THERMOCAPILLARY MIGRATION OF A COLLECTION OF SPHERICAL BUBBLES

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Outline

1) Capillary effects. Thermocapillary migration of one bubble

2) Challenging issues for a $N-$bubble cluster

3) Adopted framework. Governing equations

4) Available results. Review

5) Auxiliary Stokes flows and relevant surface quantities

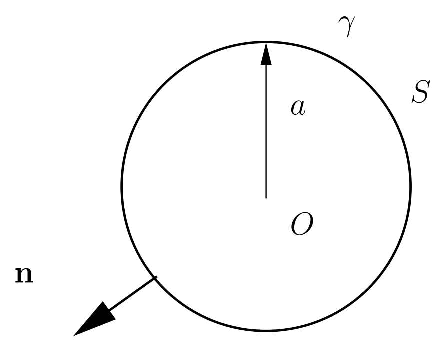
6) Advocated boundary-integral equations

7) Numerical implementation

8) Numerical comparisons and results

9) Concluding remarks

Capillary effects. Case of a single bubble



- A surface tension γ on S

For γ uniform: no tangential stress on S and motionless bubble

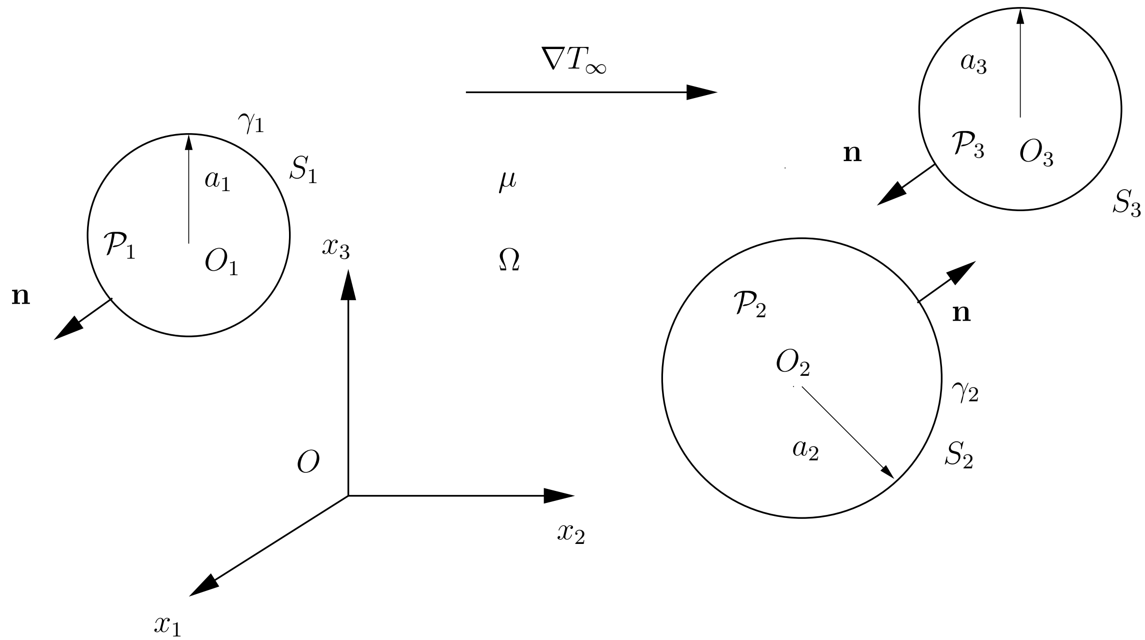
If γ becomes non uniform (for instance when temperature dependent)?

- A tangential stress on S which induces a motion of the liquid in the direction of the colder pole

- A resulting and so-called thermocapillary migration of the bubble in the direction of the applied ambient temperature gradient

General assumptions

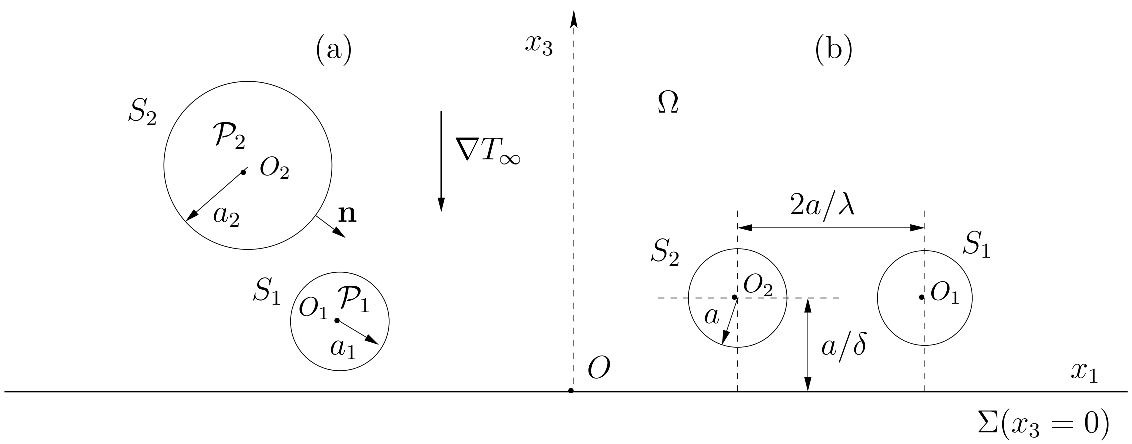
Unbounded liquid



- Newtonian liquid with uniform viscosity μ and density ρ
- External temperature gradient ∇T_∞
- N spherical bubbles $\mathcal{P}_n(O_n, a_n)$ with surface S_n

General assumptions

Bounded liquid



- Newtonian liquid with uniform viscosity μ and density ρ
- N spherical bubbles $\mathcal{P}_n(O_n, a_n)$ with surface S_n
- External temperature gradient ∇T_∞
- Σ : solid plane wall with prescribed and uniform temperature (∇T_∞ is normal to Σ)

Assumptions

- S_n has surface tension $\gamma_n(T)$ with $\gamma_n' = d\gamma_n/dT$ uniform
- S_n is non conducting. If T is the disturbed temperature one thus obtains the boundary condition

$$\nabla T.\mathbf{n} = 0 \quad \text{on} \quad S_n$$

- Torque-free bubbles
- Force-free bubbles or given buoyancy net forces on the bubbles for a prescribed uniform gravity field $\mathbf{g}$

Problem

- $\mathbf{U}^{(n)}$: translational velocity of $\mathcal{P}_n$?
- Accurate procedure to compute those velocities at a reasonable cpu time cost?

Governing equations

- Quasi-steady flow with velocity field $\mathbf{u}$ (typical scale V), pressure field $p + \rho \mathbf{g}.\mathbf{x}$ and disturbed temperature field $T = T_\infty + T'$
- Vanishing capillary number $Ca = \mu V/\gamma$ with γ a characteristic value of γ_n : negligible deformation of the bubbles
- For a the length scale and K the uniform liquid thermal diffusivity, the energy equation is

$$\nabla^2 T = Ma \nabla T.\mathbf{u}, \qquad Ma = aV/K$$

- Assuming vanishing Marangoni number

the Disturbed temperature $T = T_\infty + T'$ obeys

$$\nabla^2 T' = 0 \quad \text{in } \Omega, \qquad \nabla T' \to \mathbf{0} \qquad \text{as } r \to \infty,$$

$$\nabla T'.\mathbf{n} = -\nabla T_\infty.\mathbf{n} \;\; \text{on} \;\; S_n, \quad T' = 0 \;\; \text{on} \;\; \Sigma$$

- The Reynolds number Re is:

$$\mathrm{Re} = \rho V a/\mu, \quad \text{convection/diffusion}$$

- Neglecting inertial effects ($\mathrm{Re} \to 0$),

the Low-Reynolds-number flow $(\mathbf{u}, p)$ with stress tensor $\boldsymbol{\sigma}$ obeys

$$\mu \nabla^2 \mathbf{u} = \nabla p \text{ and } \nabla . \mathbf{u} = 0, \;\; (\mathbf{u}, p) \to (\mathbf{0}, 0) \text{ as } r \to \infty,$$

$$\mathbf{u} = \mathbf{0} \;\; \text{on} \;\; \Sigma$$

$$\mathbf{u}.\mathbf{n} = \mathbf{U}^{(n)}.\mathbf{n} \;\; \text{on} \;\; S_n$$

$$\boldsymbol{\sigma}.\mathbf{n} - [\mathbf{n}.\boldsymbol{\sigma}.\mathbf{n}]\mathbf{n} = -\gamma_n' \nabla_s (T_\infty + T') \;\; \text{on} \;\; S_n$$

- $3N$ additional conditions (uniform gravity field $\mathbf{g}$)

$$\int_{S_n} \boldsymbol{\sigma}.\mathbf{n} dS_n = 4\pi a_n^3 \rho \mathbf{g}/3$$

Available results?

