The coherent vortex extraction

Wavelet methods for the simulation of turbulence in fluid mechanics

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IMPAN - Numerical Analysis

Fluid Mechanics Seminar
Warsaw - 27th June 2007

Outline

1. Divergence-free wavelets for turbulent flows
   - Search for coherent structures
   - Divergence-free wavelets
2. Coherent Vortex Extraction
   - Principle
   - Numerical results
3. Simulation of Navier-Stokes equations
   - Numerical scheme
   - Adaptivity
   - Numerical experiments

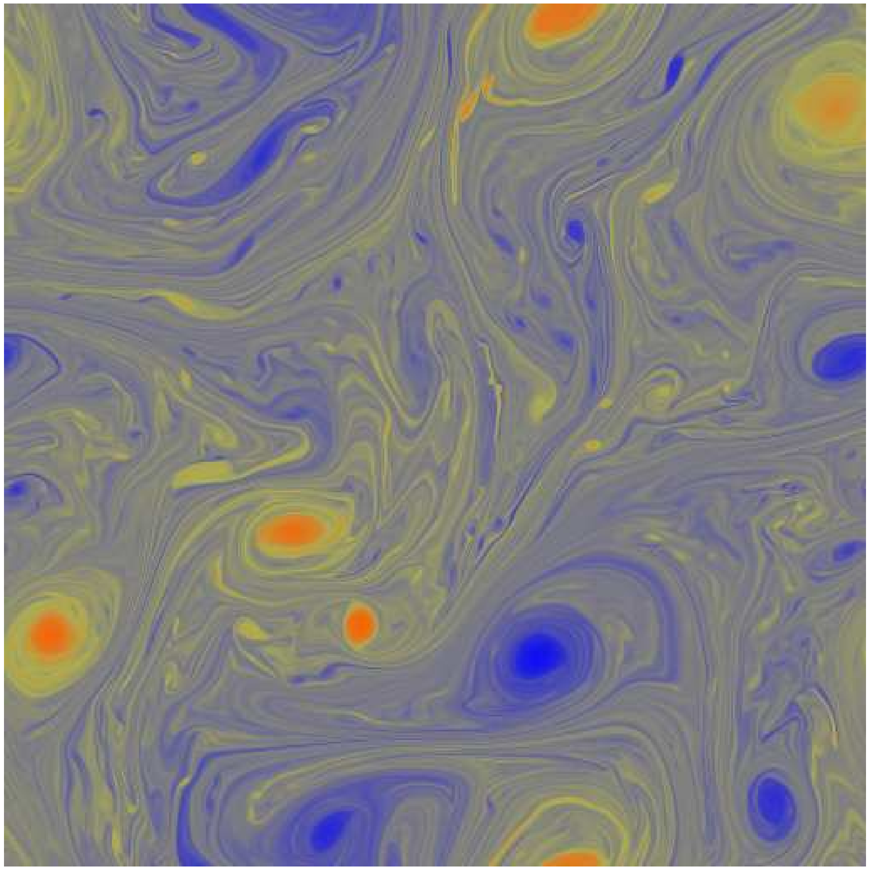
Turbulence

Richardson’s parody of Swift’s poem:

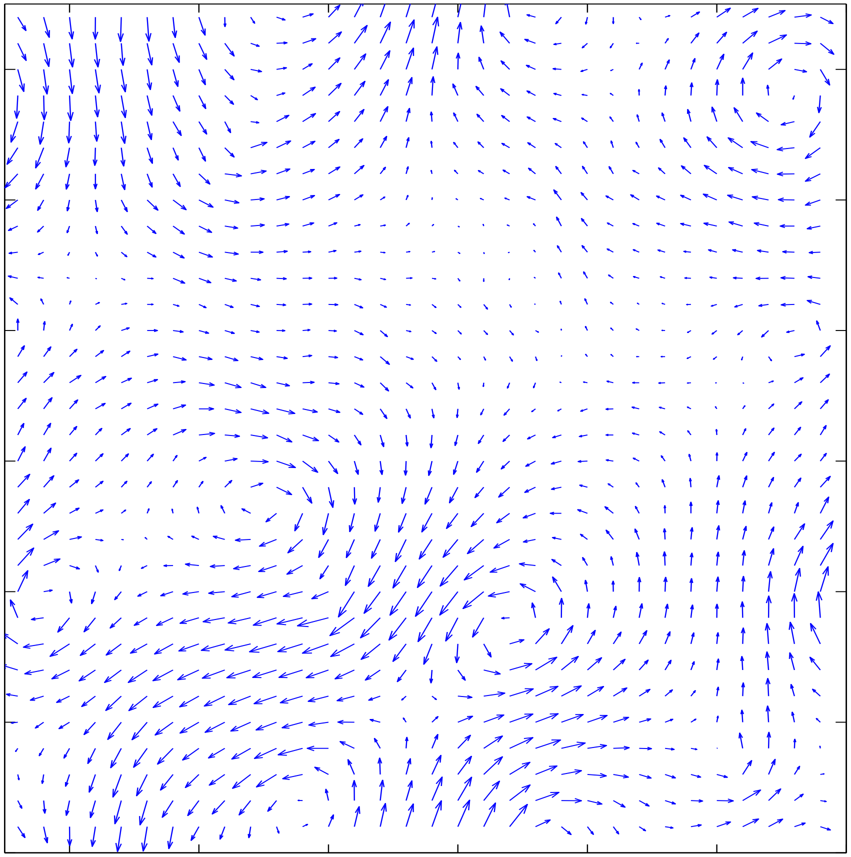
« Greater whirls have lesser whirls
that feed on their velocity
and lesser whirls have smaller whirls
and so on to viscosity»

