[1]Autor 1 (dr hab.) **Wasyl Kowalczuk** (ORCID.org/0000-0001-5391-2706)

[1]Akademia Techniczno-Artystyczna Nauk Stosowanych w Warszawie,

[1]Instytut Podstawowych Problemów Techniki Polskiej Akademii Nauk,

[1]E-mail: vkoval@ippt.pan.pl

EULER'S ELASTICAS IN NONLOCAL THEORY OF ELASTICITY

Streszczenie (abstrakt):

A generalization of the Euler's elastic problem, i.e., finding stationary configurations (planar elasticas) of the Bernoulli's thin ideal elastic rod with boundary conditions defined through fixed endpoints and/or tangents at the endpoints, for nonlocal stress tensors and the corresponding nonlocal differential constitutive stress-strain relations (nonlocal theory of elasticity) is considered. In the classical (local) Euler-Bernoulli's beam model the solutions of the governing equations for bending moments and shear forces with static boundary conditions in the case of large deformations can be obtained using Jacobi elliptic functions and incomplete elliptic integrals. It can be shown that even for a simplified nonlocal beam model proposed by Eringen the governing differential equations have much more elaborated form comparing to the local case, which makes the problem of finding the exact analytical solutions of the boundary value problems being quite a challenging task. Nevertheless, some approach based on the iterative integration method of finding an analytical form of the solution is proposed as well as the strongly nonlinear differential equation on the tangent slope angle for the Euler's elasticas has been derived and analysed.

Słowa kluczowe: Euler's elasticas; Euler-Bernoulli's beam model; nonlocal differential constitutive stress-strain relations; nonlocal theory of elasticity; incomplete elliptic integrals of the first, second, and third kind; Jacobi elliptic functions

1. Introduction

The classical Euler-Bernoulli's model was originally developed in the XVIII-th century in order to describe the large deformations of plane curved beams based on the local elasticity theory. It can be shown that the exact solution of this problem can be written using the Jacobi elliptic functions and incomplete elliptic integrals of the first, second, and third kind (see, e.g., the paper of Huo, Y.-L., Pei, X.-S., & Li, M.-Y. [3] in application to the analysis of the lightweight shock absorbing structures formed from many arc-curved beams placed between two flat platforms, where the curved beams can store more energy and produce less reaction forces compared to the ordinary elastic structures).

The Euler-Bernoulli's theory can be applied also to very small objects, e.g., nanobeams (including nanowires, nanotubes, and nanorods) considered as beams with very small length scale (nanoscale) that can be deformed with bending moments, shear and axial forces. Such objects exhibit extraordinary physical and mechanical characteristics, e.g., high aspect ratio, high flexibility, high tensile and shear strength, and high modulus of elasticity. For instance, for carbon nanotubes we have that [6]

$$\rho = 2300\,\frac{\mathrm{kg}}{\mathrm{m}^3}, \qquad E = 1000\ \mathrm{GPa}, \qquad \nu = 0.19, \qquad G = 420\ \mathrm{GPa}, \tag{1}$$

$$d = 1.0\ \mathrm{nm}, \qquad A = 0.785\ \mathrm{nm}^2, \qquad I = \frac{\pi d^4}{64} = 0.0491\ \mathrm{nm}^4, \qquad l_i = 1.5\ \mathrm{nm}, \tag{2}$$

where ρ is the density, E is the Young's (elastic) modulus, ν is the Poisson's ratio, G is the Kirchhoff's (shear) modulus, d is the diameter of nanotubes, A is their section area, I is the moment of inertia of their section, and l_i is the internal characteristic length.

In order to describe nanobeams not only the classical beam theories are used, but also the nonlocal elasticity (in relation to the small-scale effects) is applied that allows us to investigate, for instance, the problems of static bending, free vibration analysis, and also elastic buckling of carbon nanotubes (see, e.g., the papers of Reddy, J.N. [5], Reddy, J.N., Pang, S.D. [6], as well as Thongyothee, C., Chucheepsakul, S. [7] where was investigated the postbuckling of unknown-length nanobeams that is based on the concept of variable-arc-length (VAL) beams).

2. Euler's elasticas in classical local theory of elasticity – explicit analytical solutions

The formal mathematical description of the elastic line or Euler's elastica can be found, e.g., in the papers of Djondjorov, P.A., Hadzhilazova, M.Ts., Mladenov, I.M., & Vassilev, V.M. [1-2] or Pulov, V.I., Hadzhilazova, M.Ts., & Mladenov, I.M. [4], where the general solutions in terms of elliptic functions and explicit parametrizations for free elastica as well as the elastica with tension are presented.

