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Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics
Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

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RECEIVED
11 July 2023

REVISED
23 November 2023

ACCEPTED FOR PUBLICATION
1 February 2024

PUBLISHED
26 February 2024

PAPER

Quantum speed limits for change of basis

Moein Naseri[1,*], Chiara Macchiavello[2,3], Dagmar Bruß[4], Paweł Horodecki[5,6] and Alexander Streltsov[1]

1 Centre for Quantum Optical Technologies IRAU, Centre of New Technologies, University of Warsaw, Warsaw, Poland
2 Dipartimento di Fisica, Università di Pavia, via Bassi 6, I-27100 Pavia, Italy
3 INFN Sezione di Pavia, via Bassi 6, I-27100 Pavia, Italy
4 Institut für Theoretische Physik III, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany
5 International Centre for Theory of Quantum Technologies, University of Gdańsk, Wita Stwosza 63, 80-308 Gdańsk, Poland
6 Faculty of Applied Physics and Mathematics, National Quantum Information Centre, Gdańsk University of Technology, Gabriela Narutowicza 11/12, 80-233 Gdańsk, Poland
* Author to whom any correspondence should be addressed.

E-mail: m.naseri@cent.uw.edu.pl

Keywords: quantum speedlimit, speedlimit for change of basis, quantum coherence, unbiased basis, speedlimits for resource generation

Abstract

Quantum speed limits provide ultimate bounds on the time required to transform one quantum state into another. Here, we introduce a novel notion of quantum speed limits for collections of quantum states, investigating the time for converting a basis of states into an unbiased one as well as basis permutation. Establishing an unbiased basis, we provide tight bounds for the systems of dimension smaller than 5, and general bounds for multi-qubit systems and the Hilbert space dimension *d*. For two-qubit systems, we show that the fastest transformation implements two Hadamards and a swap of the qubits simultaneously. We further prove that for qutrit systems the evolution time depends on the particular type of the unbiased basis. Permuting a basis, we obtain the exact expression for the Hilbert space of dimension *d*. We also investigate speed limits for coherence generation, providing the minimal time to establish a certain amount of coherence with a unitary evolution.

1. Introduction

Striving for quantum advantages, such as an increased speed of a computation, has become a competitive goal. However, nature has established a fundamental speed limit, via a minimal time that is necessary for the unitary evolution of an initial quantum state to a final quantum state, as pointed out in [1, 2]. In a geometric approach [3–6], the quantum speed limit is linked to the length of the shortest path between initial and final state, which can be quantified via a suitable distance measure. The work [7] characterizes the quantum speed limit by rate of change of phases in quantum systems with Hermitian and non-Hermitian dynamics. In [8] the authors discussed the structure of speedlimits for state transformation and provide a comparison between the classical and quantum speedlimits. For a recent review of quantum speed limits, see [9].

The standard approach to quantum speed limits assumes that a quantum state $|\psi\rangle$ is transformed into another state $|\phi\rangle$ via a unitary evolution $U = e^{-iHt}$. The task is to determine the optimal evolution time for the transition $|\psi\rangle \to |\phi\rangle$, with respect to the energy scale of the Hamiltonian H. First results in this direction were presented for orthogonal states, and are known as Mandelstam-Tamm bound [1]:

$$T_\perp \geqslant \frac{\pi}{2\Delta E_\psi}\,, \tag{1}$$

where $(\Delta E_\psi)^2 = \langle H^2\rangle_\psi - \langle H\rangle_\psi^2$ is the energy variance. Another bound was derived later by Margolus and Levitin [2], giving

$$T_\perp \geqslant \frac{\pi}{2E_\psi}\,, \tag{2}$$

with the mean energy $E_\psi = \langle H \rangle_\psi - E_0$, and E_0 is the ground state energy. Note that the speed limits (1) and (2) differ only by the different choice of the energy scale. For transition between mixed states $\rho \to \sigma$ generalized quantum speed limits have been presented [5, 6, 10, 11]:

$$T(\rho \to \sigma) \geqslant \frac{\arccos F(\rho,\sigma)}{\min\{\Delta E_\rho, E_\rho\}} \tag{3}$$

with fidelity $F(\rho,\sigma) = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$.

While the original approaches [1, 2] studied the speed limit for unitary transitions between two quantum states, more general versions of the speed limit have been developed in the last years. This includes investigation of quantum speed limits for non-unitary and open system dynamics [12–20], as well as speed limits for the evolution of observables in the Heisenberg picture [21], and the study of speed limit for a bounded energy spectrum [22]. A theoretical approach for measuring quantum speed limits in an ultracold gas has been proposed recently in [23]. Speed limits for generating quantum resources have also been considered [24], allowing to determine optimal rates for generating quantum entanglement [25], quantum asymmetry and coherence [19, 26–28], and quantum discord [29, 30]. A recent work [31] also applied the notion of speed limit for distinguishing unitary channels using the properties of the diamond norm [32].

However, the previous approaches to the notion of speed limits consider transformations of one state of a quantum system to another one. Here in this letter, we open a new avenue by introducing and constructing a novel and well defined notion of speed limit on the space of bases of quantum states rather than the space of quantum states itself. We also prove theorems and provide bounds regarding the minimal time of transformation of a basis to another.

2. Notion of speed limit for change of basis

The early approaches [1, 2] studied the speed limit for transforming *one* quantum state into another one. However, many quantum technological applications require to transform a collection of states. An important example is quantum computation where a common operation is a change of basis, e.g. by applying the well-known Hadamard gate which transforms the computational qubit basis $\{|0\rangle, |1\rangle\}$ into $\{|+\rangle, |-\rangle\}$, with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

Which fundamental speed limits hold for such a basis transformation? We address this question in this Letter, investigating bounds on the time that is necessary to perform the transformation of a collection of states, i.e. a simultaneously transformation of an ordered set of quantum states to another class of ordered set of quantum states, minimized over all Hamiltonians. Constructing the notion of speed limit for change of a basis (a set of orthonormal states) to another basis or class of bases by unitary evolution, we aim for quantum speed limits of the form

$$T\left(|\psi_j\rangle \to |\phi_j\rangle\right) \geqslant \frac{g}{E}, \tag{4}$$

where $\{|\psi_j\rangle\}, \{|\phi_j\rangle\}$ are two ordered sets of orthonormal states, with $j = 1, \ldots, d$, where d is the dimension of the Hilbert space, and g can in general depend on the sets $\{|\psi_j\rangle\}$ and $\{|\phi_j\rangle\}$ and the relation between them. The quantity E in equation (4) denotes a notion of energy which is Hamiltonian dependent. Note that E cannot be state dependent (such as E_ψ) otherwise it would be meaningless as we are speaking of change of a whole basis. It is also worth to mention that generally the notion of speed limit (either classical or quantum) without having any constraint on the energy is also meaningless as having access to arbitrary large amount of energies we can conduct state transformation in arbitrary small interval of time.

Since E represents some notion of energy, we require to have the following properties:

(i) E is independent on the particular choice of basis $\{|\psi_j\rangle\}$.
(ii) E is additive for non-interactive Hamiltonians of the form $H^{AB} = H^A \otimes I^B + I^A \otimes H^B$:

$$E_{AB} = E_A + E_B, \tag{5}$$

where E_A and E_B are the amount of the function E corresponding to H^A and H^B, respectively.

In regard of the introduced notions, we have the following general theorem:

Theorem 1. *Let $\{|\psi_j\rangle\}$ and $\{|\phi_j\rangle\}$ be two complete orthonormal bases. A speed limit of the form*

$$T\left(|\psi_j\rangle \to |\phi_j\rangle\right) \geqslant \frac{g}{E} \tag{6}$$

directly leads to a speed limit for any basis which can be obtained from $\{|\phi_j\rangle\}$ *via a unitary* $V = \sum_j e^{i\alpha_j}|\psi_j\rangle\langle\psi_j|$:

$$T\left(|\psi_j\rangle \to V|\phi_j\rangle\right) \geqslant \frac{g}{E}. \tag{7}$$

The speed limit (7) is tight whenever equation (6) is tight.

We refer to the appendix B for the proof of the theorem.

A natural choice of E fulfilling the properties (i) and (ii) would be

$$E = \frac{1}{d}\sum_j \langle\psi_j|H|\psi_j\rangle - E_0, \tag{8}$$

which is a natural analogy of the mean energy E_ψ appearing in the Margolous–Levitin bound (2) as they both have the concept of mean in themselves (but they are two different quantities). From now on, we consider E to be the quantity defined in the equation (8) and we refer to it as mean energy. We also note that the mean energy (8) is equivalent to $E = \mathrm{Tr}[H/d] - E_0$.

In the following, we consider the interesting cases of the speed limit for transformation of a basis to an unbiased one as well as basis permutation. In addition to investigating the speed limits for change of basis, we also study speed limits for coherence generation. In particular, we consider the maximal coherence which can be established within a certain time, given some Hamiltonian with mean energy E. These results are highly relevant in the context of the resource theory of quantum coherence [26, 33, 34], taking into account that several recent works suggest that quantum coherence is more suitable than entanglement to capture the performance of certain quantum algorithms [35–37].

3. Speed limits for unbiased bases

In the following, we will determine speed limits for basis change from the computational basis $\{|n\rangle\}$ into an unbiased basis $\{|n_+\rangle\}$ with $|\langle n|n_+\rangle|^2 = 1/d$. By basis change we mean that all the vectors in the initial basis will convert to the corresponding vectors in the target basis simultaneously, see also figure 1. In the following, we investigate the single qubit, two qubits and qutrit scenario. We also present some bounds for the speed limits regarding the quantum system with arbitrary Hilbert space's dimension.