1. Coherent structures

vorticity



velocity



Example of 2D turbulent flow

3d structures

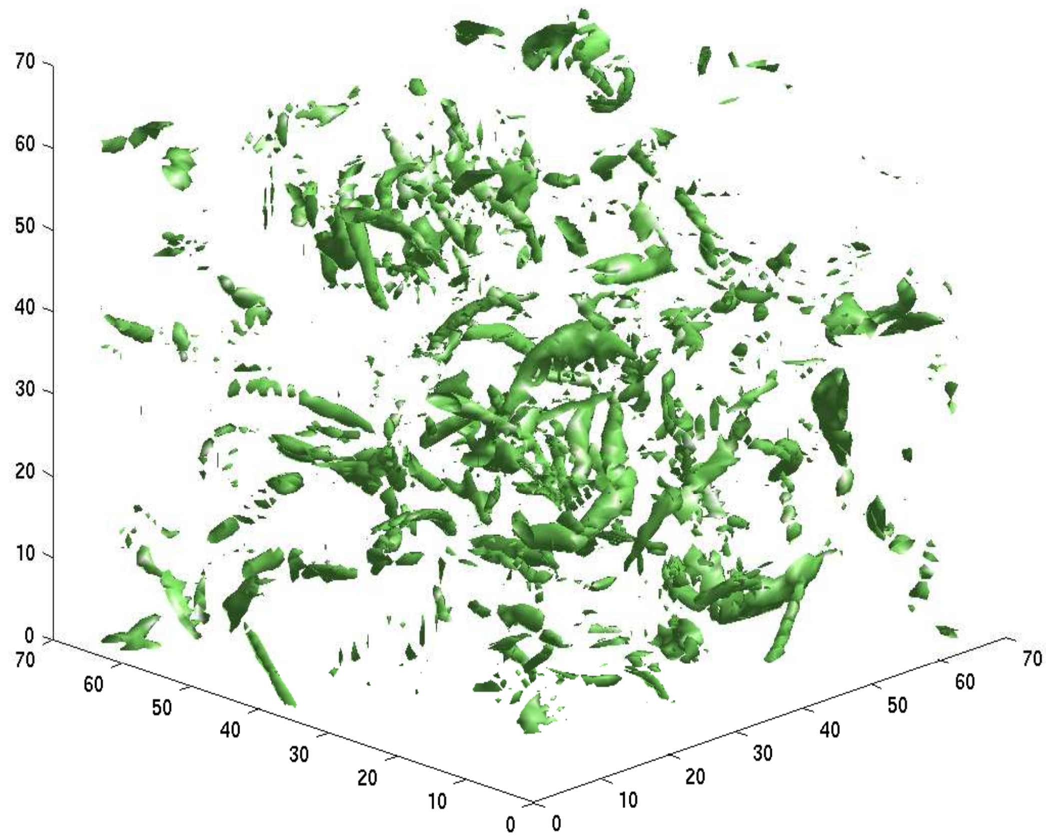
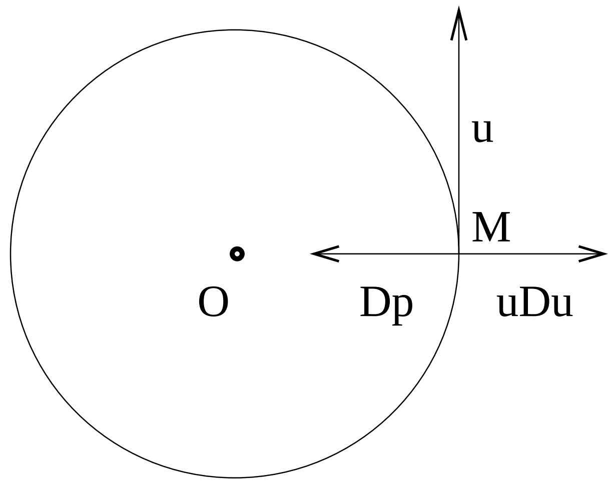


Figure 1: Example of 3D turbulent flow.

The Oseen vortex



$$\omega_{\text{Oseen}}(t,x) = \frac{K\pi}{(2\pi)^2\nu t} e^{\frac{-|x|^2}{4\nu t}}$$

Theorem: Let a Navier-Stokes solution $\omega(t,x) \in C^0([0,\infty), L^1(\mathbb{R}^2)) \cap C^0((0,\infty), L^\infty(\mathbb{R}^2))$ with initial condition $\omega(0) = \omega \in L^1(\mathbb{R}^2)$, then the quantity

$$\int_{\mathbb{R}^2} \omega(t,x)\, dx = \int_{\mathbb{R}^2} \omega_0(x)\, dx\,, \quad t \geq 0$$

is preserved in time.
The solution goes asymptotically to the Oseen vortex with constant $K = \int_{\mathbb{R}^2} \omega_0(x)\, dx$.

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \|\omega(t,x) - \omega_{\text{Oseen}}(t,x)\|_{L^p_x} = 0\,, \quad \text{for} \quad 1 \leq p \leq \infty,$$

Coherent structures in 3d

Stability of the Burgers votices [Thierry Gallay]:

Under the condition "background straining flow"

$$\mathbf{u}^s(x) = \begin{bmatrix} -\frac{\gamma}{2}\, x_1 \\ -\frac{\gamma}{2}\, x_2 \\ \gamma\, x_3 \end{bmatrix} \quad , \quad p^s(x) = -\frac{1}{2}\left(\frac{\gamma}{2}^2 x_1^2 + \frac{\gamma}{2}^2 x_2^2 + \gamma^2 x_3^2\right),$$

we observe the stability of the vortex

$$\Omega^B(x,t) = \frac{K\gamma}{4\pi\nu} e^{\frac{-\gamma(x_1^2+x_2^2)}{4\nu}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Isotropic 2D divergence-free wavelets

Isotropic divergence-free 2D scale function and wavelets ($\mathrm{div}\,\mathbf{u} = 0$) [Lem92] :