1) Case of a single bubble $\mathcal{P}_1(O_1, a_1)$

- If ∇T_∞ is uniform (Young *et al.* 1959)

$$\mathbf{U}_s^{(1)} = -\frac{a_1\gamma_1'}{2\mu}\nabla T_\infty$$

- If ∇T_∞ is divergence-free and arbitrary (Subramanian 1985)

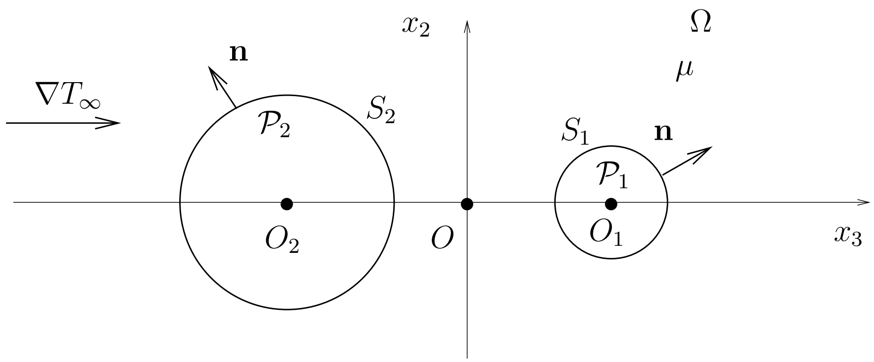
$$\mathbf{U}^{(1)} = -\frac{a_1\gamma_1'}{2\mu}\nabla T_\infty(O_1)$$

2) Case of N *equivalent* bubbles (Acrivos *et al.* 1990)

$$a_1\gamma_1' = ... = a_N\gamma_N'$$

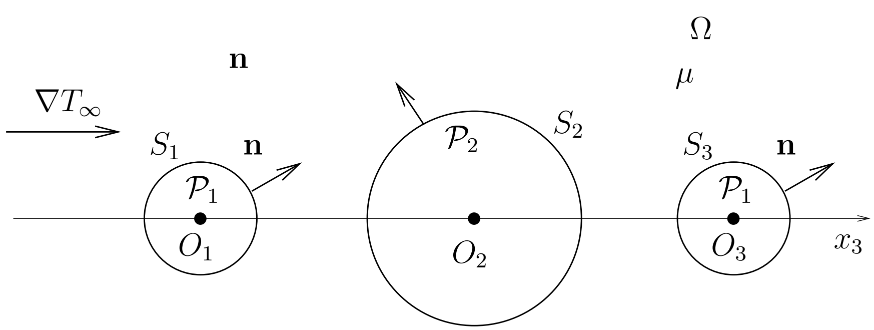
$$\mathbf{U}^{(n)} = -\frac{a_n\gamma_n'}{2\mu}\nabla T_\infty = \mathbf{U}_s^{(n)}, \;\; \nabla T_\infty \quad \text{uniform}$$

3) Case of 2 non-equivalent bubbles



- Bipolar coordinates (Meyyappan *et al.* 1983, Feuillebois 1989, Keh & Chen 1990)
- Reflections (Anderson 1985, Meyyappan & Subramanian 1984, Sun & Hu 2002)
- Multipole expansions (Satrape 1992, Wang *et al.* 1994)

4) Case of a chain of 3 non-equivalent bubbles



Multipole expansions (Keh & Chen 1992,
Wei & Subramanian 1993, Keh & Chen 1993)

5) Case of a single bubble

Meyyapan *et al.* (1981), Meyyapan *et al.* (1987), Kasumi *et al.* (2000)

Use of bipolar coordinates

- If ∇T_∞ is uniform (Young *et al.* 1959)

$$\mathbf{U}_s^{(1)} = -\frac{a_1 \gamma_1'}{2\mu} \nabla T_\infty$$

6) Case of 2 identical bubbles Kasumi *et al.* (2004)

- Finite Element Method
- bubbles touching the wall (not freely-suspended ones)
- experiments

Drawbacks

- Evaluate $(\mathbf{u}, p)$ and T in the unbounded fluid domain
- Not fully 3-D configurations of at least 3 non-equivalent bubbles

Advocated approach

- Introduce $3N$ steady Stokes flows $(\mathbf{u}_i^{(n)}, p_i^{(n)})$ with stress tensor $\boldsymbol{\sigma}_i^{(n)}$ such that

$$\mu\nabla^2\mathbf{u}_i^{(n)} = \nabla p_i^{(n)}, \;\; \nabla.\mathbf{u}_i^{(n)} = 0, \;\; (\mathbf{u}_i^{(n)}, p_i^{(n)}) \to (\mathbf{0}, 0) \;\text{ if }\; r \to \infty,$$

$$\mathbf{u}_i^{(n)}.\mathbf{n} = \delta_{nm}\mathbf{e}_i.\mathbf{n}, \;\; \boldsymbol{\sigma}_i^{(n)}.\mathbf{n} - [\mathbf{n}.\boldsymbol{\sigma}_i^{(n)}.\mathbf{n}]\mathbf{n} = \mathbf{0} \;\text{ on }\; S_m$$

$$\mathbf{u}_i^{(n)} = \mathbf{0} \;\text{ on }\; \Sigma$$

- Basic tool: the reciprocal identity

$$\int_S \mathbf{u}.\boldsymbol{\sigma}_i^{(n)}.\mathbf{n}dS = \int_S \mathbf{u}_i^{(n)}.\boldsymbol{\sigma}.\mathbf{n}dS, \;\; S = \cup_{m=1}^{N} S_m$$

- Express the required conditions

$$\mathbf{e}_i.\int_{S_n} \boldsymbol{\sigma}.\mathbf{n}dS_n = 4\pi a_n^3 \rho \mathbf{g}.\mathbf{e}_i/3$$

Key linear system

3N linear equations for $U_j^{(m)} = \mathbf{U}^{(m)}.\mathbf{e}_j$

$$\sum_{m=1}^{N}\sum_{j=1}^{3} A_{ij}^{(n),(m)} U_j^{(m)} = \sum_{m=1}^{N} \int_{S_m} \gamma'_m(\delta_{nm}\mathbf{e}_i - \mathbf{u}_i^{(n)}).\nabla_s[T_\infty + T']dS_m - 4\pi a_n^3 \rho \mathbf{g}.\mathbf{e}_i/3$$

$$A_{ij}^{(n),(m)} = \int_{S_m} (\mathbf{e}_j.\mathbf{n})(\mathbf{n}.\boldsymbol{\sigma}_i^{(n)}.\mathbf{n})dS_m$$

Well-posed problem (unique solution)