3.1. Single qubit system

A general single-qubit Hamiltonian has the form

$$H = E_+|E_+\rangle\langle E_+| + E_-|E_-\rangle\langle E_-|, \tag{9}$$

where the eigenvalues $E_\pm$ and eigenstates $|E_\pm\rangle$ can be parametrized as

$$E_\pm = (G \pm E), \quad |E_\pm\rangle\langle E_\pm| = \frac{1}{2}(I \pm \boldsymbol{n}\cdot\boldsymbol{\sigma}). \tag{10}$$

Here, G and $E \geqslant 0$ are real numbers, $\boldsymbol{n} = (n_x, n_y, n_z)$ is a normalized vector, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ contains the three Pauli operators. The Hamiltonian (9) can thus be equivalently expressed as

$$H = (E\boldsymbol{n}\cdot\boldsymbol{\sigma} + GI). \tag{11}$$

Note that E corresponds to the mean energy of the Hamiltonian:

$$E = \frac{1}{2}\mathrm{Tr}[H] - E_-. \tag{12}$$

Equipped with these tools, we will now present a bound for the evolution time between any two single-qubit states.

Proposition 2. *The time for converting a single-qubit state* ρ_0 *into the state* ρ_1 *via unitary evolution* $U = e^{-iHt}$ *is bounded as*

$$T(\rho_0 \to \rho_1) \geqslant \frac{1}{2E}\arccos\left(\frac{\boldsymbol{r}_0\cdot\boldsymbol{r}_1}{|\boldsymbol{r}_0||\boldsymbol{r}_1|}\right), \tag{13}$$

where $\boldsymbol{r}_i$ *is the Bloch vector of the state* ρ_i.

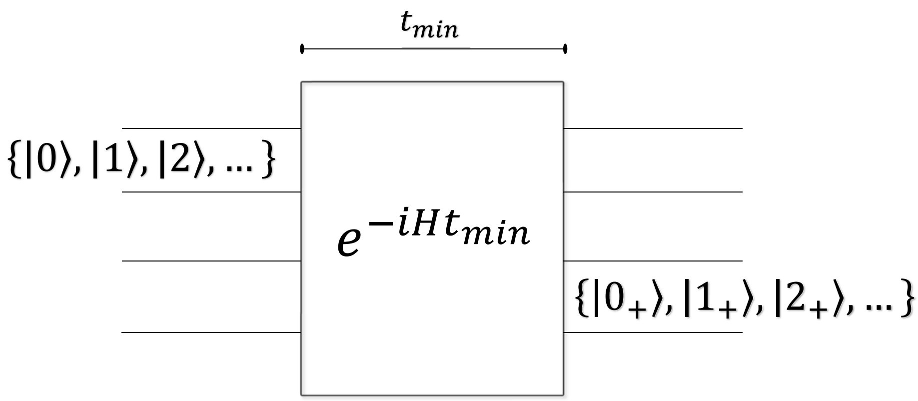

Figure 1. Generation of an unbiased basis $\{|n_+\rangle\}$ from the computational basis $\{|n\rangle\}$ via a unitary evolution $e^{-iHt_{\min}}$. All the vectors in the computational basis will transform to their corresponding vectors in the unbiased basis within the time $t_{\min}$. Note that the time $t_{\min}$ is the same for all the transformations $|n\rangle \to |n_+\rangle$.

Proof. Note that the unitary

$$U(t) = e^{-iHt} = e^{-iGt} e^{-iEt\boldsymbol{m}\cdot\boldsymbol{\sigma}} \tag{14}$$

can be interpreted as a rotation by an angle $2Et$ about the axis $\boldsymbol{n}$ of the Bloch sphere. The minimal value for Et is achieved by choosing the rotation axis $\boldsymbol{n}$ to be orthogonal to both Bloch vectors $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$:

$$\boldsymbol{n} = \frac{\boldsymbol{r}_0 \times \boldsymbol{r}_1}{|\boldsymbol{r}_0 \times \boldsymbol{r}_1|}, \tag{15}$$

$$Et = \frac{1}{2}\arccos\left(\frac{\boldsymbol{r}_0 \cdot \boldsymbol{r}_1}{|\boldsymbol{r}_0||\boldsymbol{r}_1|}\right). \tag{16}$$

This completes the proof of the proposition. □

Noting that $\mathrm{Tr}[\rho_i\rho_j] = (1 + \boldsymbol{r}_i \cdot \boldsymbol{r}_j)/2$ we can reformulate equation (13) as follows:

$$T(\rho_0 \to \rho_1) \geqslant \frac{1}{2E}\arccos\left(\frac{2\mathrm{Tr}\left[\rho_0\rho_1\right] - 1}{\sqrt{\left(2\mathrm{Tr}\left[\rho_0^2\right] - 1\right)\left(2\mathrm{Tr}\left[\rho_1^2\right] - 1\right)}}\right). \tag{17}$$

The proof of proposition 2 implies that this bound is tight, i.e. for any two single qubit-states ρ_0 and ρ_1, there exists a Hamiltonian with mean energy E saturating equation (17). For pure qubit states this expression simplifies to the tight bound

$$T(|\psi_0\rangle \to |\psi_1\rangle) \geqslant \frac{1}{2E}\arccos\left(2|\langle\psi_0|\psi_1\rangle|^2 - 1\right). \tag{18}$$

For single-qubit systems, any unitary transforming $|0\rangle$ into $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ also transforms $|1\rangle$ into $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. For a transition from the computational basis $\{|0\rangle, |1\rangle\}$ to an unbiased qubit basis we thus obtain the minimal time of transformation

$$T_{\text{unbiased}} \geqslant \frac{\pi}{4E}. \tag{19}$$

3.2. Qutrit system

It is now intuitive to assume that for $d > 2$ the evolution time into an unbiased basis increases, compared to the qubit setting. To support this intuition, consider a two-qubit system AB, and let H^A and H^B be qubit Hamiltonians which bring $\{|0\rangle, |1\rangle\}$ into $\{|+\rangle, |-\rangle\}$ within minimal time $\pi/(4E_A)$ and $\pi/(4E_B)$, respectively. If we set $E_A = E_B$, the Hamiltonian $H^{AB} = H^A \otimes I^B + I^A \otimes H^B$ achieves the transformation

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \;\to\; \{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\} \tag{20}$$

within time $\pi/(4E_A) = \pi/(2E)$, where $E = 2E_A$ is the mean energy of the total Hamiltonian H^{AB}. From this argument, we see that for $d = 4$ an unbiased basis can be achieved within time $\pi/(2E)$, which is longer compared to the single-qubit setup.

As we will see in the following, this intuition is not correct. For this, we will first focus on qutrit systems. As we show in the appendix A, a general unbiased qutrit basis can be obtained via a diagonal unitary

$$V = \sum_j e^{i\alpha_j}|j\rangle\langle j| \tag{21}$$

from one of the following two bases (denoted by $\{|n_+\rangle\}$ and $\{|\tilde{n}_+\rangle\}$, respectively):

$$|0_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\frac{2}{3}\pi}|1\rangle + e^{i\frac{4}{3}\pi}|2\rangle\right), \tag{22a}$$

$$|1_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle + |2\rangle\right), \tag{22b}$$

$$|2_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{-i\frac{2}{3}\pi}|1\rangle + e^{-i\frac{4}{3}\pi}|2\rangle\right), \tag{22c}$$

and

$$|\tilde{0}_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{-i\frac{2}{3}\pi}|1\rangle + e^{-i\frac{4}{3}\pi}|2\rangle\right), \tag{23a}$$

$$|\tilde{1}_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle + |2\rangle\right), \tag{23b}$$

$$|\tilde{2}_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\frac{2}{3}\pi}|1\rangle + e^{i\frac{4}{3}\pi}|2\rangle\right). \tag{23c}$$

Note that these two sets of basis states are odd permutations of each other. According to the theorem 1, this implies that speed limits for the transitions $\{|n\rangle\} \to \{|n_+\rangle\}$ and $\{|n\rangle\} \to \{|\tilde{n}_+\rangle\}$ will also lead to speed limits for general unbiased qutrit bases $\{|n\rangle\} \to \{V|n_+\rangle\}$ and $\{|n\rangle\} \to \{V|\tilde{n}_+\rangle\}$ with a diagonal unitary V. Equipped with these tools, we now present the first main result of this Letter.

Theorem 3. *The time for converting a qutrit basis onto an unbiased basis is bounded below as*

$$T_{\text{unbiased}} \geqslant \frac{2\pi}{9E}. \tag{24}$$

We refer to the appendix C for the proof.

Having established a speed limit for basis change it is natural to ask whether this bound is tight, i.e. whether for any unbiased basis there exists a Hamiltonian H with mean energy E saturating the bound (24). Recalling the definition of the unbiased bases $\{|n_+\rangle\}$ and $\{|\tilde{n}_+\rangle\}$ in equations (22) and (23), we answer this question in the following proposition.

Proposition 3. *The speed limit (24) is tight for the basis $\{|n_+\rangle\}$ and is saturated by the Hamiltonian $H = |\alpha\rangle\langle\alpha|$ with*

$$|\alpha\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{-i\frac{2}{3}\pi}|1\rangle + |2\rangle\right), \tag{25}$$

but not tight for the basis $\{|\tilde{n}_+\rangle\}$.

We refer to the appendix D for the proof of the proposition.

