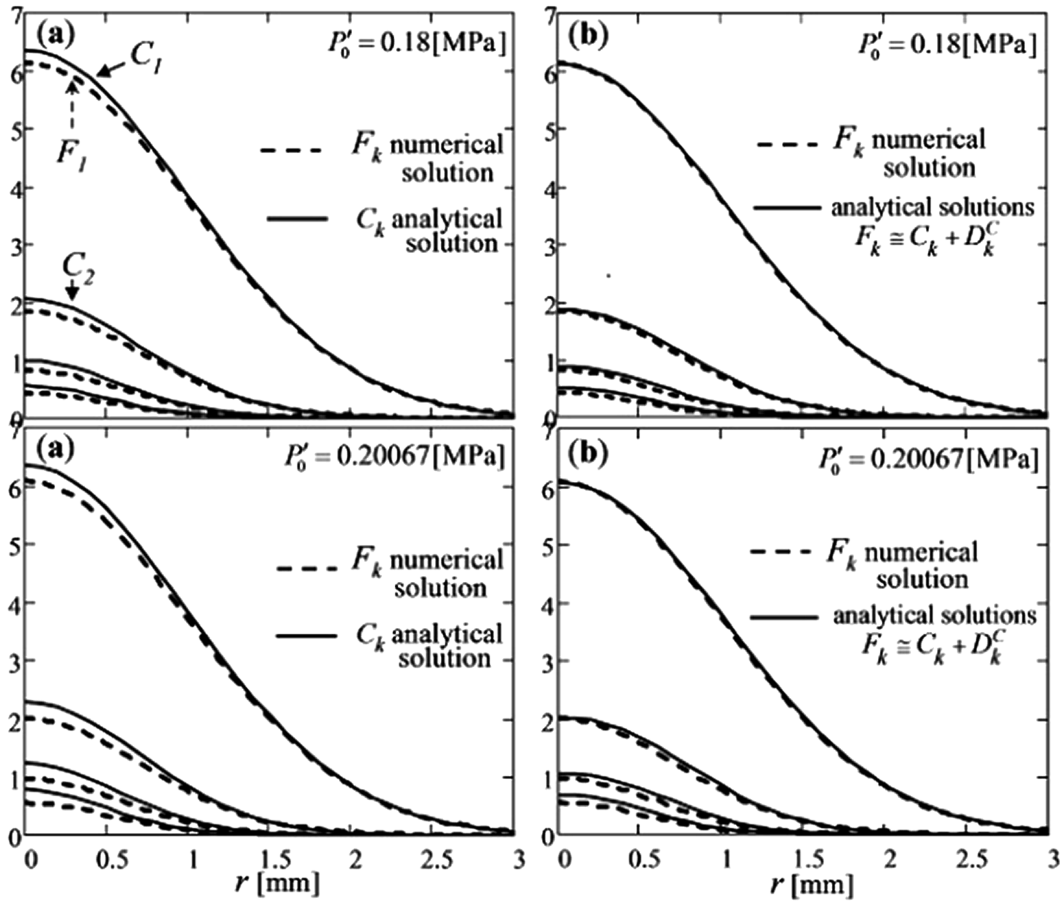

Fig. 8. Comparison radial Fourier components of the normalized pressure distributions in the focal plane $z' = z_e' = 80$ mm, without correction (**a**) and with correction (**b**), for boundary pressures $P_0' = 0.18$ MPa on a top figures and $P_0' = 0.20067$ MPa on the down. The boundary values for $z = 0$ corresponds to 1.