- $\mathbf{A}$ is real-valued, symmetric and negative-definite matrix

- one solely needs to determine surface quantities

Relevant boundary-integral equations

- Fredholm boundary-integral equation of the second kind for T' on the entire surface S

$$-4\pi T'(\mathbf{x}) + \int_{S\setminus S_m} T'(\mathbf{y})\frac{(\mathbf{x}-\mathbf{y}).\mathbf{n}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}dS(\mathbf{y}) + \int_{S_m}[T'(\mathbf{y}) - T'(\mathbf{x})]\frac{(\mathbf{x}-\mathbf{y}).\mathbf{n}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}dS(\mathbf{y})$$

$$-\int_S T'(\mathbf{y})\frac{(\mathbf{x}'-\mathbf{y}).\mathbf{n}(\mathbf{y})}{|\mathbf{x}'-\mathbf{y}|^3}dS(\mathbf{y}) = \int_S[\nabla T_\infty.\mathbf{n}](\mathbf{y})[\frac{1}{|\mathbf{x}'-\mathbf{y}|} - \frac{1}{|\mathbf{x}-\mathbf{y}|}]dS(\mathbf{y}), \quad \mathbf{x} \text{ on S}$$

Numerical evaluation of $\nabla T'$ with $\nabla T'.\mathbf{n} = \nabla T_\infty.\mathbf{n}$

- For the specific Stokes flows $(\mathbf{u}_i^{(n)}, p_i^{(n)})$

$a_i^{(n)} = \boldsymbol{\sigma}_i^{(n)}.\mathbf{n}/\mu, \ \mathbf{a}_i^{(n)} = \mathbf{u}_i^{(n)} - (\mathbf{u}_i^{(n)}.\mathbf{n})\mathbf{n}$ unknown quantities on S_m

$d_i^{(n)} = \mathbf{u}_i^{(n)}.\mathbf{n}$ given quantity on S_m

Issue: how to compute $\mathbf{a}_i^{(n)}$ and $a_i^{(n)}$?

Velocity integral representation

$$u_j(\mathbf{x}) = \frac{1}{8\pi}\int_S \Big\{ u_k(\mathbf{y}) T'_{kjl}(\mathbf{y},\mathbf{x}) n_l(\mathbf{y}) - G'_{kj}(\mathbf{y},\mathbf{x})[\frac{\mathbf{e}_k.\boldsymbol{\sigma}.\mathbf{n}}{\mu}](\mathbf{y})\Big\} dS$$

with, for $\mathbf{x}'(x_1, x_2, -x_3)$, $c_1 = c_2 = 1$ and $c_3 = -1$,

$$G'_{kj}(\mathbf{y},\mathbf{x}) = \frac{\delta_{kj}}{|\mathbf{x}-\mathbf{y}|} + \frac{[(\mathbf{x}-\mathbf{y}).\mathbf{e}_k][(\mathbf{x}-\mathbf{y}).\mathbf{e}_j]}{|\mathbf{x}-\mathbf{y}|^3} - \frac{2c_j(\mathbf{x}.\mathbf{e}_3)}{|\mathbf{x}'-\mathbf{y}|^3}\Big\{\delta_{k3}(\mathbf{x}'-\mathbf{y}).\mathbf{e}_j$$
$$-\delta_{j3}(\mathbf{x}'-\mathbf{y}).\mathbf{e}_k + (\mathbf{y}.\mathbf{e}_3)[\delta_{kj} - \frac{3[(\mathbf{x}'-\mathbf{y}).\mathbf{e}_k][(\mathbf{x}'-\mathbf{y}).\mathbf{e}_j]}{|\mathbf{x}'-\mathbf{y}|^2}]\Big\}$$
$$-\{\frac{\delta_{kj}}{|\mathbf{x}'-\mathbf{y}|} + \frac{[(\mathbf{x}'-\mathbf{y}).\mathbf{e}_k][(\mathbf{x}'-\mathbf{y}).\mathbf{e}_j]}{|\mathbf{x}'-\mathbf{y}|^3}\},$$

$$T'_{kjl}(\mathbf{y},\mathbf{x}) = \frac{6[(\mathbf{x}-\mathbf{y}).\mathbf{e}_k][(\mathbf{x}-\mathbf{y}).\mathbf{e}_j][(\mathbf{x}-\mathbf{y}).\mathbf{e}_l]}{|\mathbf{x}-\mathbf{y}|^5} - \frac{6[(\mathbf{x}'-\mathbf{y}).\mathbf{e}_k][(\mathbf{x}'-\mathbf{y}).\mathbf{e}_j][(\mathbf{x}'-\mathbf{y}).\mathbf{e}_l]}{|\mathbf{x}'-\mathbf{y}|^5}$$
$$-\frac{12c_j(\mathbf{x}.\mathbf{e}_3)}{|\mathbf{x}'-\mathbf{y}|^5}\Big\{\delta_{kl}[(\mathbf{x}'-\mathbf{y}).\mathbf{e}_j][(\mathbf{x}'-\mathbf{y}).\mathbf{e}_3] - \delta_{j3}[(\mathbf{x}'-\mathbf{y}).\mathbf{e}_k][(\mathbf{x}'-\mathbf{y}).\mathbf{e}_l]$$
$$-(\mathbf{y}.\mathbf{e}_3)\Big(\delta_{kl}[(\mathbf{x}'-\mathbf{y}).\mathbf{e}_j] + \delta_{kj}[(\mathbf{x}'-\mathbf{y}).\mathbf{e}_l] + \delta_{jl}[(\mathbf{x}'-\mathbf{y}).\mathbf{e}_k]$$
$$+\frac{5[(\mathbf{x}'-\mathbf{y}).\mathbf{e}_k][(\mathbf{x}'-\mathbf{y}).\mathbf{e}_j][(\mathbf{x}'-\mathbf{y}).\mathbf{e}_l]}{|\mathbf{x}'-\mathbf{y}|^5}\Big)\Big\}$$

Associated boundary-integral equations

$a = \boldsymbol{\sigma}.\mathbf{n}/\mu, \;\; \mathbf{a} = \mathbf{u} - (\mathbf{u}.\mathbf{n})\mathbf{n} = a_k\mathbf{e}_k$ unknown quantities on S_m

$d = \mathbf{u}.\mathbf{n}, \;\; \mathbf{d} = [\boldsymbol{\sigma}.\mathbf{n} - (\boldsymbol{\sigma}.\mathbf{n}).\mathbf{n}]/\mu = d_k\mathbf{e}_k$ given quantity on S_m

Coupled boundary-integral equations on S

$$
\begin{aligned}
&[\mathbf{a}.\mathbf{n}](\mathbf{x}) = [a_k n_k](\mathbf{x}) = 0 \text{ on } S_m, \\
&8\pi a_j(\mathbf{x}) - \int_{S_m} [a_k(\mathbf{y}) - a_k(\mathbf{x})] T'_{kjl}(\mathbf{y},\mathbf{x}) n_l(\mathbf{y}) dS \\
&- \int_{S\setminus S_m} a_k(\mathbf{y}) T'_{kjl}(\mathbf{y},\mathbf{x}) n_l(\mathbf{y}) + \int_S G'_{kj}(\mathbf{y},\mathbf{x}) n_k(\mathbf{y}) a(\mathbf{y}) dS \\
= &\int_{S\setminus S_m} [dn_k](\mathbf{y}) T'_{kjl}(\mathbf{y},\mathbf{x}) n_l(\mathbf{y}) - \int_S G'_{kj}(\mathbf{y},\mathbf{x}) n_k(\mathbf{y}) d_k(\mathbf{y}) dS \\
&\int_{S_m} \{[dn_k](\mathbf{y}) - [dn_k](\mathbf{x})\} T'_{kjl}(\mathbf{y},\mathbf{x}) n_l(\mathbf{y}) dS - 8\pi [dn_j](\mathbf{x}), \mathbf{x} \text{ on } S_m.
\end{aligned}
$$