The equilibrium equations of the small segment of the beam in the theory of elasticity are given as

$$\frac{dM}{ds} = Q, \qquad \frac{dQ}{ds} = -\kappa N, \qquad \frac{dN}{ds} = \kappa Q, \tag{3}$$

where M is the bending moment, N and Q are the axial and shear forces, and κ is the curvature of the elastica that is parameterized by the so-called arclength parameter s changing from 0 to L, where L is the total length of the elastica (sometimes, without loss of generality, the unit elastica is considered for which $L = 1$).

If we define $\theta(s)$ as the tangent slope angle at any point $P(x(s), y(s))$ of the elastica that is situated along the x-axis and subjected to the compressive load F also directed along the x-axis, then the buckling (e.g., the transversal deflection) of the Euler's elastica will happen along the y-axis. In such a situation the axial and shear forces, the curvature, and the geometrical conditions will be given as

$$N = -F\cos\theta\,, \qquad Q = F\sin\theta\,, \qquad \frac{d\theta}{ds} = \kappa, \qquad \frac{dx}{ds} = \cos\theta\,, \qquad \frac{dy}{ds} = \sin\theta\,. \tag{4}$$

In the classical (local) beam model the bending moment at any point of the elastica is proportional to its curvature, i.e., the bending moment-curvature relation is governed by the Euler-Bernoulli's law as

$$M = -EI\kappa = -EI\frac{d\theta}{ds} \tag{5}$$

with the flexural rigidity EI being expressed through the Young's modulus E and the moment of inertia I.

Additionally, it can be shown [2] that the bending moment M and the axial force N acting along the elastic curve are related through the equation

$$\frac{M^2}{2EI} = |N| \tag{6}$$

that can be understood as the manifestation of the demand that in the state of mechanical equilibrium we have that the sum of forces at all point of the elastica should be zero.

Using (3)-(5) we can obtain straightforwardly the governing equation as

$$\frac{d^2\theta}{ds^2} = -\frac{1}{EI}\frac{dM}{ds} = -\frac{Q}{EI} = -\frac{F}{EI}\sin\theta = -\alpha^2\sin\theta\,, \qquad \alpha^2 = \frac{F}{EI}. \tag{7}$$

The above second-order ordinary differential equation on the function $\theta(s)$ can be once integrated when we multiply the left- and right-hand sides of (7) by the term $2\frac{d\theta}{ds}$. Then we will obtain that

$$\frac{d}{ds}\left[\left(\frac{d\theta}{ds}\right)^2\right] = 2\alpha^2\frac{d}{ds}[\cos\theta] \tag{8}$$

can be integrated as

$$\frac{d\theta}{ds} = \sqrt{2\alpha^2\cos\theta + C} = \frac{2\alpha}{k}\sqrt{1 - k^2\sin^2\frac{\theta}{2}}, \qquad k^2 = \frac{4\alpha^2}{C + 2\alpha^2}, \tag{9}$$

where C is the first integration constant and k can be interpreted as the elliptic modulus.

In the case when $k^2 \leq 1$, the general solution of (9) can be written as [3]

$$\theta(s) = 2\arcsin\left[\mathrm{sn}\left(\frac{\alpha s}{k} + D, k\right)\right], \qquad \frac{d\theta}{ds} = \frac{2\alpha}{k}\mathrm{dn}\left(\frac{\alpha s}{k} + D, k\right), \tag{10}$$

where $\mathrm{sn}()$ and $\mathrm{dn}()$ are the Jacobi elliptic sine and delta functions and D is the second integration constant. Both integration constants C and D can be defined through the application of the corresponding boundary conditions (clamped, simply supported, etc.) for the boundary value problems (BVPs) of the curved beam.

Using (10) we can obtain that

$$\sin\theta = 2\mathrm{sn}\left(\frac{\alpha s}{k} + D, k\right)\mathrm{cn}\left(\frac{\alpha s}{k} + D, k\right), \qquad \cos\theta = 1 - 2\mathrm{sn}^2\left(\frac{\alpha s}{k} + D, k\right), \tag{11}$$

where $\mathrm{cn}()$ is the Jacobi elliptic cosine function.

