Chapter

Homogenizing Electromagnetic Fields for Microperiodic Dielectrics

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Abstract

Homogenization of the electromagnetism equations expressed by two fields, a scalar potential and a vector potential in a microperiodic inhomogeneous dielectric, is carried out. Both fields after homogenization are coupled, unlike in the case of a homogeneous medium, when they are separated. The homogenized medium is characterized by three coefficients: the effective dielectric coefficient, the average dielectric coefficient, and the root mean square dielectric coefficient. Two examples of determining the electrical permittivity are given.

Keywords: scalar and vector potentials, effective dielectric coefficient, cellular medium, asymptotic homogenization, chessboard composite

1. Introduction

Classical electrodynamics [1–3] in a dielectric with a cellular structure is considered. The length of the cell is small compared to the size of the medium and the length of the electromagnetic wave, and therefore we speak of a microperiodic medium. We use the method of homogenization of microperiodic media described in books [4–7]. We have used this method earlier to solve partial electrical problems related to piezoelectricity and magneto-electro-elastic media [8–10].

There are other methods for finding the effective constants of a non-homogeneous medium: the method of Γ-convergence [11] or based upon our compact group approach and the Hashin-Shtrikman variational theory proposed in [12], cf. also [13, 14].

The homogenization method describes fields in a non-homogeneous cellular medium using two independent spatial variables, a macroscopic variable x and a microscopic variable y. The smallness parameter λ allows us to isolate and describe the fields prevailing inside the cell. The fields also depend on the time variable t, but homogenization is not performed with respect to this variable.

In order to homogenize the electromagnetic field, we apply the formulation of the field equations using the potentials: vector A and scalar φ. Unlike the original Maxwell equations, which are first-order partial equations, the potential equations are second-order partial equations, which facilitates their homogenization.

We used this type of approach in Ref. [15] to describe the behavior of an electromagnetic wave in a medium with a variable coefficient of dielectric permeability and magnetic susceptibility. However, in this case we were unable to prove that the zero term $A^{(0)}$ of the vector field expansion does not depend on the microscopic variable y; we accepted this fact ad hoc. In this work we limit ourselves to a dielectric medium for which we can assume that the magnetic susceptibility is constant, $\mu = 1$. With such a restriction we can prove that not only the zero term $\varphi^{(0)}$ of the scalar potential expansion but also the zero term $A^{(0)}$ of the vector potential expansion do not depend on y.

In a body placed in an electric field, dipole moments are created. In a dielectric, the shifting of charges under the influence of the field takes place only within the molecule. Each molecule under the influence of the field becomes a dipole. There are also known dielectrics whose particles are naturally dipoles. These are molecules with an ionic bond, e.g., a water molecule. In a dielectric of this type not placed in an electric field, the directions of the dipole moments are scattered chaotically. As a result, the total dipole moment of such a dielectric is zero. The electric field arranges the dipoles and aligns them in its direction. From this interpretation, a conclusion is drawn as to the meaning of the term "microperiodic." The size of the microperiod must be large enough not to reveal the individual properties of the molecules and their dipoles.

In the next section, we give equations for potentials in a non-homogeneous medium, and in Section 3 we perform homogenization of our problem. In Section 3.1, we give the principles of homogenization theory. In Section 3.2, we deal with the homogenization of the Lorentz condition, and in Sections 3.3 and 3.4, we deal with the homogenization of potential fields. Due to the coupling of equations, we carry out homogenization gradually, referring at each stage to the results from neighboring points. In Section 4, we collect the most important results and draw some conclusions. Finally, we provide two examples of determining the effective electrical permittivity.

Initially, we conduct considerations in general, taking into account the possibility of changing magnetic permeability, but in the next steps we limit ourselves to the dielectric.

2. Basic equations

Let us consider a non-homogeneous body with a cellular structure characterized by the electrical permittivity ε, the magnetic permeability μ, and the electric conductivity σ, which vary from point to point according to the periodic structure of the material. The coefficients ε, μ, and σ are assumed to be positive. The position is given by vector $x = (x_1, x_2, x_3,)$ and the time by t.

The electrodynamics equations in differential form in the Gaussian system of units are as follows:

$$\epsilon_{kls}\, E_{s,l} = -\frac{1}{c}\,\frac{\partial B_k}{\partial t} \tag{1}$$

$$\epsilon_{kls}\, H_{s,l} = \frac{1}{c}\,\frac{\partial D_k}{\partial t} + \frac{4\pi}{c}\, j_k \tag{2}$$

$$B_{k,k} = 0 \tag{3}$$

$$D_{k,k} = 4\pi\rho \tag{4}$$

Here E and H are the electric and magnetic fields, D and B are the electric and magnetic inductions, while ρ and j denote the electric charge and current densities, respectively. The constant c is the velocity of light in a vacuum.

