

# Effect of Cross-Sectional Area Discontinuities in Closed Hard-Walled Ducts on Frequency of Longitudinal Modes

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A simple analytical method is developed to estimate frequencies of longitudinal modes in closed hard-walled ducts with discontinuities in a cross-sectional area. The approach adopted is based on a general expression for the acoustic impedance for a plane wave motion in a duct and conditions of impedance continuity at duct discontinuities. Formulae for mode frequencies in a form of transcendental equations were found for one, two and three discontinuities in a duct cross-section. An accuracy of the method was checked by a comparison of analytic predictions with calculation data obtained by use of numerical implementation based on the forced oscillator method with a finite difference algorithm.

**Keywords:** longitudinal duct modes, discontinuity in duct cross-section, contraction ratio.

## 1. Introduction

In acoustical systems like air supply ducts, there are often many abrupt variations in a size of the duct or junctions of ducts with different cross-sections, often called duct discontinuities, that produce partial reflection of sound waves. In some systems, variations in duct size are accompanied by rapid changes in duct shape causing a formation of resonant structure resembling closed duct with cross-sectional discontinuities. A behaviour of sound waves in duct discontinuities has been studied with the aid of numerous methods (MILES, 1946a, 1946b; KARAL, 1953; KERGOMARD, GARCIA, 1987; SAHASRABUDHE, MUNJAL, 1995; PAGNEUX *et al.*, 1996; MUEHLEISEN, SWANSON, 2002). For low frequencies an analysis, using only planar zero order mode, is employed. This method assumes pressure and velocity fields which are uniform throughout a duct cross-section, or equivalently, describes the evolution of the pressure and velocity mean values over the cross-section (MATSUI, 2010).

Similarly, a plane wave assumption is a basis of the method presented in this paper. In a theoretical model, a general expression for the acoustic impedance and conditions of impedance continuity at duct discontinuities are used to find a resonant condition for longitudinal modes. A precision of the theoretical method is verified by a comparison of resonant frequency predictions with calculation data obtained numerically for ducts with one and two discontinuities in a cross-section. A numerical approach, developed in the paper, is a generalization of the method presented by the author (MEISSNER, 2009).

## 2. Analytical formulae for frequency of longitudinal modes

Geometries of ducts under consideration are depicted in Fig. 1. A cross-sectional shape of ducts is assumed to be circular or rectangular. Viscothermal losses inside the ducts and a friction loss along duct walls are ignored. It is also postulated that amplitudes of a sound pressure are small, thus contraction effects (MEISSNER, 1999, 2000), such as a separation of acoustic flow and a generation of vortical perturbations at duct discontinuities, were neglected.

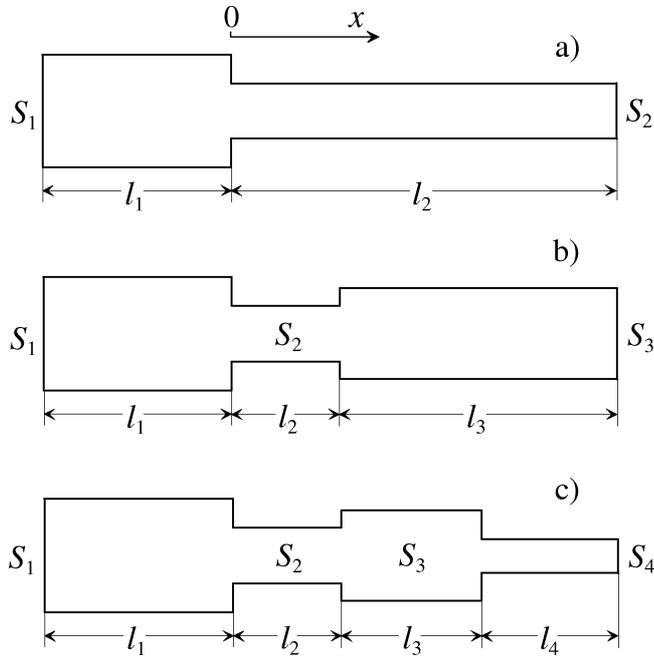


Fig. 1. Analysed geometries of hard-walled ducts having one, two and three discontinuities in cross-section. Surfaces of cross-section in different parts of ducts are denoted by  $S_1$ – $S_4$ .