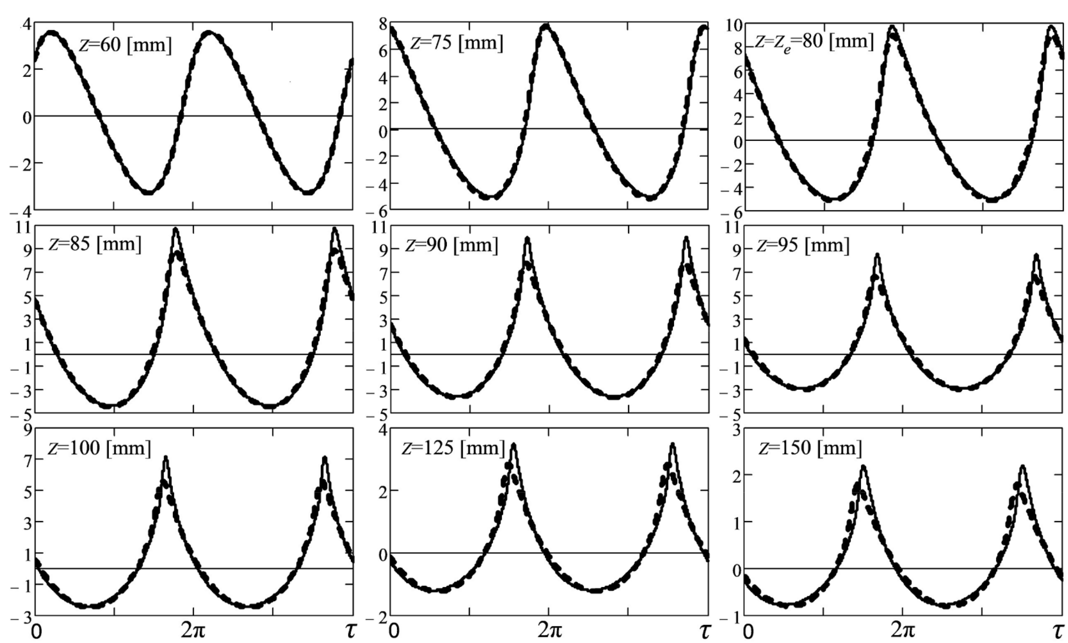

Fig. 9. On-axis Waveforms for boundary pressure $P_0' = 0.18$ MPa (=1 in normalized units). Analytical solutions - continuous lines, numerical solutions - dashed lines.

pressures that give large or borderline values of the classically understood convergence condition of the solving series Eq.(41). The analysis of the spectrum in Fig. 5.b (down) and the time profiles in Fig. 10 (the borderline case of classical convergence) suggests that the solution in the spatial regions where $\varepsilon_{k\to\infty} > 1$ can be represented as a sum of the regular functions and distribution. The obtained results show that, due

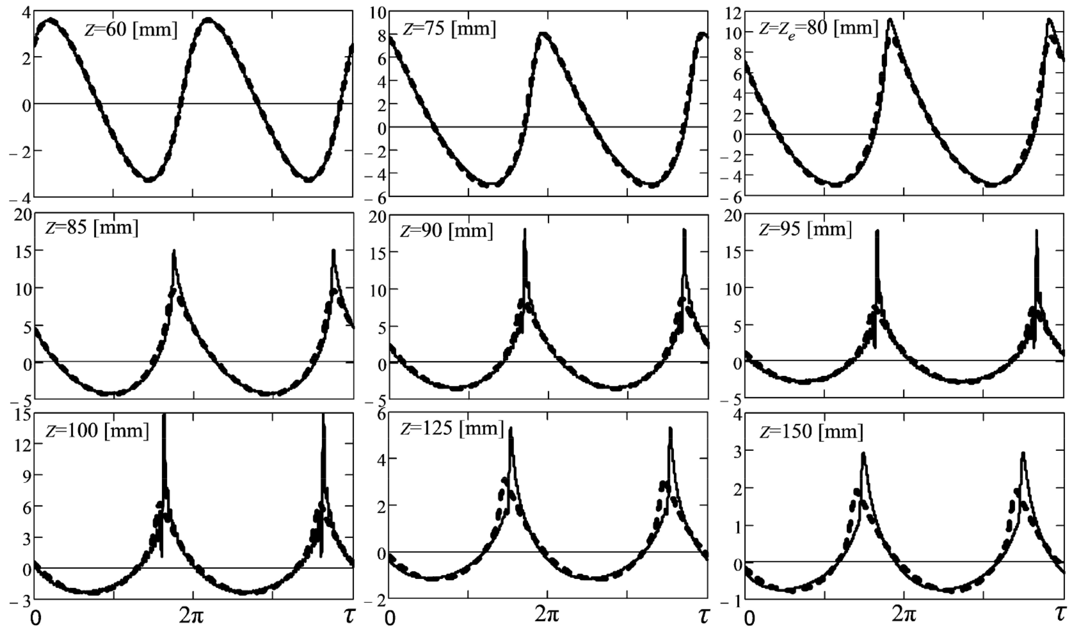

Fig. 10. On-axis waveforms for boundary pressure $P_0^{'} = 0.20067$ MPa (=1 in normalized units). Analytical solutions - continuous lines, numerical solutions - dashed lines

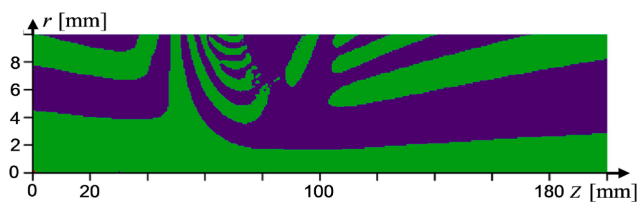

Fig. 11. Map of energy exchange between the fundamental component F_1 (initiating) and harmonics. Green - area of energy transfer to harmonics; deep blue - from harmonics to fundamental component.

to the relations between the orders of C_k and D_k, even the complete omission of D_k leads to a stronger solution than the asymptotically analytical one. That is $C \Rightarrow F$ for $q \rightarrow 0, q \neq 0$ (instead of $C \rightarrow F$).