The partial derivative of function $f = f(x,t)$ with respect to spatial coordinate is denoted by a comma $f_{,k} = \frac{\partial f}{\partial x_k}$ and the repeated indices mean summation $f_{,kk} = \sum_{k-1}^{3} \frac{\partial^2 f}{\partial x_k \partial x_k}$

The Levi-Civita completely antisymmetric symbol ϵ_{ijk} is plus 1, if (i, j, k) is an even permutation of (1, 2, 3), minus 1 if it is an odd permutation, and 0 if any index is repeated. The following relation is important: $\epsilon_{abs}\ \epsilon_{spq} = \delta_{ap}\ \delta_{ap} - \delta_{aq}\ \delta_{bp}$

Eqs. (1)–(4) are supplemented by a description of the material properties of the medium:

$$D_s = \varepsilon \tag{5}$$

$$B_s = \mu\, H_s \tag{6}$$

$$j_s = \sigma\, E_s \tag{7}$$

In addition, two more potentials are introduced, the vector potential A and the scalar potential φ with the relations:

$$B_s = \epsilon_{slk}\, A_{k,l} \tag{8}$$

$$E_k = -\frac{1}{c}\frac{\partial A_k}{\partial t} - \varphi_{,k} \tag{9}$$

These potentials satisfy Eqs. (1) and (3) identically, and Eqs. (2) and (4) expressed in terms of the potentials take the form:

$$\left(\frac{1}{\mu}\, A_{k,l}\right)_{,l} - \left(\frac{1}{\mu}\, A_{l,k}\right)_{,l} - \frac{\varepsilon}{c}\left(\frac{1}{c}\frac{\partial^2 A_k}{\partial t^2} + \frac{\partial\, \varphi_{,k}}{\partial t}\right) = -\frac{4\pi}{c}\, j_k \tag{10}$$

$$\varepsilon_{,k}\left(\frac{1}{c}\frac{\partial A_k}{\partial t} + \varphi_{,k}\right) + \varepsilon\left(\frac{1}{c}\frac{\partial A_{k,k}}{\partial t} + \varphi_{,kk}\right) = -4\pi\rho \tag{11}$$

The current density vector j can be considered either as given by relation (7) or as a given vector function of position x and time t. The charge density ρ should be considered as a given function of x and t.

The potentials A and φ are not uniquely defined by formulas (8) and (9). It is seen from them that the divergence of vector A can be assumed arbitrarily. Usually, the Lorentz condition is assumed:

$$A_{k,k} + \frac{\varepsilon\, \mu}{c}\frac{\partial \varphi}{\partial t} = 0 \tag{12}$$

If the potentials satisfy this condition, we can write Eqs. (10) and (11) in the form:

$$-\frac{\partial(1/\mu)}{\partial x_l}\frac{\partial A_l}{\partial x_k} + \frac{1}{\mu}\frac{\partial(\varepsilon\mu/c)}{\partial x_k}\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x_l}\left(\frac{1}{\mu}\frac{\partial A_k}{\partial x_l}\right) - \frac{\varepsilon}{c^2}\frac{\partial^2 A_k}{\partial t^2} = -\frac{4\pi}{c}\, j_k \tag{13}$$

$$\frac{1}{c}\frac{\partial \varepsilon}{\partial x_k}\frac{\partial A_k}{\partial t} + \frac{\partial}{\partial x_k}\left(\varepsilon\frac{\partial \varphi}{\partial x_k}\right) - \frac{\varepsilon^2\, \mu}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -4\pi\rho \tag{14}$$

This is a system of two equations for A_k and φ. In the case of a homogeneous medium, the two equations separate into two inhomogeneous wave equations, one for the vector potential and the other for the scalar one, as shown in Eqs. (55) and (57).

We will homogenize these equations, i.e., we will give the material coefficients of a homogeneous medium approximating a microperiodic medium.

In our considerations, all scalar and vector fields depend on position x and time t. We do not reveal the dependence on time t in the notation to save space.

3. Homogenization

3.1 Description of the method

The microperiodic material is a bounded set $\overline{\Omega}$ of three-dimensional space (the bar denotes the closure of the set), where Ω is a sufficiently regular domain with the boundary $\partial\Omega$.

We assume that the medium reveals a microperiodic structure, cf. [15, 16].

In the sequel, we apply the method of two-scale asymptotic expansion [1–4].

Let a microperiodic structure of body considered be λY -periodic, where $\lambda > 0$ is a small parameter and $Y = \prod_{i=1}^{3}(0, y_i)$ is the so-called basic cell. For a fixed λ the material functions:

$$\varepsilon^\lambda = \varepsilon\left(\frac{x}{\lambda}\right), \ \ \mu^\lambda = \mu\left(\frac{x}{\lambda}\right), \qquad \sigma^\lambda = \sigma\left(\frac{x}{\lambda}\right), \ \ x \in \Omega \tag{15}$$

are λY–periodic.

We introduce a microscopic variable:

$$y = \frac{x}{\lambda} \tag{16}$$

Then material functions are: $\varepsilon^\lambda = \varepsilon(y), \ \ \mu^\lambda = \mu(y), \qquad \sigma^\lambda = \sigma(y), \ \ y \in Y$
Additionally, we observe that for a quasi-periodic structure we would have:

$$\varepsilon^\lambda = \varepsilon(x, y), \ \ \mu^\lambda = \mu(x, y), \qquad \sigma^\lambda = \sigma(x, y), \ \ x \in \Omega, \quad y \in Y \tag{17}$$

where the functions $\varepsilon^\lambda = \varepsilon(x, .), \ \ \mu^\lambda = \mu(x, .), \qquad \sigma^\lambda = \sigma(x, .)$ are y-periodic, and $x \in \Omega$ is the macroscopic variable.

The basic cell Y has the form of a cube. Its volume is $|Y|$. The domain Ω is assumed to have an λY-periodic structure. The set Ω is covered with a regular mesh of size λ, each cell being a cube Y.