The above results imply that there are two different classes of unbiased bases for qutrits: bases of the form $\{V|n_+\rangle\}$ can be obtained from the computational basis at time $T = 2\pi/9E$, while bases of the form $\{V|\tilde{n}_+\rangle\}$ require an evolution time $T > 2\pi/9E$, where V is an arbitrary diagonal unitary. For the second class $\{V|\tilde{n}_+\rangle\}$ we have numerical evidence that a tight speed limit is given as (see figure 2)

$$T(|n\rangle \to |\tilde{n}_+\rangle) \geqslant \frac{4\pi}{9E}. \tag{26}$$

To see this, note that any unitary achieving the transformation $|n\rangle \to |\tilde{n}_+\rangle$ must be of the form

$$U = \sum_{n=0}^{2} e^{i\phi_n}|\tilde{n}_+\rangle\langle n| \tag{27}$$

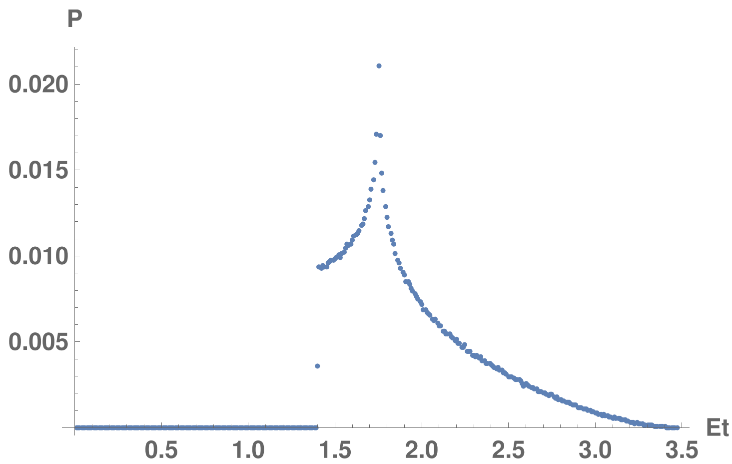

Figure 2. Numerical test of equation (26). We sample 10^6 unitaries of the form (27) with random phases $0 \leqslant \phi_n \leqslant 2\pi$ and evaluate Et using equation (8). The plot shows the numerical probability as a function of Et. As a numerical bound, we obtain $Et \geqslant \frac{4}{9}\pi + \varepsilon$ with $\varepsilon \leqslant 10^{-5}$, in good agreement with equation (26).

with some phases ϕ_n. Let now $\lambda_j = e^{-i\alpha_j}$ be the eigenvalues of U, such that the phases α_j are in increasing order and $-\pi \leqslant \alpha_j \leqslant \pi$. For a given set of such phases $\{\alpha_j\}$, there exists a Hamiltonian implementing the unitary $U = e^{-iHt}$ such that

$$E_j t = \alpha_j \ \text{ or } \ E_j t = \alpha_j + 2\pi, \tag{28}$$

where E_j are the eigenvalues of H. The mean energy of the numerically obtained Hamiltonian then fulfills

$$Et = \frac{1}{3}\sum_j E_j t - E_0 t. \tag{29}$$

Using these results, we can test equation (26), by numerically sampling random phases $0 \leqslant \phi_n \leqslant 2\pi$ and evaluating Et via equation (29). The choice of $E_j t$ as in equation (28) guarantees that the numerical Hamiltonians obtained in this way contain Hamiltonians with the minimal value of Et.

In figure 2 we show the numerical probability for obtaining a certain value of Et for 10^6 samples. As we see the minimum Et with nonzero probability occurs at around $Et \approx 1.4$. The numerical results suggest the following lower bound for Et:

$$Et \geqslant \frac{4}{9}\pi + \varepsilon, \tag{30}$$

where ε is numerically upper bounded as $\varepsilon \leqslant 10^{-5}$, in good agreement with equation (26). A Hamiltonian saturating the bound (26) is given by $\tilde{H} = -|\tilde{\alpha}\rangle\langle\tilde{\alpha}|$ with

$$|\tilde{\alpha}\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\frac{2}{3}\pi}|1\rangle + |2\rangle\right). \tag{31}$$

A direct comparison of theorem 3 with the corresponding qubit bound (19) shows that establishing an unbiased qutrit basis requires less time, compared to an unbiased qubit basis for the same mean energy E. In the following, we will discuss the main differences between the qubit and the qutrit setting.

If a single-qubit unitary $U = e^{-iHt}$ is optimal for rotating the basis $\{|0\rangle, |1\rangle\}$ onto an unbiased basis, then the unitary $U^2 = e^{-2iHt}$ permutes the basis elements $\{|0\rangle, |1\rangle\}$. This is no longer the case in the qutrit setting. For this, note that an optimal Hamiltonian for the qutrit transition $|n_+\rangle = e^{-iHt}|n\rangle$ (transition to the first class of the qutrit basis $\{|n\rangle\} \to \{|n_+\rangle\}$) is given by $H = |\alpha\rangle\langle\alpha|$, with

$$|\alpha\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{-i\frac{2}{3}\pi}|1\rangle + |2\rangle\right). \tag{32}$$

For the optimal Hamiltonian we can evaluate the fidelity between the initial state $|0\rangle$ and the time-evolved state $e^{-iHt}|0\rangle$:

$$|\langle 0|e^{-iHt}|0\rangle|^2 = \frac{1}{9}\left[5 + 4\cos(t)\right]. \tag{33}$$

Note that the right-hand side of equation (33) is never zero, which means that the evolution never permutes $|0\rangle$ with another basis element, and the same can be shown for the states $|1\rangle$ and $|2\rangle$.

Moreover, if consider the converse situation that the single-qubit unitary U permutes the basis states $\{|0\rangle, |1\rangle\}$, then $\sqrt{U}$ always rotates the $\{|0\rangle, |1\rangle\}$ basis onto an unbiased basis. This is no longer the case in the qutrit setting, as can be seen by inspection, with the permutation $U = \sum_{n=0}^{2} |(n+1) \mod 3\rangle\langle n|$. We further obtain

$$\sqrt{U} = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}, \tag{34}$$

and thus $\sqrt{U}|n\rangle$ is not a maximally coherent state for any $0 \leqslant n \leqslant 2$. It can be verified by inspection that also $U^{1/3}$ does not transform any of the states $|n\rangle$ into a maximally coherent state.

3.3. Two qubit system

So far, we considered systems of dimension 2 and 3. We will now go one step further, giving the minimal evolution time for an unbiased basis for two-qubit systems.

Theorem 5. *The time for establishing an unbiased two-qubit basis is bounded below as*

$$T_{\mathrm{unbiased}} \geqslant \frac{\pi}{4E}. \tag{35}$$

There exists a two-qubit Hamiltonian achieving this bound.

Remarkably, this bound is the same as for single-qubit systems, see equation (19). The Hamiltonian saturating equation (35) is given as

$$H = -\sigma_x \otimes \sigma_z + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_x. \tag{36}$$

The eigenvalues of this Hamiltonian are $3, -1, -1, -1$, and the mean energy of H is given as $E = 1$. Note that although the Hamiltonian H performs the task of transformation within the minimal time, it is an interactive Hamiltonian and it may be costly to implement it practically (as it generally causes the qubits to interact with each other). For $t = \pi/4$ we now define the unitary $U = e^{-itH}$. The action of this unitary onto the computational basis of two qubits is as follows:

$$U(|0\rangle|0\rangle) = e^{i\pi/4}|+\rangle|+\rangle, \tag{37a}$$

$$U(|0\rangle|1\rangle) = e^{i\pi/4}|-\rangle|+\rangle, \tag{37b}$$

$$U(|1\rangle|0\rangle) = e^{i\pi/4}|+\rangle|-\rangle, \tag{37c}$$

$$U(|1\rangle|1\rangle) = e^{i\pi/4}|-\rangle|-\rangle. \tag{37d}$$

This shows that the Hamiltonian in equation (36) indeed transforms a two-qubit basis onto an unbiased basis within time $\pi/(4E)$. We refer to the appendix E for the proof of theorem 5 and more details.

Comparing the results above for one qubit, two qubits and qutrit basis change, one might intuitively guess some patterns with respect to oddness or primeness of the numbers 3 and 5 that for $d = 5$, the minimal time of transformation would be less than $\pi/4E$ which is the minimal time of transformation for $d = 4$ (as for $d = 3$ the minimal time is less than $\pi/4E$ which is the minimal time for $d = 2$). However, we show in the appendix F that $T > \frac{\pi}{4E}$ for the Hilbert space of the dimension 5.

3.4. *n*-qubits and *d*-dimensional systems

The results presented so far show that the optimal time for transformation onto an unbiased basis is the same for single-qubit and two-qubit systems, and in both cases given by $\pi/(4E)$. For a qutrit system we have a shorter time $2\pi/(9E)$. We will now extend these results to many-qubit systems. As we will see, there exists a universal bound for n-qubit systems, allowing us to establish an unbiased basis within finite time.

Theorem 6. *For systems with n qubits, the minimal time for establishing an unbiased basis is bounded above as*

$$T_{\mathrm{unbiased}} \leqslant \frac{\pi}{2E}. \tag{38}$$

Proof. Consider the n qubit Hamiltonian

$$H_n = V^{\otimes n}, \tag{39}$$

where V is the Hadamard gate. Note that the mean energy of H_n is given as $E = 1$. We now define the unitary $U_n(t) = e^{-iH_n t}$. Using the fact that $H_n^2 = I$ it follows that

$$U_n(t) = \cos(t)\, I - i\sin(t)\, H_n. \tag{40}$$

For $t = \pi/2$ we obtain

$$U_n(\pi/2) = -i V^{\otimes n}. \tag{41}$$

This unitary transforms the computational basis of n qubits into an unbiased basis, and the proof is complete. □

Theorem 6 shows that it is possible to establish an unbiased basis of n qubits within time $\pi/(2E)$. We demonstrated this explicitly by presenting a Hamiltonian, which introduced interactions between all the qubits. Without interactions, i.e. if each of the qubits evolves independently, the optimal evolution time is given by $n\pi/(4E)$.

In the following, we present a general lower bound for the time required for establishing an unbiased basis for any d-dimensional system.