Equation (31) (recursive solution) describing the generation of new harmonics from existing ones, reveals a new surprising aspect of the influence of dispersion. We can speak of external absorption finally suppressing the beam (real part of the exponential factor in Eq.(8)) and internal absorption expressed by the coefficient $a_g(k,l) = \mathrm{Re}(\widehat{a}_g(k,l))$, enhancing the harmonic generations by amplifying the coefficients $\widetilde{b}_k(z)$. According to the results of work [11], an increase in the number of harmonics increases the dissipation of the acoustic field energy. Dispersion, in addition to influencing the harmonic generation level itself, incorporates, through $w(z)$ functions, the geometric parameters of the beam z_0 and z_e into the description. In the case of linear frequency-dependent absorption, characteristic or similar to the absorption of a number of organic substances, intrinsic absorption does not occur $a_g(k, l) \equiv 0$(as in a lossless medium). However, there is a spatial modulation of the phase of the components describing nonlinear interactions in Eq.(31).

It is interesting that $\widetilde{b}_k(z)$ Eqs.(26),(31) have a minimum with respect to the index (wave number), which with increasing z shifts towards smaller k. Note that Eqs.(26),(30), (31) are invariant due to transformations $\widetilde{b}_k(z) = \widetilde{b}_k^{'}(z) \cdot c^k$, $c \neq 0 + i0$ is a complex constant.

The transverse sizes of the harmonic beams are given by $d_{0k} := d_k(z_e) = d_0/\sqrt{k}$, which follows from the property of Gaussian beams. Nevertheless, the property of $d_{0k} := d_k(z_e) \sim 1/\sqrt{k}$ for other types of beams seems to be preserved, which is confirmed by the results of numerical harmonic calculations.

In the case of impulsive boundary conditions, the form of the solution and the procedure for its factorization are identical to those presented above. However, the sense of the indices in Eq.(3) change, since the boundary spectrum is composite due to the limited duration and the relationship between pulse duration and repetition time. The harmonic bars are similarly complex. The index corresponds to the pulse repetition frequency ($\omega_1^{'}/2\pi$). The carrier frequency (envelope filling) corresponds to some index $k = k_c$ (pulsation $\omega_c^{'} = k_c \omega_1^{'}$ in not normalized units). The time spectrum of the excitation and the initiating pulse is centered around k_c. The harmonic bars are clustered around $k = k_m := m \cdot k_c$, $m = 1, 2, 3, \ldots$, although these are not necessarily local maxima of the spectrum see [11] (overtone generation).

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

I sincerely thank Dr. Norbert Żołek for help in editing this work. Thank you very much to the anonymous reviewers. Fragments of this work appeared thanks to their suggestions and comments, enhancing the value of the work.

Appendix A

In the following formulas, we can either assume that the dependent and independent variables are dimensional or, for $c_0 = 1$, assume that they are normalized as shown in Chapter 2.

For a Maxwell medium that remembers viscous stresses, we have

$$AP = \frac{-\eta}{2c_0 t_r}\int_{-\infty}^{t} e^{-(t-t')/t_r}\Delta P(\mathbf{x},t')dt' A(\mathbf{x},t) := \frac{-\eta}{2c_0 t_r}e^{-t/t_r}\Delta\delta(\mathbf{x}). \tag{A.1}$$

For $t_r \to 0$, $(1/t_r)exp(-t/t_r)\to\delta(t)$. By calculating the generalized Fourier transform Eq.(A.1)

$$\mathrm{F}^{\mathbf{x},t}[A] = \widehat{a}^{\varsigma,\omega}(\varsigma,\widehat{\omega}) = \frac{\eta}{c_0}\frac{\varsigma^2}{1-i\widehat{\omega}t_r} \tag{A2}$$

The dispersion equation for the D'Alambert wave equation with dispersion has the form

$$\varsigma^2 - \frac{\widehat{\omega}^2}{c_0^2}\left[1 + i2\frac{c_0\widehat{a}^{\varsigma,\omega}}{\widehat{\omega}}\right] = 0 \tag{A3}$$

Its solution $\varsigma(\widehat{\omega})^2$ for $\widehat{a}^{\varsigma,\omega}$ of the form Eq.(A.2) is

$$\varsigma(\widehat{\omega})^2 = \frac{\widehat{\omega}^2}{c_0^2}\frac{1}{1-i2c_0\alpha_2\cdot\widehat{\omega}/(1-i\widehat{\omega}t_r)}\alpha_2 := \eta\Big/c_0^3 \tag{A.4}$$