From the mathematical point of view, homogenization means the passage with λ to zero in an appropriate meaning, cf. [4, 5].

According to the two-scale asymptotic approach, instead of one space variable x, we introduce two variables, macroscopic x and microscopic y, where $y = x/\lambda$, and instead of a $f(x)$, consider the function $f(x, y)$. Taking into account the formula for the total derivative (known as the chain rule), we have:

$$\frac{\partial f(x, y)}{\partial x_i} = \frac{\partial f(x, y)}{\partial x_i} + \frac{1}{\lambda}\frac{\partial f(x, y)}{\partial y_i} \quad \text{with} \ \ y = \frac{x}{\lambda} \tag{18}$$

Finally, the asymptotic expansion is introduced, e.g.:

$$f(x) = f^{(0)}(x,y) + \lambda f^{(1)}(x,y) + \lambda^2 \; f^{(2)}(x,y) + \; \dots \tag{19}$$

where functions $f^{(i)}, i = 0, 1, 2, \; \dots$ are Y-periodic.

According to the method of asymptotic expansions, we compare the terms associated with the same power of λ.

3.2 Homogenization of the Lorentz condition

Let us start the homogenization from the Lorentz condition (12). We now replace the spatial partial derivatives according to the chain rule (18) and expand the potential into series by powers of λ. We get

$$\begin{aligned}&\left(\frac{\partial}{\partial x_k}\right) + \frac{1}{\lambda}\frac{\partial}{\partial y_k}\Big)\left(A_k^{(0)}(x,y) + \lambda\, A_k^{(1)}(x,y) + \lambda^2\, A_k^{(2)}(x,y) + \; \dots \right)\\ &\qquad + \frac{\varepsilon(y)\mu(y)}{c}\left(\varphi^{(0)}(x,y) + \lambda\varphi^{(0)}(x,y) + \lambda^2\varphi^{(0)}(x,y) + \; \dots \right) = 0\end{aligned} \tag{20}$$

We equate the terms of the order 1/λ and obtain

$$\frac{\partial A_k^{(0)}(x,y)}{\partial y_k} = 0 \tag{21}$$

We will use this result in the next section to prove that the zero component, $\varphi^{(0)}$ of the scalar potential expansion does not depend on y.

The terms of order $\lambda^0 = 1$ are

$$\frac{\partial\, A_k^{(0)}(x)}{\partial x_k} + \frac{\partial A_k^{(1)}(x,y)}{\partial y_k} + \frac{\varepsilon(y)\mu(y)}{c}\,\frac{\partial \varphi^{(0)}(x)}{\partial t} = 0 \tag{22}$$

In the last equation we used the results (27) and (40) about the independence of $\varphi^{(0)}$ and $A_k^{(0)}$ from the variable y. Therefore, this equation is valid for dielectrics.

3.3 Homogenization of the scalar potential equation

According to the homogenization rules presented above, we replace in Eq. (14) the spatial partial derivatives according to the chain rule (18) and expand the potential into series by powers of λ.

$$\begin{aligned}&\frac{1}{\lambda}\frac{1}{c}\frac{\partial \varepsilon(y)}{\partial y_k}\frac{\partial}{\partial t}\left(A_k^{(0)}(x,y) + \lambda A_k^{(1)}(x,y) + \lambda^2\, A_k^{(2)}(x,y) + \; \dots \right)\\ &+\left(\frac{\partial}{\partial x_k} + \frac{1}{\lambda}\frac{\partial}{\partial y_k}\right)\left[\varepsilon(y)\left(\frac{\partial}{\partial x_k} + \frac{1}{\lambda}\frac{\partial}{\partial y_k}\right)\left(\varphi^{(0)}(x,y) + \lambda\varphi^{(1)}(x,y) + \lambda^2\varphi^{(2)}(x,y)\dots\right)\right]\\ &- \frac{\varepsilon(y)^2\mu(y)}{c^2}\,\frac{\partial^2}{\partial t^2}\left(\varphi^{(0)}(x,y) + \lambda\,\varphi^{(1)}(x,y) + \lambda^2\,\varphi^{(2)}(x,y) + \quad \dots \quad\right) = -4\pi\,\rho(y)\end{aligned} \tag{23}$$

According to the method of asymptotic expansions, we successively compare the terms associated with the same power of λ. We obtain at λ^{-2}

$$\frac{\partial}{\partial y_k}\left[\varepsilon(y)\,\frac{\partial\,\varphi^{(0)}(x,y)}{\partial y_k}\right] = 0 \tag{24}$$

We multiply the last equation by $\varphi^{(0)}$ on both sides, integrate by parts over the Y cell, use periodic boundary conditions on the cell, and get

$$\int_Y \varepsilon(y)\,\frac{\partial\varphi^{(0)}(x,y)}{\partial\, y_k}\,\frac{\partial\varphi^{(0)}(x,y)}{\partial\, y_k}\,\mathrm{d}\,Y = 0 \tag{25}$$

The left-hand side is always positive. It is clear that the function $\varphi^{(0)}$ cannot depend on y,

$$\frac{\partial\varphi^{(0)}(x,y)}{\partial\, y_k} = 0 \tag{26}$$

We therefore write

$$\varphi^{(0)} = \varphi^{(0)}(x) \tag{27}$$

According to this result, the zero term $\varphi^{(0)}$ of the expansion with respect to the parameter λ does not depend on the microscopic variable y.