Next, integrating the last two equations from (4) we obtain the coordinates of an arbitrary points on the elastic line representing the deformed beam as

$$x(s) = \int_0^s \cos\theta\, ds = \left(1 - \frac{2}{k^2}\right)s + \frac{2}{\alpha k}\mathrm{E}\left[\mathrm{am}\left(\frac{\alpha s}{k} + D, k\right), k\right] - \frac{2}{\alpha k}\mathrm{E}[\mathrm{am}(D,k),k], \tag{12}$$

$$y(s) = \int_0^s \sin\theta\, ds = \frac{2}{\alpha k}\mathrm{dn}(D,k) - \frac{2}{\alpha k}\mathrm{dn}\left(\frac{\alpha s}{k} + D, k\right), \tag{13}$$

where $\mathrm{E}()$ is the incomplete elliptic integral of the second kind and $\mathrm{am}()$ is the Jacobi elliptic amplitude.

Similarly, in the case when $k^2 > 1$, the general solution of (9) is given as [3]

$$\theta(s) = 2\arcsin\left[\frac{1}{k}\mathrm{sn}\left(\alpha s + D, \frac{1}{k}\right)\right], \qquad \frac{d\theta}{ds} = \frac{2\alpha}{k}\mathrm{cn}\left(\alpha s + D, \frac{1}{k}\right). \tag{14}$$

Using (10) we can again obtain that

$$\sin\theta = \frac{2}{k}\mathrm{sn}\left(\alpha s + D, \frac{1}{k}\right)\mathrm{dn}\left(\alpha s + D, \frac{1}{k}\right), \qquad \cos\theta = 1 - \frac{2}{k^2}\mathrm{sn}^2\left(\alpha s + D, \frac{1}{k}\right). \tag{15}$$

Finally, integrating the last two equations from (4) we obtain the coordinates of an arbitrary points on the deformed elastica as

$$x(s) = \int_0^s \cos\theta\, ds = -s + \frac{2}{\alpha}\mathrm{E}\left[\mathrm{am}\left(\alpha s + D, \frac{1}{k}\right), \frac{1}{k}\right] - \frac{2}{\alpha}\mathrm{E}\left[\mathrm{am}\left(D, \frac{1}{k}\right), \frac{1}{k}\right], \tag{16}$$

$$y(s) = \int_0^s \sin\theta\, ds = \frac{2}{\alpha k}\mathrm{cn}\left(D, \frac{1}{k}\right) - \frac{2}{\alpha k}\mathrm{cn}\left(\alpha s + D, \frac{1}{k}\right). \tag{17}$$

3. Euler's elasticas in nonlocal theory of elasticity – a toy differential model

The small-scale effects (in our case nonlocality) can be combined with the classical theory of elasticity using the corresponding modifications (in the integral or differential forms) of the constitutive relations between the normal stress and strain of the beams or nanobeams. For instance, in one-dimensional analysis of nonlocal elastic materials we can assume that the nonlocal constitutive relations have the form

$$\sigma_{xx} - \mu\frac{d^2\sigma_{xx}}{ds^2} = E\varepsilon_{xx}, \tag{18}$$

i.e., we can assume that the normal (i.e., along the x-axis) stress σ_{xx} at some point $P(s)$ with the arclength parameter s depends not only on the normal strain ε_{xx} at point $P(s)$, but also on the normal strain's values at all other points of the elastica. This fact can be equivalently written in the integral (with some kernel function) and differential forms (see, e.g., [6-7]), so in the further discussion we will use the latter one.

The scaling factor $\mu = \epsilon_0^2 l_i^2$ in (18) can be understood as the parameter describing the degree of nonlocality (presented in the dimension of the squared length) which is defined as the function of the material parameter ϵ_0 and the internal characteristic length l_i (e.g., the lattice parameter, the granular size, the distance between C-C bonds, etc.) [6-7].

The above nonlocal constitutive relation (18) applied to the Euler-Bernoulli's beam theory leads us to modification of the bending moment-curvature relation (5) that for our toy model can be now expressed as

$$M - \mu\frac{d^2M}{ds^2} = -EI\kappa = -EI\frac{d\theta}{ds}. \tag{19}$$

Then from (3)-(4) we can derived that

$$\frac{d^2M}{ds^2} = \frac{dQ}{ds} = -\kappa N = F\cos\theta\frac{d\theta}{ds}. \tag{20}$$

Substituting (20) into (19) we will obtain that

$$M = -EI(1 - \mu\alpha^2 \cos\theta)\frac{d\theta}{ds}, \qquad \alpha^2 = \frac{F}{EI} \tag{21}$$

which reduces to (5) when the nonlocality parameter μ approaches 0.