Thus

$$\widehat{a}^{\omega}(\widehat{\omega}) := \widehat{a}^{\varsigma,\omega}(\varsigma(\widehat{\omega}),\widehat{\omega}) = \frac{\alpha_2\widehat{\omega}^2}{1-i\widehat{\omega}(2c_0\alpha_2+t_r)} \tag{A.5}$$

and

$$\widehat{a}(\omega) = \frac{\alpha_2\omega^2(1+i\omega(2c_0\alpha_2+t_r))}{1+\omega^2(2c_0\alpha_2+t_r)^2} \tag{A.6}$$

Based on Eq.(A.1) for $t_r = 0$ operator $A^{x,t}\to A^x = -(\eta/c_0)\Delta$ is local in space and corresponds to the Navier-Stokes model with hybrid kinematic viscosity η, (accounting for thermal conductivity) and thermo viscous stress $c_0A^x = -\eta\Delta$. While the operator A^t corresponding to $\widehat{a}^{\omega}$ is not local in time. However, the nonlocality and full dispersion associated with the $h(\omega)$ coefficient is of order α^2 and can almost always be neglected. This justifies (for $t_r = 0$) the commonly used model of classical absorption $A^t := \alpha_2\partial_{tt} + o(\alpha^2)$, which is in fact not analytically correct. For the expressions defined below Eq.(10), in normalized units $c_0 = 1$, $\omega\to k$, l, we obtain from Eq.(A.6), $\widehat{a}_g := a_g + ih_g$, $\widehat{a}_d := a_d + ih_d$ where: $a_g(k,l) \cong \alpha_2 2l(k-l)$, $h_g(k,l) \cong 6\alpha_2^2 kl(k-l)$, $h_g/a_g \cong 3\alpha_2 k$, $a_d(k,l) \cong \alpha_2 2l(k+l)$, $h_d(k,l) \cong 2a_2^2 l(3k^2+3kl+l^2)$, $h_d/a_d \approx 3\alpha_2 k$, $l<k$. For water $\alpha_2 = 2.5\cdot10^{-14}$Np/(m·Hz^2) $2c_0\alpha_2 \simeq 7.5\cdot10^{-10}$ s. In normalized units and for $\Omega_0'/2\pi = 3$MHz $\alpha_2 = 1.79\cdot10^{-5}$. Thus, in the band up to 100 MHz ($k \cong = 35$) $h_{(g,d)}/a_{(g,d)} \cong 3\alpha_2 k \leq 5.4\cdot10^{-3}$ and the amplitude modulation effects of nonlinear interactions far outweigh those of phase modulation. On this basis, the calculations for water omit h_g and h_d.

For many organic (biological) media, the following interpolation of the absorption coefficient is assumed from measured data $a(\omega) = \alpha_1|\omega|$, $\omega \in [0,\infty)$. We will determine the full dispersion using the method of analytical extensions (see [32]). We have $|\omega| = \omega\cdot sig(\omega) = \omega(-H(-\omega)+H(\omega))$, $H(\cdot)$ is the Heaviside distribution. Based on [32] $\widehat{H}(\widehat{\omega}) = -(1/2\pi i)ln(-\widehat{\omega})$ and $\widehat{H}(\widehat{\omega}) = (1/2\pi i)ln(\widehat{\omega})$ are analytic extensions of $H(\omega)$ and $H(-\omega)$ respectively. Thus analytical form of $a(\omega) = \alpha_1|\omega|$ is

$$\widehat{a}^{\omega}(\widehat{\omega}) = -\alpha_1(\widehat{\omega}/2\pi i)(ln(\widehat{\omega}) + ln(-\widehat{\omega})) \tag{A.6}$$

The real and imaginary part of the dispersion coefficient is obtained by determining the jump of the dispersion function on the real axis (see [32]),

$$a(\omega) = \lim_{\varepsilon\to0}(\widehat{a}^{\omega}(\omega+i\varepsilon) - \widehat{a}(\omega-i\varepsilon)) = \alpha_1|\omega| \tag{A.7}$$

$$h(\omega) = \lim_{\varepsilon\to0}(\widehat{a}^{\omega}(\omega+i\varepsilon) + \widehat{a}(\omega-i\varepsilon)) = (2\alpha_1/\pi)\omega ln(|\omega|) \tag{A.8}$$

In the works [11,26] it was shown that the absorption operator in spatial coordinates, (corresponding to $\widehat{a}^{\varsigma}(\varsigma)$) has the form $A^x = (\alpha_1/\pi)\nabla\cdot\left[\left(\mathbf{x}/|\mathbf{x}|^4\right)\underset{\mathbf{x}}{\otimes}\cdots\right]$.

Analogously to classical absorption we have $\widehat{a}_g = 0 + ih_g$, where taking into account Eq.(A.7) and Eq.(A.8) $h_g(k,l) = \alpha_1(2/\pi)ln\left[(k/k-l)^k\cdot(k-l/l)^l\right] \leq \alpha_1(2/\pi)kln(2)$, $\widehat{a}_d(k,l) = 2\alpha_1 l + i(2\alpha_1/\pi)ln\left[(k+l)^{k+l}l^l/k^k\right]$, $h_d < (2\alpha_1/\pi)\cdot(2k-1)ln(2k-1)$, $1 \leq l < k$. For blood $\alpha_1 \cong 5\cdot10^{-6}$ Np/(m·Hz). In normalized variables $\alpha_1 \cong 1.2\cdot10^{-3}$.