Advocated numerical implementation

- Isoparametric triangular curvilinear Boundary Elements on each S_n
- $\mathbf{c}_m$ unit vector on S_m with $\mathbf{t}_1 = \mathbf{n} \wedge \mathbf{c}_m \neq \mathbf{0}, \mathbf{t}_2 = \mathbf{n} \wedge \mathbf{t}_1 \neq \mathbf{0}$ and $\mathbf{a} = a^1\mathbf{t}_1 + a^2\mathbf{t}_2$
- Solve discretized systems $AX = Y$ by Gaussian elimination
- For a $N-$bubble cluster solve $3N + 1$ linear systems

Mobility coefficients

For ∇T_∞ uniform

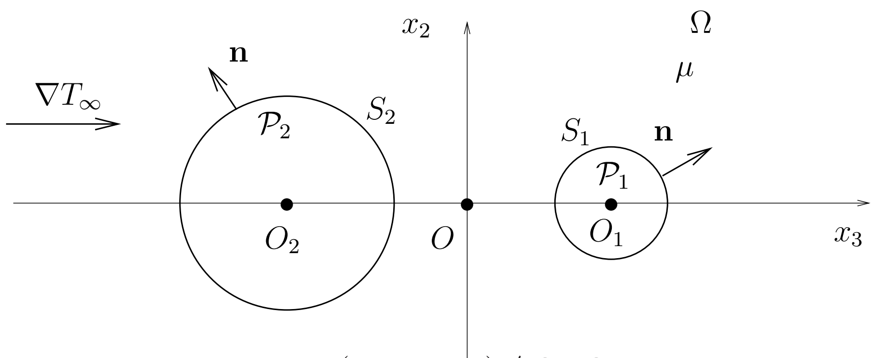
$$\mathbf{U}^{(n)} = \sum_{m=1}^{N} \mathbf{M}^{(n),(m)}.\mathbf{U}_s^{(m)}, \ \mathbf{U}_s^{(m)} = -\frac{\gamma'_m a_m}{2\mu}\nabla T_\infty$$

$$\mathbf{M}^{(n),(m)} = \mathbf{M}_{ij}^{(n),(m)}\mathbf{e}_i \otimes \mathbf{e}_j$$

Properties

$$\sum_{m=1}^{N} \mathbf{M}_{ii}^{(n),(m)} = 1, \ \sum_{m=1}^{N} \mathbf{M}_{ij}^{(n),(m)} = 0 \ \text{ if } \ j \neq i$$

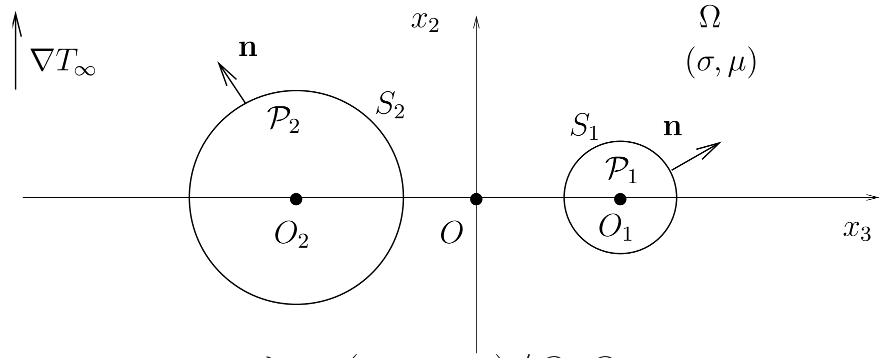
Numerical comparisons for 2 bubbles: axisymmetric Case



$$0 \leq \lambda = (a_1 + a_2)/O_1O_2 < 1$$

N	λ	M_{11}	M_{12}	M_{21}	M_{22}
74	3/23	0.97201	0.00116	0.00015	0.97203
242	3/23	0.99535	0.00074	0.00012	0.99592
1058	3/23	0.99903	0.00067	0.00008	0.99960
M[83]	3/23	0.99934	0.00066	0.00008	0.99992
74	0.6	0.90966	0.06562	0.00989	0.96343
242	0.6	0.93081	0.06545	0.00919	0.98691
1058	0.6	0.93418	0.06548	0.00912	0.99057
M[83]	0.6	0.93447	0.06553	0.00912	0.99088
74	10/11	0.59198	0.15698	0.08536	0.95447
242	10/11	0.73274	0.25997	0.04577	0.95183
1058	10/11	0.73099	0.26856	0.04699	0.95266
S[92]	10/11	0.73136	0.26864	0.04700	0.95300
M[83]	10/11	0.73106	0.26894	0.04650	0.95350

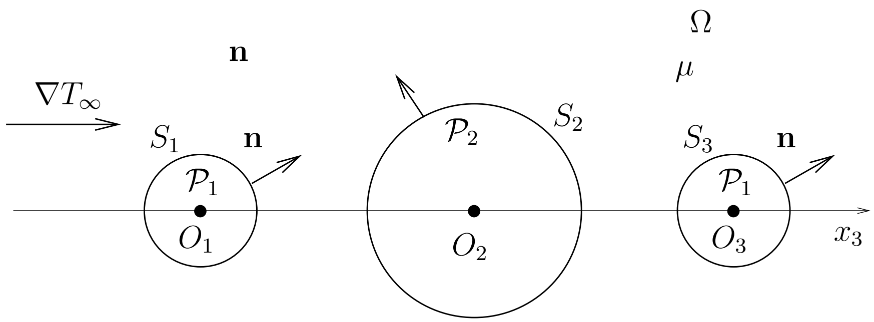
Numerical comparisons for 2 bubbles: asymmetric Case



$$0 \leq \lambda = (a_1 + a_2)/O_1O_2 < 1$$

N	λ	$M_{11}^{(1)}$	$M_{12}^{(1)}$	$M_{11}^{(2)}$	$M_{12}^{(2)}$	$M_{21}^{(2)}$	$M_{22}^{(2)}$
242	0.2	0.9969	-0.0004	0.9976	-0.0011	-0.0001	0.9965
1058	0.2	1.0002	-0.0005	1.0009	-0.0012	-0.0001	0.9999
K[93]	0.2	1.0005	-0.0005	1.0012	-0.0012	-0.0002	1.0002
242	0.6	1.0100	-0.0134	1.0284	-0.0315	-0.0040	1.0005
1058	0.6	1.0134	-0.0137	1.0319	-0.0321	-0.0041	1.0038
K[93]	0.6	1.0137	-0.0137	1.0322	-0.0322	-0.0041	1.0041
242	0.9	1.0447	-0.0477	1.1063	-0.1090	-0.0149	1.0116
1058	0.9	1.0483	-0.0486	1.1105	-0.1107	-0.0153	1.0150
K[93]	0.9	1.0486	-0.0486	1.1109	-0.1109	-0.0153	1.0153
242	0.95	1.0543	-0.0573	1.1270	-0.1298	-0.0181	1.0149
1058	0.95	1.0580	-0.0583	1.1312	-0.1314	-0.0185	1.0183
K[93]	0.95	1.0583	-0.0583	1.1317	-0.1317	-0.0185	1.0185

