This article has been accepted for publication in IEEE Transactions on Information Theory. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TIT.2025.3526419

Multipartite entanglement theory with entanglement-nonincreasing operations

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A key problem in quantum information science is to determine optimal protocols for the interconversion of entangled states shared between remote parties. While for two parties a large number of results in this direction is available, the multipartite setting still remains a major challenge. In this article, this problem is addressed by extending the resource theory of entanglement for multipartite systems beyond the standard framework of local operations and classical communication. Specifically, we consider transformations capable of introducing a small, controllable increase of entanglement of a state, with the requirement that the increase can be made arbitrarily small. We demonstrate that in this adjusted framework, the transformation rates between multipartite states are fundamentally dictated by the bipartite entanglement entropies of the respective quantum states. Remarkably, this approach allows the reduction of tripartite entanglement to its bipartite analog, indicating that every pure tripartite state can be reversibly synthesized from a suitable number of singlets distributed between pairs of parties.

I. INTRODUCTION

Quantum entanglement, one of the most intriguing and fundamental phenomena in quantum mechanics, has been extensively studied for its potential to revolutionize our understanding of the physical world and to bring forth a new era of technological advancements. When two distant parties share an entangled quantum state, they acquire the ability to execute certain tasks that would be unattainable without this type of correlations [1]. This unique feature of quantum entanglement dramatically expands the operational capabilities of remote parties, enabling phenomena such as quantum teleportation [2] and quantum key distribution [3] that fundamentally transcend the limitations of classical physics.

While the capabilities and limitations associated with entanglement in a two-party setup have been extensively studied [1], the emergence of large-scale quantum networks necessitates an understanding of states entangled across multiple parties. The significance of such multipartite entanglement is underscored by its role in various quantum protocols such as multipartite remote state preparation [4] and quantum secret sharing [5, 6]. In the latter protocol, a confidential message is disseminated among several parties in a manner that requires their collective cooperation for the retrieval of the message. To fully leverage the extensive potential of multipartite entangled states, it is essential to gain a comprehensive understanding of how these states can be manipulated.

Local operations and classical communication (LOCC) present a fundamental operational framework in the exploration of quantum entanglement [1, 7, 8]. At its core, LOCC involves two distinct classes of actions. The first class, local operations, pertains to actions performed independently on each subsystem of an entangled state. These actions encompass unitary transformations, measurements, and the incorporation of ancillary systems, which can be performed locally by the distant parties. The second class, classical communication, enables the remote parties to distribute the results of their local operations through classical channels. In essence, LOCC can be considered the most comprehensive class of protocols that can be executed by remote parties without the need for exchanging quantum particles. Consequently, LOCC protocols cannot create entanglement, thereby rendering any entangled state a valuable resource within this framework.

In the bipartite setting, our comprehension of the potential and limitations inherent in LOCC is notably advanced, especially regarding transformations involving pure states. Given any pair of pure states $|\psi\rangle$ and $|\phi\rangle$, shared between Alice and Bob, it is possible to verify whether the transformation $|\psi\rangle \rightarrow |\phi\rangle$ is feasible under LOCC [9]. Furthermore, in the asymptotic regime — where many copies of $|\psi\rangle$ are at our disposal — we have precise knowledge of the transformation rates, which are intricately tied to the entanglement entropies of the involved quantum states [10].

In contrast to the well-established findings in bipartite systems, the multipartite setting presents a substantially more intricate landscape. Even when considering pure states of a few qubits, the current understanding is characterized by isolated results [11–19]. Given two arbitrary four-qubit states $|\psi\rangle^{ABCD}$ and $|\phi\rangle^{ABCD}$ our current knowledge neither permits us to verify conclusively whether $|\psi\rangle$ can be transformed into $|\phi\rangle$ via LOCC with unit probability, nor allows us to determine the optimal asymptotic transformation rate for such a conversion. Needless to say, this situation presents a compelling challenge.

Nevertheless, recent research in the field has revealed a promising strategy: some problems in quantum information science can be effectively addressed by carefully relaxing conventional constraints. Notable instances include entanglement catalysis [20], where a meaningful relaxation of the standard restrictions has been shown to considerably simplify the problem under

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This article has been accepted for publication in IEEE Transactions on Information Theory. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TIT.2025.3526419

investigation [21–26]. Another intriguing example explores the overlap between theories of entanglement and thermodynamics, with a particular focus on probing the potential existence of a principle in entanglement theory that parallels the second law of thermodynamics. This investigation essentially hinges on the query of reversibility in transformations between entangled states. In other words, it questions the feasibility of performing lossless transformations between any given entangled state ρ and another entangled state σ when considered in an asymptotic context. In the bipartite setting, the resource theory of entanglement is inherently asymptotically irreversible under LOCC [27]. This irreversibility persists even when the LOCC framework is relaxed, provided that the modified set of operations still remains incapable of generating entanglement [28]. However, there are indications that reversibility can be established when considering broader classes of operations, e.g., those capable of generating small, controllable amounts of entanglement [29–32], or involving ancillary particles acting as an entanglement battery [33].

In this article, we propose a relaxation of the LOCC framework for the multipartite setting. Specifically, we explore transformations that can increase the relative entropy of entanglement of any state by a small amount ε, with the requirement that ε can be made arbitrarily small. This relaxation of the LOCC framework uncovers an interesting pattern; the transformation rates between multipartite states are intrinsically governed by the bipartite entanglement entropies of the quantum states involved. Implications of these findings are also discussed.

II. PRELIMINARIES

Here, we introduce the notation and definitions used throughout this article. In general, a state of a quantum system is described by a density matrix ρ, i.e., a positive semi-definite matrix acting on a Hilbert space $\mathcal{H}$. An important quantity in quantum information theory is the von Neumann entropy, which is defined as

$$S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho). \tag{1}$$

For two quantum states ρ and σ on the same Hilbert space, the distance between the states can be quantified as $\|\rho - \sigma\|_1$ with the trace norm

$$\|M\|_1 = \mathrm{Tr}\sqrt{M^\dagger M}. \tag{2}$$

It holds that $\|\rho - \sigma\|_1 \geq 0$ with equality if and only if $\rho = \sigma$. Other useful quantities in this context are the fidelity [34, 35]

$$F(\rho,\sigma) = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2, \tag{3}$$

the Bures distance

$$D_B(\rho,\sigma) = \sqrt{2 - 2\sqrt{F(\rho,\sigma)}}, \tag{4}$$

and the quantum relative entropy [36]

$$S(\rho\|\sigma) = \mathrm{Tr}(\rho \log_2 \rho) - \mathrm{Tr}(\rho \log_2 \sigma). \tag{5}$$

It holds that $D_B(\rho,\sigma) \geq 0$ and $S(\rho\|\sigma) \geq 0$, with equality in both cases if and only if $\rho = \sigma$. Moreover, the Bures distance fulfills the triangle inequality, i.e., for any three quantum states ρ, σ, and τ it holds that

$$D_B(\rho,\tau) \leq D_B(\rho,\sigma) + D_B(\sigma,\tau). \tag{6}$$

For two parties, Alice and Bob, the joint Hilbert space is defined as a tensor product of the individual Hilbert spaces: $\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B$. For a quantum state ρ^{AB} on the total Hilbert space $\mathcal{H}^{AB}$, the reduced state on Alice's part is defined as $\rho^A = \mathrm{Tr}_B[\rho^{AB}]$, where Tr_B denotes the partial trace. A quantum state ρ^{AB} is called separable [1, 37] if it can be written as

$$\rho^{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B$$

with probabilities p_i and quantum states ρ_i^A and ρ_i^B on $\mathcal{H}^A$ and $\mathcal{H}^B$, respectively. Any state which is not separable is called entangled [1, 37]. This definition can be extended to more than 2 parties in a straightforward way. A tripartite state ρ^{ABC} is called fully separable if it can be written as $\rho^{ABC} = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C$, and similarly for any number of parties n.