Theorem 7. *The time for establishing an unbiased basis for a system of dimension d is bounded below by*

$$T_{\text{unbiased}} > \frac{\pi(d-1)}{4Ed}. \tag{42}$$

As we see, for large Hilbert space dimension $d \to \infty$ the lower bound converges to $\pi/4E$. For systems of dimension 6 this bound can be improved slightly to $T \geqslant 0.227/E$. We refer to the Appendix I for the proof of the theorem and more details. Comparing this lower bound with the bound in the theorem 6, we see that in the limit $n \to \infty$ the minimal time T for establishing an unbiased basis of n qubits fulfills $\pi/4E \leqslant T \leqslant \pi/2E$.

4. Speed limits for basis permutation

It is instrumental to compare the above results to the speed limits for permuting the basis $\{|n\rangle\}$:

$$U|n\rangle = |(n+1) \mod d\rangle \tag{43}$$

for all $0 \leqslant n \leqslant d-1$. In this regard, we have the following proposition.

Proposition 8. *The time for permuting a basis is bounded below by*

$$T_{\text{perm}} \geqslant \frac{\pi(d-1)}{dE}. \tag{44}$$

Proof. As we discuss in the G, the eigenvalues of the permutation unitary (G.1) have the form

$$\lambda_j = e^{-i\frac{2\pi j}{d}}, \tag{45}$$

where integer j is in the range $0 \leqslant j \leqslant d-1$. It follows that for any permutation unitary $U = e^{-iHt}$ it must hold that

$$t\sum_j E_j = \sum_j \frac{2\pi j}{d} = \pi(d-1). \tag{46}$$

The proof of the proposition is complete by noting that $E = \sum_j E_j/d$. □

Interestingly, for a given Hamiltonian H there are only two options: either the unitary $U = e^{-iHt}$ leads to permutation with $t = \pi(d-1)/(dE)$, or the Hamiltonian never leads to a basis permutation. We further note that our analysis applies only to permutations of the form (43).

5. Speed of evolution for coherence generation

We will now present speed limits for the creation of quantum coherence under unitary evolution. In particular, we are interested in the maximal value of coherence $C_{\max}$ which can be achieved from a given state ρ within a fixed time t:

$$C_{\max}(\rho,t) = \max_H \left\{ C\left(e^{-iHt}\rho e^{iHt}\right) \right\}, \tag{47}$$

and the maximization is performed over all Hamiltonians H with average energy $E = \mathrm{Tr}[H]/d - E_0$. As a quantifier of coherence we use the ℓ_1-norm of coherence [26, 33]

$$C(\rho) = \sum_{i\neq j} |\rho_{ij}|, \tag{48}$$

which can be estimated efficiently in experiments by using collective measurements [38, 39].

We will first discuss the single-qubit setting. Recall that in this case the unitary $U(t) = e^{-iHt}$ can be interpreted as a rotation by an angle $2Et$ about the axis $\boldsymbol{n}$ of the Bloch sphere. As for single-qubit states the amount of coherence C corresponds to the Euclidean distance to the incoherent axis, $C_{\max}(\rho,t)$ corresponds to the largest distance from the incoherent axis, maximized over all rotations with a fixed angle $2Et$. The optimal rotation axis $\boldsymbol{n}$ is orthogonal to the Bloch vector $\boldsymbol{r}$ and the incoherent axis, and $C_{\max}$ takes the following form:

$$C_{\max}(\rho,t) = |\boldsymbol{r}|\cos\left(\arcsin\left[\frac{|r_z|}{|\boldsymbol{r}|}\right] - 2Et\right). \tag{49}$$

Note that $C_{\max}$ cannot be larger than $|\boldsymbol{r}|$, and this value is attained for the time

$$T_{\mathrm{mc}} = \frac{1}{2E}\arcsin\frac{|r_z|}{|\boldsymbol{r}|}, \tag{50}$$

in which case the final state is in the maximally coherent plane. If the initial state is pure, it can be parameterized as

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle, \tag{51}$$

and the maximal amount of coherence achievable in a given time t takes the form

$$C_{\max}(|\psi\rangle,t) = \cos(\arcsin[\cos\theta] - 2Et). \tag{52}$$

In the next step we will consider systems of arbitrary dimension $d \geqslant 2$ and evaluate the minimal time for converting a pure state $|\psi\rangle$ into a maximally coherent state of the form

$$|+\rangle_d = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{i\phi_j}|j\rangle \tag{53}$$

with phases ϕ_j. The following proposition gives a bound for the evolution time $T(|\psi\rangle \to |+\rangle_d)$.

Proposition 9. *The time for converting a state $|\psi\rangle$ into a maximally coherent state $|+\rangle_d$ via unitary evolution $U = e^{-iHt}$ is bounded as*

$$T(|\psi\rangle \to |+\rangle_d) \geq \frac{1}{dE}\arccos\left[\frac{2}{d}\left(\sum_j |\langle\psi|j\rangle|\right)^2 - 1\right]. \tag{54}$$

We refer to the appendix J for the proof. For $d = 2$ and in the σ_z basis, we have $\frac{2}{d}\left(\sum_j |\langle\psi|j\rangle|\right)^2 - 1 = \sin(\theta)$ and $\cos(\theta) = \frac{r_z}{|\boldsymbol{r}|}$, thus we obtain:

$$T(|\psi\rangle \to |+\rangle_d) = \frac{1}{2E}\arccos\left[\sqrt{1 - \frac{r_z^2}{|\boldsymbol{r}|^2}}\right] = \frac{1}{2E}\arcsin\frac{|r_z|}{|\boldsymbol{r}|} \tag{55}$$

which is the same as T_{mc} in the equation (50).

6. Conclusions and outlook

We have introduced and investigated a novel notion of speed limits for basis change via unitary evolutions, proving general theorems as well as providing bounds on the evolution time which are optimal for several interesting scenarios. Basis change is of importance in quantum computational tasks and mixed states transformations.

For dimensions $d \leqslant 4$ we found the optimal evolution time required to convert the computational basis into an unbiased, i.e. maximally coherent basis. Perhaps surprisingly, the minimal evolution times coincide for $d = 2$ and $d = 4$, when Hamiltonians with the same mean energy E are considered. Moreover, for $d = 3$ the saturation of the speed limit prefers a special ordering of the basis that is unbiased with respect to the computational basis. We also showed that an n-qubit Hadamard gate can be implemented within time $\pi/2E$. This proves that in multi-qubit systems, a maximally coherent basis can be established within a period of time which is independent on the number of qubits. These results further imply that in multi-qubit systems interactive Hamiltonians can significantly reduce the evolution time, compared to the time for establishing an unbiased basis by evolving each qubit independently. We further showed that in the limit $d \to \infty$ the time for establishing an unbiased basis is at least $\pi/4E$. Speed limits for basis permutation are also discussed.

We have also investigates speed limits for generating a certain amount of quantum coherence, as well as minimal time to convert a pure state into a maximally coherent one. We expect that our methods can also be used to derive minimal transformation times for general bases and other quantum resources, such as quantum entanglement and imaginarity [40–42].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

This work was supported by the National Science Centre Poland (Grant No. 2022/46/E/ST2/00115) and the ‘Quantum Coherence and Entanglement for Quantum Technology’ project, carried out within the First Team programme of the Foundation for Polish Science co-financed by the European Union under the European Regional Development Fund. P H acknowledges support by the Foundation for Polish Science (IRAP project, ICTQT, Contract No. 2018/MAB/5, co-financed by EU within Smart Growth Operational Programme). C M and D B acknowledge support by the EU QuantERA Project QuICHE. C M acknowledges support by the PNRR MUR Project PE0000023-NQSTI.

Appendix A. Unbiased bases for qutrits

Up to an overall phase for each basis element, an arbitrary unbiased basis (w.r.t. the computational basis) for a qutrit can be written as

$$|0_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\alpha_{0,1}}|1\rangle + e^{i\alpha_{0,2}}|2\rangle\right), \tag{A.1a}$$

$$|1_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\alpha_{1,1}}|1\rangle + e^{i\alpha_{1,2}}|2\rangle\right), \tag{A.1b}$$

$$|2_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\alpha_{2,1}}|1\rangle + e^{i\alpha_{2,2}}|2\rangle\right), \tag{A.1c}$$

where the phases $\alpha_{i,j}$ need to fulfill the condition

$$1 + e^{i(\alpha_{k,1}-\alpha_{l,1})} + e^{i(\alpha_{k,2}-\alpha_{l,2})} = 3\delta_{k,l}. \tag{A.2}$$

This condition determines the form of the basis to be either

$$|0_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\left(\alpha_{0,1}+\frac{2}{3}\pi\right)}|1\rangle + e^{i\left(\alpha_{0,2}+\frac{4}{3}\pi\right)}|2\rangle\right), \tag{A.3a}$$

$$|1_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\alpha_{0,1}}|1\rangle + e^{i\alpha_{0,2}}|2\rangle\right), \tag{A.3b}$$

$$|2_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\left(\alpha_{0,1}-\frac{2}{3}\pi\right)}|1\rangle + e^{i\left(\alpha_{0,2}-\frac{4}{3}\pi\right)}|2\rangle\right), \tag{A.3c}$$

or

$$|0_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\left(\alpha_{0,1}-\frac{2}{3}\pi\right)}|1\rangle + e^{i\left(\alpha_{0,2}-\frac{4}{3}\pi\right)}|2\rangle\right), \tag{A.4a}$$

$$|1_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\alpha_{0,1}}|1\rangle + e^{i\alpha_{0,2}}|2\rangle\right), \tag{A.4b}$$

$$|2_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\left(\alpha_{0,1}+\frac{2}{3}\pi\right)}|1\rangle + e^{i\left(\alpha_{0,2}+\frac{4}{3}\pi\right)}|2\rangle\right). \tag{A.4c}$$

If we now introduce the unbiased bases

$$|0_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\frac{2}{3}\pi}|1\rangle + e^{i\frac{4}{3}\pi}|2\rangle\right), \tag{A.5a}$$

$$|1_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle + |2\rangle\right), \tag{A.5b}$$

$$|2_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{-i\frac{2}{3}\pi}|1\rangle + e^{-i\frac{4}{3}\pi}|2\rangle\right), \tag{A.5c}$$

and

$$|\tilde{0}_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{-i\frac{2}{3}\pi}|1\rangle + e^{-i\frac{4}{3}\pi}|2\rangle\right), \tag{A.6a}$$

$$|\tilde{1}_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle + |2\rangle\right), \tag{A.6b}$$

$$|\tilde{2}_+\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\frac{2}{3}\pi}|1\rangle + e^{i\frac{4}{3}\pi}|2\rangle\right), \tag{A.6c}$$

we see that any basis of the form (A.3) or (A.4) can be obtained from the basis (A.5) or (A.6), respectively, by using the diagonal unitary $V = |0\rangle\langle 0| + e^{i\alpha_{0,1}}|1\rangle\langle 1| + e^{i\alpha_{0,2}}|2\rangle\langle 2|$.