Now, in this section we restrict the considerations to dielectrics, for which $\mu = 1$.

At λ^{-1} we get

$$\frac{1}{c}\,\frac{\partial\varepsilon(y)}{\partial y_k}\,\frac{\partial A_k^{(0)}(x)}{\partial t} + \frac{\partial}{\partial y_k}\left[\varepsilon(y)\left(\frac{\partial\varphi^{(0)}(x)}{\partial x_k} + \frac{\partial\varphi^{(1)}(x,y)}{\partial y_k}\right)\right] = 0 \tag{28}$$

In this equation the result (40) of the next section is used, that $A_k^{(0)}$ does not depend on y.

This equation will be satisfied if we assume:

$$\varphi^{(1)}(x,y) = \alpha_l(\mathrm{y})\left[\frac{1}{c}\,\frac{\partial A_l^{(0)}(x)}{\partial t} + \frac{\partial\varphi^{(0)}(x)}{\partial x_l}\right] \tag{29}$$

Here the vector function $\alpha_l(\mathrm{y})$ is Y-periodic solution to the following local equation:

$$\frac{\partial}{\partial y_k}\left[\varepsilon(y)\left(\delta_{kl} + \frac{\partial\alpha_l(\mathrm{y})}{\partial y_k}\right)\right] = 0 \tag{30}$$

Finally, the coefficients at $\lambda^0 = 1$ give

$$\frac{1}{c}\,\frac{\partial\varepsilon(y)}{\partial y_k}\,\frac{\partial A_k^{(1)}(x,y)}{\partial t} + \frac{\partial}{\partial x_k}\left[\varepsilon(y)\left(\frac{\partial\varphi^{(0)}(x)}{\partial x_k} + \frac{\partial\varphi^{(1)}(x,y)}{\partial y_k}\right)\right] - \frac{\varepsilon(y)^2}{c^2}\,\frac{\partial^2\varphi^{(0)}(x)}{\partial t^2} = 0 \tag{31}$$

and after using the solution (29):

$$\frac{1}{c}\frac{\partial\varepsilon(y)}{\partial y_k}\frac{\partial A_k^{(1)}(x,y)}{\partial t}+\frac{\partial}{\partial x_k}\left[\varepsilon(y)\left(\delta_{kl}+\frac{\partial\alpha_l(\mathrm{y})}{\partial y_k}\right)\frac{\partial\varphi^{(0)}(x)}{\partial x_l}+\varepsilon(y)\frac{\partial\alpha_l(\mathrm{y})}{\partial y_k}\frac{\partial A_l^{(0)}(x)}{c\,\partial t}\right]$$
$$-\frac{\varepsilon(y)^2}{c^2}\frac{\partial^2\varphi^{(0)}(x)}{\partial t^2}=-4\pi\,\rho(y) \tag{32}$$

We have an identity

$$\frac{\partial\varepsilon(y)}{\partial y_k}\frac{\partial A_k^{(1)}(x,y)}{\partial t}=\frac{\partial}{\partial y_k}\left[\varepsilon(y)\frac{\partial A_k^{(1)}(x,y)}{\partial t}\right]-\varepsilon(y)\frac{\partial^2 A_k^{(1)}(x,y)}{\partial y_k\partial t}$$

Integrating by parts, we have

$$\int_Y\frac{\partial\varepsilon(y)}{\partial y_k}\frac{\partial A_k^{(1)}(x,y)}{\partial t}\mathrm{d}y=-\int_Y\varepsilon(y)\frac{\partial^2 A_k^{(1)}(x,y)}{\partial y_k\,\partial t}\mathrm{d}y$$

and according to Eq. (22):

$$\frac{\partial A_k^{(1)}(x,y)}{\partial y_k}=-\frac{\partial A_k^{(0)}(x)}{\partial x_k}-\frac{\varepsilon(y)}{c}\frac{\partial\varphi^{(0)}(x)}{\partial t} \tag{33}$$

After integrating (32) and rearranging the terms, we get

$$\int_Y\varepsilon(y)\left(\delta_{kl}+\frac{\partial\alpha_l(\mathrm{y})}{\partial y_k}\right)\mathrm{d}y\left[\frac{\partial^2\varphi^{(0)}(x)}{\partial x_k\partial x_l}+\frac{\partial^2 A_l^{(0)}(x)}{\partial x_k\,\partial t}\right]+\int_Y\frac{\varepsilon(y)^2}{c^2}\mathrm{d}y\left[\frac{\partial^2\varphi^{(0)}(x)}{\partial t^2}-\frac{\partial^2\varphi^{(0)}(x)}{\partial t^2}\right]$$
$$=-4\pi\int_Y\rho(y)\mathrm{d}y \tag{34}$$

3.4 Homogenization of the vector potential equation

We replace in Eq. (13) the spatial partial derivatives according to the chain rule (18) and expand the vector potential A into a series by powers of λ.