There are many possible ways to quantify the amount of entanglement in a quantum state [1]. In this article we will use the generalized robustness of entanglement R_g and the relative entropy of entanglement E_r defined as [38–41]

$$R_\mathrm{g}(\rho) = \min_\sigma \left\{ s \geq 0 : \frac{\rho + s\sigma}{1+s} \in \mathcal{S} \right\}, \tag{7}$$

$$E_\mathrm{r}(\rho) = \min_{\sigma_s \in \mathcal{S}} S(\rho\|\sigma_s), \tag{8}$$

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where $\mathcal{S}$ is the set of bipartite separable states. A closely related entanglement quantifier which will also be used in the following is the regularized relative entropy of entanglement

$$E_\infty(\rho) = \lim_{n\to\infty} \frac{1}{n} E_{\mathrm{r}}(\rho^{\otimes n}). \tag{9}$$

A quantum operation describes the most general transformation that a quantum system can undergo. Quantum operations correspond to completely positive trace preserving maps $\Lambda[\rho] = \sum_i K_i \rho K_i^\dagger$ with Kraus operators K_i having the property $\sum_i K_i^\dagger K_i = \mathbb{1}$. An important class of operations in the bipartite setting is known as local operations and classical communication (LOCC) [7, 8]. As mentioned in the introduction, these are transformations which can be implemented by local actions on each of the systems and a classical communication channel. Any entanglement E measure does not increase under LOCC [1, 42], i.e.,

$$E(\Lambda[\rho]) \leq E(\rho) \tag{10}$$

for any LOCC protocol Λ. This also applies for E_{g} and E_{r} defined above. Another important property of E_{r} which we will use in this article is its behavior on states of the form $\sum_i p_i \rho_i^{AB} \otimes |i\rangle\langle i|^{A'}$, where the particle A' is in Alice's lab. In particular, it holds [43]

$$E_{\mathrm{r}}\left(\sum_i p_i \rho_i^{AB} \otimes |i\rangle\langle i|^{A'}\right) = \sum_i p_i E_{\mathrm{r}}\left(\rho_i^{AB}\right). \tag{11}$$

III. ASYMPTOTICALLY ENTANGLEMENT-NONINCREASING OPERATIONS

A key question since the early days of entanglement theory is to describe all state transformations possible within the LOCC setting. In the asymptotic setup, a transformation $\rho \to \sigma$ can be achieved with rate r if for any $\varepsilon > 0$ there exists some n and an LOCC protocol Λ_{LOCC} such that

$$\left\| \Lambda_{\mathrm{LOCC}}\left[\rho^{\otimes n}\right] - \sigma^{\otimes \lfloor rn \rfloor} \right\|_1 < \varepsilon. \tag{12}$$

The supremum of such feasible rates r is known as the transformation rate $R(\rho \to \sigma)$.

As previously stated, for pure bipartite states $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$, the transformation rate is closely related to the entanglement entropies of the quantum states involved. Specifically, the transformation rate from $|\psi\rangle^{AB}$ to $|\phi\rangle^{AB}$ is given by the ratio of their entanglement entropies: $R(\psi^{AB} \to \phi^{AB}) = E(\psi^{AB})/E(\phi^{AB})$ [10]. Here, the entanglement entropy E is defined as the von Neumann entropy of the reduced state, i.e.,

$$E(\psi^{AB}) = S(\psi^A) = -\mathrm{Tr}(\psi^A \log_2 \psi^A). \tag{13}$$

When extending this scenario to multipartite pure states, it becomes evident that the bipartite entanglement entropies serve as upper bounds for the transformation rates via multipartite LOCC [16]:

$$R(\psi \to \phi) \leq \min_T \frac{E^{T|\overline{T}}(\psi)}{E^{T|\overline{T}}(\phi)} = \min_T \frac{S(\psi^T)}{S(\phi^T)}, \tag{14}$$

where the minimum is taken over all subsets T of all the parties, and $\overline{T}$ is the complement of T. As we will see in the following, this inequality can be turned into an equality, if the set of LOCC operations is extended accordingly.

To extend the set of LOCC operations, we will consider transformations that allow the injection of a small, controllable amount of entanglement into the system, similar to the approach followed in [29–31]. While these transformations may be more challenging to implement compared to LOCC, they provide valuable insights into the limits of entanglement manipulation. A guiding principle in our framework is to preserve the resource-like nature of entanglement. To achieve this, our extension ensures that the asymptotic transformation rates between any two entangled states, ρ and σ, remain bounded. In the multipartite case, this implies that every multipartite entangled state allows for a finite rate of singlet extraction between any pair of parties. This suggests that our extension of the LOCC paradigm yields a theory where singlets maintain their fundamental role as valuable resource states.

We say that a quantum operation Λ on a bipartite Hilbert space $\mathcal{H}^{AB}$ is *ε-entanglement-nonincreasing* if

$$E_{\mathrm{r}}(\Lambda[\rho]) - E_{\mathrm{r}}(\rho) \leq \varepsilon \tag{15}$$

for all states ρ acting on $\mathcal{H}^{AB}$, see also Fig. 1. A sequence of operations $\{\Lambda_n\}$ is called *asymptotically entanglement-nonincreasing (AEN)* if for any $\varepsilon > 0$ the operations Λ_n are ε-entanglement-nonincreasing for all n large enough. We further say that ρ^{AB} can

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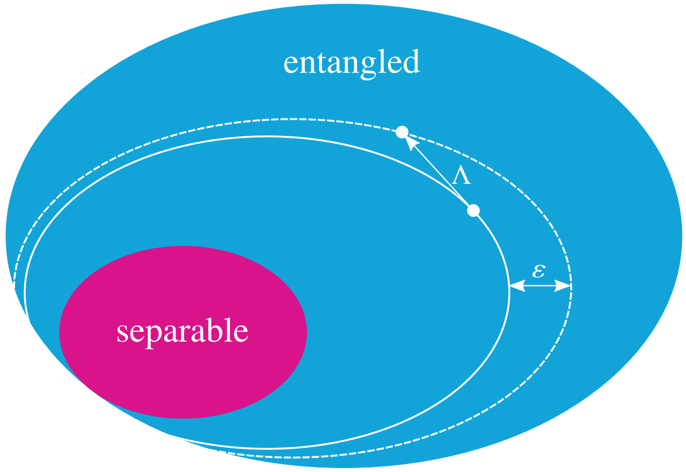