Finally, note that dispersion is a linear phenomenon. However, it occurs under non-linear propagation conditions and one should speak rather about quasi-dispersion. Solutions of the quasi-dispersion equations corresponding to NAE are of the form $\varsigma = \varsigma(\widehat{\omega}) + o(q)$ [11,26]. So if the dispersion is not a given function of only frequencies, then an attempt to go to such a description in the indicated manner may require the omission of non-linear terms. However, please note that it may be $o(\widehat{a}) \sim o(q)$.

Appendix B

For finite radius r_g of circular area integration (cut beam) and for on the axial distributions $r = 0$ the Eq.(14) takes the form

$$\Psi_k^H(z) := \Psi_k^I(z) - \Psi_k^{II}(z)$$

$$= -\frac{z_N}{\varsigma(0)} ze^{-ikz} exp\left(\frac{kz^2)}{2\varsigma(0)}\right)\cdot\left(\int_z^\infty - \int_{z_g}^\infty exp\left(\frac{-kR^2}{2\varsigma(0)}\right)\partial_R \frac{e^{ikR}}{R} dR\right) \tag{B.1}$$

where $z_g = z_g(z) := \sqrt{z^2 + r_g^2}$. Integrating by parts we obtain for the first integral in Eq.(B.1)

$$\Psi_k^I(z) := \frac{z_N}{\varsigma(0)}\left(1 - \frac{zk}{\varsigma(0)} exp\left(\frac{kz^2}{2\varsigma(0)} - ikz\right)\right)\int_z^\infty exp\left(\frac{-kR^2}{2\varsigma(0)} + ikR\right) dR \tag{B.2}$$

$$= \frac{z_N}{\varsigma(0)}\left(1 - \frac{zk}{\varsigma(0)}\sqrt{\frac{2\varsigma(0)}{k}} exp\left(\frac{-k\varsigma(z)^2}{2\varsigma(0)}\right)\int_{-i\sqrt{k/\varsigma(0)}\varsigma(z)}^\infty exp(-\xi^2)d\xi\right)d\xi$$

$$= \frac{z_N}{\varsigma(0)}\left(1 - z\sqrt{\frac{k}{2\varsigma(0)}} exp\left(\frac{-k\varsigma(z)^2}{2\varsigma(0)}\right)\sqrt{\pi}\cdot \mathrm{erfc}\left(-i\sqrt{\frac{k}{\varsigma(0)}}\varsigma(z)\right)\right)$$

Similarly for $\Psi_k^{II}(z)$

$$\Psi_k^{II}(z) = \frac{z_N}{\varsigma(0)}\left(\frac{z}{z_g} exp\left(\frac{k\left(\varsigma(z_g)^2 - \varsigma(z)^2\right)}{2\varsigma(0)}\right)\right.$$

$$\left. - z\sqrt{\frac{k}{2\varsigma(0)}} exp\left(\frac{-k\varsigma(z)^2}{2\varsigma(0)}\right)\sqrt{\pi}\cdot \mathrm{erfc}\left(-i\sqrt{\frac{k}{2\varsigma(0)}}\varsigma(z_g)\right)\right) \tag{B.3}$$

For finite $r_g = r_t := d_k(0) = |\varsigma(0)|\sqrt{2/z_0 k}$ the Eq.(B.1) represents realistic on axis field distribution generated by source radius r_t, appodized by parabolic phase lens with focal length $R(0)$. With growth of r_g, the function $\Psi_k^{II}(z)$ quickly disappears and for $r_g > 2r_t$, $\Psi_k^{II}(z) \cong 0$. Using asymptotical expansion [31]

$$\mathrm{erfc}(-i\xi) = \frac{2}{\sqrt{\pi}} exp(\xi^2)\frac{1}{-2i\xi}\left(1 + \sum_{l=1}\frac{(2l-1)!}{\xi^{2l}2^l(l-1)!}\right) \tag{B.4}$$

we obtain for $|\xi|^2 \gg 1$

$$\Psi_k^H(z) = \Psi_k^I(z) = \frac{z_N}{\varsigma(z)}\left(1 + \frac{\varsigma(0)}{k\varsigma(z)^2} + \dots\right) = \Psi_k(z)\left(1 + \frac{\varsigma(0)}{k\varsigma(z)|\varsigma(0)|}\Psi_k(z) + \dots\right) \tag{B.5}$$

$$\Psi_k^{II}(z) = \frac{z_N z}{\varsigma(z_g)z_g} exp\left(\frac{k\left(\varsigma(z_g)^2 - \varsigma(z)^2\right)}{2\varsigma(0)}\right)\left(1 + \frac{\varsigma(0)}{k\varsigma(z_g)^2} + \dots\right) \tag{B.6}$$