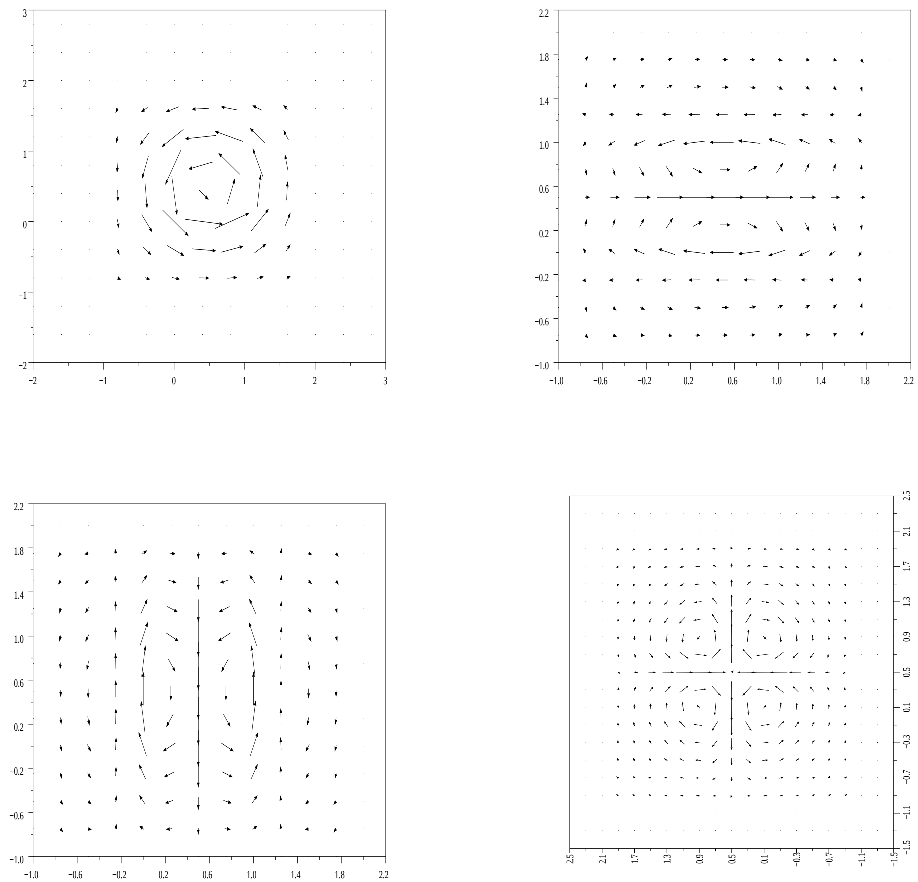
$$\Phi^{\mathrm{div}}(x_1, x_2) = \left| \begin{array}{l} \varphi_1(x_1)\varphi_1'(x_2) \\ -\varphi_1'(x_1)\varphi_1(x_2) \end{array} \right.$$

$$\Psi^{\mathrm{div}\,(1,0)}(x_1, x_2) = \left| \begin{array}{l} \psi_1(x_1)\varphi_1'(x_2) \\ -\psi_1'(x_1)\varphi_1(x_2) \end{array} \right.$$

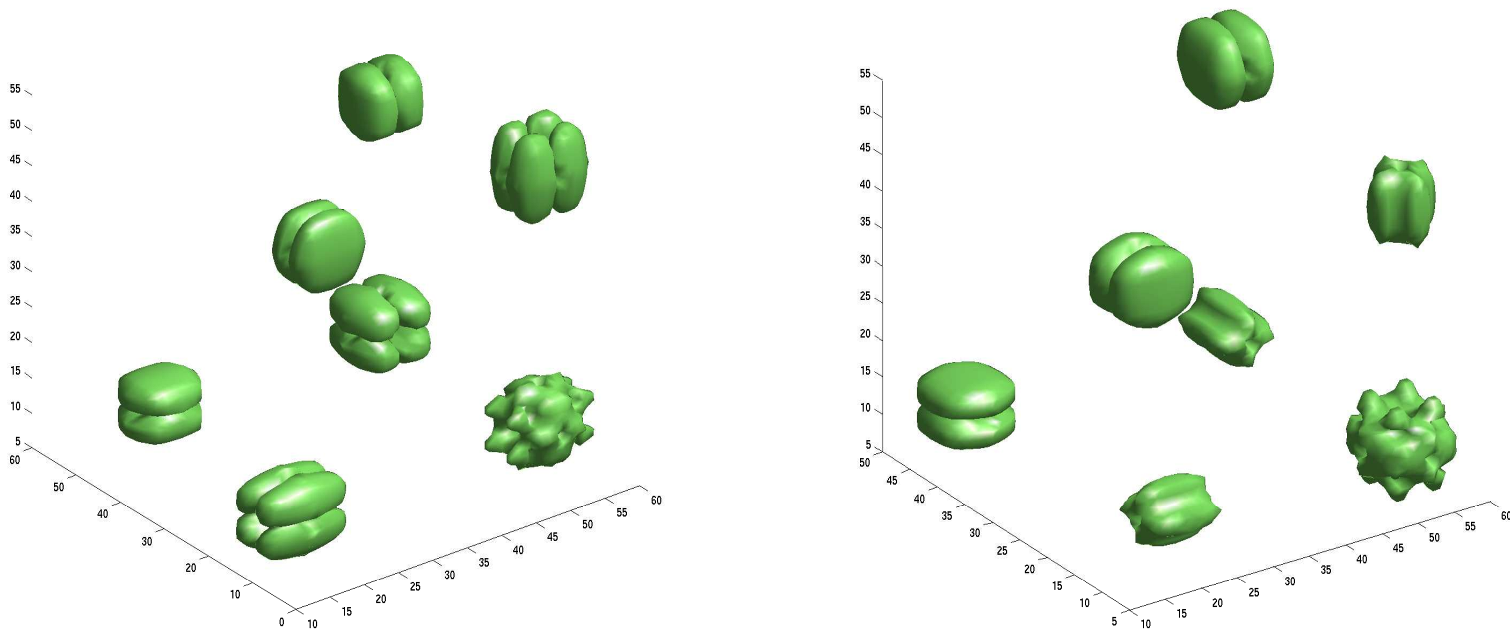
$$\Psi^{\mathrm{div}\,(0,1)}(x_1, x_2) = \left| \begin{array}{l} -\varphi_1(x_1)\psi_1'(x_2) \\ \varphi_1'(x_1)\psi_1(x_2) \end{array} \right.$$