Figure 1. An operation Λ is called ε-entanglement-nonincreasing if it can increase the relative entropy of entanglement of any state by at most ε. The figure shows states with a constant relative entropy of entanglement $E_{\mathrm{r}} = c$ (solid curve), and states with $E_{\mathrm{r}} = c + \varepsilon$ (dashed curve). An ε-entanglement-nonincreasing operation cannot transform states inside the solid boundary outside of the dashed boundary.

be converted into σ^{AB} with rate r via AEN operations if for any $\delta > 0$ there exists a monotonic infinite integer sequence $\{k_n\}$ and an AEN sequence $\{\Lambda_k\}$ such that

$$\frac{1}{2}\left\|\Lambda_{k_n}\left[\rho^{\otimes k_n}\right] - \sigma^{\otimes \lfloor rk_n \rfloor}\right\|_1 < \delta \tag{16}$$

holds true for all n large enough. Here, Λ_{k_n} is a quantum operation with the input Hilbert space being k_n copies of $\mathcal{H}^{AB}$, and the output Hilbert space is $\lfloor rk_n \rfloor$ copies of $\mathcal{H}^{AB}$. The supremum of all rates achievable in this setup will be denoted by $R_{\mathrm{AEN}}(\rho \to \sigma)$. It is clear that

$$R_{\mathrm{AEN}}(\rho \to \sigma) \geq R(\rho \to \sigma). \tag{17}$$

As we show in the following proposition, R_{AEN} can be upper bounded in terms of the regularized relative entropy of entanglement.

Proposition 1. *For any two bipartite states ρ and σ the conversion rate via asymptotically entanglement-nonincreasing operations is bounded as*

$$R_{\mathrm{AEN}}(\rho \to \sigma) \leq \frac{E_\infty(\rho)}{E_\infty(\sigma)}. \tag{18}$$

Proof. Let r be a feasible rate for the conversion $\rho \to \sigma$, i.e., for any $\delta > 0$ there exists an integer sequence $\{k_n\}$ and an AEN sequence $\{\Lambda_k\}$ such that Eq. (16) holds for all n large enough. Using asymptotic continuity of the relative entropy of entanglement [44, 45] we obtain

$$E_{\mathrm{r}}\left(\sigma^{\otimes \lfloor rk_n \rfloor}\right) - E_{\mathrm{r}}\left(\Lambda_{k_n}\left[\rho^{\otimes k_n}\right]\right) \leq \delta \log_2 d^{rk_n} + (1+\delta) h\left(\frac{\delta}{1+\delta}\right), \tag{19}$$

where $d = d_A = d_B$ denotes the local dimension of $\rho = \rho^{AB}$ and $\sigma = \sigma^{AB}$. Using the fact that the sequence $\{\Lambda_k\}$ is AEN we further have

$$E_{\mathrm{r}}\left(\sigma^{\otimes \lfloor rk_n \rfloor}\right) \leq E_{\mathrm{r}}\left(\rho^{\otimes k_n}\right) + \varepsilon + \delta \log_2 d^{rk_n} + (1+\delta) h\left(\frac{\delta}{1+\delta}\right) \tag{20}$$

for all n large enough. Dividing both sides of the inequality by k_n and taking the limit $n \to \infty$ we find

$$rE_\infty(\sigma) \leq E_\infty(\rho) + \delta r \log_2 d. \tag{21}$$

Since we can choose arbitrary $\delta > 0$, it follows that $r \leq E_\infty(\rho) / E_\infty(\sigma)$ and the proof is complete. □

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An important implication of this result is that the AEN setting ensures a bounded singlet distillation rate, as $R_{\mathrm{AEN}}(\rho \to \psi^-) \leq E_\infty(\rho)$ for the singlet state $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$.

It is useful to compare the class of operations discussed above with those introduced in [29], where the authors examined ε-non-entangling operations in the bipartite setting. These are operations Λ that can transform a separable state ρ_s into an entangled state such that $R_{\mathrm{g}}(\Lambda[\rho_s]) \leq \varepsilon$, where R_{g} denotes the generalized robustness of entanglement which has been defined in Section II. The authors of [29] also defined a sequence of operations $\{\Lambda_n\}$ to be asymptotically non-entangling if each Λ_n is ε_n-non-entangling and $\lim_{n\to\infty} \varepsilon_n = 0$. We observe that any ε-non-entangling operation is also ε-entanglement-nonincreasing. Furthermore, a sequence of operations that is asymptotically non-entangling is also AEN. However, it remains an open question whether these two notions coincide, i.e., whether every AEN sequence is also asymptotically non-entangling.

IV. AEN IN MULTIPARTITE SYSTEMS

For multipartite systems, we will consider operations which are ε-entanglement-nonincreasing in any bipartition. As an example, for three parties, A, B, and C, we require that Eq. (15) is fulfilled for the bipartite relative entropy of entanglement in all cuts $A|BC$, $B|AC$, and $C|AB$. This definition is then extended to more than 3 parties in a straightforward way, i.e., it is required that

$$E_{\mathrm{r}}^{T|\overline{T}}(\Lambda[\rho]) - E_{\mathrm{r}}^{T|\overline{T}}(\rho) \leq \varepsilon \tag{22}$$

holds true for any bipartition $T|\overline{T}$ of the total multipartite system. Correspondingly, a sequence of multipartite operations $\{\Lambda_n\}$ will be called asymptotically entanglement-nonincreasing, if it is AEN in any bipartition.

Asymptotic conversion rates via multipartite AEN are then defined analogously to the bipartite setting. From Eq. (18) it is clear that the optimal rate is not larger than

$$R_{\mathrm{AEN}}(\rho \to \sigma) \leq \min_T \frac{E_\infty^{T|\overline{T}}(\rho)}{E_\infty^{T|\overline{T}}(\sigma)}. \tag{23}$$

This means that multipartite AEN operations lead to bounded distillation rates, when it comes to distilling singlets between any pair of parties. Since there exist multipartite entangled states which are separable in all bipartitions [46], AEN operations can in principle create large amounts of multipartite entanglement. Nevertheless, Eq. (23) ensures that the theory obtained in this way does not trivialize, preserving the valuable role of entanglement as a resource.

Equipped with these tools, we are ready to present the main result of this article.