In the case of a sound propagation through a duct discontinuity, partial reflections of sound waves take place. When a cross-size of duct is small in comparison

with a wavelength, there are only plane waves propagating in both of its parts. However, in a close proximity of discontinuity higher order modes are excited. In low-frequency range these modes form the evanescent waves which decay exponentially with a distance from a discontinuity. Thus, for a sake of simplicity, it is assumed in the further analysis that higher order modes may be ignored and on both sides of discontinuity only plane waves propagate.

Under the plane wave assumption, a velocity potential associated with the longitudinal resonant modes can be described by

$$\phi(x, t) = [A \sin(kx) + B \cos(kx)]e^{j\omega t}, \quad (1)$$

where  $A$  and  $B$  are unknown amplitudes and  $k = \omega/c$  is a wave number. Using Eq. (1) a general expression for the acoustic impedance can be found

$$Z(x) = -jZ_0 \frac{A \sin(kx) + B \cos(kx)}{A \cos(kx) - B \sin(kx)}, \quad (2)$$

where  $Z_0 = \rho c/S$  is a wave impedance,  $\rho$  is an air density and  $S$  is a surface of a duct cross-section. For waves travelling along  $x$ -axis, the duct system shown in Fig. 1a represents a connection of two ducts with different cross-sections, thus at a junction the following continuity condition of acoustic impedance must be satisfied

$$Z(0^-) = Z(0^+), \quad (3)$$

where notations  $a^-$  and  $a^+$  denote values of  $x$  immediately smaller and immediately greater than  $a$ , respectively. Equation (3) together with Eq. (2) leads to the following expression

$$S_2[Z(-l_1) - jZ_1 \tan(kl_1)] [Z(l_2) \tan(kl_2) - jZ_2] - S_1[Z(-l_1) \tan(kl_1) + jZ_1] [Z(l_2) + jZ_2 \tan(kl_2)] = 0, \quad (4)$$

where  $Z_1 = \rho c/S_1$  and  $Z_2 = \rho c/S_2$  are wave impedances in left and right parts of the duct. Since the duct is closed on both sides by rigid walls, the impedances  $Z(-l_1)$  and  $Z(l_2)$  are infinite, thus from Eq. (4) we obtain the expression

$$\frac{\cot(kl_1)}{S_1} + \frac{\cot(kl_2)}{S_2} = 0 \quad (5)$$

from which frequencies of longitudinal modes can be calculated. These frequencies depend on the duct lengths  $l_1$  and  $l_2$  as well as the contraction ratio  $S_2/S_1$ . An application of the same method to the duct having two discontinuities in a cross-section (Fig. 1b) gives

$$Z(0^-) [Z(l_2^+) - jZ_2 \cot(kl_2)] + jZ_2 [jZ_2 + Z(l_2^+) \cot(kl_2)] = 0, \quad (6)$$

where the impedances  $Z(0^-)$  and  $Z(l_2^+)$  are expressed by

$$\begin{aligned} Z(0^-) &= jZ_1 \frac{Z(-l_1) - jZ_1 \tan(kl_1)}{Z(-l_1) \tan(kl_1) + jZ_1}, \\ Z(l_2^+) &= -jZ_3 \frac{Z(l_2 + l_3) + jZ_3 \tan(kl_3)}{Z(l_2 + l_3) \tan(kl_3) - jZ_3}, \end{aligned} \quad (7)$$

where  $Z_3 = \rho c/S_3$ . Taking into account that impedances at  $x = -l_1$  and  $x = l_2 + l_3$  are infinite, from Eq. (6) the following resonant condition for longitudinal modes can be obtained

$$\frac{\cot(kl_1) \cot(kl_2)}{S_1 S_2} + \frac{\cot(kl_1) \cot(kl_3)}{S_1 S_3} + \frac{\cot(kl_2) \cot(kl_3)}{S_2 S_3} = \frac{1}{S_2^2}, \quad (8)$$

which is somewhat more complex than Eq. (5) because in this situation a modal frequency is influenced by three duct lengths  $l_1$ – $l_3$ , and two contraction ratios:  $S_2/S_1$  and  $S_3/S_1$ .

