ABSTRACT
An incremental formulation based on the Taylor's expansion for nonlinear design sensitivity problems is developed. Computational aspects are discussed. A few numerical benchmark problems illustrate the paper.

1. INTRODUCTION
Design sensitivity analysis of linear structures has been by now well established in the literature. One of the first papers considering this type of analysis was that of Zienkiewicz, [1]. The next important contribution was due to Haug and Arora [2] where the adjoint variable and direct differentiation methods were presented. The variational approach to design sensitivity was contained in the papers by Dems and Mróz [3,4] and Haug, Choi, Komkov [5]. The latter paper discussed examples which have became valuable benchmarks for sensitivity computation of complex structures using the finite element method.

Nonlinear sensitivity was considered by Haftka and Mróz [6] and Mróz, Kamat, Plant [7]. A comprehensive discussion of computer implementation aspects was presented by Arora, Cardoso [8] who introduced algorithms for the design sensitivity analysis employed within the ADINA system. However, the approach is not in fact incremental and thus not general enough to deal with any nonlinear behaviour.

Many results considering computational aspects of static, dynamic and stochastic design sensitivity contain papers by Hien and Kleiber [9,10,11]. In the present work the incremental approach based on the first order expansion is given. This is the most general and consistent method which involves tangent stiffness matrix and thus it is suitable for both geometric and material nonlinear problems.

2. PROBLEM STATEMENT.
Consider the structural response functional at time t+Δt for a spatially
discretized system with \( N \) degrees of freedom given by

\[
\psi_{(t+\Delta t)} - \psi_{(t)} = G [q (b), b].
\]  

(1)

The equilibrium equation at time \( t+\Delta t \) is expressed in the incremental form as

\[
\begin{align*}
\Delta q_b \cdot \Delta e & = \Delta Q_a (b), \\
\end{align*}
\]

where \( b = \{ q_b \}, e = 1, \ldots, n; q_e = \{ q_e \}, \) and \( \Delta q = \{ \Delta q \}, a = 1, \ldots, N \) denote the vectors of design variables, nodal displacements and displacement increments, respectively. The objective is to evaluate the sensitivity gradient coefficients of the response functional with respect to design variables at \( t+\Delta t \) assuming \( \delta \psi / \delta e \) at time \( t \) to be given. We introduce the following notation for partial derivatives

\[
\begin{align*}
\frac{\partial (\cdot)}{\partial b} & = \delta (\cdot)/\delta b, \\
\frac{\partial (\cdot)}{\partial q} & = \delta (\cdot)/\delta q. \\
\end{align*}
\]

(3)

Differentiation of Eq. (1) with respect to \( e \) using the chain rule leads to

\[
\begin{align*}
\psi_{(t+\Delta t)} - \psi_{(t)} & = G_e e + G_{q_e} \Delta q_e, \\
\end{align*}
\]

(4)

To express the foregoing equation explicitly in terms of the design variable variation let us first make the first order expansion of the function \( \psi \) about \( \Delta q \) to obtain

\[
\begin{align*}
\psi_{(t+\Delta t)} & = \psi_{(t)} + (G_e + G_{q_e} \Delta q_e) \Delta q_e + \Delta q_{a_e} + \cdots, \\
\end{align*}
\]

(5)

Since the configuration, displacement field and sensitivity gradient coefficients are all known at \( t \), the only term which is necessary to determine is the derivative of the displacement increment with respect to design variables, i.e. \( \Delta q_{a_e} \). This is done by differentiating eq. (2) with respect to \( e \) to get

\[
\begin{align*}
\Delta q_{a_e} = K_{ab} \cdot \Delta q_{b_e} & = \left\{ K_{p_{11}} \Delta q_{p_{11}} + K_{p_{12}} \Delta q_{p_{12}} + \cdots \right\} \Delta q_{p_{11}}, \\
\end{align*}
\]

(6)

Substituting the above equation into Eq. (5) yields the expression for sensitivity at \( t+\Delta t \). This is known as the direct differentiation technique.

Let us now introduce the adjoint technique. Consider the last term in Eq. (5). Introducing Eq. (6) into this term we define an adjoint vector \( \lambda = \{ \lambda_b \} \) which is independent of \( b \) as follows.

\[
\begin{align*}
\Delta q_{a_e} & = K_{ab} [\Delta q_{b_e} - (K_{p_{11}} \Delta q_{p_{11}} + K_{p_{12}} \Delta q_{p_{12}} + \cdots) \Delta q_{p_{11}}].
\end{align*}
\]
If the system stiffness matrix is symmetric the equilibrium equation of the adjoint system can be written as

$$K_{op}^{\lambda} = G_{a} + G_{ab} \Delta q_{p}$$

Consequently, the expression for design sensitivity at time instant $t + \Delta t$ takes the form

$$\psi = \psi_{e} + (G_{a} + G_{ab} q_{p}) \Delta q_{a} + \lambda (\Delta Q_{a} \cdot e - (K_{aq} q_{e} + K_{ap} \Delta q_{p})$$

In contrast to the incremental equations of the primary system, the adjoint equation Eq. (8) is linear.