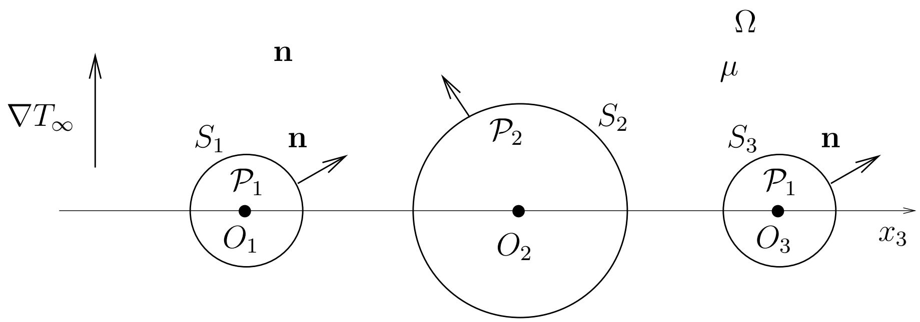
Numerical comparisons for a 3-bubble chain: axisymmetric Case



$$0 \leq \lambda = (a_1 + a_2)/O_1O_2 < 1,\ \ O_1O_2 = O_2O_3,\ \ a_1 = a_3$$

(a_1, a_2, a_3)	N	λ	M_{11}	M_{21}	M_{31}	M_{12}	M_{22}
$(1, 1, 1)$	242	0.9	0.8612	0.1170	0.0444	0.0894	0.7609
$(1, 1, 1)$	1058	0.9	0.8635	0.1186	0.0453	0.0908	0.7624
$(1, 1, 1)$	K[92]	0.9	0.8639	0.1186	0.0453	0.0909	0.7627
$(1, 2, 1)$	242	0.9	0.7421	0.0426	0.0226	0.2294	0.9134
$(1, 2, 1)$	1058	0.9	0.7410	0.0435	0.0235	0.2351	0.9126
$(1, 2, 1)$	K[92]	0.9	0.7413	0.0435	0.0235	0.2352	0.9130
$(2, 1, 2)$	242	0.4	0.9912	0.0191	0.0026	0.0024	0.9582
$(2, 1, 2)$	1058	0.4	0.9949	0.0190	0.0025	0.0023	0.9617
$(2, 1, 2)$	K[92]	0.4	0.9952	0.0190	0.0025	0.0023	0.9620
$(2, 1, 2)$	242	0.9	0.9248	0.2233	0.0545	0.0173	0.5457
$(2, 1, 2)$	1058	0.9	0.9257	0.2284	0.0564	0.0176	0.5426
$(2, 1, 2)$	K[92]	0.9	0.9261	0.2285	0.0564	0.0176	0.5430

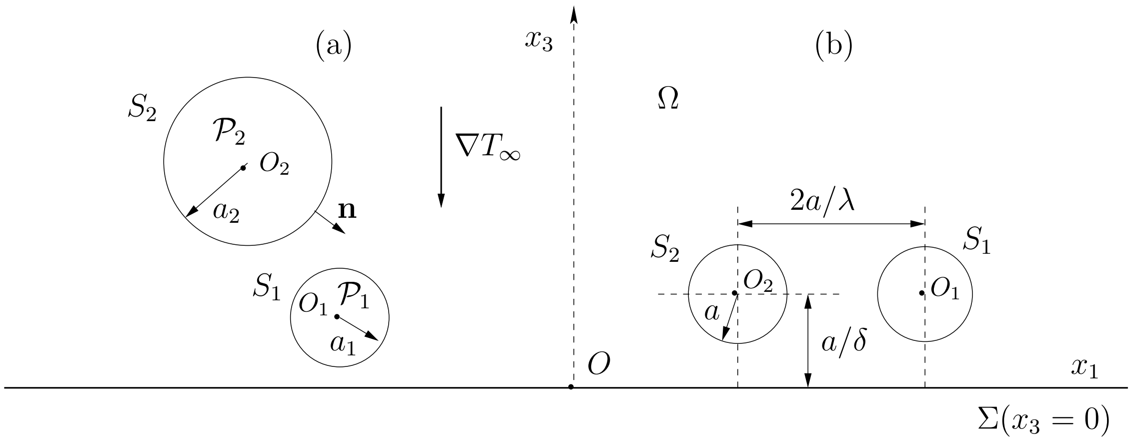
Numerical comparisons for a 3-bubble chain: asymmetric Case



$$0 \leq \lambda = (a_1 + a_2)/O_1O_2 < 1, \;\; O_1O_2 = O_2O_3, \;\; a_1 = a_3$$

(a_1, a_2, a_3)	N	λ	M_{11}	M_{21}	M_{31}	M_{12}	M_{22}
$(1, 1, 1)$	242	0.9	1.0519	-0.0483	-0.0066	-0.0482	1.0940
$(1, 1, 1)$	1058	0.9	1.0556	-0.0491	-0.0068	-0.0490	1.0980
$(1, 1, 1)$	K[93]	0.9	1.0559	-0.0492	-0.0068	-0.0491	1.0983
$(1, 2, 1)$	242	0.9	1.1087	-0.0149	-0.0026	-0.1088	1.0269
$(1, 2, 1)$	1058	0.9	1.1129	-0.0153	-0.0027	-0.1103	1.0303
$(1, 2, 1)$	K[93]	0.9	1.1132	-0.0153	-0.0027	-0.1105	1.0306
$(2, 1, 2)$	242	0.4	0.9988	-0.0093	-0.0011	-0.0011	1.0153
$(2, 1, 2)$	1058	0.4	1.0021	-0.0095	-0.0012	-0.0012	1.0187
$(2, 1, 2)$	K[93]	0.4	1.0024	-0.0095	-0.0012	-0.0012	1.0190
$(2, 1, 2)$	242	0.9	1.0269	-0.1127	-0.0138	-0.0161	1.2233
$(2, 1, 2)$	1058	0.9	1.0304	-0.1143	-0.0141	-0.0165	1.2286
$(2, 1, 2)$	K[93]	0.9	1.0307	-0.1145	-0.0141	-0.0165	1.2290

Numerical results for a bounded liquid



2 equal bubbles (radius a) with $\gamma_2' = \eta \gamma_1'$

$$\delta = a/\mathbf{OO_1}.\mathbf{e}_3, \ \lambda = 2a/O_1O_2$$

Normalized velocities $\mathbf{v}^{(n)}$

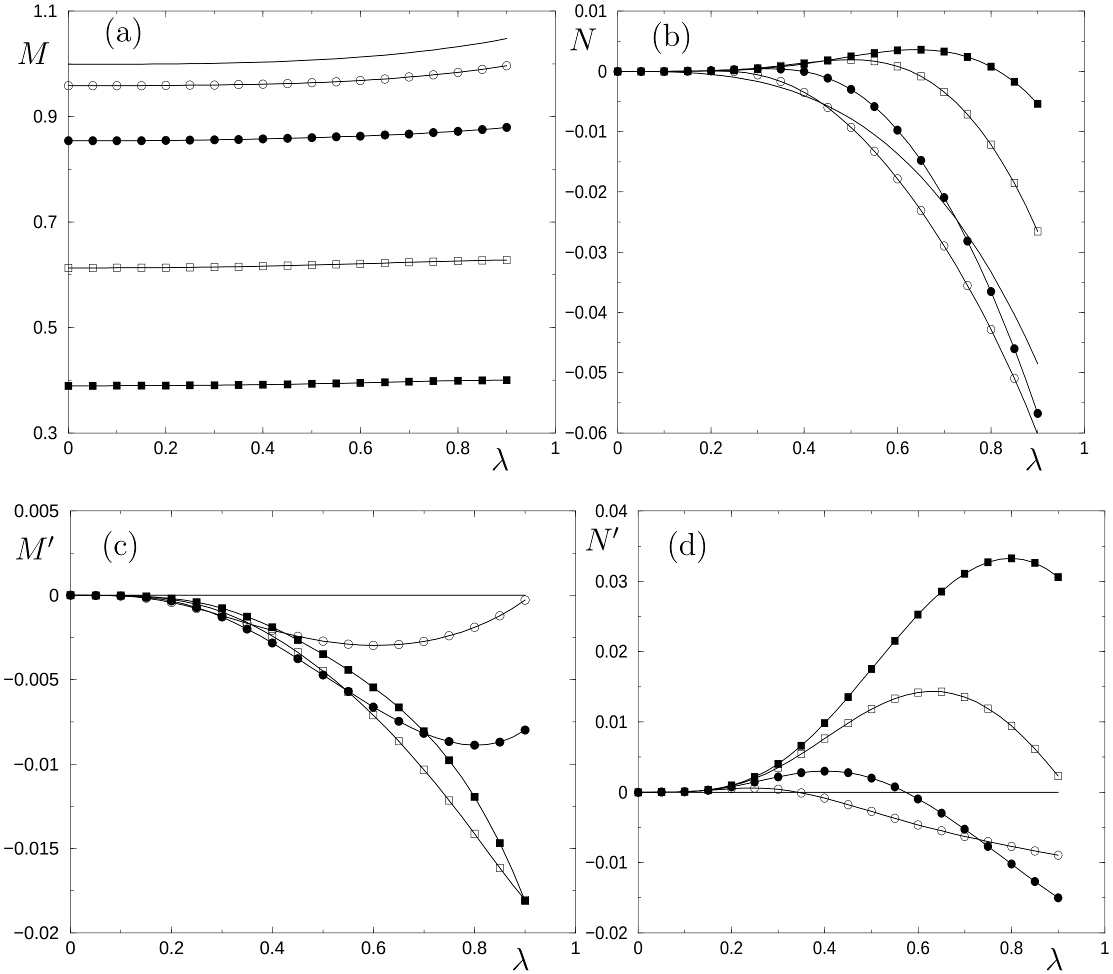
$$\mathbf{v}^{(n)} = -2\mu\mathbf{U}^{(n)}/[\gamma_n' a_n \nabla T_\infty.\mathbf{e}_3]$$