$$-\frac{1}{\lambda}\frac{\partial[1/\mu(y)]}{\partial y_l}\left(\frac{\partial}{\partial x_k}+\frac{1}{\lambda}\frac{\partial}{\partial y_k}\right)\left(A_l^{(0)}(x,y)+\lambda A_l^{(1)}(x,y)+\lambda^2A_l^{(2)}(x,y)+\ \dots\right)$$
$$+\frac{1}{\lambda}\frac{1}{c\mu(y)}\frac{\partial[\varepsilon(y)\mu(y)]}{\partial y_k}\frac{\partial}{\partial t}\left(\varphi^{(0)}(x)+\lambda\varphi^{(1)}(x,y)+\lambda^2\,\varphi^{(2)}(x,y)+\ \dots\right)$$
$$+\left(\frac{\partial}{\partial x_l}+\frac{1}{\lambda}\frac{\partial}{\partial y_l}\right)\left[\frac{1}{\mu(y)}\left(\frac{\partial}{\partial x_l}+\frac{1}{\lambda}\frac{\partial}{\partial y_l}\right)\left(A_k^{(0)}+\lambda A_k^{(1)}+\lambda^2A_k^{(2)}+\ \dots\right)\right]$$
$$-\frac{\varepsilon(y)}{c^2}\frac{\partial^2}{\partial t^2}\left(A_k^{(0)}+\lambda A_k^{(1)}+\lambda^2A_k^{(2)}+\ \dots\right)=-\frac{4\pi}{c} \tag{35}$$

We now restrict homogenizing to fields in dielectrics for which the coefficient μ can be set equal to 1. Then Eq. (35) reduces to

$$\begin{aligned}
&\frac{1}{\lambda}\frac{1}{c}\frac{\partial[\varepsilon(y)]}{\partial y_k}\frac{\partial}{\partial t}\left(\varphi^{(0)}(x)+\lambda\varphi^{(1)}(x,y)+\lambda^2\varphi^{(2)}(x,y)+\dots\right)\\
&+\left(\frac{\partial}{\partial x_l}+\frac{1}{\lambda}\frac{\partial}{\partial y_l}\right)\left[\left(\frac{\partial}{\partial x_l}+\frac{1}{\lambda}\frac{\partial}{\partial y_l}\right)\left(A_k^{(0)}+\lambda A_k^{(1)}+\lambda^2 A_k^{(2)}+\dots\right)\right]\\
&-\frac{\varepsilon(y)}{c^2}\frac{\partial^2}{\partial t^2}\left(A_k^{(0)}+\lambda A_k^{(1)}+\lambda^2 A_k^{(2)}+\dots\right)=-\frac{4\pi}{c}j_k
\end{aligned}\tag{36}$$

Again, compare the terms associated with the same power of λ. We obtain at λ^{-2}:

$$\frac{\partial^2 A_k^{(0)}(x,y)}{\partial y_l \partial y_l}=0\tag{37}$$

Similarly, as before with the scalar potential, we multiply the last equation by $A_k^{(0)}$ on both sides, integrate by parts over the Y cell, use periodic boundary conditions on the cell, and get

$$\int_Y \frac{\partial A_k^{(0)}(x,y)}{\partial y_l}\frac{\partial A_k^{(0)}(x,y)}{\partial y_l}\,\mathrm{d}\,Y=0\tag{38}$$

Under the integral it is the sum of non-negative components, and so it could be equal to zero; each component should vanish. Hence,

$$\frac{\partial A_k^{(0)}(x,y)}{\partial y_l}=0\tag{39}$$

So $A_k^{(0)}$ does not depend on y,

$$A_k^{(0)}=A_k^{(0)}(x)\tag{40}$$

Next, taking into account the result (40), we equate the coefficients at λ^{-1}, and get

$$\frac{1}{c}\frac{\partial[\varepsilon(y)]}{\partial y_k}\frac{\partial\varphi^{(0)}(x)}{\partial t}+\frac{\partial}{\partial y_l}\left(\frac{\partial A_l^{(0)}(x)}{\partial x_k}+\frac{\partial A_l^{(1)}(x,y)}{\partial y_k}\right)=0\tag{41}$$

or, evidently

$$\frac{1}{c}\frac{\partial[\varepsilon(y)]}{\partial y_k}\frac{\partial\varphi^{(0)}(x)}{\partial t}+\frac{\partial^2 A_l^{(1)}(x,y)}{\partial y_l\partial y_k}=0\tag{42}$$

or

$$\frac{\partial}{\partial y_k}\left[\frac{\varepsilon(y)}{c}\frac{\partial\varphi^{(0)}(x)}{\partial t}+\frac{\partial A_l^{(1)}(x,y)}{\partial y_l}\right]=0\tag{43}$$

But, from the Lorentz condition (22) for dielectrics,

$$\frac{\partial\, A_k^{(0)}(x)}{\partial x_k}+\frac{\partial A_k^{(1)}(x,y)}{\partial y_k}+\frac{\varepsilon(y)}{c}\frac{\partial \varphi^{(0)}(x)}{\partial t}=0 \tag{44}$$

it is clear that the interior of the square bracket in Eq. (43) does not depend on y, and this equation is satisfied identically.