### 3. COMPUTER IMPLEMENTATION.

The adjoint variable technique is chosen here for computing the design derivatives. There are two sets of equations to be solved for primary and adjoint structures, respectively. The adjoint set is linear with respect to the adjoint variable at time $t$; it is thus possible to solve the equations for both systems in parallel.

The incremental set of Eqs. (2) may be solved for displacements by any nonlinear algorithm, such as Newton - Raphson [12, 13] schemes, for instance, and then the adjoint vector may be calculated exploiting the triangularized tangent stiffness matrix of the primary system at time $t$. The next step is to evaluate the design sensitivity gradient coefficients according to Eq. (9). For simple response functions frequently used in engineering practice such as

$$\psi = \frac{|q_{a}|}{q_{a}} - 1 < 0,$$

in which the quantity $q_{a}$ may be understood as an allowable displacement, Eq. (9) is reduced to

$$\psi_{e} = \psi_{e} + \lambda (\Delta Q_{a} \cdot e - (K_{aq} q_{e} + K_{ap} \Delta q_{p})$$

An important consideration is to choose an effective method of determining the derivatives of the stiffness matrix with respect to displacements and to adjoint variables. The derivatives with respect to design variables can be calculated explicitly by direct differentiation, or implicitly by using a finite difference
technique [14] or by the least square fit method. In the finite element context
the derivatives with respect to displacements can be preferably evaluated by an
implicit technique since the stiffness matrix is generally an implicit function of
displacements. Both methods have been employed in our study. The derivatives are
calculated explicitly for truss, beam and quadrilateral flat plate-shell elements
and implicitly for isoparametric Ahmad-type shell element.

4. FINITE ELEMENT PACKAGE CODE POLSAP AND NUMERICAL EXAMPLES.

The finite element system POLSAP is a considerable extended version of the well
known program SAP - IV [15]. The current linear version of POLSAP provides 14
types of analysis including static and dynamic sensitivity of deterministic and
stochastic response [11]. Any arbitrarily complex beam-plate-shell structures can
be analyzed. The structural response and constraints can be assumed as functions
of displacements and stresses. Design variables may be taken as cross-sectional
areas, Young modula, lengths (for beams and bars), thickness (for plates and
shells) and material densities of structural members.

The program is operative on IBM-PC and compatible computers under DOS or UNIX
systems. The work to extend POLSAP into nonlinear sensitivity range based on the
approach presented here is underway.

The numerical results shown below have been computed for linear structures. The
nonlinear counterparts will be discussed during the conference presentation. The
response functions are expressed by Eq. (10). The design derivatives are
calculated with respect to cross-sectional areas for bars and with respect to
thickness for plate. All of the design constraints considered are imposed on nodal
displacements along z-axes and have the same value of 0.01 m.

Example 1.

The classical example is the von Mises two-bar truss shown in Fig. 1.

![Fig. 1. Two-bar truss.](image)

The nonlinear static results are presented in the work of Kotula and Kleiber [16].
The constraint is imposed at node 2. The sensitivity gradient coefficient for both bars $db/db$ is equal to $-7.374043$.

**Example 2.**

The second example is that of a space bar structure (Fig. 2), for which nonlinear static results were given in [17].

![Space truss diagram](image)

**Fig. 2. Space truss.**

The results of the sensitivity analysis are assembled in Tab. 1 which reflects the change of design sensitivity for ideal and slightly imperfect structure. An error of 0.2 m along x-axis in position of the apex node is assumed. The design constraint is imposed at node 1.

**Table 1.**

<table>
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<th>element number</th>
<th>Perfect structure</th>
<th>Imperfect structure</th>
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<td>-2.6292336493E-03</td>
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<tr>
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<tr>
<td>6</td>
<td>-2.6082236449E-03</td>
<td>-2.6185512616E-03</td>
</tr>
</tbody>
</table>
Example no 3.

A quarter of rectangular clamped plate is considered next (Fig. 3).

The solution was obtained using two types of elements. Flat plate-shell quadrilateral and isoparametric 9-node Ahmad type shell elements. The derivatives with respect to thickness were calculated explicitly for quadrilaterals and implicitly by finite difference technique for isoparametric elements. 100 elements for both cases. The constraint is imposed at node A. Comparison of results for linear case is shown in Fig 4.
5. FINAL REMARKS.

The application of the first order expansion only for all the functions involved seems to be consistent with the first order sensitivity analysis. The incremental formulation proposed has significant advantages and no drawbacks when compared against other formulations which as a rule require also finding of the internal forces. The adjoint equation is linear. Computational algorithms based on the formulation employ only the tangent stiffness matrix, which is consistent with the general "rate - philosophy" in the nonlinear structural analysis.

REFERENCES


STRESZCZENIE