Mobility coefficients M, N, M' and N'

$$\mathbf{v}^{(1)} = [M' + \eta N']\mathbf{e}_1 + [M + \eta N]\mathbf{e}_3, \quad \mathbf{v}^{(2)} = -[M' + N'/\eta]\mathbf{e}_1 + [M + N/\eta]\mathbf{e}_3$$

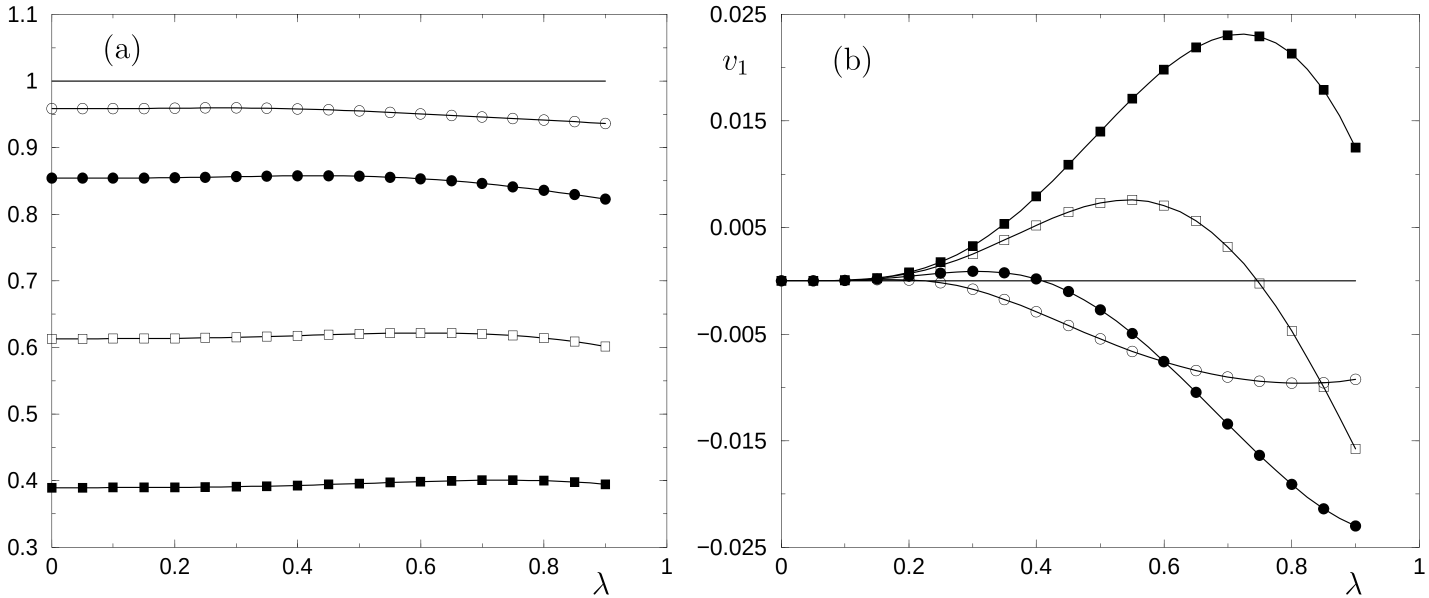
Comparisons ($\lambda_3 = 1.0811$)

(M, N)	(λ, δ)	$N_1 = 74$	$N_1 = 242$	$N_1 = 530$	$N_1 = 1058$	[previous]
M	(0,0.2)	0.9664	0.9909	0.9943	0.9947	0.9950
M	(0,2/3)	0.7724	0.7914	0.7943	0.7945	0.7948
M	$(0, \lambda_3)$	0.3388	0.3197	0.3139	0.3139	0.3145
M	(0.2,0)	0.9726	0.9965	0.9998	1.0002	1.0005
N	(0.2,0)	-0.0002	-0.0005	-0.0005	-0.0005	-0.0005
M	(0.6,0)	0.9852	1.0096	1.0130	1.0134	1.0137
N	(0.6,0)	-0.0124	-0.0135	-0.0137	-0.0137	-0.0137
M	(0.9,0)	1.0194	1.0441	1.0479	1.0483	1.0486
N	(0.9,0)	-0.0464	-0.0485	-0.0485	-0.0486	-0.0486



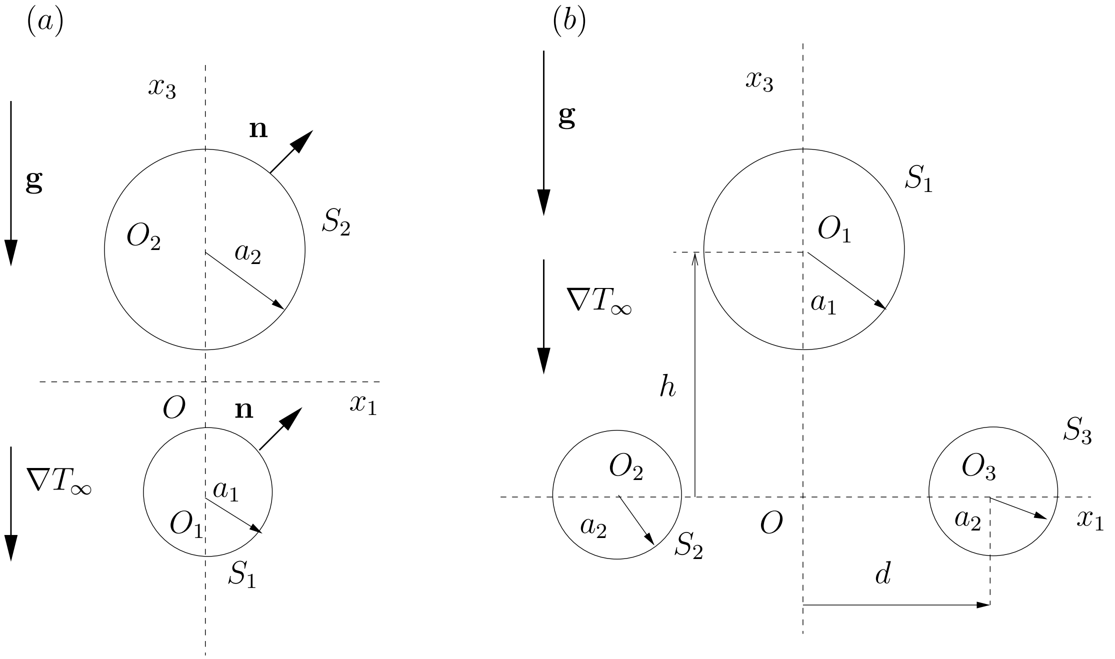
Coefficients M, N, M' and N' for $\delta = 0$ (solid curves), $\delta = 0.4\,(\circ), \delta = 0.6\,(\bullet), \delta = 0.8\,(\square)$ and $\delta = 0.9\,(\blacksquare)$.