Theorem 2. *For any two multipartite pure states $|\psi\rangle$ and $|\phi\rangle$ the optimal conversion rate via asymptotically entanglement-nonincreasing operations is given by*

$$R_{\mathrm{AEN}}(\psi \to \phi) = \min_T \frac{E^{T|\overline{T}}(\psi)}{E^{T|\overline{T}}(\phi)} = \min_T \frac{S(\psi^T)}{S(\phi^T)}. \tag{24}$$

Proof. From Eq. (23) we see that the maximal rate cannot exceed $\min_T S(\psi^T)/S(\phi^T)$. We will now present a sequence of AEN operations which achieves conversion at this rate. Consider the sequence

$$\Lambda_n^{\psi,\phi}[\rho] = \mathrm{Tr}\left[|\psi\rangle\langle\psi|^{\otimes n}\rho\right]|\phi\rangle\langle\phi|^{\otimes\lfloor rn\rfloor} \otimes |0\rangle\langle 0|^K + \mathrm{Tr}\left[\left(\mathbb{1} - |\psi\rangle\langle\psi|^{\otimes n}\right)\rho\right]\mu_s \otimes |1\rangle\langle 1|^K, \tag{25}$$

with some fully separable state μ_s, and the register K is in possession of Alice. When applied onto the state $\rho = \psi^{\otimes n}$, we obtain

$$\Lambda_n^{\psi,\phi}\left[\psi^{\otimes n}\right] = \phi^{\otimes\lfloor rn\rfloor} \otimes |0\rangle\langle 0|^K. \tag{26}$$

Tracing out the register K, we see that $|\psi\rangle$ can be converted into $|\phi\rangle$ with rate r in this setting. We show in Proposition 8 that in the bipartite setting the sequence $\Lambda_n^{\psi,\phi}$ is AEN whenever $r < S(\psi^A)/S(\phi^A)$. This means that in the multipartite setting, the sequence $\Lambda_n^{\psi,\phi}$ is AEN in the bipartition $T|\overline{T}$ whenever $r < S(\psi^T)/S(\phi^T)$. Thus, for

$$r < \min_T \frac{S(\psi^T)}{S(\phi^T)} \tag{27}$$

the sequence $\Lambda_n^{\psi,\phi}$ is AEN in any bipartition. In summary, this shows that it is possible to convert $|\psi\rangle$ into $|\phi\rangle$ via AEN operations at any rate r which fulfills Eq. (27). □

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The above theorem leads to several implications, which we will discuss in the following. Within the LOCC framework, an asymptotic transformation between two states, ρ and σ, is dubbed reversible if the transformation rates satisfy the condition

$$R(\rho \to \sigma) = R(\sigma \to \rho)^{-1}. \tag{28}$$

As mentioned earlier, a salient characteristic of entanglement theory is the general irreversibility of asymptotic transformations via LOCC [27], implying the existence of states that violate Eq. (28). Though in the bipartite context, asymptotic LOCC transformations between pure entangled states are reversible [10], this ceases to hold when extended to scenarios involving more than two parties, even for pure states [12].

In the AEN setting, we say that a reversible transformation between ρ and σ is possible if $R_{\mathrm{AEN}}(\rho \to \sigma) = R_{\mathrm{AEN}}(\sigma \to \rho)^{-1}$. From Theorem 2 we immediately see that a reversible transformation between multipartite pure states is possible if and only if

$$\frac{S(\psi^T)}{S(\phi^T)} = \frac{S(\psi^{T'})}{S(\phi^{T'})} \tag{29}$$

for any two subsets of all the parties T and T'. This directly implies that asymptotic transformations between multipartite pure states are irreversible in general even under AEN operations.

On the other hand, reversibility can be established for some important classes of states. In particular, our framework enables reversible transformations between GHZ and W states, i.e., 3-qubit states of the form

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \tag{30}$$

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{31}$$

By symmetry of the states, it is straightforward to see that Eq. (29) is fulfilled in this case. For the transformation rate we obtain

$$R_{\mathrm{AEN}}(|W\rangle \to |\mathrm{GHZ}\rangle) = h(1/3) \approx 0.92 \tag{32}$$

with the binary entropy $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$. Note that a reversible conversion between these states is not possible via LOCC, and even in an extended setting allowing for all operations which preserve the positivity of the partial transpose in any bipartition [13].

Another interesting case when the AEN setting allows for reversible transformations is a conversion between two GHZ states and three singlets in a tripartite configuration. In more detail, each of the three parties, Alice, Bob and Charlie, are holding two qubits and wish to convert the following states into each other:

$$|\psi\rangle^{ABC} = |\mathrm{GHZ}\rangle^{A_1B_1C_1} \otimes |\mathrm{GHZ}\rangle^{A_2B_2C_2}, \tag{33}$$

$$|\phi\rangle^{ABC} = |\psi^-\rangle^{A_1B_2} \otimes |\psi^-\rangle^{B_1C_2} \otimes |\psi^-\rangle^{C_1A_2}. \tag{34}$$

Also in this case it is straightforward to see that Eq. (29) holds, and moreover in the AEN setting these states can be asymptotically interconverted with unit rate, as follows directly from Theorem 2. Note that a reversible interconversion between these states is not possible in the LOCC framework [12, 47].

We further notice that in the tripartite AEN setting singlets shared by each of the two parties form a reversible entanglement generating set (REGS). In more detail, we will show that for any tripartite state $|\psi\rangle^{ABC}$ and any $\varepsilon > 0$ there exists a state $|\psi_\varepsilon\rangle^{ABC}$ such that $|\langle\psi|\psi_\varepsilon\rangle|^2 > 1-\varepsilon$ and integers n, m_1, m_2 and m_3 such that

$$R_{\mathrm{AEN}}\left(|\psi_\varepsilon\rangle^{\otimes n} \to |\psi^-\rangle_{A_1B_2}^{\otimes m_1} \otimes |\psi^-\rangle_{B_1C_2}^{\otimes m_2} \otimes |\psi^-\rangle_{C_1A_2}^{\otimes m_3}\right) = R_{\mathrm{AEN}}\left(|\psi^-\rangle_{A_1B_2}^{\otimes m_1} \otimes |\psi^-\rangle_{B_1C_2}^{\otimes m_2} \otimes |\psi^-\rangle_{C_1A_2}^{\otimes m_3} \to |\psi_\varepsilon\rangle^{\otimes n}\right) = 1. \tag{35}$$

This means that every tripartite pure state can be approximated by a state which is reversibly interconvertible into singlets shared between the parties. We note that the concept of REGS has been first discussed in [12].