It follows from Eq.(B.5) that the difference and relative difference (on the axis) between the solutions of the Helmholtz and "wave diffusion" equations of Eq.(13) have estimation

$$\left|\Psi_k^H(z) - \Psi_k(z)\right| \le \frac{z_N}{kz_0^2}|\Psi_k(z)| \tag{B.7}$$

$$\left|\frac{\Psi_k^H(z) - \Psi_k(z)}{\Psi_k^H(z)}\right| \le \left|\frac{\varsigma(0)\Psi_k(z)}{|\varsigma(0)|k\varsigma(z)}\right| \le \frac{1}{kz_0}|\Psi_k(z)| \tag{B.8}$$

for every z. The maximum of the deviation of $\Psi_k(z,0)$ from $\Psi_k^H(z,0)$ is reached at $z = z_e$ and quickly goes to zero for $z\to 0$ and $z\to\infty$.

Appendix C

Based on the relation $|\partial_z W_k/2k| \sim o(1/z_0)|W_k|$ we request that $z_0\cdot|\partial_z W_k/2k| \le |W_k|$. Hence, based on Eq.(28) we have

$$\left|\frac{(k-1)z_0}{2k\varsigma(z)}\right| \le \left|ln\left[1 + \frac{iz}{\varsigma(0)}\right]\right| \tag{C.1}$$

It is easy to see that for sufficiently small z this inequality will not be satisfied. Using the expansion of the logarithm with respect to $iz/\varsigma(0)$ we have (for sure) for $z/z_N = z/|\varsigma(0)| < 1$

$$\frac{z}{\sqrt{z_0^2 + z_e^2}} < \left|ln\left[1 + \frac{iz}{\varsigma(0)}\right]\right| \tag{C.2}$$

Hence, if

$$\frac{(k-1)z_0}{2k\sqrt{z_0^2 + (z - z_e)^2}} \le \frac{z}{\sqrt{z_0^2 + z_e^2}} \tag{C.3}$$

then all the more Eq.(C.1). Assuming the equality in Eq.(C.3), we obtain an equation that allows us to determine the limiting value of z_b for which it is satisfied. This equation can be strictly solved, however, it misses the point. It can be seen that inequality (C.3) is satisfied for $z = z_0$. Thus we consider small values of z compared to z_N. For $0 < z < z_0$ the left-hand side of Eq.(C.3) is almost constant compared to the right-hand side. By substituting $z = 0$ into the left-hand side of Eq.(C.3) we obtain, as a first and very good approximation

$$z \equiv z_b = \frac{(k-1)z_0}{2k} \leq \frac{z_0}{2} \tag{C.4}$$

The inequality (C.3) is thus satisfied for $z_b < z$. Accordingly, for the second derivative of $z_0 \cdot |\partial_{zz} W_k / 2k| \leq |\partial_z W_k|$ we obtain

$$\left| \frac{((k-2) - ln(1 + iz/\varsigma(0)))z_0}{2k\varsigma(z)} \right| \leq \left| ln\left[1 + \frac{iz}{\varsigma(0)}\right] \right| \tag{C.5}$$

For $k = 2$, the inequality Eq.(C.5) is satisfied for each $z > 0$. For $k \geq 3$ we obtain the problem which was solved above for the first derivative.

We can calculate the integral in Eq.(21) analytically for the above-used expansion of the logarithm power with respect to $iz/\varsigma(0)$. Leaving aside the details we obtain a very good estimation

$$\left| \int_0^z e^{-2ik(z-z')} \partial_{z'} \widetilde{W}_k dz' \right| \leq \frac{k-1}{2kz} \left| \widetilde{W}_k \right| k > 2 \tag{C.6}$$

Hence if z_0 is a large parameter (however if $z_0 \ll z_N = |\varsigma(0)|$) then for $z > z_0/2$ the integral in Eq.(21) can be neglected and $W_k = \widetilde{W}_k + o(1/z_0)$. According to the introduced normalizations, $z_0 := K_0' z_0'$ is the number of wavelengths of the fundamental disturbance $(k = 1)$ k = 1 with pulsation $\Omega_0' = c_0' K_0'$ on the section z_0' multiplied by 2π. It should be noted, however, that the restrictions on the magnitude of the second derivative of the function W_k are not significant because in the region $0 \leq z \leq z_0/2$ in which they are not satisfied, for $k > 2, W_k \sim (z/z_N)^{k-1}$ are close to zero and do not differ much from each other see Fig. 2. .

