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Parametric version of the Minty–Browder Theorem and its applications

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Plan

Avramescu Theorem

Theorem 1 (Avramescu Theorem)

Let (X, d) be a complete metric space, C – a closed convex subset of a normed space $(Y, \|\cdot\|)$. Moreover let $F: X \times C \longrightarrow X$ and $G: X \times C \longrightarrow C$ be continuous mappings. Assume that the following conditions are satisfied:

- *There is a constant $L \in [0, 1)$, such that:*

$$d\big(F(u, v), F(w, y)\big) \leq Ld(u, w) \quad \text{for all } u, w \in X \text{ and } v \in V;$$

- *$G(X \times C)$ is a relatively compact subset of Y.*

Then, there exists $(u_0, v_0) \in X \times C$ satisfying

$$\begin{cases} F(u_0, v_0) = u_0, \\ G(u_0, v_0) = v_0. \end{cases}$$

C. Avramescu, "Some remarks on a fixed point theorem of Krasnoselskii," Electron. J. Qual. Theory Differ. Equ., vol. 2003, p. 15, 2003, Id/No 5

I. Benedetti, T. Cardinali, and R. Precup, "Fixed point-critical point hybrid theorems and application to systems with partial variational structure," *J. Fixed Point Theory Appl.*, vol. 23, no. 4, p. 19, 2021, Id/No 63

Avramescu Theorem

Theorem 2

Let (X, d) *be a complete metric space,* C *– a closed convex subset of a normed space* $(Y, \|\cdot\|)$*. Assume that* $F: X \times C \longrightarrow X$ and $G: X \times C \longrightarrow C$ *are continuous. Assume that the following conditions are satisfied:*

- *there is a continuous function* $L: C \longrightarrow (0,1)$*, such that:*

$$d\big(F(u,v), F(w,v)\big) \leq L(v)\, d(u,w) \quad \textit{for all } u, w \in X \textit{ and } v \in C;$$

- $G(X \times C)$ *is a relatively compact subset of* Y*;*

Then, there exists $(u_0, v_0) \in X \times C$ *with:*

$$\begin{cases} F(u_0, v_0) = u_0, \\ G(u_0, v_0) = v_0. \end{cases}$$

Avramescu Theorem

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Let (X, d) be a complete metric space, C – a closed convex subset of a normed space $(Y, \|\cdot\|)$. Assume that $F : X \times C \longrightarrow X$ and $G : X \times C \longrightarrow C$ are continuous. Assume that the following conditions are satisfied:

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- *$G(X \times C)$ is a relatively compact subset of Y;*

Then, there exists $(u_0, v_0) \in X \times C$ with:

$$\begin{cases} F(u_0, v_0) = u_0, \\ G(u_0, v_0) = v_0. \end{cases}$$

Proof.

For any $v \in C$, by the Banach Contraction Principle, there exists a unique $S(v) \in X$ such that

$$F(S(v), v) = S(v).$$

Moreover, the mapping $S : C \longrightarrow X$ is continuous. Applying the Schauder Fixed Point Theorem to an operator $G(S(\cdot), \cdot) : C \longrightarrow C$ we get the assertion. □

Parametric Browder-Minty Theorem

Theorem 3 (Parametric Browder-Minty Theorem)

Assume that Y is a metric space, while X is a reflexive Banach space. If $A\colon X \times Y \longrightarrow X^*$ is an operator such that:

- *$A(\cdot, y)$ is radially continuous and strictly monotone for all $y \in Y$;*
- *$A(u, \cdot)$ is continuous for every $u \in X$;*
- *for every $y_0 \in Y$ there exists an open neighbourhood V of y_0 and a coercive function $\gamma\colon [0, \infty) \longrightarrow \mathbb{R}$ such that*

$$\langle A(u, y), u\rangle \geq \gamma(\|u\|)\|u\| \quad \text{for all } y \in V \text{ and } u \in X.$$

Then for every $y \in Y$ there exists a unique u_y such that $A(u_y, y) = 0$. Moreover $y_n \to y$ in Y implies $u_{y_n} \rightharpoonup u_y$ in X.

M. Bełdziński, M. Galewski, and I. Kossowski, "Dependence on parameters for nonlinear equations—abstract principles and applications," Mathematical Methods in the Applied Sciences, vol. 45, no. 3, pp. 1668–1686, 2021

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M. Bełdziński, M. Galewski, and I. Kossowski, "Dependence on parameters for nonlinear equations—abstract principles and applications," Mathematical Methods in the Applied Sciences, vol. 45, no. 3, pp. 1668–1686, 2021

Lemma 4

Assume that X is a real and reflexive Banach space. Then there exists a demicontinuous, bounded, coercive and strictly monotone operator $J\colon X \longrightarrow X^*$ such that $J(0) = 0$.