$$\Psi^{\mathrm{div}\,(1,1)}(x_1, x_2) = \left| \begin{array}{l} \psi_1(x_1)\psi_1'(x_2) \\ -\psi_1'(x_1)\psi_1(x_2) \end{array} \right.$$

Example of 2D divergence-free wavelets



Example of 3D divergence-free wavelets



Isosurfaces of modulus of vorticity of the 14 isotropic divergence-free 3D wavelets

Anisotropic wavelets

Expressions

- divergence-free wavelets:

$$\Psi^{\mathrm{div}}_{\mathbf{j},\mathbf{k}}(x_1,x_2) = \left| \begin{array}{l} 2^{j_2}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2) \\ -2^{j_1}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2) \end{array} \right.$$

- gradient wavelets:

$$\Psi^{\mathrm{curl}}_{\mathbf{j},\mathbf{k}}(x_1,x_2) = \left| \begin{array}{l} 2^{j_1}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2) \\ 2^{j_2}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2) \end{array} \right.$$

with scale $\mathbf{j}=(j_1,j_2)\in\mathbb{Z}^2$ and position $\mathbf{k}=(k_1,k_2)\in\mathbb{Z}^2$.

2. Coherent Vortex Extraction

Aim: Perform a coherent vortex extraction of 3D turbulent field thanks to divergence-free wavelets

Criteria:

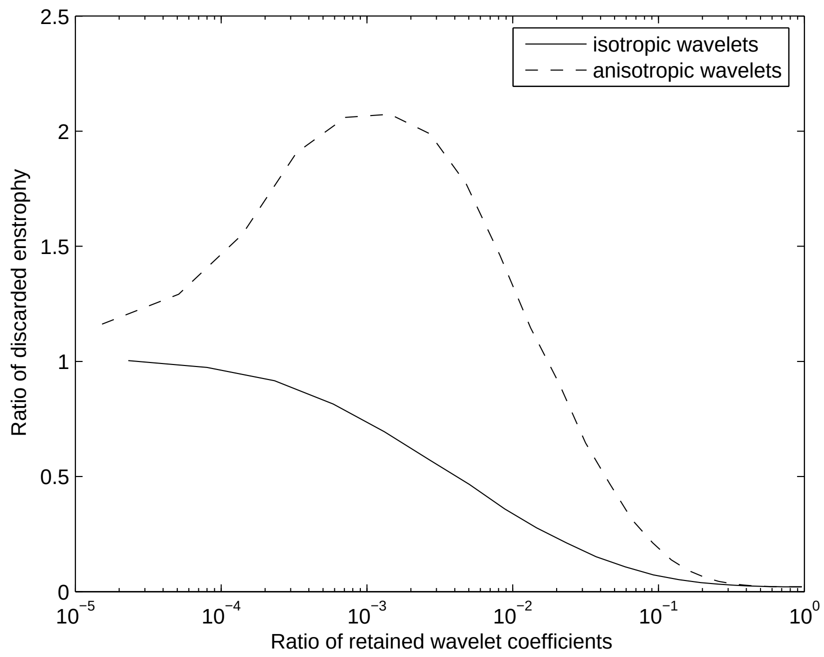
- Error in enstrophy:
  $\|\omega_{\text{total}}\|^2 = \|\omega_{\text{coh}}\|^2 + \|\omega_{\text{incoh}}\|^2 + 2\times < \omega_{\text{coh}}, \omega_{\text{incoh}} >$
- Error in energy (idem)
- Visualization, cheking for structures in incoherent part
- Fourier spectra to check the energy repartition

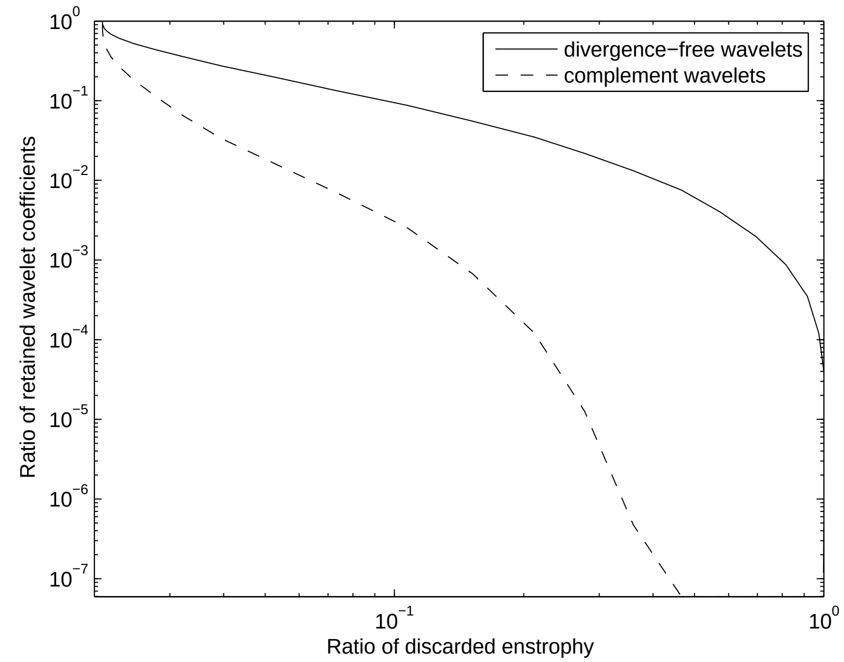
Numerical results: compression tables

We used the first one for compression:

$Decomp.field$ $Vorticity$	$Total$	$Coherent$	$Incoherent$	$Correlation$
$\%coef$	100%	3%	97%	
$Enstrophy$	71.2	73	23	-12.4
$Enstrophy(\%)$	100%	102.5%	32.3%	-34.8%
$orthogonalcase$ $Vorticity$	$Total$	$Coherent$	$Incoherent$	$Correlation$
$Enstrophy(\%)$	100%	75.5%	24.5%	0%
$biorthogonalcase$ $Vorticity$	$Total$	$Coherent$	$Incoherent$	$Correlation$
$Enstrophy(\%)$	100%	69.0%	27.3%	3.7%

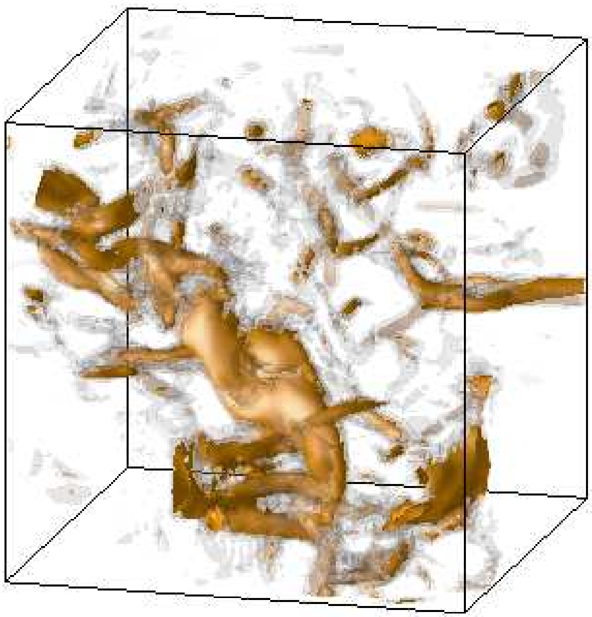
Numerical results: compression graph





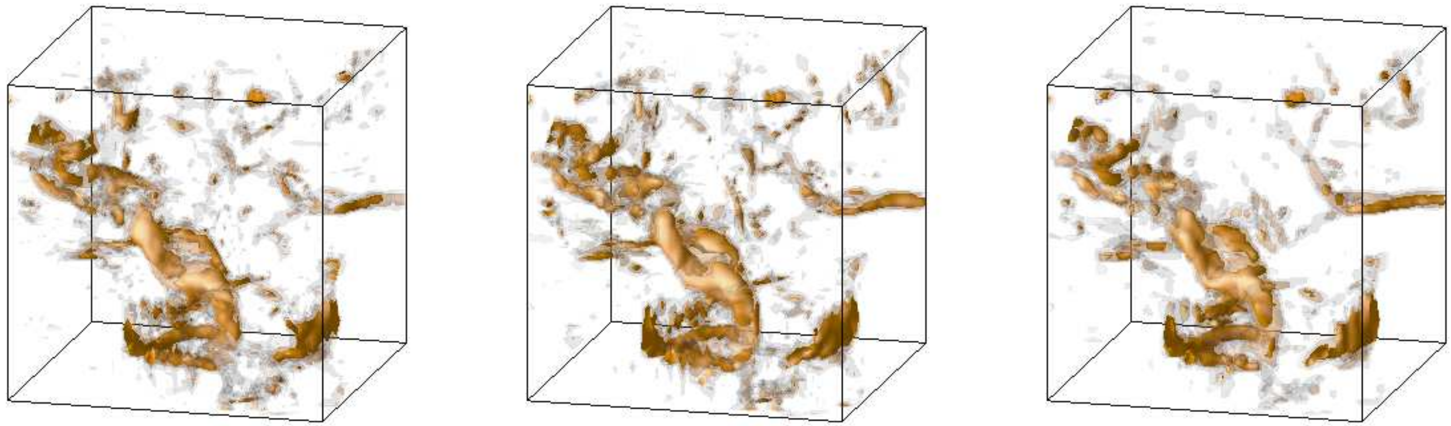
On the left, comparison between isotropic and anisotropic div-free wavelet compression in semi-log scale. On the right, contributions from div-free coefficients and complement coefficients.

Numerical results: visualization



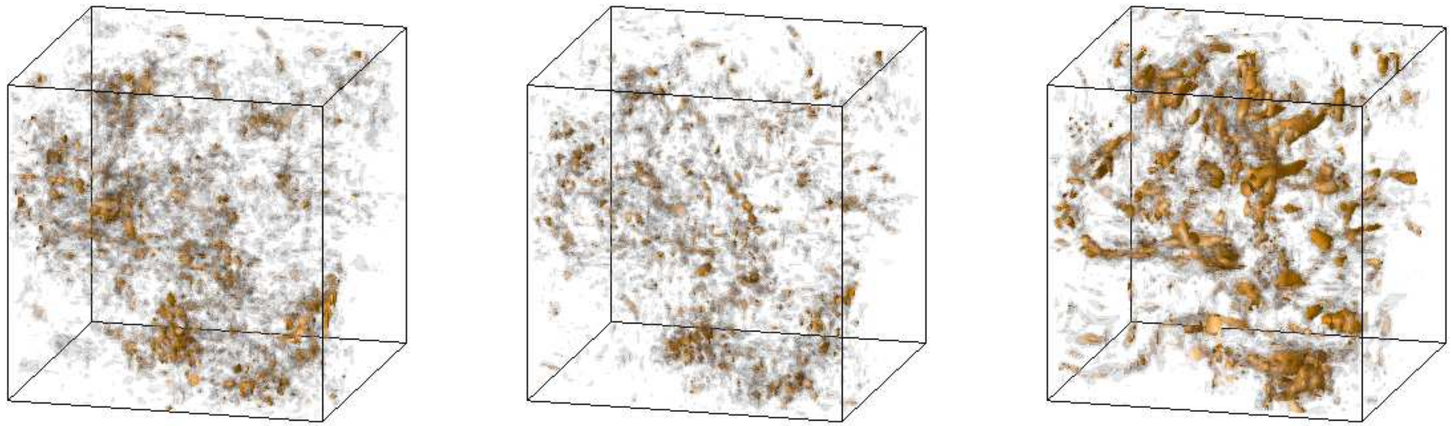
A sub-cube 64^3 *of the initial turbulent field* 256^3 *(Re=260)*