Differentiating (21) with respect to the arclength parameter s and using (3)-(4) again, we will obtain that the modification of the governing equation (7) is now given as

$$(1 - \mu\alpha^2 \cos\theta)\frac{d^2\theta}{ds^2} + \mu\alpha^2 \sin\theta \left(\frac{d\theta}{ds}\right)^2 = -\alpha^2 \sin\theta. \tag{22}$$

If we will define $P(\theta) = 1 - \mu\alpha^2 \cos\theta$, then (22) can be rewritten as

$$P(\theta)\frac{d^2\theta}{ds^2} + P'(\theta)\left(\frac{d\theta}{ds}\right)^2 = -\frac{1}{\mu}P'(\theta). \tag{23}$$

The above second-order ordinary differential equation can be once integrated when we multiply the left- and right-hand sides of (23) by the term $2P(\theta)\frac{d\theta}{ds}$. Then we will obtain that

$$\frac{d}{ds}\left[P^2(\theta)\left(\frac{d\theta}{ds}\right)^2\right] = -\frac{1}{\mu}\frac{d}{ds}[P^2(\theta)] \tag{24}$$

can be integrated as

$$\frac{d\theta}{ds} = \sqrt{\frac{1}{\mu}\left(\frac{C}{P^2(\theta)} - 1\right)} = \sqrt{\frac{1}{\mu}\left(\frac{C}{(1 - \mu\alpha^2 \cos\theta)^2} - 1\right)} = \frac{1}{\sqrt{\mu}}\frac{\sqrt{C - (1 - \mu\alpha^2 \cos\theta)^2}}{1 - \mu\alpha^2 \cos\theta}, \tag{25}$$

where C is the first integration constant.

Let us notice the interesting fact that whereas (22) can be quite easily reduced to (7) when the nonlocality parameter μ approaches 0, the connection between (25) and (9) is not so obvious (division by 0).

Substituting (25) into (23) we will also obtain the concise expression for the second derivative of θ, i.e.,

$$\frac{d^2\theta}{ds^2} = -\frac{C}{\mu}\frac{P'(\theta)}{P^3(\theta)} = -\frac{C\alpha^2 \sin\theta}{(1 - \mu\alpha^2 \cos\theta)^3}. \tag{26}$$

Next, we will separate the variables in (25) and integrate the obtained expressions, then

$$s(\theta) = \sqrt{\mu}\int \frac{(1 - \mu\alpha^2 \cos\theta)d\theta}{\sqrt{C - (1 - \mu\alpha^2 \cos\theta)^2}} \tag{27}$$

Which leads us also to the corresponding expressions for the coordinates of the deformed elastica, i.e.,

$$x(\theta) = \int \cos\theta\, ds = \sqrt{\mu}\int \frac{(1 - \mu\alpha^2 \cos\theta)\cos\theta\, d\theta}{\sqrt{C - (1 - \mu\alpha^2 \cos\theta)^2}}, \tag{28}$$

$$y(\theta) = \int \sin\theta\, ds = \sqrt{\mu}\int \frac{(1 - \mu\alpha^2 \cos\theta)\sin\theta\, d\theta}{\sqrt{C - (1 - \mu\alpha^2 \cos\theta)^2}}. \tag{29}$$

Let us notice that the last integral can be quite easily integrated using the obvious substitution, i.e., introducing the new variable $\xi = C - (1 - \mu\alpha^2 \cos\theta)^2$, then $d\xi = 2\mu\alpha^2(1 - \mu\alpha^2 \cos\theta)\sin\theta\, d\theta$ and

$$y(\theta) = \frac{1}{2\sqrt{\mu}\alpha^2}\int \frac{d\xi}{\sqrt{\xi}} = y_0 + \frac{\sqrt{\xi}}{\sqrt{\mu}\alpha^2} = y_0 + \frac{1}{\sqrt{\mu}\alpha^2}\sqrt{C - (1 - \mu\alpha^2 \cos\theta)^2}, \tag{30}$$

where y_0 is the integration constant.

4. Final remarks

Unfortunately, contrary to the expression (30) for $y(\theta)$, expressions for $s(\theta)$ and $x(\theta)$ are not so simple, but they can be also written through corresponding combinations of the incomplete elliptic integrals of the first, second, and third kind. In order to do this we can use the substitution $t = \pm\csc\theta - \cot\theta$, then