J. Lindenstrauss, "On nonseparable reflexive Banach spaces," Bull. Am. Math. Soc., vol. 72, pp. 967–970, 1966

General assumptions

1. X and Y are real normed spaces, while X is reflexive.
2. Operator $F: X \times Y \longrightarrow X^*$ satisfies the following assumptions:
   - $F(\cdot, v)$ is monotone and radially continuous for every fixed $v \in Y$;
   - $F(u, \cdot)$ is continuous for every $u \in X$;
   - there exists function $\gamma : [0, \infty)^2 \longrightarrow \mathbb{R}$ such that

     $$\langle F(u, v), u\rangle \geq \gamma(\|u\|, \|v\|)$$

     and that

     $$\lim_{x\to\infty} \gamma(x, y) = \infty$$

     uniformly with respect to y on every bounded interval.
3. Operator $G: X \times Y \longrightarrow Y$ is compact and

$$\left.\begin{array}{l} u_n \rightharpoonup u \text{ in } X \\ v_n \to v \text{ in } Y \end{array}\right\} \Longrightarrow G(u_n, v_n) \to G(u, v) \text{ in } Y.$$

Recall that operator $T: E \longrightarrow F$ is called *compact* if it is continuous and if it maps bounded subsets of E onto relatively compact subsets of F.

Main Result

Theorem 5

Assume that there exists a bounded and convex set $C \subset Y$ *such that* $G(X \times C) \subset C$*. Then there exists at least one solution to*

$$\begin{cases} F(u,v) = 0, \\ G(u,v) = v \end{cases} \tag{P}$$

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Proof.

Let $J : X \longrightarrow X^*$ be as in Lemma 4. For $n \in \mathbb{N}$ we define $F_n : X \times Y \longrightarrow X^*$ by $F_n(u, v) = F(u, v) + \frac{1}{n} J(u)$. For every $n \in \mathbb{N}$, $v \in Y$ there exists a unique $u_n(v)$ such that $F_n(u_n(v), v) = 0$. Moreover $Y \ni v \longmapsto u_n(v) \in X$ is strong-weak continuous. Moreover

$$\gamma(\|u_n(v)\|, \|v\|) \leq 0 \quad \text{for all } v \in Y \text{ and } n \in \mathbb{N}.$$

Hence $S := \{u_n(v) : n \in \mathbb{N}, v \in C\}$ is a bounded set. By the Schauder Fixed Point Theorem, applied to $G(u_n(\cdot), \cdot) : C \longrightarrow C$, there exists $u_n \in X$, $v_n \in Y$ such that

$$F(u_n, v_n) = -\frac{1}{n} J(u_n) \quad \text{and} \quad G(u_n, v_n) = v_n.$$

We get $v_n \to v$ (up to subsequence) and (also up to subsequence) $u_n \rightharpoonup u$. Then $G(u, v) = v$. Relation $F(u, v) = 0$ follows by the Minty Trick. □

J. Franců, "Monotone operators. A survey directed to applications to differential equations," *Applications of Mathematics*, vol. 35, no. 4, pp. 257–301, 1990

M. Galewski, *Basic monotonicity methods with some applications* (Compact Textb. Math.). Cham: Birkhäuser, 2021

Relations to the Krasnoselskii fixed point theorem

Theorem 6 (Krasnoselskii)

Let D be a closed bounded convex subset of a Banach space X, $A\colon\, D \longrightarrow X$ a contraction and $B\colon\, D \longrightarrow X$ a continuous mapping with $B(D)$ relatively compact. If

$$A(x) + B(y) \in D \text{ for every } x, y \in D$$

then the mapping $A + B$ has at least one fixed point.