Appendix B. Speed limits for unitary rotated bases

Let $\{|\psi_j\rangle\}$ and $\{|\phi_j\rangle\}$ be two complete orthonormal bases. A speed limit of the form

$$T\left(|\psi_j\rangle \to |\phi_j\rangle\right) \geqslant \frac{g}{E} \tag{B.1}$$

directly leads to a speed limit for any basis which can be obtained from $\{|\phi_j\rangle\}$ via a unitary $V = \sum_j e^{i\alpha_j}|\psi_j\rangle\langle\psi_j|$:

$$T\left(|\psi_j\rangle \to V|\phi_j\rangle\right) \geqslant \frac{g}{E}. \tag{B.2}$$

The speed limit (B.2) is tight whenever equation (B.1) is tight. To prove this, let H be a Hamiltonian such that

$$e^{-iHt}|\psi_j\rangle = |\phi_j\rangle. \tag{B.3}$$

Then the Hamiltonian $H' = VHV^\dagger$ achieves the transformation

$$e^{-iH't}|\psi_j\rangle = e^{-i\alpha_j}V|\phi_j\rangle, \tag{B.4}$$

which can be seen by using the expression $e^{-iH't} = Ve^{-iHt}V^\dagger$. Noting that H and H' have the same mean energy E, we see that equation (B.1) implies the speed limit (B.2) for any unitary V which is diagonal in the $\{|\psi_j\rangle\}$ basis. Moreover, the speed limit (B.2) is tight for all diagonal unitaries V whenever equation (B.1) is tight.

As we have seen in the appendix A, any unbiased basis of a qutrit can be created from the basis $\{|n_+\rangle\}$ or $\{|\tilde{n}_+\rangle\}$ (see equations (A.5) and (A.6)) via a diagonal unitary V. In combination with the arguments mentioned above, this implies that speed limits for the transitions $\{|n\rangle\} \to \{|n_+\rangle\}$ and $\{|n\rangle\} \to \{|\tilde{n}_+\rangle\}$ will also lead to speed limits for general unbiased qutrit bases $\{|n\rangle\} \to \{V|n_+\rangle\}$ and $\{|n\rangle\} \to \{V|\tilde{n}_+\rangle\}$.

Appendix C. Proof of theorem 3

Before we focus on the case $d=3$ we will discuss the problem for general d. For this, let $U = e^{-iHt}$ be a unitary achieving the transformation $\{|n\rangle\} \to \{|n_+\rangle\}$, where $\{|n_+\rangle\}$ is now a maximally coherent basis of dimension d. Any unitary achieving the desired transformation must be of the form

$$U = \sum_{n=0}^{d-1} e^{i\phi_n} |n_+\rangle\langle n| \tag{C.1}$$

with some phases ϕ_n. We further obtain

$$\mathrm{Tr}\left[U + U^\dagger\right] = \sum_{n=0}^{d-1} \left(e^{i\phi_n}\langle n|n_+\rangle + e^{-i\phi_n}\langle n_+|n\rangle\right). \tag{C.2}$$

Noting that $\langle n|n_+\rangle = e^{i\gamma_n}/\sqrt{d}$ with some phases γ_n we arrive at the inequality

$$-2\sqrt{d} \leqslant \mathrm{Tr}\left[U + U^\dagger\right] \leqslant 2\sqrt{d}. \tag{C.3}$$

On the other hand, recalling that $U = e^{-iHt}$ with a Hamiltonian H we obtain

$$\mathrm{Tr}\left[U + U^\dagger\right] = 2\sum_i \cos(E_i t), \tag{C.4}$$

where E_i are the eigenvalues of the Hamiltonian. In summary, for any unitary transformation $U = e^{-iHt}$ leading to the transformation $\{|n\rangle\} \to \{|n_+\rangle\}$ it must hold that

$$-\sqrt{d} \leqslant \sum_i \cos(E_i t) \leqslant \sqrt{d}. \tag{C.5}$$

We will now consider $d=3$. In this case, we will show that any unitary $U = e^{-iHt}$ leading to the transformation $\{|n\rangle\} \to \{|n_+\rangle\}$ fulfills

$$Et \geqslant \frac{2}{9}\pi. \tag{C.6}$$

Assuming that E_i are in increasing order, we see that $E \geqslant (E_2 - E_0)/3$. Thus, for proving equation (C.6) it is enough to prove that

$$(E_2 - E_0)\, t \geqslant \frac{2}{3}\pi. \tag{C.7}$$

We will prove this by contradiction, assuming that the transformation is possible with a unitary violating equation (C.7). Violation of equation (C.7) implies that

$$(E_1 - E_0)\, t \leqslant \frac{\pi}{3} \quad \text{or} \quad (E_2 - E_1)\, t \leqslant \frac{\pi}{3}. \tag{C.8}$$

In the first case $(E_1 - E_0)t \leqslant \pi/3$, we can set (without loss of generality) $E_0 t = -\pi/6$, which implies the inequalities

$$|E_1 t| \leqslant \frac{\pi}{6}, \quad E_2 t < \frac{\pi}{2}. \tag{C.9}$$

It follows that

$$\sum_i \cos(E_i t) > 2\cos\left(\frac{\pi}{6}\right), \tag{C.10}$$

which is a contradiction to equation (C.5). The remaining case $(E_2 - E_1)t \leqslant \pi/3$ can be treated similarly, by choosing (without loss of generality) $E_2 t = \pi/6$, thus obtaining the following inequalities:

$$|E_1 t| \leqslant \frac{\pi}{6}, \quad E_0 t > -\frac{\pi}{2}. \tag{C.11}$$

Also in this case we obtain the inequality (C.10), in contradiction to equation (C.5). This completes the proof of the bound (C.6). Since the methods presented above apply for any qutrit basis which is unbiased with respect to the computational basis, this completes the proof of Theorem 3.

Appendix D. Proof of proposition 3 of the main text

According to theorem 3, we have the following inequalities for transition into the bases (A.5) and (A.6):

$$T(|n\rangle \to |n_+\rangle) \geqslant \frac{2\pi}{9E}, \tag{D.1a}$$

$$T(|n\rangle \to |\tilde{n}_+\rangle) \geqslant \frac{2\pi}{9E}. \tag{D.1b}$$

As can be checked by inspection, equation (D.1*a*) is saturated for the basis (A.5) by the Hamiltonian $H = |\alpha\rangle\langle\alpha|$ with

$$|\alpha\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{-i\frac{2}{3}\pi}|1\rangle + |2\rangle\right). \tag{D.2}$$

We will now prove that the inequality (D.1*b*) is strict for the basis (A.6), i.e. there is no evolution e^{-iHt} leading to the transformation $|n\rangle \to |\tilde{n}_+\rangle$ within the time $t = 2\pi/(9E)$. Assume—by contradiction—that the bound is saturated for some unitary $U = e^{-iHt}$:

$$|\tilde{n}_+\rangle = e^{-iHt}|n\rangle, \quad t = \frac{2\pi}{9E}. \tag{D.3}$$

Recalling that E_i are in decreasing order and following the arguments from the proof of Theorem 3, it must be that

$$E_1 = E_0, \tag{D.4}$$

$$(E_2 - E_0)\,t = \frac{2}{3}\pi. \tag{D.5}$$

Without loss of generality we can choose

$$E_0 t = E_1 t = -\frac{\pi}{6}, \quad E_2 t = \frac{\pi}{2}. \tag{D.6}$$

Summarizing these arguments, there exists a unitary $U = e^{-iHt}$ fulfilling equation (D.3) and having eigenvalues

$$\lambda_0 = \lambda_1 = e^{i\frac{\pi}{6}}, \quad \lambda_2 = e^{-i\frac{\pi}{2}}, \tag{D.7}$$

which implies that it fulfills

$$\mathrm{Tr}\left[U + U^\dagger\right] = 2\sqrt{3}. \tag{D.8}$$

On the other hand, the unitary also admits the form

$$U = \sum_{n=0}^{2} e^{i\phi_n}|\tilde{n}_+\rangle\langle n|, \tag{D.9}$$

with some phases ϕ_n. We find that

$$\mathrm{Tr}\left[U + U^\dagger\right] = \frac{2}{\sqrt{3}}(\cos\phi_0 + \cos\phi_1) - \frac{1}{\sqrt{3}}\cos\phi_2 + \sin\phi_2. \tag{D.10}$$

Together with equation (D.8) we obtain

$$\frac{2}{\sqrt{3}}\left(\cos\phi_0+\cos\phi_1\right)-\frac{1}{\sqrt{3}}\cos\phi_2+\sin\phi_2=2\sqrt{3}. \tag{D.11}$$

This equation has a unique solution in the range $0\leqslant\phi_i\leqslant 2\pi$, given by

$$\phi_0=\phi_1=0,\ \ \phi_2=\frac{2}{3}\pi. \tag{D.12}$$

This implies that the eigenvalues of U must be

$$\mu_0=\mu_1=e^{-i\frac{\pi}{6}},\ \ \mu_2=e^{i\frac{\pi}{2}}, \tag{D.13}$$

which is a contradiction to equation (D.7). This completes the proof of the proposition.