To prove this, we will first show that any tripartite state $|\psi\rangle^{ABC}$ is reversibly interconvertible into a state comprising bipartite entangled states shared between each pair of parties (see also Fig. 2). In more detail, we will show that for any tripartite pure state $|\psi\rangle^{ABC}$ there exists a state of the form

$$|\phi\rangle^{ABC} = |\phi_1\rangle^{A_1B_2} \otimes |\phi_2\rangle^{B_1C_2} \otimes |\phi_3\rangle^{C_1A_2} \tag{36}$$

having the same von Neumann entropies of the subsystems as $|\psi\rangle^{ABC}$, i.e.,

$$S(\psi^A) = S(\phi^A), \tag{37a}$$

$$S(\psi^B) = S(\phi^B), \tag{37b}$$

$$S(\psi^C) = S(\phi^C). \tag{37c}$$

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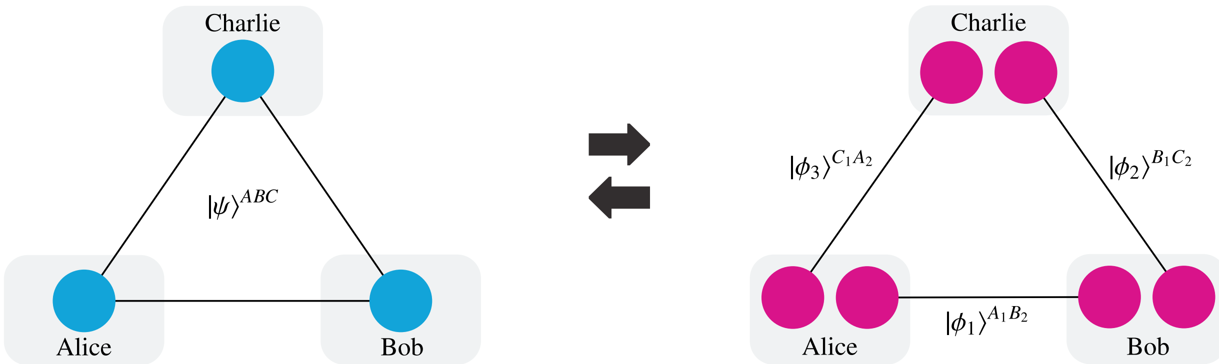

Figure 2. Through the application of AEN operations, we can execute a reversible conversion of any tripartite pure state $|\psi\rangle^{ABC}$ (as shown in the left part of the figure) into a pure state consisting of bipartite entangled states between each pair of parties (represented in the right part of the figure).

For this, let us denote the entanglement entropies of the states $|\phi_i\rangle$ by $s_i = E(\phi_i)$. For a given state $|\psi\rangle^{ABC}$ there exists a state $|\phi\rangle^{ABC}$ with the claimed features whenever there exist nonnegative numbers s_i fulfilling the conditions

$$S(\psi^A) = s_1 + s_3, \tag{38a}$$
$$S(\psi^B) = s_1 + s_2, \tag{38b}$$
$$S(\psi^C) = s_2 + s_3. \tag{38c}$$

Solving these equations for s_i we obtain

$$s_1 = \frac{1}{2}\left[S(\psi^A) + S(\psi^B) - S(\psi^C)\right], \tag{39a}$$
$$s_2 = \frac{1}{2}\left[S(\psi^B) + S(\psi^C) - S(\psi^A)\right], \tag{39b}$$
$$s_3 = \frac{1}{2}\left[S(\psi^A) + S(\psi^C) - S(\psi^B)\right]. \tag{39c}$$

By the subadditivity of the von Neumann entropy we immediately see that each term on the right-hand side of Eqs. (39) is nonnegative. This proves that the state $|\phi\rangle^{ABC}$ with the claimed features exists for any tripartite pure state $|\psi\rangle^{ABC}$. Using Theorem 2, we see that the conversion rates fulfill

$$R_{\mathrm{AEN}}(|\psi\rangle \to |\phi\rangle) = R_{\mathrm{AEN}}(|\phi\rangle \to |\psi\rangle) = 1 \tag{40}$$

which means that $|\psi\rangle^{ABC}$ and $|\phi\rangle^{ABC}$ are reversibly interconvertible via AEN operations.

Assume now that the state $|\psi\rangle^{ABC}$ is chosen such that s_1, s_2, and s_3 in Eqs. (39) are rational. Then, there exists an integer n such that ns_1, ns_2, and ns_3 are integers. Moreover, Theorem 2 directly implies that $|\psi\rangle^{\otimes n}$ can be reversibly interconverted into the state

$$|\phi'\rangle = |\psi^-\rangle_{A_1B_2}^{\otimes ns_1} \otimes |\psi^-\rangle_{B_1C_2}^{\otimes ns_2} \otimes |\psi^-\rangle_{C_1A_2}^{\otimes ns_3} \tag{41}$$

via AEN operations, i.e.,

$$R_{\mathrm{AEN}}(|\psi\rangle^{\otimes n} \to |\phi'\rangle) = R_{\mathrm{AEN}}(|\phi'\rangle \to |\psi\rangle^{\otimes n}) = 1. \tag{42}$$

This proves that singlets form a REGS in this case.

If s_1, s_2, and s_3 are not rational, then for any $\varepsilon > 0$ there exists a state $|\psi_\varepsilon\rangle^{ABC}$ such that $|\langle\psi|\psi_\varepsilon\rangle|^2 > 1 - \varepsilon$ and

$$s_1' = \frac{1}{2}\left[S(\psi_\varepsilon^A) + S(\psi_\varepsilon^B) - S(\psi_\varepsilon^C)\right], \tag{43a}$$
$$s_2' = \frac{1}{2}\left[S(\psi_\varepsilon^B) + S(\psi_\varepsilon^C) - S(\psi_\varepsilon^A)\right], \tag{43b}$$
$$s_3' = \frac{1}{2}\left[S(\psi_\varepsilon^A) + S(\psi_\varepsilon^C) - S(\psi_\varepsilon^B)\right] \tag{43c}$$

This article has been accepted for publication in IEEE Transactions on Information Theory. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TIT.2025.3526419

are all rational. Similarly as above, it follows that

$$R_{\mathrm{AEN}}(|\psi_\varepsilon\rangle^{\otimes n} \to |\phi'\rangle) = R_{\mathrm{AEN}}(|\phi'\rangle \to |\psi_\varepsilon\rangle^{\otimes n}) = 1 \tag{44}$$

holds true for some n. This proves that singlets form a REGS for AEN transformations in the tripartite setting. Note that in the LOCC framework singlets are not enough to generate all states reversibly for more than two parties [12, 47].

At this stage, one might wonder whether the choice of the relative entropy of entanglement in Eq. (15) is essential, or if other entanglement measures could similarly yield a result analogous to Theorem 2. Although we cannot fully resolve this question at present, we note that the proof of our main result in Theorem 2 relies on several specific properties of the relative entropy of entanglement. Since other entanglement quantifiers may not share all of the properties used in the proofs, it appears unlikely that our approach can be directly extended to other entanglement measures.

V. METHODS

In this section we give more details on the technical methods used in the proof of Theorem 2. The key element of the proof is Proposition 8. As a preparation for the proof of Proposition 8 we need the following results.

We will first consider the following sequence of transformations:

$$\Lambda_n[\rho] = \mathrm{Tr}\left[|\psi^-\rangle\langle\psi^-|^{\otimes n}\rho\right]|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor rn\rfloor}\otimes|0\rangle\langle 0|^K + \mathrm{Tr}\left[\left(\mathbb{1}-|\psi^-\rangle\langle\psi^-|^{\otimes n}\right)\rho\right]\mu_s\otimes|1\rangle\langle 1|^K \tag{45}$$

with some fully separable state μ_s, and K is a register in the possession of Alice. We will first show that this transformation generates a vanishingly small amount of entanglement for large n. As entanglement quantifiers we will use the generalized robustness of entanglement R_{g} and the relative entropy of entanglement E_{r} defined in Section II.