I. Benedetti, T. Cardinali, and R. Precup, "Fixed point-critical point hybrid theorems and application to systems with partial variational structure," J. Fixed Point Theory Appl., vol. 23, no. 4, p. 19, 2021, Id/No 63

M. A. Krasnosel'skij, "Some problems of nonlinear analysis. Translat. by H. P. Thielman," Transl., Ser. 2, Am. Math. Soc., vol. 10, pp. 345–409, 1958

We say that $A\colon\, H \longrightarrow H$ is *one-sided contraction* if there exists $m < 1$ such that

$$\langle A(u) - A(w), u - w \rangle \leq m\|u - w\|^2 \quad \text{for all } u, w \in H. \tag{1}$$

Relations to the Krasnoselskii fixed point theorem

Theorem 7

Assume that $A\colon H \longrightarrow H$ *is a radially continuous one-sided contraction and* $B\colon H \longrightarrow H$ *is continuous and compact. Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda \in [0, \lambda_0]$ *the mapping* $u \longmapsto A(u) - \lambda B(u)$ *has a fixed point, or in other words, equation*

$$A(u) = \lambda B(u) + u \tag{2}$$

has a solution.

Relations to the Krasnoselskii fixed point theorem

Theorem 7

Assume that $A: H \longrightarrow H$ is a radially continuous one-sided contraction and $B: H \longrightarrow H$ is continuous and compact. Then there exists $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0]$ the mapping $u \longmapsto A(u) - \lambda B(u)$ has a fixed point, or in other words, equation

$$A(u) = \lambda B(u) + u \tag{2}$$

has a solution.

Proof.

Take $r = \frac{\|A(0)\|}{1-m}$ and define $P_r: H \longrightarrow H$ by

$$P(u) := \begin{cases} u & \text{if } \|u\| \leq r, \\ \frac{u}{\|u\|} & \text{if } \|u\| > r. \end{cases}$$

Put $\lambda_0 := \frac{1}{1+\sup_{\|u\| \leq r} \|B(u)\|}$ and fix $\lambda < \lambda_0$. Let $X = Y = H$, $F(u,v) = u - A(u) - v$, and $G(u,v) = \lambda B(P_r(u))$. Identifying H with H^* via the Riesz Representation, we can apply Theorem 5 and obtain $u_0 \in H$ such that $u_0 - A(u_0) = \lambda B(P_r(u_0))$. Therefore using (1) we get

$$\|u_0\|((1-m)\|u_0\| - \|A(0)\|) \leq \langle u_0 - A(u_0), u_0 \rangle = \lambda \langle B(P_r(u_0)), u_0 \rangle \leq \|u_0\|,$$

which gives $\|u_0\| \leq r$ and hence u_0 solves (2). Since λ was taken arbitrary from $[0, \lambda_0]$, we get the assertion. □

Relations to the Krasnoselskii fixed point theorem

Theorem 8

Let C be a closed convex subset of a Hilbert space H, $A\colon H \longrightarrow H$ is a radially continuous one-sided contraction and $B\colon C \longrightarrow H$ is continuous with $B(C)$ relatively compact. If

$$u = A(u) + B(v) \text{ and } v \in C \text{ imply that } u \in C, \tag{3}$$

then the mapping $A + B$ has at least one fixed point.

Relations to the Krasnoselskii fixed point theorem

Theorem 8

Let C be a closed convex subset of a Hilbert space H, $A: H \longrightarrow H$ is a radially continuous one-sided contraction and $B: C \longrightarrow H$ is continuous with $B(C)$ relatively compact. If

$$u = A(u) + B(v) \text{ and } v \in C \text{ imply that } u \in C, \tag{3}$$

then the mapping $A + B$ has at least one fixed point.

Proof.

By the Strongly Monotone Principle, a mapping $I - A: H \longrightarrow H$ is a bijection with a continuous inverse. Let $T: C \longrightarrow X$ be given by

$$T(u) := (I - A)^{-1}\big(B(u)\big).$$

By (3) we have $T(C) \subset C$. Since $T(C)$ is relatively compact, the assertion follows by the Schauder Fixed Point Theorem. □

Main Result

Theorem 9

Assume that there exists a function $\psi : [0,\infty)^2 \longrightarrow [0,\infty)$ *such that*

$$\|G(u,v)\| \leq \psi(\|u\|, \|v\|);$$

If there exists $R > 0$ *such that*

$$\gamma(x,y) \leq 0 \text{ and } y \leq R \text{ imply that } \psi(x,y) \leq R. \tag{4}$$

then system (P) *has at least one solution.*

Main Result

Theorem 9

Assume that there exists a function $\psi : [0,\infty)^2 \longrightarrow [0,\infty)$ *such that*

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If there exists $R > 0$ *such that*

$$\gamma(x,y) \le 0 \text{ and } y \le R \text{ imply that } \psi(x,y) \le R. \tag{4}$$

then system (P) *has at least one solution.*

Proof.