Appendix D

After substituting $D_l = D_l^C + D_l^D$ into Eq.(37) and calculating the integral with respect to ρ of the expression containing D_l^C, we obtain

$$D_k^D(z,r) = q\frac{k}{2i} \int_0^z \int_0^\infty \mathrm{G}_k(z-s,r,\rho) \sum_{l=1}^{k-1} D_l^D(s,\rho) C_{k-l}(s,\rho) e^{\widehat{a_g}(k,l)s} \rho d\rho ds \tag{D.1}$$

$$-\frac{(qz_N)^{k+1}}{4i} \int_0^z \frac{exp\left(\frac{ikr^2}{2(z-s)}\right)}{z-s} \sum_{l=1}^{k-1} l \cdot \widetilde{b}_{k-l}(s) w(s)^{k-l-1} \int_0^s \frac{w(s')}{|\varsigma(s')|^2} \cdot$$

$$\cdot \sum_{m=1}^{K-l} (qz_N|w(s')|)^{2(m-1)} \widetilde{b}_{k+m}(s') \widetilde{b}_m^*(s') e^{-\widehat{a_d}(l,m)s'} \mathrm{G}_{k,l,m}(z,s,s',r) ds' ds$$

$$\widehat{G}_{k,l,m}(z,s,s',r) := k^2 \int_0^\infty \mathrm{G}_k(z-s,r,\rho) \Psi_{k-l}(s,\rho) G_{l,m}(s,s',\rho) \rho d\rho \tag{D.2}$$

$$\widehat{G}_{k,l,m}(z,s,s',r) = \frac{z_N}{\varsigma(s)} \frac{|\varsigma(s')|^2}{Z_{l,m}(s,s')\xi_{l,m}(s')} \cdot \frac{k^2}{2\kappa(z,s,s')^2} exp\left(\frac{-k^2 r^2}{4(z-s)^2 \kappa(z,s,s')^2}\right) \tag{D.3}$$

$$\kappa(z,s,s')^2 := \left(\frac{k}{Z_{l,m}(s,s')} + \frac{k-l}{2\varsigma(s)} - \frac{ik}{2(z-s)}\right) \tag{D.4}$$

Since, $D_1^D(s,\rho) \equiv 0$ then for $k = 2$, the first integral in Eq.(D.1) is equal to zero and we obtain the factorization $D_2 = D_2^C + D_2^D$ where D_2^D depends on two integrals. Of course, writing out the explicit forms of Eq.(D.1) for $k > 2$ makes no sense due to the rapidly increasing size of the expressions. Moreover, explicitly $o(D_k^D) = q^{k+1}$.

References

[1] J. Wójcik, A. Nowicki, P.A. Lewin, P.E. Bloomfield, T. Kujawska, L. Filipczyński, Wave envelopes method for description of nonlinear acoustic wave propagation, Ultrasonics 44 (3) (2006) 310–329, https://doi.org/10.1016/j.ultras.2006.04.001.

[2] R.T. Bayer, Nonlinear Acoustics in Fluids, Edited by R.T. Bayer, Part II ,Van Nostrand Reihold Company Inc., Library of Congress Catalog Card Number : 84-3610. ISBN: 0-442- 21182-1, © 1984.

[3] R.T. Bayer, Nonlinear acoustics, Written for the Naval Systems Command Department of the Navy, 1974.

[4] L.K. Zarembo, W.A. Krasilnikow, Introduction to nonlinear acoustics, Publisher "Science" The main editorial office of math.- phys. literature, Nauka Moscow (1966).

[5] M.F. Hamilton, D.T. Blackstock, Nonlinear Acoustics, © Academic Press (1998).

[6] O.V. Rudenko, S.I. Soluyan (Eds.), Theoretical Foundations of Nonlinear Acoustics, Springer US, Boston, MA, 1977.

[7] S. Earnshaw, Phil. Trans. Roy. Soc. (London) 150 (1860) 133.

[8] L.D. Landau, E.M. Lifshitz, Fluid Mechanics, Pergamon, Oxford, second edition, 1987.

[9] T. Lappas, A. Leonard, P.E. Dimotakis, Riemann Invariant Manifolds for the Multidimensional Euler Equations, SIAM J. Sci. Comput. 20 (4) (1999) 1481–1512.

[10] E.F. Toro (Ed.), Riemann Solvers and Numerical Methods for Fluid Dynamics, Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[11] J. Wójcik, Conservation of energy and absorption in acoustic fields for linear and nonlinear propagation, J. Acoust. Soc. Am. 104 (5) (1998) 2654–2663, https://doi.org/10.1121/1.423849.

[12] S. Makarov, M. Ochman, Nonlinear and Thermoviscous Phenomena in Acoustics, Part II, Acustica 83 (2) (1997) 197–222.

[13] V.P. Kuznetsov, Equations of nonlinear acoustics, Akusticheskii Zhurnal 16 (4) (1970) 548–553.