Visualization: coherent part



Coherent part with the div-free wavelets (left), the orthogonal Coifman 12 wavelets (center) and the biorthogonal Harten wavelets (right).

Visualization: incoherent part



Incoherent part with the div-free wavelets (left), the orthogonal Coifman 12 wavelets (center) and the biorthogonal Harten wavelets (right).

3. Navier-Stokes scheme

Wavelet decomposition of the NS solution

- Incompressible Navier-Stokes equations:

$$(\text{N-S}) \begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \\ \mathrm{div}\mathbf{u} = 0, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) \end{cases}$$

- Decomposition of $\mathbf{u}$ in a wavelet basis of $\mathbf{H}_{div,0}$:

$$\mathbf{u}(t,x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} d_{j,k}(t)\Psi_{j,k}^{\mathrm{div}}(x)$$

- Then we numerically solve ($\mathbb{P} =$ Leray projector):

$$\partial_t \mathbf{u} + \mathbb{P}\left[\mathbf{u}\cdot\nabla\mathbf{u}\right] - \nu\Delta\mathbf{u} = \mathbb{P}(\mathbf{f})$$

Adams-Bashford order 2 in time

Semi-implicit wavelet scheme:

- Heat kernel implicit in time.
- Wavelet discretization $\mathbf{u}(n\delta t, x) = \sum_\lambda c_{n,\lambda}\, \Psi_\lambda^{\mathrm{div}}(x)$
- At each time step n, we solve:
  intermediate step $\mathbf{u}_{n+1/2}$

$$\left(Id - \nu\frac{\delta t}{2}\Delta\right)\mathbf{u}_{n+1/2} = \mathbf{u}_n - \frac{\delta t}{2}\mathbb{P}\left[(\mathbf{u}_n \cdot \nabla)\mathbf{u}_n\right]$$

then

$$\left(Id - \nu\frac{\delta t}{2}\Delta\right)\mathbf{u}_{n+1} = \mathbf{u}_n + \delta t\left(\frac{\nu}{2}\Delta\mathbf{u}_n - \mathbb{P}\left[(\mathbf{u}_{n+1/2} \cdot \nabla)\mathbf{u}_{n+1/2}\right]\right)$$

Adaptive scheme

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f} & t \in [0,T],\ x \in \mathbb{R}^d,\ d = 2 \text{ or } 3 \\ \text{div } \mathbf{u} = \nabla \cdot \mathbf{u} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \end{cases}$$

Λ_n= set of active wavelet coefficients.

$$\mathbf{u}_N(n\delta t, x) = \sum_{\lambda \in \Lambda_n} c_{n,\lambda}\ \Psi_\lambda^{\text{div}}(x)$$

with $\#(\Lambda_n) = N$ (the set Λ_n has N elements) and $\Psi_\lambda^{\text{div}} \in \mathrm{H}_{\text{div},0} = \{\mathbf{u} \in L^2,\ \text{div}(\mathbf{u}) = 0\}$.

