ADVANCED BIFURCATION ANALYSIS USING MODIFIED STIFFNESS METHOD OF GROUP THEORETIC IMPERFECTIONS

Ichiro Ario*, Dong Ma*, Haicheng Ma*[a)], M. Zawidzki**
*Graduate School of Advanced Science and Engineering, Hiroshima University, Japan
**Institute of Fundamental Technological Research, Polish Academy of Sciences, Poland

Summary Multiple bifurcations[1] due to symmetry are often encountered when analyzing nonlinear motifs of nano-mechanics or structures with axial multiple symmetry. The location of multiple bifurcation points and bifurcation path tracing become problems in numerical analysis. In this paper, as a solution to this problem, the a priori information of group-theoretic bifurcation theory is applied. By utilizing the irreducible representation of the dihedral group, we propose to represent the initial imperfection vectors according to the difference in symmetry of the structural system. A part of the tangent stiffness matrix is corrected by coordinate transformations, and the modified stiffness method is proposed to separate the multiple bifurcation points to a single point in the direction of the bifurcation path. As a numerical analysis example, a bifurcation analysis of a fullerene structure is performed to demonstrate the feasibility of the presented method.

BACKGROUND AND PURPOSE

In bifurcation analysis of nonlinear post-buckling behaviors in axisymmetric structures, multiple bifurcations are frequently observed attributed to the symmetry inherent in these structures. A significant challenge in numerical analysis is accurate determination of the bifurcation point locations and effectively tracing the bifurcation paths from these points. Typically, the bifurcation path direction is inferred from eigenvectors corresponding to zero eigenvalues. However, accurate estimation of the branching direction becomes complex in cases where multiple bifurcation points have a high degree of multiplicity[1]. Additionally, leveraging the mathematical principles of symmetry, the bifurcation patterns can be represented by the irreducible representations of group theory application[2].

This research incorporates the modified stiffness method, wherein the adjustment to the tangent stiffness matrix is guided by a group-theoretic imperfection vector. This vector, integrated with the irreducible representations of dihedral group, corresponds to bifurcation modes with varying symmetries. Essentially, this method synergizes the a priori group-theoretic principles outlined in [4] with the modified stiffness approach.

THE SEARCH OF THE SINGULAR POINTS

We denote the total potential energy function of a discretized system as $\mathcal{U}(\boldsymbol{f}, \boldsymbol{u}, \boldsymbol{\epsilon})$. $\boldsymbol{f} \in \mathbf{R}^N$ is the the load pattern vector, $\boldsymbol{u} \in \mathbf{R}^N$ is the displacement vector, and $\boldsymbol{\epsilon} \in \mathbf{R}^N$ is the imperfection vector. The (N-dimensional) equilibrium equation of the perfect system is expressed as follows:

$$\left(\frac{\partial \mathcal{U}}{\partial \boldsymbol{u}}\right)^{\mathrm{T}} \equiv \boldsymbol{F}(\boldsymbol{u}, f, \boldsymbol{\epsilon}) = \mathbf{0} \tag{1}$$

where, $\boldsymbol{F} \in \mathbf{R}^N$ is a vector function, $\boldsymbol{u}$ is the displacement vector, f is the load parameter and $\boldsymbol{\epsilon}$ is the imperfection vector. For $\boldsymbol{\epsilon} = \mathbf{0}$, this system has no imperfections. The solutions$(\boldsymbol{u}, f, \boldsymbol{\epsilon})$ that satisfy this equilibrium equation form a curve. The Jacobian matrix is obtained by partially differentiating the equilibrium equation by the displacement $\boldsymbol{u}$, i.e. $J = J(\boldsymbol{u}, f, \boldsymbol{\epsilon}) \equiv \partial \boldsymbol{F}/\partial \boldsymbol{u}$.

MODIFIED STIFFNESS METHOD WITH INITIAL IMPERFECTIONS HAVING SYMMETRIES

Let $\boldsymbol{\epsilon}_i \in \mathbf{R}^N$ be any initial imperfection displacement, and let $\boldsymbol{u} \in \mathbf{R}^N$ be the displacement vector of the complete system. We define the displacement associated with $\boldsymbol{\epsilon}_i$, and the nonlinear equilibrium equation of the system (in N dimensions) is given as:

$$\boldsymbol{u}' = \boldsymbol{u} + \boldsymbol{\epsilon}_i, \quad \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{\epsilon}, f) \simeq \boldsymbol{F}(\boldsymbol{u} + \boldsymbol{\epsilon}_i, f) = \mathbf{0}, \quad {}^{\exists}\boldsymbol{\epsilon}_i \in \boldsymbol{\epsilon} \tag{2}$$

where f is the load parameter. To investigate the properties of an equilibrium point $(\boldsymbol{u}', f)$, we locally linearize Eq.(2) in its neighborhood. The incremental equilibrium equation is given as follows:

$$\begin{aligned} \widetilde{\boldsymbol{F}}(\widetilde{\boldsymbol{u}'}, \widetilde{f}) &\equiv \boldsymbol{F}(\boldsymbol{u}' + \widetilde{\boldsymbol{u}'}, f + \widetilde{f}) - \boldsymbol{F}(\boldsymbol{u}', f) \\ &\simeq (J + J(\boldsymbol{\epsilon}_i))\widetilde{\boldsymbol{u}} + \frac{\partial \boldsymbol{F}}{\partial f}\widetilde{f} = \mathbf{0} \end{aligned} \tag{3}$$