[14] E. A. Zabolotskaya and R. V. Khokhlov, Quasi-Plane Waves in the Nonlinear Acoustics of Confined Beams, Akusticheskii Zhurnal, Vol. 15, No.1, (1969) 40-47. [Sov. Phys. Acoust. 15, (1969), 35 -40.]; [Sov. Phys. Acoust. 16, 39 (1970)].

[15] G. Du and M.A. Breazeale, Theoretical description of a focused Gaussian ultrasonic beam in a nonlinear medium, J. Acoust. Soc. Am., 81, (1), (1987) 51-57.
[16] S.I. Soluyan, R.V. Khokhlov, Propagation of acoustic waves of finite amplitude in a dissipative medium, Vestnik. Moskovskogo Univ. 3 (1961) 52–61.
[17] S. Saha Ray, New exact solutions of nonlinear fractional acoustic wave equations in ultrasound, Computers and Mathematics with Applications 71 (2016) 859–868, https://doi.org/10.1016/j.camwa.2016.01.001.
[18] J. Wójcik, B. Gambin, Theoretical and numerical aspects of nonlinear reflection–transmission phenomena in acoustics, Applied Mathematical Modelling 42 (2017) 100–113, https://doi.org/10.1016/j.apm.2016.10.026.
[19] V. V. Lychagin, Contact geometry and non-linear second order differential equations, Uspekhi Mat. Nauk, Example I. pp.156, Volume 34, Issue 1(205), (1979) 137–165.
[20] K. Naugolnykh, S.I. Soluyan, R.V. Khokhlov, Sfiericzeskije wolny koniecznoj amplitudy w wiaskoju tieploprowodijuszcziej srede, Akusticheskii Zhurnal 9 (1) (1963) 54–60.
[21] B.B. Cary, Prediction of Finite-Amplitude Waveform Distortion with Dissipation and Spreading Loss, J. Acoust. Soc. Am. 43 (1968) 1364–1372.
[22] M.S. Bruzon, M.L. Gandarias, M. Torrisi, R. Tracinà, Some traveling wave solutions for the dissipative Zabolotskaya-Khokhlov equation, Journal of Mathematical Physics 50 (103504) (2009) 1–12, https://doi.org/10.1063/1.3204507.
[23] A.M. Kamchatnov, M.V. Pavlov, On exact solutions of nonlinear acoustic equations, Wave Motion 67 (2016) 81–88, https://doi.org/10.1016/j.wavemoti.2016.07.009.
[24] S. V. Manakov and P. M. Santini, On the dispersionless Kadomtsev–Petviashvili equation in n+1 dimensions: exact solutions, the Cauchy problem for small initial data and wave breaking, J. Phys. A: Math. Theor. 44, (2011) 1-15. 405203, © 2011 IOP Publishing Ltd Printed in the UK &the USA. doi:10.1088/1751-8113/44/40/405203.
[25] Ş. Akçağıl T. Aydemir, New exact solutions for the Khokhlov-Zabolotskaya Kuznetsov, the Newell-Whitehead-Segel and the Rabinovich wave equations by using a new modification of the tanh-coth method, Cogent Mathematics 3: 1193104, (2016) 1-12. 10.1080/23311835.2016.1193104.
[26] J. Wójcik, Dispersion and energy losses, Solid State Phys. 89 (2003) 335–342, https://doi.org/10.4028/www.scientific.net/SSP.89.335.
[27] M. O'Donnell, E.T. Jaynes, J.G. Miller, Kramers-Kronig relationships between ultrasonic attenuation and phase velocity, J. Acoust. Soc. Am. 69 (3) (1981) 696–701, https://doi.org/10.1121/1.385566.
[28] P. Christopher, K. Parker, New approaches to nonlinear diffractive field propagation, J. Acoust. Soc. Am. 90 (1) (1991) 488–499, https://doi.org/10.1121/1.401274.
[29] E.G. Radulescu, J. Wójcik, P.A. Lewin, A. Nowicki, Nonlinear Propagation Model for Ultrasound Hydrophones Calibration in Frequency Range up to 100 MHz, Ultrasonics 41 (4) (2003) 239–245, https://doi.org/10.1016/S0041-624X(03)00124-0.
[30] H. Kogelnik, T. Li, Laser Beams and Resonators, Appl. Opt. 5 (10) (1966) 1550–1567, https://doi.org/10.1364/AO.5.001550.
[31] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover Publications, Inc. New York (1965), Library of Congress Catalog Card Number: 65-12253.
[32] H. Bremerman, Distributions, Complex Variables and Fourier Transform, Addison-Wesley Publisching Company, Inc., Reading Massachusets, Chap. 7.12, (1965) H. Bremerman, Distributions, Complex Variables and Fourier Transform, MIR Publishing, Moskva, Chap. 7.12, (1968), 99.