Proposition 3. *For any separable state ρ_s and any $r > 0$ it holds that*

$$R_{\mathrm{g}}(\Lambda_n[\rho_s]) \leq \frac{1}{2^{n(1-r)}}. \tag{46}$$

Proof. By convexity of the generalized robustness of entanglement [39] we obtain the following bound for any separable state ρ_s:

$$R_{\mathrm{g}}(\Lambda_n[\rho_s]) \leq \mathrm{Tr}\left[|\psi^-\rangle\langle\psi^-|^{\otimes n}\rho_s\right]R_{\mathrm{g}}\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor rn\rfloor}\right). \tag{47}$$

Recalling that the generalized robustness of a pure state is given by [39, 40]

$$R_{\mathrm{g}}(\psi) = \left(\sum_i c_i\right)^2 - 1, \tag{48}$$

where c_i are the Schmidt coefficients (i.e. $|\psi\rangle = \sum_i c_i |ii\rangle$), we obtain

$$R_{\mathrm{g}}\left(|\psi^-\rangle\langle\psi^-|^{\otimes m}\right) = \left(\sum_{i=0}^{2^m-1}\frac{1}{\sqrt{2^m}}\right)^2 - 1 = 2^m - 1. \tag{49}$$

Moreover, for any separable state ρ_s it holds that [48]

$$\mathrm{Tr}\left[|\psi^-\rangle\langle\psi^-|^{\otimes n}\rho_s\right] \leq \frac{1}{2^n}. \tag{50}$$

Collecting these results, we arrive at the desired inequality:

$$R_{\mathrm{g}}(\Lambda_n[\rho_s]) \leq \mathrm{Tr}\left[|\psi^-\rangle\langle\psi^-|^{\otimes n}\rho_s\right]R_{\mathrm{g}}\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor rn\rfloor}\right) \leq \frac{2^{\lfloor rn\rfloor}-1}{2^n} \leq \frac{1}{2^{n(1-r)}}. \tag{51}$$

□

We will further make use of the following proposition which has been proven in [31].

Proposition 4. *For any quantum operation Λ fulfilling*

$$\max_{\rho_s\in\mathcal{S}} R_{\rm g}\left(\Lambda\left[\rho_s\right]\right) \leq \varepsilon \tag{52}$$

the following holds for any quantum state ρ:

$$E_{\rm r}\left(\Lambda\left[\rho\right]\right) - E_{\rm r}\left(\rho\right) \leq \log_2\left(1+\varepsilon\right). \tag{53}$$

In the next step, we will provide an upper bound for the fidelity between n singlets and any state ρ on the same Hilbert space.

Proposition 5. *The following inequality holds for all states ρ, all n, and all $r > 0$:*

$$F\left(|\psi^-\rangle\langle\psi^-|^{\otimes n}, \rho\right) \leq \frac{E_{\rm r}(\rho) + \log_2\left(1+\frac{1}{2^{n(1-r)}}\right)}{\lfloor rn\rfloor}. \tag{54}$$

Proof. Due to Propositions 3 and 4, we find that

$$E_{\rm r}(\Lambda_n[\rho]) \leq E_{\rm r}(\rho) + \log_2\left(1+\frac{1}{2^{n(1-r)}}\right) \tag{55}$$

holds for all states ρ, where Λ_n is the transformation defined in Eq. (45). In the next step, observe that[1]

$$E_{\rm r}(\Lambda_n[\rho]) = \mathrm{Tr}\left[|\psi^-\rangle\langle\psi^-|^{\otimes n}\rho\right] E_{\rm r}\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor rn\rfloor}\right), \tag{56}$$

which follows directly from the definition of Λ_n and Eq. (11). This means that

$$F\left(|\psi^-\rangle\langle\psi^-|^{\otimes n}, \rho\right) = \mathrm{Tr}\left[|\psi^-\rangle\langle\psi^-|^{\otimes n}\rho\right] \leq \frac{E_{\rm r}(\rho) + \log_2\left(1+\frac{1}{2^{n(1-r)}}\right)}{E_{\rm r}\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor rn\rfloor}\right)}. \tag{57}$$

The proof is complete by noting that $E_{\rm r}(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor rn\rfloor}) = \lfloor rn\rfloor$. □

Another useful tool which will be applied in this article is a variation of Corollary 7 in [49]. In particular, we consider LOCC protocols which convert $|\psi\rangle^{\otimes n}$ into a maximally entangled state of local dimension L_n, i.e., the target state is

$$|\phi_{L_n}\rangle = \frac{1}{\sqrt{L_n}}\sum_{i=0}^{L_n-1}|ii\rangle. \tag{58}$$

We assume that the transformation is exact but probabilistic, having success probability P_n. Moreover, let p be a probability distribution containing the Schmidt coefficients of $|\psi\rangle$, i.e., $|\psi\rangle = \sum_i \sqrt{p_i}|ii\rangle$. For $r > 0$ we define the function [49]

$$E(r) = \min_{q:D(q\|p)\leq r}\left\{D(q\|p) + H(q)\right\}, \tag{59}$$

where q is a probability distribution, $H(q) = -\sum_i q_i\log_2 q_i$ is the Shannon entropy, and $D(q\|p) = \sum_i q_i \log_2\frac{q_i}{p_i}$ is the relative entropy. The following proposition is a direct consequence of Corollary 7 in [49].

Proposition 6. *For any $r > 0$, $\varepsilon > 0$, and $\delta > 0$ there exists a sequence of LOCC protocols converting $|\psi\rangle^{\otimes n}$ into the state $|\phi_{L_n}\rangle$ with success probability P_n such that the following inequalities hold for all n large enough:*

$$r - \delta \leq -\frac{1}{n}\log_2(1-P_n), \tag{60}$$

$$\frac{1}{n}\log_2 L_n \geq E(r) - \varepsilon. \tag{61}$$

With this result, we are now able to prove the following proposition.

---

[1] Note that this equality would not hold if one would define Λ_n without the register K.

Proposition 7. *For any $R < S(\psi^A)$ there exists $\alpha > 0$ and a sequence of LOCC protocols Φ_n such that*

$$F\left(\Phi_n\left[\psi^{\otimes n}\right], |\phi^+\rangle\langle\phi^+|^{\otimes\lfloor Rn\rfloor}\right) \geq 1 - 2^{-\alpha n} \tag{62}$$

for all n large enough.