[a)] Corresponding author. E-mail: madonglucas@gmail.com

Here, $\widetilde{\boldsymbol{u}'} = (\widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{\epsilon}_i})$ is the incremental displacement. The difference of tangent stiffness matrix due to the initial imperfection is:

$$\Delta\tilde{J} \equiv J(\boldsymbol{u}') - J(\boldsymbol{u}) = J(\boldsymbol{\epsilon}_i) \tag{4}$$

We consider the tangent stiffness matrix $J(\boldsymbol{\epsilon}_i)$ of the initial imperfection vector $\boldsymbol{\epsilon}_i$ with respect to the tangent stiffness matrix $J = J(\boldsymbol{u})$:

$$J(\boldsymbol{\epsilon}_i) = T(\boldsymbol{\epsilon}_i)^{\mathrm{T}}\, J\, T(\boldsymbol{\epsilon}_i) \tag{5}$$

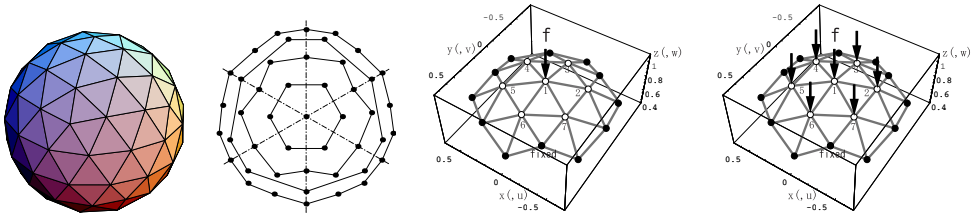
A this point, we use a transformation method based on a matrix consisting of the initial imperfection vectors $\boldsymbol{\epsilon}_i \in \boldsymbol{\epsilon}$. We also utilize a transformation method based on a matrix comprising initial imperfection vectors, denoted as ϵ_i, within the set $\boldsymbol{\epsilon}$.

In this paper, we align the initial imperfection vector with an orthogonal matrix that reflects the symmetry of the structure [4]. This alignment is then associated with a standard measure of initial imperfection size, $\bar{\epsilon}_0$. We apply the modified tangent stiffness matrix, defined in Eq.(4), to explore various bifurcation path-following strategies. These strategies are based on perturbations in the initial imperfections, corresponding to subgroups of the symmetry group of the symmetric structure.

EXAMPLES OF NUMERICAL ANALYSIS

This research investigates the post-buckling behavior of nonlinear finite element method (FEM) for the cap part of a fullerene structure. The analysis considers two types of loading patterns: single loads and distributed loads. In the analytical model, the member stiffness, denoted as EA, is assumed uniform for all members.

The block diagonalization method (BDM) is used to separate complex eigenvalues into different blocks, leveraging the symmetry information of the system. This enables the identification of individual bifurcation mechanisms in the axisymmetric system. The analysis focuses solely on vectors in the z-direction. This paper particularly emphasizes the vertical displacement of nodes in the pentagonal structure of a fullerene. By substituting imperfection vectors, the vertical coordinates of each node are accordingly modified.



(a) Fullerene (b) Orbits (c) Single loading (d) Distributed loading

Figure 1. Multiple bifurcation analysis for family of fullerene structures

CONCLUSIONS

This research demonstrates that by integrating group-theoretic imperfections with the modified stiffness method for eigenvalue separation, it is possible to effectively distinguish the multiple bifurcation points along the decomposing bifurcation path of a single bifurcation point. The analytical results affirm the suitability of this method for addressing the proposed singularity challenges.

In this theoretical approarch, the fullerene structure is discretized for analysis. The mechanical behavior of these structures is explored through three-dimensional nonlinear finite displacement analysis. By categorizing different symmetries, fullerenes with various configurations are reduced to distinct groups characterized by initial defects. Through static analysis, numerous singular points and concealed bifurcation paths are uncovered in nano-tube and/or fullerene structures exhibiting high symmetry.

References

[1] I.Ario, and M. NAKAZAWA.: Analysis of multiple bifurcation behaviour for periodic structures, *ARCHIVES OF MECH.*, 72 (4), 283–306, 2020.

[2] C. Yao, S. Qiuzhi , F. Jian.: Group-theoretical form-finding of cable-strut structures based on irreducible representations for rigid-body translations, *Int. J. of Mechanical Sciences*, 144(2018), 205-215.

[3] Ikeda, K. and Murota, K.: Imperfect bifurcation in structures and materials, *(Applied mathematical sciences, v.149) Springer*, 2019.

[4] I. Ario and M. Zawidzki: Application of Group Theory to Symmetric Structures, CRC Press, Taylor & Francis Group, 2024.