The coefficients at $\lambda^0=1$ give

$$\frac{1}{c}\frac{\partial \varepsilon(y)}{\partial y_k}\frac{\partial \varphi^{(1)}(x)}{\partial t}+\frac{\partial}{\partial x_l}\left(\frac{\partial A_k^{(0)}}{\partial x_l}+\frac{\partial A_k^{(1)}}{\partial y_l}\right)-\frac{\varepsilon(y)}{c^2}\frac{\partial^2 A_k^{(0)}(x)}{\partial t^2}=-\frac{4\pi}{c}j_k(x,y) \tag{45}$$

By the relation (29) we get

$$\begin{aligned}&\frac{1}{c}\frac{\partial \varepsilon(y)}{\partial y_k}\frac{\partial}{\partial t}\left\{\alpha_l(\mathrm{y})\left[\frac{1}{c}\frac{\partial A_l^{(0)}(x)}{\partial t}+\frac{\partial \varphi^{(0)}(x)}{\partial x_l}\right]\right\}+\frac{\partial}{\partial x_l}\left(\frac{\partial A_k^{(0)}}{\partial x_l}+\frac{\partial A_k^{(1)}}{\partial y_l}\right)\\&-\frac{\varepsilon(y)}{c^2}\frac{\partial^2 A_k^{(0)}(x)}{\partial t^2}=-\frac{4\pi}{c}j_k(x,y)\end{aligned} \tag{46}$$

or

$$\begin{aligned}&\frac{1}{c}\frac{\partial \varepsilon(y)}{\partial y_k}\alpha_l(\mathrm{y})\frac{\partial^2 \varphi^{(0)}(x)}{\partial x_l \partial t}+\frac{\partial}{\partial x_l}\left(\frac{\partial A_k^{(0)}}{\partial x_l}+\frac{\partial A_k^{(1)}}{\partial y_l}\right)\\&+\frac{1}{c^2}\left\{\frac{\partial}{\partial y_k}\left[\varepsilon(y)\alpha_l(\mathrm{y})\right]-\left[\varepsilon(y)\left(\delta_{kl}+\frac{\partial \alpha_l(\mathrm{y})}{\partial y_k}\right)\right]\right\}\frac{\partial^2 A_l^{(0)}(x)}{\partial t^2}=-\frac{4\pi}{c}j_k(x,y)\end{aligned} \tag{47}$$

Integrate both sides of the last equation over cell Y. After use of the periodic boundary,

$$\begin{aligned}&\frac{1}{c}\int_Y \frac{\partial \varepsilon(y)}{\partial y_k}\alpha_l(y)\,\mathrm{d}y\,\frac{\partial^2 \varphi^{(0)}(x)}{\partial x_l \partial t}+\frac{\partial^2 A_k^{(0)}}{\partial x_l \partial x_l}-\frac{1}{c^2}\int_Y\left[\varepsilon(y)\left(\delta_{kl}+\frac{\partial \alpha_l(\mathrm{y})}{\partial y_k}\right)\right]\mathrm{d}y\,\frac{\partial^2 A_l^{(0)}(x)}{\partial t^2}\\&=-\frac{4\pi}{c}\int_Y j_k(x,y)\,\mathrm{d}y\end{aligned} \tag{48}$$

4. Results and conclusions

Let us summarize the most important results of our calculations. We introduce the abbreviation,

$$\langle(\ldots)\rangle=\frac{1}{|\mathrm{Y}|}\int_Y \mathrm{d}y\,(\ldots) \tag{49}$$

and notation for the effective dielectric coefficient,

$$\varepsilon_{kl}^{\mathrm{eff}}=\left\langle \varepsilon(y)\left(\delta_{kl}+\frac{\partial \alpha_l(\mathrm{y})}{\partial y_k}\right)\right\rangle \tag{50}$$

Now, the equation (34) is written in the form

$$\varepsilon_{kl}^{\text{eff}} \left[\frac{\partial^2 \varphi^{(0)}(x)}{\partial x_k \partial x_l} + \frac{\partial^2 A_l^{(0)}(x)}{\partial x_k \partial t} \right] + \frac{\left\langle \varepsilon(y)^2 \right\rangle}{c^2} \left[\frac{\partial^2 \varphi^{(0)}(x)}{\partial t^2} - \frac{\partial^2 \varphi^{(0)}(x)}{\partial t^2} \right] = -4\pi \left\langle \rho(y) \right\rangle \quad (51)$$

In essence of course, it is the equation

$$\varepsilon_{kl}^{\text{eff}} \left[\frac{\partial^2 \varphi^{(0)}(x)}{\partial x_k \partial x_l} + \frac{\partial^2 A_l^{(0)}(x)}{\partial x_k \partial t} \right] = -4\pi \left\langle \rho(y) \right\rangle \quad (52)$$

We have kept the second square bracket in (51) for a moment to show the transition to the case of a homogeneous medium with dielectric constant ε.

Namely, in a homogeneous medium:

$$\left\langle \varepsilon(y)^2 \right\rangle = \varepsilon^2 \quad (53)$$

and

$$\varepsilon_{kl}^{\text{eff}} = \varepsilon \quad (54)$$

Then,

$$\varepsilon_{kl}^{\text{eff}} \frac{\partial^2 A_l^{(0)}(x)}{\partial x_k \partial t} + \frac{\left\langle \varepsilon(y)^2 \right\rangle}{c^2} \frac{\partial^2 \varphi^{(0)}(x)}{\partial t^2} = \frac{\partial}{\partial t} \left[\varepsilon_{kl}^{\text{eff}} \frac{\partial A_l^{(0)}(x)}{\partial x_k} + \frac{\left\langle \varepsilon(y)^2 \right\rangle}{c^2} \frac{\partial \varphi^{(0)}(x)}{\partial t} \right] = 0$$

in virtue of Eqs. (53) and (54) and the Lorentz condition (12).