Appendix E. Proof of theorem 5

We will now focus on the case $d=4$. For this case we will prove the lower bound

$$Et\geqslant\frac{\pi}{4}. \tag{E.1}$$

We will prove this by contradiction, assuming that there exists a unitary $U=e^{-iHt}$ transforming $\{|n\rangle\}$ onto a maximally coherent basis with

$$Et<\frac{\pi}{4}. \tag{E.2}$$

Without loss of generality we can assume that $E_0=0$, which implies $E=(E_1+E_2+E_3)/4$.

We now define $\alpha_i=E_i t$. Note that $\pi>\alpha_i\geqslant 0$. Due to equation (E.2) we have $\alpha_3<\pi-\alpha_1-\alpha_2$, which further implies

$$\cos\left(\alpha_3\right)>\cos\left(\pi-\alpha_1-\alpha_2\right)=-\cos\left(\alpha_1+\alpha_2\right). \tag{E.3}$$

It follows that

$$\cos\left(\alpha_1\right)+\cos\left(\alpha_2\right)+\cos\left(\alpha_3\right)>\cos\left(\alpha_1\right)+\cos\left(\alpha_2\right)-\cos\left(\alpha_1+\alpha_2\right). \tag{E.4}$$

We will now investigate closer the right-hand side of equation (E.4), defining

$$f(\boldsymbol{\alpha})=\cos\left(\alpha_1\right)+\cos\left(\alpha_2\right)-\cos\left(\alpha_1+\alpha_2\right). \tag{E.5}$$

In particular, we will show that $f(\boldsymbol{\alpha})\geqslant 1$ holds true whenever

$$\alpha_i\geqslant 0, \tag{E.6a}$$

$$\alpha_1+\alpha_2\leqslant\pi. \tag{E.6b}$$

For this, we evaluate the partial derivatives of f with respect to α_i:

$$\frac{\partial f}{\partial\alpha_1}=\sin\left(\alpha_1+\alpha_2\right)-\sin\left(\alpha_1\right), \tag{E.7}$$

$$\frac{\partial f}{\partial\alpha_2}=\sin\left(\alpha_1+\alpha_2\right)-\sin\left(\alpha_2\right). \tag{E.8}$$

To find local extrema of f we set $\partial f/\partial\alpha_i=0$, which implies $\sin(\alpha_1)=\sin(\alpha_2)$. This means that $\alpha_1=\alpha_2$, or $\alpha_1=\pi-\alpha_2$. With the condition $\alpha_1=\alpha_2$ we further obtain $\sin(2\alpha_2)=\sin(\alpha_2)$, with the solutions

$$\alpha_1=\alpha_2=0, \tag{E.9a}$$

$$\alpha_1=\alpha_2=\frac{\pi}{3}. \tag{E.9b}$$

On the other hand, the condition $\alpha_1 = \pi - \alpha_2$ together with $\partial f/\partial\alpha_i = 0$ leads to $\sin(\alpha_1) = \sin(\alpha_2) = 0$, with the solutions

$$\alpha_1 = 0, \alpha_2 = \pi, \tag{E.10a}$$

$$\alpha_1 = \pi, \alpha_2 = 0. \tag{E.10b}$$

For proving that $f(\boldsymbol{\alpha}) \geqslant 1$ we evaluate $f(\boldsymbol{\alpha})$ at the extrema (E.9) and (E.10), and also at the boundary of the region defined in equation (E.6). For the solutions (E.9) we obtain $f(\boldsymbol{\alpha}) = 1$ and $f(\boldsymbol{\alpha}) = 3/2$, respectively. Moreover, the solutions (E.10) give $f(\boldsymbol{\alpha}) = 1$.

It remains to show that $f(\boldsymbol{\alpha}) \geqslant 1$ also at the boundary of the region defined in equation (E.6). For a given value of $\alpha_1 \in [0,\pi]$, the boundary is attained for $\alpha_2 = 0$ or $\alpha_2 = \pi - \alpha_1$. As one can verify by inspection, $f(\boldsymbol{\alpha}) = 1$ in both cases. In summary, this proves that $f(\boldsymbol{\alpha}) \geqslant 1$ within the region (E.6).

Collecting the above arguments, equation (E.2) implies that there is a unitary $U = e^{-iHt}$ achieving the transformation $\{|n\rangle\} \to \{|n_+\rangle\}$ with $\sum_i \cos(E_i t) > 2$, in contradiction to equation (C.5). This completes the proof of the lower bound (E.1).

As is explained in the main text, it is indeed possible to achieve the transformation $\{|n\rangle\} \to \{|n_+\rangle\}$ within time $t = \pi/(4E)$. This completes the proof of the theorem.

Appendix F. Comparison of speed limits for transformation to an unbiased basis for $d = 4$ and $d = 5$

Comparison of the results for the minimal time T of transformation of a basis to an unbiased basis for the Hilbert spaces of dimension $d = 2$, $d = 3$ and $d = 4$, may intuitively leads us to the pattern that for $d = 5$ which is the next prime number, the minimal time would decrease. Here, we show that this intuitive pattern is wrong and the minimal time of transformation to an unbiased basis for $d = 5$ must be greater than $\frac{\pi}{4}$ (the minimal time for $d = 2, 4$). To prove this, lets assume that there exist a Hamiltonian H with the eigenenergies $\{E_i\}_{i=1}^5$ and the mean energy E such that

$$T \leqslant \frac{\pi}{4E}. \tag{F.1}$$

Then we must have $\sum_{i=1}^5 E_i T \equiv \sum_{i=1}^5 \alpha_i \leqslant \pi + \pi/4$. Without loss of generality, we assume that the minimum eigenenergy of H is $E_{\min} = 0$. Minimizing the function

$$f(\boldsymbol{\alpha}) = \sum_{j=1}^{5} \cos\alpha_i \tag{F.2}$$

with the respect to the constraint $0 \leqslant \sum_{i=1}^5 \alpha_i \leqslant \pi + \pi/4$ we find $\min f = 1 + 4\cos\frac{5\pi}{16} \approx 3.22$ which is a contradiction because according to the equation (C.5), the maximum of the function $f(\boldsymbol{\alpha})$ for a Hamiltonian transforming a basis to an unbiased one in the Hilbert space of dimension 5 is equal to $\sqrt{5} \approx 2.2 < 3.22$.

Appendix G. Eigenvalues of permutation unitary

In the following we will determine the eigenvalues of the permutation unitary

$$U|n\rangle = |(n+1) \mod d\rangle. \tag{G.1}$$

Let $|\psi\rangle = \sum_n a_n |n\rangle$ be an eigenstate of U, i.e.

$$U|\psi\rangle = e^{i\alpha}|\psi\rangle. \tag{G.2}$$

From equation (G.1) we obtain

$$a_n = e^{i\alpha} a_{(n+1) \mod d}, \tag{G.3}$$

which implies that all coefficients a_n must have the same absolute value: $|a_n|^2 = 1/d$. Thus, any eigenstate $|\psi\rangle$ has the form

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i\phi_j}|j\rangle. \tag{G.4}$$

From this it follows that U cannot be degenerate. To prove this, assume—by contradiction—that there exists two eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ with the same eigenvalue. Then, any superposition of $|\psi_1\rangle$ and $|\psi_2\rangle$ is also an eigenstate of U. Moreover, by superposing $|\psi_1\rangle$ and $|\psi_2\rangle$ we can obtain an eigenstate which is not of the form (G.4), which is the desired contradiction. Consequently, any linear combination of $|\psi_1\rangle$ and $|\psi_2\rangle$ would also be an eigenstate of U. Furthermore, through such a linear combination, we could generate an eigenstate that does not adhere to the form (G.4), leading to the sought contradiction.

In the next step note that any permutation unitary must fulfill

$$U^d = I. \tag{G.5}$$

Together with the fact that U is non-degenerate, the eigenvalues of U must be of the form $\lambda_n = e^{i\frac{2\pi}{d}n}$, where n is an integer in the range $0 \leqslant n \leqslant d-1$.

Appendix H. Speed limits for pure states

Let H be a Hamiltonian of dimension d with eigenvalues E_i and eigenstates $|E_i\rangle$. Without loss of generality, we assume that the eigenvalues are in increasing order, and thus $E_{\max} = E_{d-1}$ and $E_{\min} = E_0$.

Suppose now that an initial state $|\psi\rangle$ evolves for the time $0 \leqslant t \leqslant \pi/E_{\mathrm{gap}}$, where $E_{\mathrm{gap}} = E_{\max} - E_{\min}$ is the energy gap of the Hamiltonian. In the following, we are interested in the minimal overlap between the initial state $|\psi\rangle$ and the time-evolved state $|\psi_t\rangle = e^{-iHt}|\psi\rangle$:

$$F_{\min} = \min_{|\psi\rangle} \left|\langle\psi|e^{-iHt}|\psi\rangle\right|, \tag{H.1}$$

minimized over all initial states $|\psi\rangle$.

Proposition 10. *For a given Hamiltonian H and evolution time $0 \leqslant t \leqslant \pi/E_{\mathrm{gap}}$ it holds that*

$$F_{\min} = \left|\langle\psi_{\min}|e^{-iHt}|\psi_{\min}\rangle\right| = \frac{1}{2}\left|e^{-iE_{\mathrm{gap}}t}+1\right| \tag{H.2}$$

with $|\psi_{\min}\rangle = \frac{1}{\sqrt{2}}(|E_0\rangle + |E_{d-1}\rangle)$.