Proof. Note that $E(r)$ is continuous and monotonically decreasing for $r > 0$, and moreover $\lim_{r\to 0} E(r) = H(p) = S(\psi^A)$ [49]. By continuity, for any $\varepsilon' > 0$ there exists $r_{\varepsilon'} > 0$ such that $E(r_{\varepsilon'}) = S(\psi^A) - \varepsilon'$. It follows that for any $\varepsilon > 0$, $\delta > 0$, and $\varepsilon' > 0$ the following inequalities hold for all n large enough:

$$P_n \geq 1 - 2^{-n(r_{\varepsilon'}-\delta)}, \tag{63}$$

$$\log_2 L_n \geq n\left[S(\psi^A) - \varepsilon' - \varepsilon\right]. \tag{64}$$

Choosing $R = S(\psi^A) - \varepsilon' - \varepsilon$, we conclude that for any $\varepsilon > 0$, $\delta > 0$, and $\varepsilon' > 0$ there exists a sequence of (deterministic) LOCC protocol Φ_n such that the following inequality holds for all n large enough:

$$F\left(\Phi_n\left[\psi^{\otimes n}\right], |\phi^+\rangle\langle\phi^+|^{\otimes\lfloor Rn\rfloor}\right) \geq P_n \geq 1 - 2^{-n(r_{\varepsilon'}-\delta)}. \tag{65}$$

Since $\varepsilon > 0$, $\delta > 0$, and $\varepsilon' > 0$ can be chosen arbitrarily, we obtain the desired inequality (62). □

We will now consider a generalization of the transformation Λ_n defined above. For two pure states $|\psi\rangle$ and $|\phi\rangle$ and some $q > 0$ we define

$$\Lambda_n^{\psi,\phi}[\rho] = \mathrm{Tr}\left[|\psi\rangle\langle\psi|^{\otimes n}\rho\right]|\phi\rangle\langle\phi|^{\otimes\lfloor qn\rfloor} \otimes |0\rangle\langle 0|^K + \mathrm{Tr}\left[\left(\mathbb{1} - |\psi\rangle\langle\psi|^{\otimes n}\right)\rho\right]\mu_s \otimes |1\rangle\langle 1|^K, \tag{66}$$

where μ_s is some fully separable state and the register K is in possession of Alice. Note that this is the type of transformations which is used in the proof of Theorem 2. The following proposition is the key element of the proof of Theorem 2.

Proposition 8. *For any two bipartite pure states $|\psi\rangle$ and $|\phi\rangle$ and any $q < S(\psi^A)/S(\phi^A)$ the sequence of operations $\Lambda_n^{\psi,\phi}$ is asymptotically entanglement-nonincreasing.*

Proof. The relative entropy of entanglement of $\Lambda_n^{\psi,\phi}[\rho]$ is given by

$$E_{\mathrm{r}}\left(\Lambda_n^{\psi,\phi}[\rho]\right) = \mathrm{Tr}\left[|\psi\rangle\langle\psi|^{\otimes n}\rho\right] \times \lfloor qn\rfloor S(\phi^A). \tag{67}$$

Consider now a sequence of states ρ_n acting on n copies of $\mathcal{H}^{AB}$. Our goal is to show that for any such sequence and any $\varepsilon > 0$ it holds that

$$E_{\mathrm{r}}\left(\Lambda_n^{\psi,\phi}[\rho_n]\right) - E_{\mathrm{r}}(\rho_n) < \varepsilon \tag{68}$$

for all n large enough.

Choose some $R > 0$. By Proposition 5, for any sequence of states σ_n, any $r > 0$ and any n it holds that

$$F\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}, \sigma_n\right) \leq \frac{E_{\mathrm{r}}(\sigma_n) + \log_2\left(1 + \frac{1}{2^{\lfloor Rn\rfloor(1-r)}}\right)}{\lfloor r\lfloor Rn\rfloor\rfloor}. \tag{69}$$

Choosing $\sigma_n = \Phi_n[\rho_n]$ with some sequence of states ρ_n and LOCC protocols Φ_n we obtain

$$\begin{aligned} F\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}, \Phi_n[\rho_n]\right) &\leq \frac{E_{\mathrm{r}}(\Phi_n[\rho_n]) + \log_2\left(1 + \frac{1}{2^{\lfloor Rn\rfloor(1-r)}}\right)}{\lfloor r\lfloor Rn\rfloor\rfloor} \\ &\leq \frac{E_{\mathrm{r}}(\rho_n) + \log_2\left(1 + \frac{1}{2^{\lfloor Rn\rfloor(1-r)}}\right)}{\lfloor r\lfloor Rn\rfloor\rfloor}, \end{aligned} \tag{70}$$

where we used the fact that the relative entropy of entanglement does not increase under LOCC [41].

In the next step, recall that it is possible to distill the states $|\psi\rangle$ into singlets at rate $S(\psi^A)$. In more detail, for any $R < S(\psi^A)$ there exists some $\alpha > 0$ and a sequence of LOCC protocols Φ_n such that the following inequality holds for all n large enough [49]:

$$F\left(\Phi_n\left[\psi^{\otimes n}\right], |\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}\right) \geq 1 - 2^{-\alpha n}, \tag{71}$$

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we also refer to Proposition 7 for more details. From Eq. (71) it follows that

$$D_B\left(\Phi_n\left[\psi^{\otimes n}\right], |\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}\right) \leq \sqrt{2-2\sqrt{1-2^{-\alpha n}}}, \tag{72}$$

where D_B is the Bures distance defined in Section II. Recalling that the Bures distance fulfills the triangle inequality and the data processing inequality, we find for any sequence of states ρ_n:

$$\begin{aligned} D_B\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}, \Phi_n\left[\rho_n\right]\right) &\leq D_B\left(\Phi_n\left[\psi^{\otimes n}\right], \Phi_n\left[\rho_n\right]\right) + D_B\left(\Phi_n\left[\psi^{\otimes n}\right], |\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}\right) \\ &\leq D_B\left(\Phi_n\left[\psi^{\otimes n}\right], \Phi_n\left[\rho_n\right]\right) + \sqrt{2-2\sqrt{1-2^{-\alpha n}}} \\ &\leq D_B\left(\psi^{\otimes n}, \rho_n\right) + \sqrt{2-2\sqrt{1-2^{-\alpha n}}}, \end{aligned} \tag{73}$$

which is equivalent to

$$D_B\left(\psi^{\otimes n}, \rho_n\right) \geq D_B\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}, \Phi_n\left[\rho_n\right]\right) - \sqrt{2-2\sqrt{1-2^{-\alpha n}}}. \tag{74}$$

Expressing this in terms of fidelity we, see that for any $R < S(\psi^A)$ there exists some $\alpha > 0$ such that for any sequence of states ρ_n the following holds for all n large enough:

$$\begin{aligned} F\left(\psi^{\otimes n}, \rho_n\right) &= \left[1-\frac{1}{2}D_B^2\left(\psi^{\otimes n}, \rho_n\right)\right]^2 \\ &\leq \left[1-\frac{1}{2}\left(D_B\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}, \Phi_n\left[\rho_n\right]\right) - \sqrt{2-2\sqrt{1-2^{-\alpha n}}}\right)^2\right]^2 \\ &= \left[1-\frac{1}{2}\left(\sqrt{2-2\sqrt{F\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}, \Phi_n\left[\rho_n\right]\right)}} - \sqrt{2-2\sqrt{1-2^{-\alpha n}}}\right)^2\right]^2. \end{aligned} \tag{75}$$

Here, Φ_n is the sequence of LOCC protocols which distills $|\psi\rangle$ into singlets, see Eq. (71).