Among the ducts studied, the latter has the most complicated configuration. The continuity conditions, which must be fulfilled in this case, are the following

$$Z(0^-) = Z(0^+), \quad Z(l_2^-) = Z(l_2^+), \quad Z[(l_2 + l_3)^-] = Z[(l_2 + l_3)^+]. \quad (9)$$

Using conditions (9) in Eq. (2) and taking into account that the duct is rigidly terminated we finally obtain

$$\begin{aligned} &\frac{\cot(kl_1) \cot(kl_2) \cot(kl_3)}{S_1 S_2 S_3} + \frac{\cot(kl_1) \cot(kl_2) \cot(kl_4)}{S_1 S_2 S_4} \\ &+ \frac{\cot(kl_1) \cot(kl_3) \cot(kl_4)}{S_1 S_3 S_4} + \frac{\cot(kl_2) \cot(kl_3) \cot(kl_4)}{S_2 S_3 S_4} \\ &= \frac{\cot(kl_1)}{S_1 S_3^2} + \frac{\cot(kl_2)}{S_2 S_3^2} + \frac{\cot(kl_3)}{S_2^2 S_3} + \frac{\cot(kl_4)}{S_2^2 S_3}. \end{aligned} \quad (10)$$

Note, when  $S_1 = S_2 = S_3 = S_4$ , the above equation yields:

$$\sin[k(l_1 + l_2 + l_3 + l_4)] = 0.$$

Using the same analytical method, the formulae for a frequency of longitudinal modes can be found for ducts having more cross-sectional discontinuities.

### 3. Numerical prediction of duct resonant frequencies

For geometries of ducts studied a more precise method for calculating duct resonant frequencies is the eigenmode method, commonly known as the modal analysis. The modal analysis has become a useful tool for studying the acoustical behaviour of rigidly bounded spaces and it is based on solving a wave equation by expanding the solution in the eigenfunctions and then finding the resonant

frequencies by solving eigenvalue equations. The eigenfunctions must satisfy certain requirements, in particular, Neumann boundary condition at rigid walls. For enclosed spaces of irregular shape, a determination of eigenfunction distribution requires the application of numerical procedures.

Computational methods commonly used to solve eigenvalue problems are the finite element method, the boundary element method and numerical implementations, such a method based on the correspondence between the wave equation and the diffusion equation (SAPOVAL *et al.*, 1997) and the method of point-matching (KANG, LEE, 2000). A numerical procedure employed in this study is the force oscillator method and it is based on the principle that a response of a linear system to a harmonic excitation is large when the driving frequency is close to the resonant frequency (NAKAYAMA, YAKUBO, 2001). In this method the eigenvalue problem is solved by use of a solution of the wave equation with a source term  $-q(\mathbf{r}) \cos(\omega t)$ , satisfying homogenous initial conditions and the Neumann boundary condition, where  $\mathbf{r} = (x, y, z)$  is a position vector and  $\omega$  is a driving frequency. A form of this solution is the following (MEISSNER, 2007)

$$p(\mathbf{r}, t) = 2\sqrt{V} \sum_{r=0}^{\infty} \frac{Q_r \Phi_r(\mathbf{r}) \sin[(\omega + \omega_r)t/2] \sin[(\omega - \omega_r)t/2]}{\omega^2 - \omega_r^2}, \quad (11)$$