Proof. Expanding the initial state in the eigenbasis of the Hamiltonian as $|\psi\rangle = \sum_j c_j|E_j\rangle$ with complex coefficients c_j allows us to write the overlap $|\langle\psi|e^{-iHt}|\psi\rangle|$ as follows:

$$\left|\langle\psi|e^{-iHt}|\psi\rangle\right| = \left|\sum_j |c_j|^2 e^{-iE_jt}\right|. \tag{H.3}$$

Noting that the coefficients c_j fulfill the condition $\sum_j |c_j|^2 = 1$, our figure of merit can be expressed as

$$F_{\min} = \min_{|\psi\rangle} \left|\langle\psi|e^{-iHt}|\psi\rangle\right| = \min_{\{p_j\}} \left|\sum_j p_j e^{-iE_jt}\right|, \tag{H.4}$$

where the minimum on the right-hand side is taken over all probability distributions $\{p_j\}$. Recalling that $E_{\mathrm{gap}}t \leqslant \pi$, it is straightforward to see that the minimum is attained for the following choice of $\{p_j\}$:

$$p_j = \begin{cases} \frac{1}{2} & \text{for } j=0 \text{ and } j=d-1, \\ 0 & \text{for } 0<j<d-1. \end{cases} \tag{H.5}$$

It follows that the optimal state $|\psi_{\min}\rangle$, minimizing the overlap $|\langle\psi|e^{-iHt}|\psi\rangle|$, can be chosen as

$$|\psi_{\min}\rangle = \frac{1}{\sqrt{2}}(|E_0\rangle + |E_{d-1}\rangle), \tag{H.6}$$

as claimed. In the last step, it is straightforward to verify that

$$\left|\langle\psi_{\min}|e^{-iHt}|\psi_{\min}\rangle\right| = \frac{1}{2}\left|e^{-iE_{\mathrm{gap}}t}+1\right| \tag{H.7}$$

which completes the proof of the proposition. □

Remarkably, $F_{\min}$ does not depend on the structure of the Hamiltonian, but only on the gap between the largest and the smallest eigenvalue E_{gap}. In the following, we will use this result to bound the evolution time between pure states.

Proposition 11. *The time for converting a pure states $|\psi_0\rangle$ into another state $|\psi_1\rangle$ via unitary evolution $U = e^{-iHt}$ is bounded as*

$$T(|\psi_0\rangle \to |\psi_1\rangle) \geqslant \frac{1}{E_{\text{gap}}} \arccos\left(2|\langle\psi_0|\psi_1\rangle|^2 - 1\right). \tag{H.8}$$

Proof. If the states $|\psi_0\rangle$ and $|\psi_1\rangle$ fulfill $|\psi_1\rangle = e^{-iHt}|\psi_0\rangle$ with $0 \leqslant t \leqslant \pi/E_{\text{gap}}$, then by Proposition 10 it follows that

$$|\langle\psi_0|\psi_1\rangle|^2 \geqslant \frac{1}{4}\left|e^{-iE_{\text{gap}}t} + 1\right|^2. \tag{H.9}$$

This inequality is equivalent to

$$t \geqslant \frac{1}{E_{\text{gap}}} \arccos\left(2\left|\langle\psi_0|\psi_1\rangle\right|^2 - 1\right). \tag{H.10}$$

On the other hand, if $|\psi_0\rangle$ and $|\psi_1\rangle$ fulfill $|\psi_1\rangle = e^{-iHt}|\psi_0\rangle$ with $t > \pi/E_{\text{gap}}$, equation (H.8) is automatically satisfied, since $\arccos(x) \leqslant \pi/2$ for $x \geqslant 0$. This completes the proof. □

Noting that $E_{\text{gap}} \leqslant dE$, where $E = \text{Tr}[H]/d - E_0$ is the average energy of the Hamiltonian, we immediately obtain the following lemma.

Lemma 1. *The time for converting a pure state $|\psi_0\rangle$ into another state $|\psi_1\rangle$ via unitary evolution $U = e^{-iHt}$ is bounded below as*

$$T(|\psi_0\rangle \to |\psi_1\rangle) \geqslant \frac{1}{dE} \arccos\left(2|\langle\psi_0|\psi_1\rangle|^2 - 1\right). \tag{H.11}$$

Moreover, for any two pure states $|\psi_0\rangle$ and $|\psi_1\rangle$ there exists a Hamiltonian H saturating equation (H.11). To see this, recall that equation (H.11) is tight for $d = 2$, see also equation (18). Let now $H = |\phi\rangle\langle\phi|$ be a Hamiltonian which saturates the inequality for $d = 2$. Note that the mean energy in this case is given by $E = 1/2$. This implies that the Hamiltonian achieves the transformation $|\psi_0\rangle \to |\psi_1\rangle$ within the time

$$t = \arccos\left(2|\langle\psi_0|\psi_1\rangle|^2 - 1\right), \tag{H.12}$$

which is the shortest possible time for $E = 1/2$. For $d > 2$ we can use the same Hamiltonian $H = |\phi\rangle\langle\phi|$ to achieve the transformation within the same time as given in equation (H.12). The mean energy is now given by $E = 1/d$, and we see that equation (H.11) is saturated.

Appendix I. Proof of theorem 7

We define $T_{\text{low}} = \frac{d-1}{dE}\frac{\pi}{4}$ and $d \geqslant 2$. Let us assume that $T \leqslant T_{\text{low}}$. Then there must exist a Hamiltonian such that:

$$ET \leqslant \frac{d-1}{d}\frac{\pi}{4}. \tag{I.1}$$

Without loss of generality, we consider $E_0 = 0$ and $E_j \geqslant 0$ for all j. Also we define $\alpha_j = E_j T$, therefore we have:

$$\sum_j \alpha_j \leqslant (d-1)\frac{\pi}{4}. \tag{I.2}$$

By equation (C.5) we must have $-\sqrt{d} \leqslant \sum_j \cos\alpha_j \leqslant \sqrt{d}$. Minimizing the function $f(\boldsymbol{\alpha}) = \sum_j \cos\alpha_j$, we show that $f(\boldsymbol{\alpha})$ is always greater than $\sqrt{d}$ in the region (I.2), hence T cannot be smaller than T_{low}. First, we find the critical points of the function $f(\boldsymbol{\alpha})$ inside the region (not on the boundary). Taking the first derivatives of the function in α_i, we obtain the following equations:

$$\sin\alpha_i = 0\ \forall i. \tag{I.3}$$

This shows that $\alpha_i = K_i\pi$ and $K_i \geqslant 0$. For these values, $\cos\alpha_i$ is either 1 or -1, thus the minimum of the function (among these critical points) occurs when we have maximum number of -1 which with respect to the constraint (I.2), $\lfloor\frac{d-1}{4}\rfloor$ number of α_i must be equal to π and the others be zero. Therefore the minimum is $d-2\lfloor\frac{d-1}{4}\rfloor$ if $\frac{d-1}{4}$ is not an integer. In the case $\frac{d-1}{4}$ is an integer, the point will be on the boundary of the region which we will consider it in the following.

Now, we find the critical points on the boundary of the region (I.2) where we have $\sum_j \alpha_j = (d-1)\frac{\pi}{4}$ and $\alpha_j \geqslant 0$. Generally, we assume that we are on the part of the boundary where x number of the $\{\alpha_i\}_{i=1}^{d-1}$ are zero. Applying the Lagrange multipliers method, we end up with the equations below:

$$\sin\alpha_i = k \ \forall i, \tag{I.4}$$

where k is the Lagrange multiplier. Equation (I.4) show that either $\alpha_i = \lambda + 2K_i\pi$ or $\alpha_i = \pi - \lambda + 2K_i'\pi$ in which $0 \leqslant \lambda \leqslant \frac{\pi}{2}$ and K_i, K_i' are non-negative integers (because $\alpha_i \geqslant 0$). Being on the part of the boundary with x number of α_i to be zero and assuming that N number of them are of the form $\alpha_i = \pi - \lambda + 2K_i'\pi$, we must have (by $\sum_j \alpha_j = (d-1)\frac{\pi}{4}$):

$$(d-x-2N)\lambda + \left(N + \sum_j K_j' + \sum_l K_l\right)\pi = (d-1)\frac{\pi}{4}. \tag{I.5}$$

We define $K \equiv \sum_j K_j' + \sum_l K_l$. If we write λ in terms of K and N we obtain:

$$\lambda = \frac{(d-1)/2 - 2(N+K)}{d-x-2N}\frac{\pi}{2} \tag{I.6}$$

and the function takes the form $x + (d-x-2N)\cos\lambda$. If we are in the domain $N < \frac{d-x}{2}$ then the function takes its minimum when λ is largest and it occurs for $K=0$ (for any x and N) . If we are in the domain $N > \frac{d-x}{2}$ then we have:

$$\lambda = \frac{N-(d-1)/4}{N-(d-x)/2}\frac{\pi}{2} + \frac{K}{N-(d-x)/2}\frac{\pi}{2}. \tag{I.7}$$

Since $x \leqslant \sqrt{d}$ (otherwise $\sum_j \cos\alpha_j > \sqrt{d}$ and the proof would be done), we can easily show that the first term in equation (I.7) is greater than $\pi/2$ as the coefficient $\frac{N-(d-1)/4}{N-(d-x)/2}$ is greater than 1:

$$N - \frac{d-1}{4} \geqslant N - \frac{d-\sqrt{d}}{2} \geqslant N - \frac{d-x}{2} \Longleftrightarrow \left(\sqrt{d}-1\right)^2 \geqslant 0. \tag{I.8}$$