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}, \qquad d\theta = \frac{2dt}{1+t^2} \tag{31}$$

and (27) and (28) can be rewritten as

$$s(t) = \frac{2\sqrt{\mu}(1+\mu\alpha^2)}{\sqrt{C-(1+\mu\alpha^2)^2}}\int\frac{Q_2(t)dt}{(1+t^2)\sqrt{P_4(t)}}, \qquad x(t) = \frac{2\sqrt{\mu}(1+\mu\alpha^2)}{\sqrt{C-(1+\mu\alpha^2)^2}}\int\frac{Q_4(t)dt}{(1+t^2)^2\sqrt{P_4(t)}}, \tag{32}$$

where the polynomials $Q_2(t)$, $Q_4(t)$, and $P_4(t)$ are defined as

$$Q_2(t) = l + t^2, \qquad Q_4(t) = l + nt^2 - t^4, \qquad l = \frac{1-\mu\alpha^2}{1+\mu\alpha^2}, \qquad n = \frac{2\mu\alpha^2}{1+\mu\alpha^2}, \tag{33}$$

$$P_4(t) = (t+\beta^+)(t+\beta^-), \qquad \beta^\pm = \frac{b \pm \sqrt{b^2 - ac}}{a} = \left\{\frac{\sqrt{C}+\mu\alpha^2-1}{\sqrt{C}-\mu\alpha^2-1}, \frac{\sqrt{C}-\mu\alpha^2+1}{\sqrt{C}+\mu\alpha^2+1}\right\}, \tag{34}$$

$$a = C-(1+\mu\alpha^2)^2, \quad b = \frac{a}{2}(\beta^+ + \beta^-) = C - 1 + \mu^2\alpha^4, \quad c = a\beta^+\beta^- = C-(1-\mu\alpha^2)^2. \tag{35}$$

Then both integrals in (32) can be rewritten as corresponding combinations (in the form of quite long and elaborate expressions) of the incomplete elliptic integrals of the first, second, and third kind, i.e.,

$$\mathrm{F}(\varphi,k) = \int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \qquad E(\varphi,k) = \int_0^\varphi \sqrt{1-k^2\sin^2\varphi}\, d\varphi, \tag{36}$$

$$\Pi(n,\varphi,k) = \int_0^\varphi \frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}, \tag{37}$$

where the Jacobi elliptic amplitude φ, the elliptic modulus k, and the characteristic n are given as

$$\varphi(\theta) = \arcsin\left[\frac{\pm\csc\theta - \cot\theta}{\sqrt{\beta^+}}\right], \qquad k = \sqrt{\frac{\beta^+}{\beta^-}}, \qquad n = \beta^+. \tag{38}$$

As further research, we plan to simplify the obtained expressions for $s(\theta)$ and $x(\theta)$ and present them in more concise forms, as well as to apply the obtained general solution of the modified governing equation (22) to the chosen boundary value problems (BVPs) of the deformed beam.

Informacja o finansowaniu

Niniejsze badanie nie było finansowane przez żadne instytucje publiczne, komercyjne ani organizacje non-profit.

Konflikt interesów

Autor deklaruje, że nie występuje konflikt interesów.

5. Bibliography

1. Djondjorov, P.A., Hadzhilazova, M.Ts., Mladenov, I.M., & Vassilev, V.M. (2008). *Explicit parametrization of Euler's elastica. Geometry, Integrability and Quantization,* 9, 175-186. https://doi.org/10.7546/giq-9-2008-175-186

2. Djondjorov, P.A., Hadzhilazova, M.Ts., Mladenov, I.M., & Vassilev, V.M. (2009). *A note on the passage from the free to the elastica with a tension. Geometry, Integrability and Quantization,* 10, 164-170. https://doi.org/10.7546/giq-10-2009-164-170

3. Huo, Y.-L., Pei, X.-S., & Li, M.-Y. (2022). *Large deformation analysis of a plane curved beam using Jacobi elliptic functions. Acta Mechanica,* 233, 3497-3510. https://doi.org/10.1007/s00707-022-03279-3

4. Pulov, V.I., Hadzhilazova, M.Ts., & Mladenov, I.M. (2016). *A new characterization of Eulerian elasticas. Geometry, Integrability and Quantization,* 17, 309-317. https://doi.org/10.7546/giq-17-2016-309-317

5. Reddy, J.N. (2007). *Nonlocal theories for bending, buckling and vibration of beams. International Journal of Engineering Science,* 45, 288-307. https://doi.org/10.1016/j.ijengsci.2007.04.004

6. Reddy, J.N., Pang, S.D. (2008). *Nonlocal continuum theories of beams for the analysis of carbon nanotubes. Journal of Applied Physics,* 103, 023511-1-16. https://doi.org/10.1063/1.2833431

7. Thongyothee, C., Chucheepsakul, S. (2015). *Postbuckling of unknown-length nanobeam considering the effects of nonlocal elasticity and surface stress. International Journal of Applied Mechanics,* 7(3), 1550042-1-26. https://doi.org/10.1142/S1758825115500428