Normalized velocities $v_i = \mathbf{v}^{(1)}.\mathbf{e}_i$ for identical bubbles $(\eta = 1)$



Normalized velocities $v_i = \mathbf{v}^{(1)}.\mathbf{e}_i$ for identical bubbles $(a_2 = a_1, \eta = 1)$ with $i = 1, 2$ and $\delta = 0$ (solid curves), $\delta = 0.4$ ($\circ$), $\delta = 0.6$ ($\bullet$), $\delta = 0.8$ ($\square$), $\delta = 0.9$ ($\blacksquare$). (a) v_3. (b) v_1.

Results for combined buoyancy and thermocapillary effects



- $\mathbf{g} = -g\mathbf{e}_3$ and $\nabla T_\infty = -\alpha \mathbf{e}_3$ with $g > 0$ and $\alpha > 0$

- Competition between gravity and thermocapillary effects ($\gamma'_n < 0$)

$$G_n = -\frac{\rho a_n g}{3\gamma'_n \alpha} > 0$$ Bond number

Single bubble

$$\mathbf{U}^{(n)} = \frac{a_n^2 \rho g}{3\mu}[1 - \frac{1}{G_n}]\mathbf{e}_3$$

- Critical value $G_n = 1/2$
- $G_n > 1/2$: gravity>thermocapillarity
- $G_n < 1/2$: gravity<thermocapillarity
- flow structure (also for $Ma > 0$) given by Merritt, Morton and Subramanian (1993)

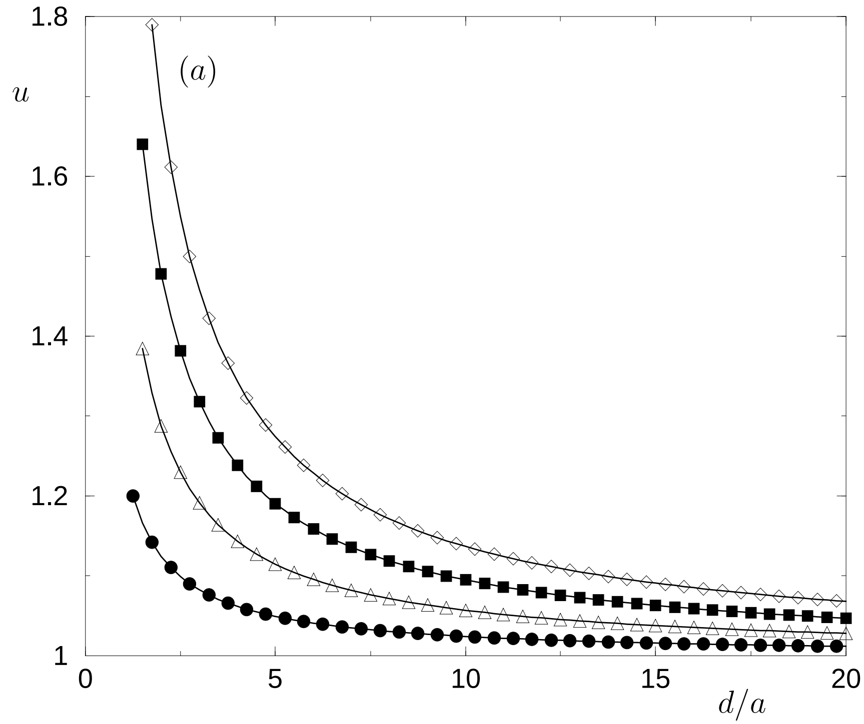
Case of two bubbles

- Treated by Wei and Subramanian (1995) for an arbitrary orientation of the line of centers with respect to the direction of gravity $\mathbf{g}$
- Results nicely recovered by the present approach

Case of $N(2, ..., 5)$ equal bubbles $(a_{(n)} = a, \gamma'_n = \gamma')$ located at the vortices of a regular horizontal polygon

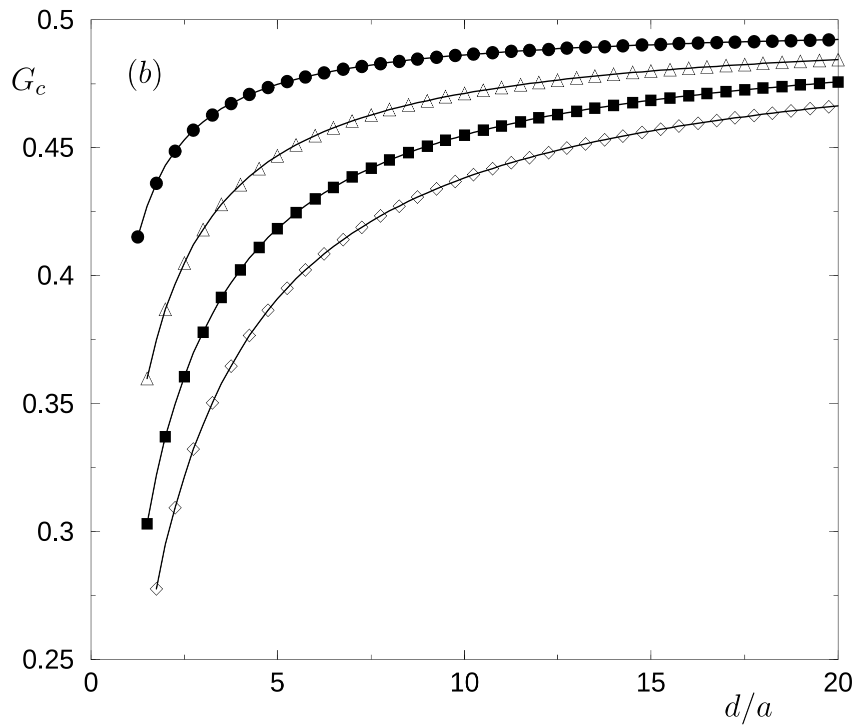
$$\mathbf{OO}_n = d(\cos\alpha_n \mathbf{e}_1 + \sin\alpha_n \mathbf{e}_2), \quad \alpha_n = 2\pi(n-1)/N$$

$$\mathbf{U}^{(n)} = a^2 \rho g u/(3\mu)\mathbf{e}_3, \qquad G = -\frac{\rho a g}{3\gamma'\alpha} > 0$$



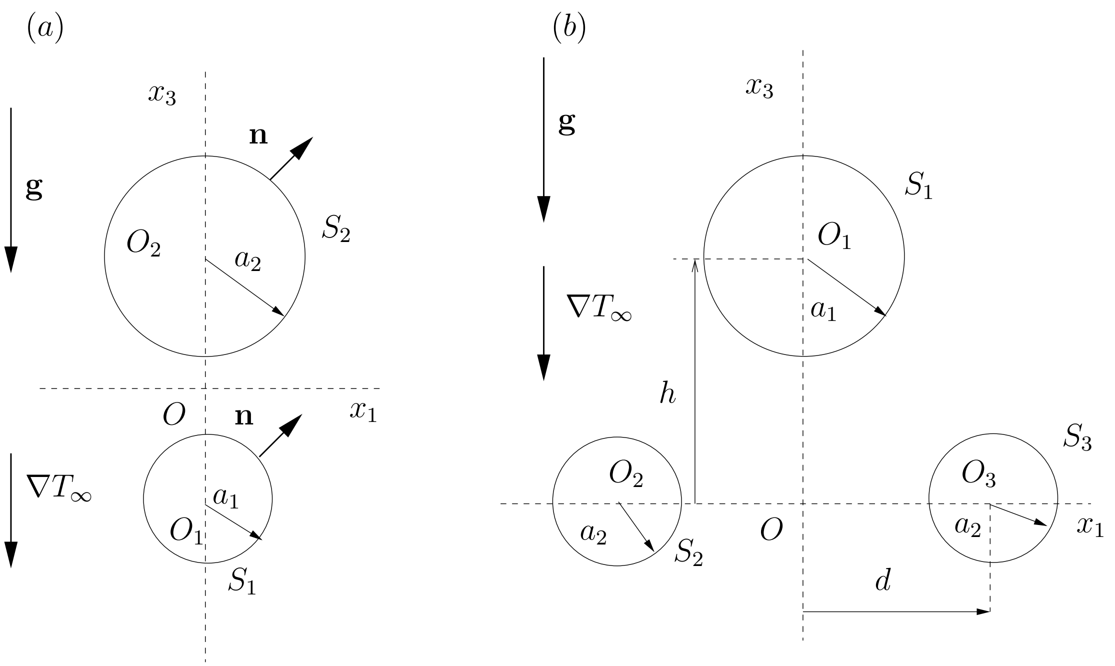
(a) Normalized velocity u for $\nabla T_\infty = \mathbf{0}$.
$N = 2\,(\bullet)$, $N = 3\,(\Delta)$, $N = 4\,(\blacksquare)$ and $N = 5\,(\diamond)$.