Using Eq. (67), we see that for any $R < S(\psi^A)$ there is some $\alpha > 0$ such that the following holds for all n large enough:

$$\begin{aligned} E_r\left(\Lambda_n^{\psi,\phi}[\rho_n]\right) &= \lfloor qn\rfloor\, S(\phi^A) F\left(\psi^{\otimes n}, \rho_n\right) \\ &\leq \lfloor qn\rfloor\, S(\phi^A)\left[1-\frac{1}{2}\left(\sqrt{2-2\sqrt{F\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}, \Phi_n\left[\rho_n\right]\right)}} - \sqrt{2-2\sqrt{1-2^{-\alpha n}}}\right)^2\right]^2 \\ &= \lfloor qn\rfloor\, S(\phi^A)\left[1-\frac{1}{2}\left(V^2+W^2-2VW\right)\right]^2, \end{aligned} \tag{76}$$

where we have defined

$$V = \sqrt{2-2\sqrt{F\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}, \Phi_n\left[\rho_n\right]\right)}}, \tag{77}$$

$$W = \sqrt{2-2\sqrt{1-2^{-\alpha n}}}. \tag{78}$$

By further defining

$$X = \sqrt{F\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}, \Phi_n\left[\rho_n\right]\right)}, \tag{79}$$

$$Y = \frac{1}{2}\left(W^2-2VW\right), \tag{80}$$

$$Z = Y^2 - 2XY \tag{81}$$

we can express Eq. (76) as follows:

$$\begin{aligned} E_r\left(\Lambda_n^{\psi,\phi}[\rho_n]\right) &\leq \lfloor qn\rfloor\, S(\phi^A)\left[\sqrt{F\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}, \Phi_n\left[\rho_n\right]\right)} - \frac{1}{2}\left(W^2-2VW\right)\right]^2 \\ &= \lfloor qn\rfloor\, S(\phi^A)\left[X-Y\right]^2 \\ &= \lfloor qn\rfloor\, S(\phi^A)\left[X^2+Y^2-2XY\right] \\ &= \lfloor qn\rfloor\, S(\phi^A)\left[F\left(|\psi^-\rangle\langle\psi^-|^{\otimes\lfloor Rn\rfloor}, \Phi_n\left[\rho_n\right]\right) + Z\right]. \end{aligned} \tag{82}$$

Using Eq. (70) we conclude that for any $r > 0$, $R < S(\psi^A)$ the following holds for all n large enough:

$$E_{\mathrm{r}}\left(\Lambda_n^{\psi,\phi}[\rho_n]\right) \leq \lfloor qn \rfloor S(\phi^A) \frac{E_{\mathrm{r}}(\rho_n) + \log_2\left(1 + \frac{1}{2^{\lfloor Rn \rfloor (1-r)}}\right)}{\lfloor r \lfloor Rn \rfloor \rfloor} + \lfloor qn \rfloor S(\phi^A) Z. \tag{83}$$

Note that for any given value of q in the range $0 < q < S(\psi^A)/S(\phi^A)$ there exist some values for r and R in the range $0 < r < 1$ and $0 < R < S(\psi^A)$ such that

$$\frac{\lfloor qn \rfloor S(\phi^A)}{\lfloor r \lfloor Rn \rfloor \rfloor} \leq 1 \tag{84}$$

for all n large enough. For any such choice of q, r and R we see that the following holds for all n large enough:

$$E_{\mathrm{r}}\left(\Lambda_n^{\psi,\phi}[\rho_n]\right) \leq E_{\mathrm{r}}(\rho_n) + \log_2\left(1 + \frac{1}{2^{\lfloor Rn \rfloor (1-r)}}\right) + \lfloor qn \rfloor S(\phi^A) Z. \tag{85}$$

In the final part of the proof we will analyze closer the term $\lfloor qn \rfloor S(\phi^A) Z$. From Eq. (81) we see that Z can be written in the form

$$Z = W\left[\frac{1}{4}\left(W^3 - 4VW^2 + 4V^2W\right) - XW + 2XV\right]. \tag{86}$$

Note that each of the terms V, W, and X is bounded for all n, and moreover $\lim_{n\to\infty} nW = 0$ for all $\alpha > 0$, which implies that $\lim_{n\to\infty} nZ = 0$. From this it is clear that the term $\lfloor qn \rfloor S(\phi^A) Z$ can be made arbitrarily small by choosing large enough n. For any $r < 1$, $R > 0$ it is further clear that

$$\lim_{n\to\infty} \log_2\left(1 + \frac{1}{2^{\lfloor Rn \rfloor (1-r)}}\right) = 0, \tag{87}$$

and the proof is complete. □

VI. CONCLUSIONS

To conclude, we have established a framework for multipartite entanglement theory that presents a comprehensive solution for asymptotic transformation rates across all multipartite pure states. Our approach uniquely incorporates a subtle relaxation to the LOCC paradigm, permitting all transformations on a multipartite system that can increase the bipartite relative entropy of entanglement of any state by ε, requiring that ε can be made arbitrarily small. The primary finding of our research is that transformation rates for all multipartite pure states are fundamentally determined by the bipartite entanglement entropies of the involved quantum states. These results underscore the centrality of entanglement entropy in governing quantum state transitions.

In the context of a tripartite system, our methodology bridges a crucial gap between tripartite and bipartite entanglement theory. Although transformations of multipartite pure states are typically irreversible, even in our setup, we demonstrate that reversibility can be achieved in certain meaningful scenarios. This extends to the reversible conversion between GHZ and W states in a three-qubit setting, as well as the reversible conversion between a pair of GHZ states and three singlets, each singlet being shared by a different pair of parties. Additionally, we establish that in the framework proposed herein, singlets can act as a reversible entanglement generating set for all tripartite pure states.

It is worth noting that the results presented in this article conclusively address an open question posed in [12] over two decades ago. The authors of [12] asked whether there exists a notion of asymptotic state transformations that would enable reversible interconversion among all multipartite pure states which possess identical entanglement entropies across all bipartitions. Our work provides a positive response to this question, significantly advancing our understanding of entanglement theory in the multipartite setting.

The results described in this article lead to several intriguing questions and promising trajectories for further investigation. One such line of research pertains to the role of catalysis within the AEN framework. Recent studies indicate that within the LOCC paradigm, asymptotic transformations bear a close relationship with single-copy transformations that involve catalysis [21–26]. Currently, it remains an open question whether this relationship extends to the AEN setting.

This work was supported by the National Science Centre Poland (Grant No. 2022/46/E/ST2/00115).

Note added: After the completion of this manuscript, two independent works [50, 51] have presented a proof of the generalized quantum Stein's lemma. These works also establish that the resource theory of entanglement becomes fully reversible in the bipartite setting when LOCC is extended to asymptotically non-entangling operations [29–31]. Given that asymptotically non-entangling operations form a subset of AEN, this result further demonstrates that AEN operations also lead to a fully reversible entanglement theory in the bipartite case.

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