where  $V$  is a volume of the duct,  $r$  is a mode number,  $\Phi_r$  are orthogonal eigenfunctions normalized in the duct volume,  $\omega_r$  are resonant frequencies and

$$Q_r = \frac{c^2}{\sqrt{V}} \int_V q(\mathbf{r}) \Phi_r(\mathbf{r}) dv \quad (12)$$

is a factor determining a sound source strength. Suppose, that the driving frequency  $\omega$  is close to the resonant frequency  $\omega_s$ , then for sufficiently large time  $t = T$  only the term connected with the mode  $s$  contributes the sum in Eq. (11), so one can write  $p(\mathbf{r}, T) \approx a\Phi_s(\mathbf{r})$ , where  $a$  is a constant. A spatial distribution of the function  $\Phi_s$  can be determined after a normalization which results in the elimination of the constant. Finally, a use of the formula

$$\omega_s = c \sqrt{-\int_V \Phi_s \nabla^2 \Phi_s dv}, \quad (13)$$

derived directly from the eigenvalue equation:  $\nabla^2 \Phi_s + (\omega_s/c)^2 \Phi_s = 0$ , enables to calculate the resonant frequency  $\omega_s$ .

#### 4. Comparison between analytical and numerical results

In order to verify the accuracy of derived analytical formulae, a comparison with numerical results has been undertaken. In the numerical study the ducts

having rectangular cross-section and one or two cross-sectional discontinuities were considered (Figs. 2, 5). Because for these ducts a dependence of eigenfunctions  $\Phi_r$  on  $z$  describes a clearly defined cosine function, the expressions for  $\Phi_r$  and the resonant frequencies  $\omega_r$  can be written as

$$\Phi_r(\mathbf{r}) \equiv \Phi_{imn}(\mathbf{r}) = \begin{cases} \Psi_{mn}(x, y)/\sqrt{h}, & i = 0, \\ \sqrt{2} \cos(i\pi z/h) \Psi_{mn}(x, y)/\sqrt{h}, & i > 0, \end{cases} \quad (14)$$

$$\omega_r \equiv \omega_{imn} = \sqrt{(i\pi c/h)^2 + \omega_{mn}^2}, \quad (15)$$

where  $i, m, n = 0, 1, 2, 3, \dots$ ,  $h$  is a height of duct,  $\Psi_{mn}$  are orthogonal eigenfunctions normalized over the surface  $S$  of duct horizontal section and frequencies  $\omega_{mn}$  are given by

$$\omega_{mn} = c \sqrt{-\int_S \Psi_{mn} \nabla^2 \Psi_{mn} dx dy}. \quad (16)$$

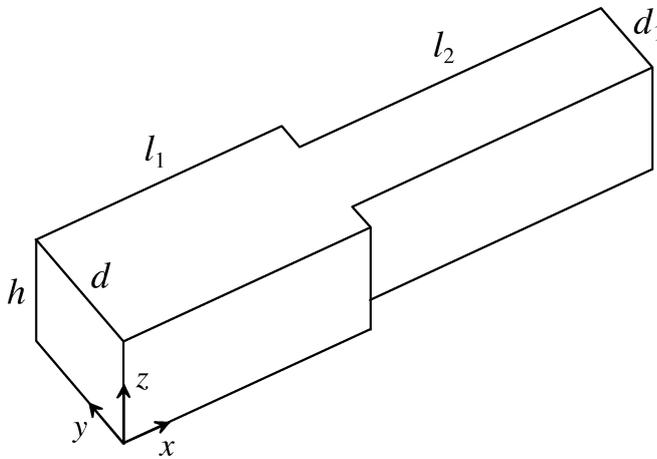


Fig. 2. Rectangular duct under consideration having one cross-sectional discontinuity. Proportion of duct dimensions:  $d/l = 0.2$ ,  $h/l = 0.15$ ,  $l_2/l = 0.7$ , where  $l = l_1 + l_2$ .