Let us denote $D_R := \{v \in Y : \|v\| \le R\}$. Define F_n and $u_n(v)$ as in the proof of Theorem 5. Let $G_n : D_R \longrightarrow Y$ be given by $G_n(v) = G(u_n(v), v)$. Then G_n is continuous. Moreover, since $\|v\| \le R$ and since γ is uniformly coercive, set $S := \{u_n(v) : n \in \mathbb{N},\ v \in D_R\}$ is bounded. Therefore

$$G_n\left(D_R\right) \subset G\left(S \times D_R\right)$$

and hence G_n is compact. Now we show that $G_n : D_R \longrightarrow D_R$. Let $\|v\| \le R$. Then

$$\|G_n(v)\| = \|G(u_n(v), v)\| \le \psi\left(\|u_n(v)\|, \|v\|\right)$$

Moreover $0 = \langle F_n(u_n(v), v), u_n(v)\rangle \ge \gamma(\|u_n(v)\|, \|v\|)$ and hence $\|G_n(v)\| \le R$ by (4). By the Schauder Fixed Point Theorem, applied to G_n, there exists $v_n \in D_R$ such that $G_n(v_n) = v_n$. Mimicking the proof of Theorem 5, we get the assertion. □

Nonlocal q-Laplace equation

Let us fix $u \in C[0,1]$ and consider the following nonlinear system

$$\begin{cases} -\dfrac{d}{dt}\left(|\dot{v}(t)|^{q-2}\dot{v}(t)\right) = g(t,u(t),v(t)) & \text{for } t \in (0,1), \\ v(0) = \displaystyle\int_0^1 h_0(v(s))dA_0(s), \quad v(1) = \int_0^1 h_1(v(s))dA_1(s), & \end{cases} \tag{P$_q$}$$

where $q > 1$, $g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$, $h_0, h_1 : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous and $A_0, A_1 : [0,1] \longrightarrow \mathbb{R}$ have bounded variations.

Lemma 10

For every $u, v \in C[0,1]$ there is exactly one $c = c(u,v)$ such that

$$\int_0^1 \psi_q^{-1}\left(c(u,v) - \int_0^s g\big(\tau, u(\tau), v(\tau)\big)d\tau\right)ds = \int_0^1 h_1(v(s))dA_1(s) - \int_0^1 h_0(v(s))dA_0(s).$$

Moreover, the mapping $C[0,1] \times C[0,1] \ni (u,v) \longmapsto c(u,v) \in \mathbb{R}$ is continuous.

Here and further on, a function associated with q-Laplacian is denoted by $\psi_q : \mathbb{R} \longrightarrow \mathbb{R}$, that is

$$\psi_q(\zeta) = |\zeta|^{q-2}\zeta.$$

Proof.

For fixed $u, v \in C[0,1]$ we define

$$\Theta_{(u,v)}(c) = \int_0^1 \psi_q^{-1}\left(c - \int_0^s g\big(\tau, u(\tau), v(\tau)\big)d\tau\right)ds - \int_0^1 h_1(v(s))dA_1(s) + \int_0^1 h_0(v(s))dA_0(s).$$

Since ψ_q is continuous and strictly increasing, then so is function $\Theta_{(u,v)}$. Moreover, we have $\lim_{|c|\to\infty} |\Theta_{(u,v)}(c)| = \infty$. Hence $\Theta_{(u,v)}(c) = 0$ for a unique $c = c(u,v)$. Next, the monotonicity of ψ_q yields

$$c(u,v) \lesseqgtr \psi_q\left(\int_0^1 h_1(v(s))dA_1(s) - \int_0^1 h_0(v(s))dA_0(s)\right) \pm \sup_{0\le t\le 1} |g(t, u(t), v(t))|$$

Now, let $u_n \to u_0$ and $v_n \to v_0$ in $C[0,1]$. Then $c(u_n, v_n) \to c$ up to the subsequence. Moreover

$$\lim_{n\to\infty} \Theta_{u_n,v_n}\big(c(u_n,v_n)\big) = \Theta_{u_0,v_0}(c),$$

which gives $c = c(u_0, v_0)$. □

Nonlocal q-Laplace equation

We define $T : C[0,1] \times C[0,1] \longrightarrow C[0,1]$ by the formula

$$T(u,v)(t) = \int_0^t \psi_q^{-1}\left(c(u,v) - \int_0^s g(\tau,u(\tau),v(\tau))d\tau\right) ds + \int_0^1 h_0(v(s))dA_0(s),$$

where c is defined in Lemma 10.