Thresholding

Let the expansions

$$\mathbf{u}_n = \sum_\lambda c_{n,\lambda}\ \Psi_\lambda^{\mathrm{div}}$$

and

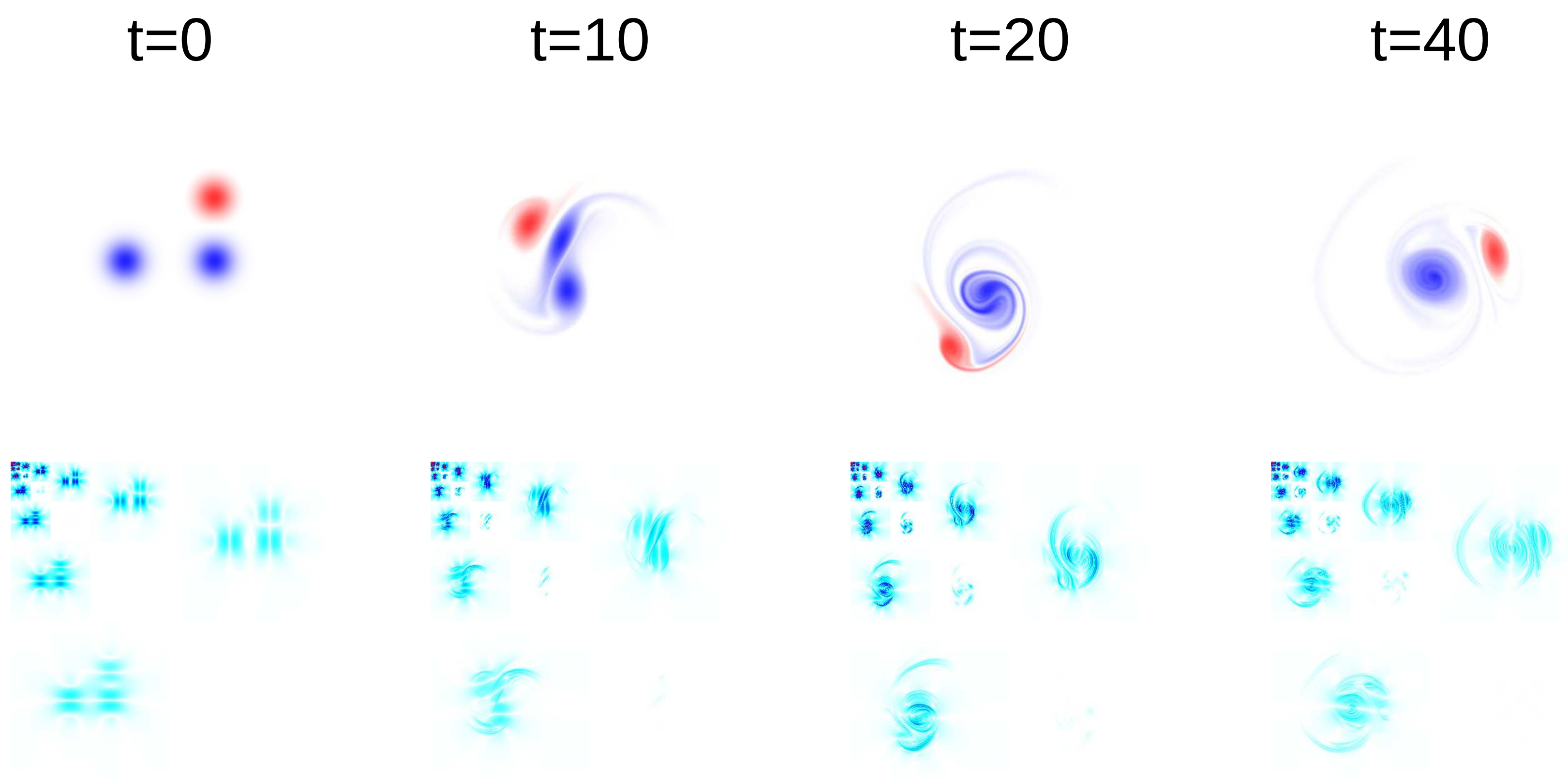
$$\mathbb{P}\left[(\mathbf{u}_n \cdot \nabla)\mathbf{u}_n - \mathbf{f}\right] = \sum_\lambda d_{n,\lambda}\ \Psi_\lambda^{\mathrm{div}}$$

And let $\sigma_{0\,n} > 0$ and $\sigma_{1\,n} > 0$ be two thresholds.

Two criteria for an element λ to be in Λ_n: $c_{n,\lambda}$ is activated if

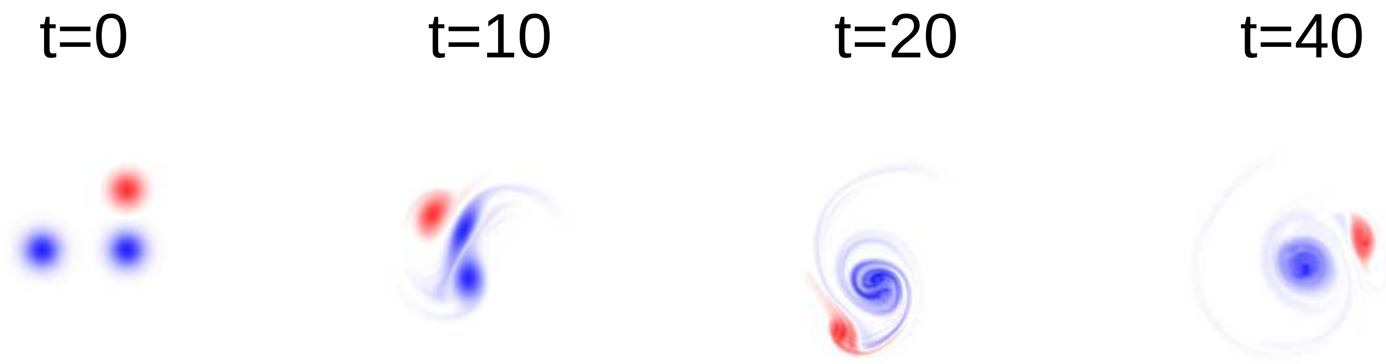
- $|c_{n,\lambda}| \geq \sigma_{0\,n}$
- or $|d_{n,\lambda}| \geq \sigma_{1\,n}$

Numerical test on the “merging of 3 vortice



Vorticity fields and wavelet coefficients on a 512^2 grid, pseudo-spectral.

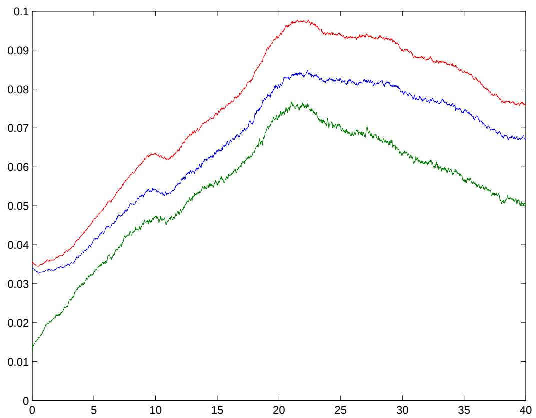
Full wavelet code



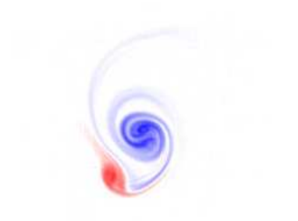
- simplest spline of degree 1 and 2 **wavelet code**
- *semi-implicit* schema of order 2 for the time evolution
- 256^2 grid, $\delta t = 0.02$ and $\nu = 5.10^{-5}$
- 7 iterations for Helmholtz, 3 for the implicit Laplacian,
- Code using **uniquely** wavelet transforms