Eq. (51) then takes the form of the wave equation from the classical field theory for a homogeneous dielectric medium:

$$\frac{\varepsilon}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x_k \partial x_k} = \frac{4\pi}{\varepsilon} \rho \quad (55)$$

We apply the same procedure to Eq. (48) describing vector potential and get:

$$\frac{1}{c} \left(\langle \varepsilon(y) \rangle \delta_{kl} - \varepsilon_{kl}^{\text{eff}} \right) \frac{\partial^2 \varphi^{(0)}(x)}{\partial x_l \partial t} + \frac{\partial^2 A_k^{(0)}}{\partial x_l \partial x_l} - \frac{1}{c^2} \varepsilon_{kl}^{\text{eff}} \frac{\partial^2 A_l^{(0)}(x)}{\partial t^2} = -\frac{4\pi}{c} \left\langle j_k(x, y) \right\rangle \quad (56)$$

In a homogeneous medium, when Eqs. (53) and (54) hold true, we get the classical wave equation:

$$\frac{\varepsilon}{c^2} \frac{\partial^2 A_k}{\partial t^2} - \frac{\partial^2 A_k}{\partial x_l \partial x_l} = \frac{4\pi}{c} j_k \quad (57)$$

Eqs. (52) and (56) are our basic results. They show that in a non-homogeneous medium:

i. There is a coupling of the equations for the potentials A and φ. Only in the homogeneous case are the equations for this field separated.

ii. The equations for both fields represent damped waves, which was, of course, to be expected.

iii. A microperiodically non-homogeneous medium is anisotropic, which is also obvious.

iv. Instead of one material constant, as in a homogeneous medium, we have three constants, one tensor $\varepsilon_{kl}^{\text{eff}}$ and two mean values $\langle \varepsilon(y) \rangle$ and $\left\langle \varepsilon(y)^2 \right\rangle$.

5. Examples

5.1 One-dimensional case

As an example, let us first consider a layered medium consisting of two types of dielectrics, cf. **Figure 1**. The basic cell consists of two materials, and the problem is one-dimensional.

Eq. (30) for the local function now takes the form:

$$\frac{\mathrm{d}}{\mathrm{d}y} \left[\varepsilon(y) \left(1 + \frac{\mathrm{d}\alpha(\mathrm{y})}{\mathrm{d}y} \right) \right] = 0 \tag{58}$$

with

$$\varepsilon(y) = \begin{cases} \varepsilon_1 & \text{for} \quad 0 < y < y_1 \\ \varepsilon_2 & \text{for} \quad y_1 \leq y < Y \end{cases} \tag{59}$$

Integrating (58) with periodic boundary values gives:

$$\alpha(y) = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2} y \qquad \text{for} \qquad 0 < y < y_1 \tag{60}$$

$$\alpha(y) = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2} y_1 + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} y \qquad \text{for} \qquad y_1 \leq y < Y \tag{61}$$

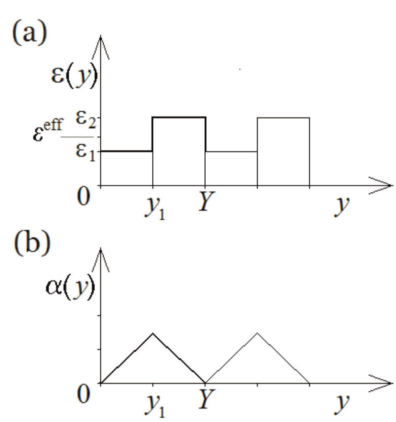

Figure 1.
Layered medium of two dielectric layers with constants ε_1 and ε_2, (a). Local function $\alpha(\mathrm{y})$ for constants $\varepsilon_1 = 1$, $\varepsilon_2 = 2$, and $y_1 = Y/2$ (b).

The effective coefficient for our case after (50) reads:

$$\varepsilon^{\text{eff}} = \frac{1}{Y}\int_0^Y \mathrm{d}y\ \varepsilon(y)\left(1 + \frac{\mathrm{d}\alpha(\mathrm{y})}{\mathrm{d}y}\right) \tag{62}$$

After performing the integration with $\alpha(y)$ given by (60) and (61) we get:

$$\varepsilon^{\text{eff}} = 2\ \frac{\varepsilon_1\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \tag{63}$$

The effective electrical permittivity is the harmonic mean of the values ε_1 and ε_2. This trivial example shows how the homogenization method works.

5.2 Two-dimensional case

Now let us consider a two-dimensional problem.

Let the dielectric coefficient have two values, ε_1 and ε_2 as before, but both values are distributed over the surface in a statistically equivalent way, i.e. the areas occupied by these values are the same. In particular, they can be distributed like the black and white squares on a chessboard, as shown in **Figure 2**. Let us write formula (30) in two dimensions

$$\frac{\partial}{\partial y_k}\ \left[\varepsilon(y)\left(\ \delta_{kl} + \frac{\partial\alpha_l(\mathrm{y})}{\partial y_k}\right)\right] = 0 \tag{64}$$

Now the indices *k, l* take only two values, 1,2. Let us denote:

$$e_k^l = \ \delta_{kl} + \frac{\partial\alpha_l(\mathrm{y})}{\partial y_k} \tag{65}$$

and

$$q_k^l = \varepsilon\ e_k^l \tag{66}$$

Based on (64) we have:

$$\frac{\partial\, q_k^l}{\partial y_k} = 0 \tag{67}$$

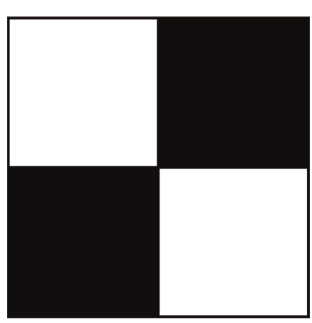

Figure 2.
Basic cell of a chessboard composite, e.g., white square $\boldsymbol{\varepsilon_1}$, *black square* $\boldsymbol{\varepsilon_2}$.