Lemma 11

For every $u \in C[0,1]$, the function $v \in C[0,1]$ is a solution to (P_q) if and only if it is a fixed point of operator $T(u,\cdot)$.

We impose the following assumptions on g:

1. there are numbers $0 \le A, B, C$, $0 < r \le (p-1)(q-1)$ and $0 < \theta < q-1$ such that

$$|g(t,u,v)| \le A|u|^r + B|v|^\theta + C \quad \text{for all } t \in [0,1] \text{ and } u,v \in \mathbb{R};$$

2. there exists a number $\alpha_j > 0$ such that $|h_j(v)| \le \alpha_j|v|$ for all $v \in \mathbb{R}$, $j = 0,1$;
3. $\left(2\alpha_0 \operatorname{Var} A_0 + \alpha_1 \operatorname{Var} A_1\right) < 1$, where $\operatorname{Var}\xi$ stands for a variation of a function ξ.

q-Laplace equation

Theorem 12

For every $u \in C[0,1]$ the problem (P_q) *admits at least one solution.*

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Proof.

Since second assumption holds, we get

$$\left| \int_0^1 h_0(v(s))dA_0(s) \right| \le \alpha_0 \|v\|_\infty \operatorname{Var} A_0. \tag{5}$$

According to the proof of Lemma 10, for every $s \in [0,1]$ we have

$$|c(u,v) - g(s,u(s),v(s))| \le \psi_q\left(\left(\alpha_1 \operatorname{Var} A_1 + \alpha_0 \operatorname{Var} A_0\right)\|v\|_\infty\right) + 2\left(A\|u\|_\infty^r + B\|v\|_\infty^\theta + C\right).$$

The above estimation combined with (5) gives

$$\|T(u,v)\|_\infty \le \left(\alpha_1 \operatorname{Var} A_1 + 2\alpha_0 \operatorname{Var} A_0\right)\|v\|_\infty + \left(2A\|u\|_\infty^r + 2C\right)^{\frac{1}{q-1}} + (2B)^{\frac{1}{q-1}}\|v\|_\infty^{\frac{\theta}{q-1}}.$$

Since the third assumption holds and $\theta < q-1$, we have $\|T(u,v)\|_\infty \le R$ whenever $\|v\|_\infty \le R$ for sufficiently large $R > 0$. Operator $T(u,\cdot)$ is completely continuous, hence the existence of solution to (P_q) is a consequence of the Schauder Fixed Point Theorem. □

Perturbed p-Laplace equation

Let $\varphi : \Omega \times \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R}$ and assume that there exist continuous function $M : [0, \infty) \longrightarrow (0, \infty)$ and constant $m > 0$, such that for a.e. $x \in \Omega$, all $y \in \mathbb{R}$ and $s \geq r \geq 0$ there is:

1. $\varphi(\cdot, y, r)$ is Lebesgue measurable;
2. $\varphi(x, \cdot, r)$ and $\varphi(x, y, \cdot)$ are continuous;
3. $m \leq \varphi(x, y, r) \leq M(|y|)$;
4. $\varphi(x, y, r)r \leq \varphi(x, y, s)s$.

For $p \geq 2$ we define $D_{p,\varphi} : W_0^{1,p}(0,1) \times C[0,1] \longrightarrow W^{-1,p'}(0,1)$ by

$$\langle D_{p,\varphi}(u, v), w\rangle = \int_0^1 \varphi\big(t, v(t), |\dot{u}(t)|^{p-1}\big)|\dot{u}(t)|^{p-2}\dot{u}(t)\dot{w}(t)dt.$$

Lemma 13

- *$D_{p,\varphi}(\cdot, v)$ is monotone and radially continuous for every fixed $v \in Y$;*
- *$D_{p,\varphi}(u, \cdot)$ is continuous for every $u \in X$;*
- *for every $u \in W_0^1(0,1)$ and $v \in C[0,1]$ we have*

$$\langle D_{p,\varphi}(u, v), u\rangle \geq m\|u\|^p_{W_0^{1,p}}.$$

Perturbed p-Laplace equation

Take a function $f : [0,1] : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that for all $u, v \in \mathbb{R}$ and a.e. $t \in [0,1]$ there is:

1. $f(\cdot, u, v)$ is Lebesgue measurable;
2. $f(t, \cdot, v)$ is continuous and nonincreasing;
3. $f(t, u, \cdot)$ is continuous;
4. there exists a nondecreasing function $\delta : [0, \infty) \longrightarrow \mathbb{R}$ satisfying

$$\delta(v) \geq \sup_{\substack{0 \leq t \leq 1 \\ -v \leq \xi \leq v}} |f(t, 0, \xi)|.$$

Define $S : W_0^{1,p}(0,1) \times C([0,1]) \longrightarrow W^{-1,p'}(0,1)$ by the formula

$$\langle S(u,v), w \rangle = \int_0^1 \varphi(t, v(t), |\dot{u}(t)|^{p-1}) |\dot{u}(t)|^{p-2} \dot{u}(t) \dot{w}(t) dt - \int_0^1 f\big(t, u(t), v(t)\big) w(t) dt.$$

Lemma 14

For all $u \in W_0^{1,p}(0,1)$ and every $v \in C[0,1]$ there is

$$\langle S(u,v), u \rangle \geq m \|u\|^p_{W_0^{1,p}} - \frac{1}{\lambda_p} \delta(\|v\|_\infty) \|u\|_{W_0^{1,p}}.$$

Here $\lambda_p = \inf \left\{ c > 0 : \int_0^1 |u(t)|^p dt \leq c \int_0^1 |\dot{u}(t)|^p dt \text{ for all } u \in W_0^{1,p}(0,1) \right\}.$

System of nonlinear equations

$$\begin{cases} -\dfrac{d}{dt}\left(\varphi\left(t, v(t), |\dot{u}(t)|^{p-1}\right)|\dot{u}(t)|^{p-2}\dot{u}(t)\right) = f(t, u(t), v(t)) & \text{for } t \in (0,1), \\ -\dfrac{d}{dt}\left(|\dot{v}(t)|^{q-2}\dot{v}(t)\right) = g(t, u(t), v(t)) & \text{for } t \in (0,1), \\ u(0) = u(1) = 0, & \\ v(0) = \displaystyle\int_0^1 h_0(v(s))dA_0(s), \quad v(1) = \int_0^1 h_1(v(s))dA_1(s). & \end{cases} \tag{$\mathrm{P}_{p,q}$}$$

Theorem 15

Assume that there exists $0 \le a, b$ *and* $\sigma < \frac{(p-1)(q-1)}{r}$ *such that*

$$|\delta(y)| \le a|y|^{\sigma} + b \quad \textit{for all } y \in \mathbb{R}.$$

Then system $(\mathrm{P}_{p,q})$ *has at least one solution.*

Proof.

We apply Theorem 9 taking $X = W_0^{1,p}(0,1)$, $Y = C[0,1]$, $F = S$ and $G = T$. □

Example

$$
\begin{cases}
-\frac{d}{dt}\left(|\dot{u}(t)|\dot{u}(t)\right) = |v(t)|^2 - v(t)^2 u(t)^5 - v(t)^4 u(t) + t^2 & \text{for } t \in (0,1), \\
-\frac{d}{dt}\left(|\dot{v}(t)|^2\dot{v}(t)\right) = v(t)\cos(v(t)) + u(t)\sqrt{|u(t)|} + \cos(u(t)) + v(t)\sin(t) & \text{for } t \in (0,1), \\
u(0) = u(1) = 0, \\
v(0) = \int_0^1 \sin\left(v(s)\right) dA_0(s), \quad v(1) = \int_0^1 \cos\left(v(s)\right) dA_1(s),
\end{cases}
\tag{6}
$$

where $A_0, A_1 \colon [0,1] \longrightarrow \mathbb{R}$ are arbitrary functions with a finite variation. To apply Theorem 15 we let $p = 3$, $q = 4$ and define $\varphi, f, g \colon [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$
\begin{aligned}
\varphi(t,u,v) &= 1, \\
f(t,u,v) &= |v|^2 - v^2 u^5 - v^4 u + t^2, \\
g(t,u,v) &= v\cos(v) + u\sqrt{|u|} + \cos(u) + v\sin(t).
\end{aligned}
$$

Then $\delta \colon [0,\infty) \longrightarrow [0,\infty)$ is given by $\delta(v) = v^2 + 1$ and

$$
|g(t,u,v)| \le 2|v| + |u|^{3/2} + 1 \quad \text{for all } t \in [0,1] \text{ and } u, v \in \mathbb{R}.
$$

Therefore solvability of system (6) follows by Theorem 15.

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