Also, The second term in (I.7) is positive. Thus, in the domain $N > \frac{d-x}{2}$, λ is greater than $\frac{\pi}{2}$ which is a contradiction to the initial assumption $\lambda \leqslant \frac{\pi}{2}$. Furthermore in the case $N = \frac{d-x}{2}$, from the equation (I.7), we get $d+1+2K = 2x$ which is a contradiction as x is a positive integer and $x \leqslant \sqrt{d}$. Therefore, $N < \frac{d-x}{2}$ and λ takes the following form for the minimum of the function:

$$\lambda = \frac{(d-1)/4 - N}{(d-x)/2 - N}\frac{\pi}{2}. \tag{I.9}$$

Moreover, from the equation (I.8), we know that $\frac{d-1}{4} \leqslant \frac{d-x}{2}$ so we must have $0 \leqslant N \leqslant \frac{d-1}{4}$ because $\lambda \geqslant 0$. Now, we should see which value of N in the domain minimizes the function. We should obtain the minimum of the function below while N varies:

$$x + (d-x-2N)\cos\left(\frac{(d-1)/4 - N}{(d-x)/2 - N}\frac{\pi}{2}\right). \tag{I.10}$$

By taking the first derivative of this function in N we can easily see that it is monotonically decreasing in the valid domain of N, hence the value $N_0 = \lfloor (d-1)/4 \rfloor$ achieves the minimum of $f(\boldsymbol{\alpha})$ with the value of $x + (d-x-2\lfloor (d-1)/4 \rfloor)\cos\left(\frac{(d-1)/4 - \lfloor (d-1)/4 \rfloor}{(d-x)/2 - \lfloor (d-1)/4 \rfloor}\frac{\pi}{2}\right)$ which is always greater than $\sqrt{d}$ for $d \geqslant 2$:

$$
\begin{aligned}
\sqrt{d} &\leqslant x\left(1-\cos\left(\frac{(d-1)/4-N_0}{(d-x)/2-N_0}\frac{\pi}{2}\right)\right) \\
&\quad+\frac{d+1}{2}\cos\left(\frac{1}{\frac{d+1}{2}-\frac{\sqrt{d}}{2}}\frac{\pi}{2}\right) \\
&\leqslant x+(d-x-2\lfloor(d-1)/4\rfloor)\cos\left(\frac{(d-1)/4-\lfloor(d-1)/4\rfloor}{(d-x)/2-\lfloor(d-1)/4\rfloor}\frac{\pi}{2}\right)
\end{aligned} \tag{I.11}
$$

where for obtaining the second inequality we used the facts that $1 \leqslant x \leqslant \sqrt{d}$ and $\frac{d-1}{4}-1 \leqslant \lfloor\frac{d-1}{4}\rfloor \leqslant \frac{d-1}{4}$. Thus, the minimum of the function $f(\boldsymbol{\alpha})$ in the region (I.2) is always greater than $\sqrt{d}$ which is a contradiction to equation (C.5), and the proof is complete.

We will now present a lower bound for the speed limit in the Hilbert space of the dimension $d=6$. We will show that the minimal time for transformation of the basis $\{|i\rangle\}_{i=0}^{5}$ to an unbiased basis via a Hamiltonian with fixed mean energy E is bounded below by

$$
\frac{1}{3E}\arccos\left(-\frac{\sqrt{6}-4}{2}\right) \leqslant T. \tag{I.12}
$$

To prove the lower bound, let assume there exist a Hamiltonian for which

$$
T < \frac{1}{3E}\arccos\left(-\frac{\sqrt{6}-4}{2}\right), \tag{I.13}
$$

thus we must have $\sum_{i=0}^{5} E_i T < 2\arccos\left(-\frac{\sqrt{6}-4}{2}\right)$. We define $E_i T = \alpha_i$ and without loss of generality we consider the minimum eigenenergy of the Hamiltonian $E_{min} = E_0 = 0$. By equation (C.5) we must have $-\sqrt{6} \leqslant \sum_j \cos\alpha_j \leqslant \sqrt{6}$. We show that the function $f(\boldsymbol{\alpha}) = \sum_j \cos\alpha_j$ is always greater than $\sqrt{d}$ in the region

$$
R = \left\{\sum_{i=0}^{5}\alpha_i < 2\arccos\left(-\frac{\sqrt{6}-4}{2}\right) \wedge \alpha_i > 0\ \forall i\right\}. \tag{I.14}
$$

Hence, T cannot be smaller than $\frac{1}{3E}\arccos\left(-\frac{\sqrt{6}-4}{2}\right)$.

We minimize the function $f(\boldsymbol{\alpha}) = \sum_{i=0}^{5}\cos\alpha_i$ in the region closure of R. First, we find all the critical points inside the region. By taking the derivatives of the function $f(\boldsymbol{\alpha})$ and equating them to zero, we obtain the critical points as $\alpha_i = K_i\pi$, $K_i \geqslant 0$ and K_i are integers. As $\cos(K_i\pi) = \pm 1$, the minimum of the function among these critical points occurs when we have the maximum number of -1 (with respect to our region R, we are allowed to have only one -1). Thus the minimum among these critical points is 4. Now, we find the minimum on the boundaries $\sum_{i=0}^{5}\alpha_i = 2\arccos\left(-\frac{\sqrt{6}-4}{2}\right)$. Let us assume (without loss of generality) that we are on the part of these boundaries such that x number of α_i are zero. Note that $0 \leqslant x \leqslant 3$ otherwise the function $f(\boldsymbol{\alpha})$ is greater than $\sqrt{6}$ and we are done with the proof according to equation (C.5). Applying Lagrange multiplier method, we obtain the following set of equations:

$$
\sin\alpha_i = k,\ \forall k \tag{I.15}
$$

where k is the multiplier. From these equations we find that α_i must be of the following form:

$$
\alpha_i = \begin{cases} \lambda + 2K_i\pi & \text{or} \\ \pi - \lambda + 2K_i'\pi, \end{cases} \tag{I.16}
$$

in which $0 \leqslant \lambda \leqslant \frac{\pi}{2}$ and K_i and K_i' are non-negative integers (they must be non-negative as α_i are non-negative). We further assume (without loss of generality) that N number of α_i are in the second form of equation (I.16). By the constraint on the border of the closure of R, we have:

$$
(6-x-N)\lambda + N(\pi-\lambda) + 2\left(\sum_i K_i + \sum_j K_j'\right) = 2\arccos -\frac{\sqrt{6}-4}{2}. \tag{I.17}
$$

Solving this equation for λ we obtain:

$$
\lambda = \frac{2\arccos\left(-\frac{\sqrt{6}-4}{2}\right) - (2K+N)\pi}{6-x-2N}. \tag{I.18}
$$

where $K = \sum_i K_i + \sum_j K'_j$. Equation (I.18) implies that $N < \frac{6-x}{2}$ otherwise $\lambda > \pi/2$ which is a contradiction (to the initial assumption that $0 \leqslant \lambda \leqslant \frac{\pi}{2}$). The function $f(\alpha)$ for the critical points on the boundary becomes $(6-x-2N)\cos\left(\frac{2\arccos\left(-\frac{\sqrt{6}-4}{2}\right)-(2K+N)\pi}{6-x-2N}\right)$. Considering that $N < \frac{6-x}{2}$ and $0 \leqslant \lambda \leqslant \frac{\pi}{2}$, it takes its minimum for any x and N when $K = 0$. Thus the minimum of the function on the boundary must be of the form $(6-x-2N)\cos\left(\frac{2\arccos\left(-\frac{\sqrt{6}-4}{2}\right)-N\pi}{6-x-2N}\right)$ which is greater than or equal $\sqrt{6}$ for any $1 \leqslant x \leqslant 3$ and $N < \frac{6-x}{2}$. Therefore, the minimum of the function $f(\alpha)$ over the region R is greater than $\sqrt{6}$ which is a contradiction to equation (C.5) and the proof is complete.

Appendix J. Proof of proposition 9 of the main text

From lemma 1 in the appendix H, it follows that the evolution time into a maximally coherent state is bounded as

$$T(|\psi\rangle \to |+\rangle_d) \geqslant \frac{1}{dE}\arccos\left(2|\langle\psi|+\rangle_d|^2 - 1\right). \tag{J.1}$$

Thus, in order to obtain a bound which is valid for all maximally coherent states, we need to estimate the maximal overlap $|\langle\psi|+\rangle_d|$ over all states of the form:

$$|+\rangle_d = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{i\phi_j}|j\rangle \tag{J.2}$$

Expanding the initial state $|\psi\rangle$ in the incoherent basis $\{|i\rangle\}$ as

$$|\psi\rangle = \sum_{j=0}^{d-1} c_j e^{i\alpha_j}|j\rangle \tag{J.3}$$

with $c_j \geqslant 0$, it is straightforward to see that the overlap $|\langle\psi|+\rangle_d|^2$ is maximized if we set $\phi_j = \alpha_j$, thus arriving at

$$\max_{|+\rangle_d} |\langle\psi|+\rangle_d|^2 = \frac{1}{d}\left(\sum_j |\langle\psi|j\rangle|\right)^2. \tag{J.4}$$

Alternatively, this result can be obtained following [43, 44], noting that $\max_{|+\rangle_d}|\langle\psi|+\rangle_d|^2$ corresponds to the maximal fidelity between the state $\Lambda[|\psi\rangle\langle\psi|]$ and the particular maximally coherent state $|+\rangle_d = \sum_j |j\rangle/\sqrt{d}$, maximized over all incoherent operations Λ. Using equation (J.4) in equation (J.1) completes the proof.

ORCID iDs

Moein Naseri https://orcid.org/0000-0001-9456-4621
Chiara Macchiavello https://orcid.org/0000-0002-2955-8759
Dagmar Bruß https://orcid.org/0000-0003-4661-2267
Paweł Horodecki https://orcid.org/0000-0002-3233-1336
Alexander Streltsov https://orcid.org/0000-0002-7742-5731

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