Similarly, using the curl operation:

$$\epsilon_{3ab} \frac{\partial e_b^l}{\partial y_a} = 0 \tag{68}$$

Because,

$$\epsilon_{3ab} \frac{\partial}{\partial y_a} \left(\delta_{bl} + \frac{\partial \alpha_l(\mathrm{y})}{\partial y_b} \right) = 0 \tag{69}$$

Eqs. (66), (67) and (68) correspond formally at the microscopic level to the macroscopic Eqs. (5), (4) and (1), respectively.

Eqs. (67) and (68) can be written as follows:

$$q_{1,1}^l + q_{2,2}^l = 0 \tag{70}$$

and

$$e_{2,1}^l - e_{1,2}^l = 0 \tag{71}$$

where the comma denotes partial differentiation with respect to the indicated component.

We are introducing new vectors:

$$q_a^{l*} = A\, R_{ab}\ e_b^l \tag{72}$$

and

$$e_a^{l*} = \frac{1}{A}\, R_{ab}\ q_b^l \tag{73}$$

where A is an unknown constant, and

$$R_{ab} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \tag{74}$$

We verify

$$q_a^{l*} = A\, R_{ab}\ e_b^l = A \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \bullet \begin{vmatrix} e_1^l \\ e_2^l \end{vmatrix} = A\left(-e_2^l, e_1^l\right) \tag{75}$$

and by (71)

$$q_{a,a}^{l*} = A\,\left(-e_{2,1}^l + e_{1,2}^l\right) = 0 \tag{76}$$

On the other hand

$$e_a^{l*} = \frac{1}{A}\, R_{ab}\ q_b^l = \frac{1}{A} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \bullet \begin{vmatrix} q_1^l \\ q_2^l \end{vmatrix} = \frac{1}{A}\left(-q_2^l, q_1^l\right) \tag{77}$$

and by (70):

$$\epsilon_{3ab}\, e^{l\,*}_{b,a} = \frac{1}{A}\left(q^l_{1,1} + q^l_{2,2}\right) = 0 \tag{78}$$

Moreover, by (66)

$$e^l_a = \frac{q^l_a}{\varepsilon} \tag{79}$$

and by (77)

$$q^{l\,*}_a = A\, R_{ab}\;\, e^l_b = A^2\, \frac{1}{\varepsilon}\,\frac{1}{A}\;\, R_{ab}\, q^l_b = A^2\, \frac{1}{\varepsilon}\, e^{l\,*}_a \tag{80}$$

We write this result as

$$q^{l\,*}_a = \varepsilon^*\; e^{l\,*}_a \tag{81}$$

We choose A in such a way that

$$\varepsilon^* = A^2\,\frac{1}{\varepsilon} = \begin{cases} \varepsilon_2 & \text{if} \quad \varepsilon = \varepsilon_1 \\ \varepsilon_1 & \text{if} \quad \varepsilon = \varepsilon_2 \end{cases} \tag{82}$$

it is

$$A^2 = \varepsilon_1\, \varepsilon_2 \tag{83}$$

The system of equations for quantities with asterisks (81) and (76, 78) is the same as for quantities without asterisks (66) and (70, 71).

Now, consider the mean values of fields:

$$\left\langle q^l_a \right\rangle = \frac{1}{Y}\int_Y q^l_a\, \mathrm{d}Y \quad \text{and} \qquad \left\langle e^l_a \right\rangle = \frac{1}{Y}\int_Y e^l_a\, \mathrm{d}Y \tag{84}$$

The effective dielectric coefficient ε^{eff} is the coefficient between means

$$\left\langle q^l_a \right\rangle = \;\; \varepsilon^{\text{eff}}\;\; \left\langle e^l_a \right\rangle \tag{85}$$

In similar manner

$$\left\langle q^{l\,*}_a \right\rangle = \;\; \varepsilon^{\text{eff}}\;\; \left\langle e^{l\,*}_a \right\rangle \tag{86}$$

and by (80) and (77)

$$A\, R_{ab}\left\langle\; e^l_b \right\rangle = \;\; \varepsilon^{\text{eff}}\;\; \frac{1}{A}\, R_{ab}\;\, \left\langle q^l_b \right\rangle \tag{87}$$

Hence,

$$A\, \left\langle\; e^l_b \right\rangle = \;\; \varepsilon^{\text{eff}}\;\; \frac{1}{A}\, \left\langle q^l_b \right\rangle \tag{88}$$

Comparing last result with (85) we get

$$\varepsilon^{\text{eff}} = A \tag{89}$$

and by (83)

$$\varepsilon^{\text{eff}} = \sqrt{\varepsilon_1 \, \varepsilon_2} \tag{90}$$

In this way, starting from the two-dimensional local function equation (64), we have recovered Dykhne's result [16]. This particular example shows the possibilities inherent in the homogenization method.

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