Critical Bond number such that $\mathbf{U}^{(n)} = \mathbf{0}$ if $\nabla T_\infty \neq \mathbf{0}$



(b) Critical parameter G_c versus d/a for $N = 2\,(\bullet)$, $N = 3\,(\Delta)$, $N = 4\,(\blacksquare)$ and $N = 5\,(\diamond)$

Case of a 3-bubble cluster of non-equivalent bubbles $(a_1 = 2a, a_2 = a_3 = a, \gamma'_n = \gamma')$



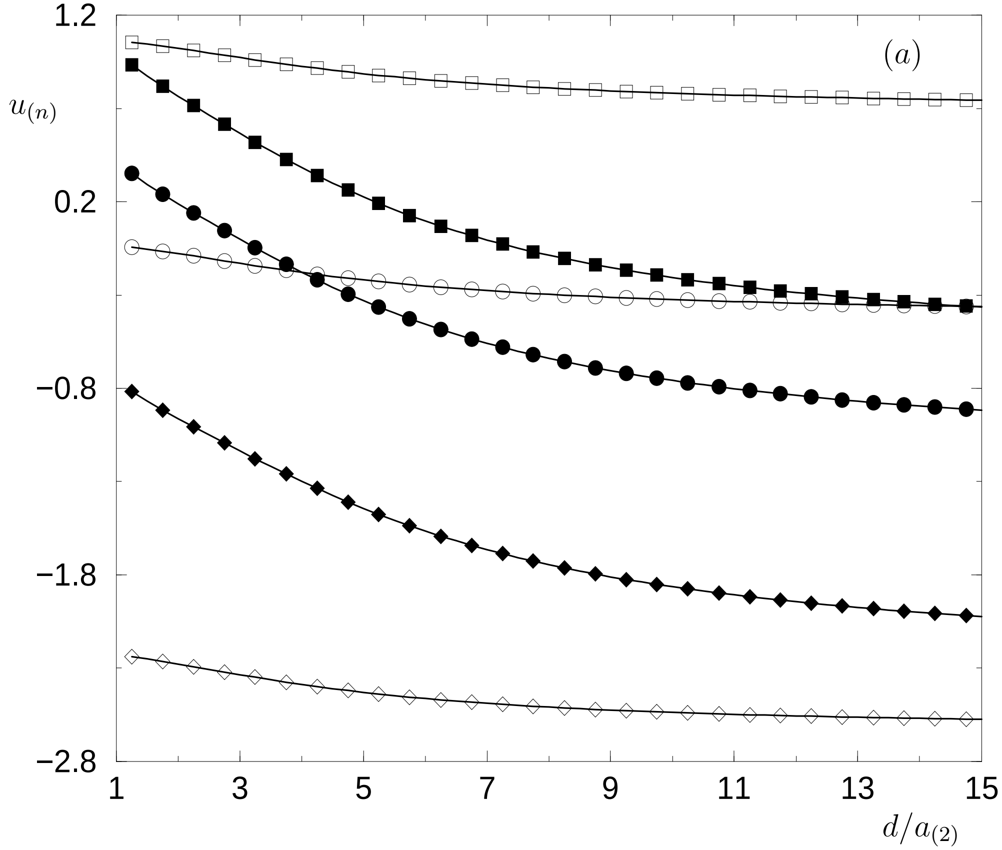
- Two-parameter geometry (d and h, large enough)

- Symmetries show that

$$\mathbf{U}^{(n)} = \frac{a^2 \rho g u}{3\mu}[v_n \mathbf{e}_1 + u_n \mathbf{e}_3], \qquad v_1 = 0, v_2 = -v_3, u_2 = u_3$$

- Adopted Bond number G_1 (big bubble)

Normalized vertical velocities u_1 and u_2 versus G_1 for $h/(2a) = 5$



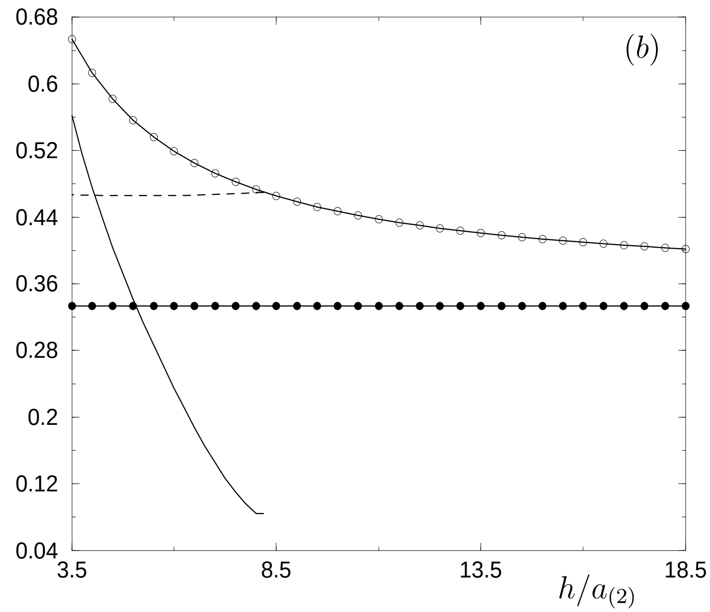
(a) $G_{(1)} = 0.30\,(u_{(1)}\,(\diamond)$ and $u_{(2)}\,(\blacklozenge))$, $G_{(1)} = 0.45\,(u_{(1)}\,(\bullet)$ and $u_{(2)}\,(\circ))$ and $G_{(1)} = 0.60\,(u_{(1)}\,(\square)$ and $u_{(2)}\,(\blacksquare))$

$u_1 - u_2$ changes sign if $G_i \leq G \leq G_s$ and vanishes for a critical spacing d_c

Values of G_i and G_s versus $h/(2a)$

- G_i for $d \to \infty$ (distant bubbles). Here $G_i = 1/3$

- Critical triplet $(d/(2a), d/h, G_c)$



(b) Values of G_i ($\bullet$), G_s ($\circ$) and G_c (dashed line). The solid line plots versus $h/a_{(2)}$ the critical ratio $d/(2h)$ at which all bubbles have a zero vertical velocity ($u_{(1)} = u_{(2)} = 0$) when $G_{(1)} = G_c$.

Concluding remarks

- No use to compute the fluid flow and the temperature disturbance in the entire unbounded fluid domain Ω!

- Valid for arbitrary $N-$bubble clusters

- The Boundary Element Method is quite suitable: it admits a straightforward implementation and allows for a reasonable cpu time cost (putting 242 collocation points on a bubble is quite sufficient)

Future work?

- Investigate the case of more confined liquids (two parallel plane solid walls, spherical solid cavity)

- Extend the advocated approach to the challenging case of clusters made of spherical drops