Pseudo-adaptive code

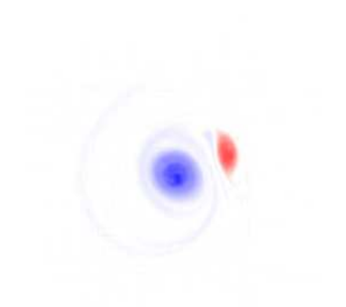
nb of active coeff



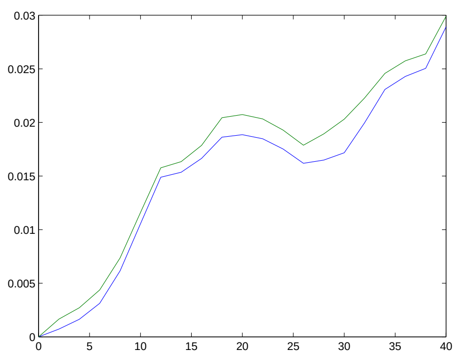
t=20



t=40



relative error



history

blue: pseudo-spectral code

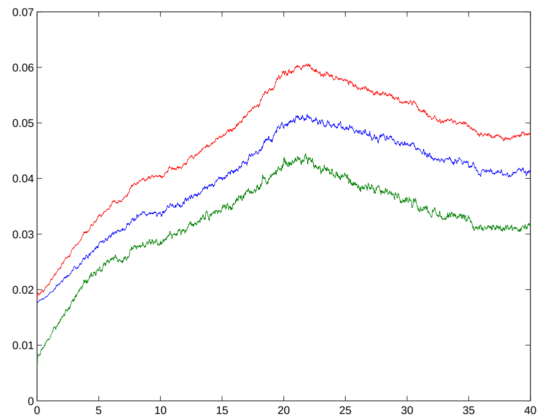
green: wavelet code

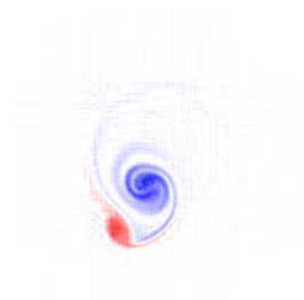
Effects of anisotropy

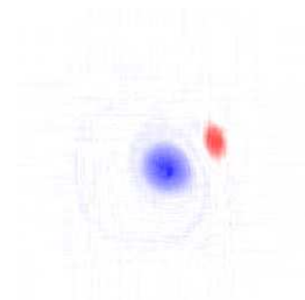
nb of active coeff

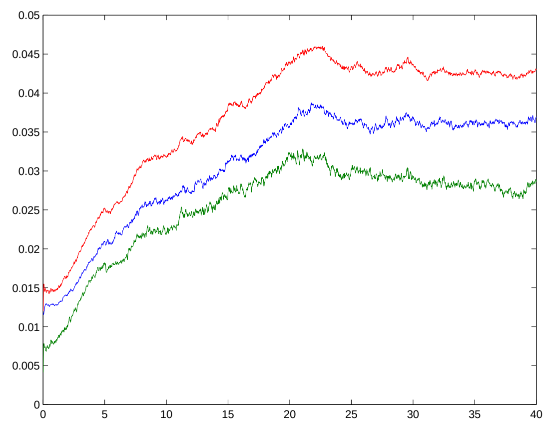
t=20

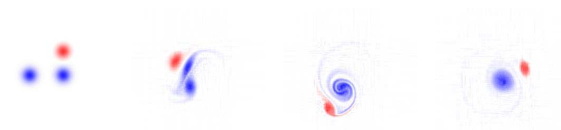
t=40

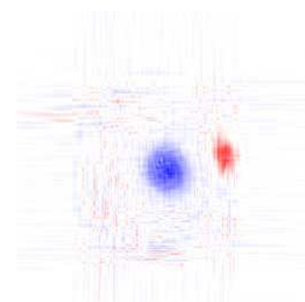








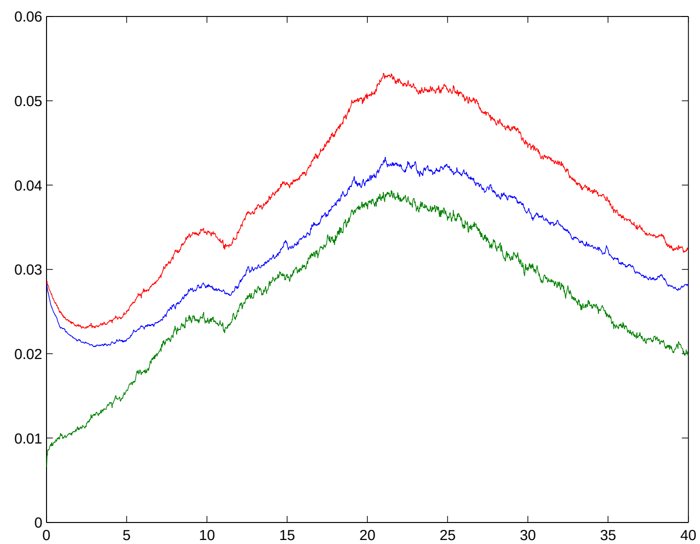




hight threshold

Generalized divergence-free wavelets

t=0 t=10 t=20 t=40



Coefficient evolution

Conclusion - Perspectives

Assets

- Computations with linear complexity ($O(n)$ operations for n the number of degrees of freedom)
- Time/frequency discretisation $\Rightarrow$ adaptivity
- Original divergence-free wavelet solver for Navier-Stokes

Perspectives

- To implement a really adaptive code $\Rightarrow$ adaptive strategy in space and in time
- make this method partially lagrangian (convecting the small vortices by the large scales of the flows)