Frequencies of duct longitudinal modes, predicted from Eqs. (5) and (8), were compared to the resonant frequencies calculated from Eq. (16), where the eigenfunctions  $\Psi_{mn}$  were computed numerically via the forced oscillator method. Calculations of  $\Psi_{mn}$  were performed in two-dimensional grid with  $80 \times 400$  elements. All frequencies were normalized by the frequency  $\omega_0 = \pi c/l$  corresponding to the fundamental resonant frequency of a duct without discontinuities, where  $l = l_1 + l_2$  for duct with one discontinuity (Fig. 2) and  $l = 2l_1 + l_2$  for duct with two discontinuities (Fig. 5). A modification of a contraction ratio was realized by a variation of the dimension  $d_1$  of ducts what is equivalent to a change in

the non-dimensional parameter  $d_1/d$ . In numerical examples, this parameter was assumed to vary from 0.05 to unity with an increment of 0.05. In the case of the contraction ratio  $d_1/d$  equal to unity, corresponding to ducts without any discontinuity in the cross-section, the eigenfunctions  $\Psi_{mn}$  and resonant frequencies  $\omega_{mn}$  are simply given by

$$\Psi_{mn}(x, y) = \begin{cases} 1/\sqrt{S}, & m = 0, n = 0, \\ \sqrt{2} \cos(m\pi x/l)/\sqrt{l}, & m > 0, n = 0, \\ \sqrt{2} \cos(n\pi y/d)/\sqrt{d}, & m = 0, n > 0, \\ \sqrt{4} \cos(m\pi x/l) \cos(n\pi y/d)/\sqrt{S}, & m > 0, n > 0, \end{cases} \quad (17)$$

$$\omega_{mn} = \pi c \sqrt{(m/l)^2 + (n/d)^2}. \quad (18)$$

For the Helmholtz mode ( $m, n = 0$ ) the eigenfunction  $\Psi_{mn}$  assumes a constant value which represents the solution of eigenvalue equation for a resonant frequency equal to zero (trivial solution of eigenvalue equation). The modes excited due to an acoustic resonance along the  $x$  and  $y$  axes correspond respectively to the longitudinal modes ( $m > 0, n = 0$ ), known also as the lengthwise axial modes, and the transverse modes ( $m = 0, n > 0$ ), called also the widthwise axial modes. The remaining modes ( $m > 0, n > 0$ ), whose shapes depend on both  $x$  and  $y$ , are called the oblique modes.

Calculations of duct resonant frequencies were performed in the frequency range where the occurrence of the first six longitudinal duct modes was noted. A comparison between analytical and numerical results obtained for the duct with one cross-sectional discontinuity is shown in Fig. 3 and as may be seen, in almost all cases the frequencies calculated from Eq. (5) agree reasonably well with numerical predictions. The interesting thing resulting from Fig. 3a-d is that the frequency of the first four longitudinal modes alternately increases and decreases with a growing contraction ratio  $d_1/d$ . In Fig. 3e the calculation data for the fifth longitudinal mode are depicted and according to the formula (5) its frequency has a constant value irrespective of the parameter  $d_1/d$ . It may be easily found that it is a consequence of a special proportion between dimensions  $l_1$  and  $l_2$  of the duct (Fig. 2). Thus, noticeable in this case differences between analytical and numerical data confirm that an acoustic motion along the duct is essentially two-dimensional, in particular in a vicinity of a duct cross-sectional discontinuity. In Fig. 3e,f the numerical data denoted by circles correspond to the frequencies  $\omega_{01}$  and  $\omega_{11}$  of the first transverse mode and the first oblique mode. For the contraction ratio  $d_1/d$  close to unity, the frequencies  $\omega_{60}$  of the sixth longitudinal mode and  $\omega_{11}$  considerably differ but large variations in  $\omega_{11}$  with decreasing value of  $d_1/d$  cause two mode re-orderings (change of mode sequence on a frequency axis). The first occurs for  $d_1/d$  close to 0.3 and the second for  $d_1/d$  equal to 0.85, approximately.

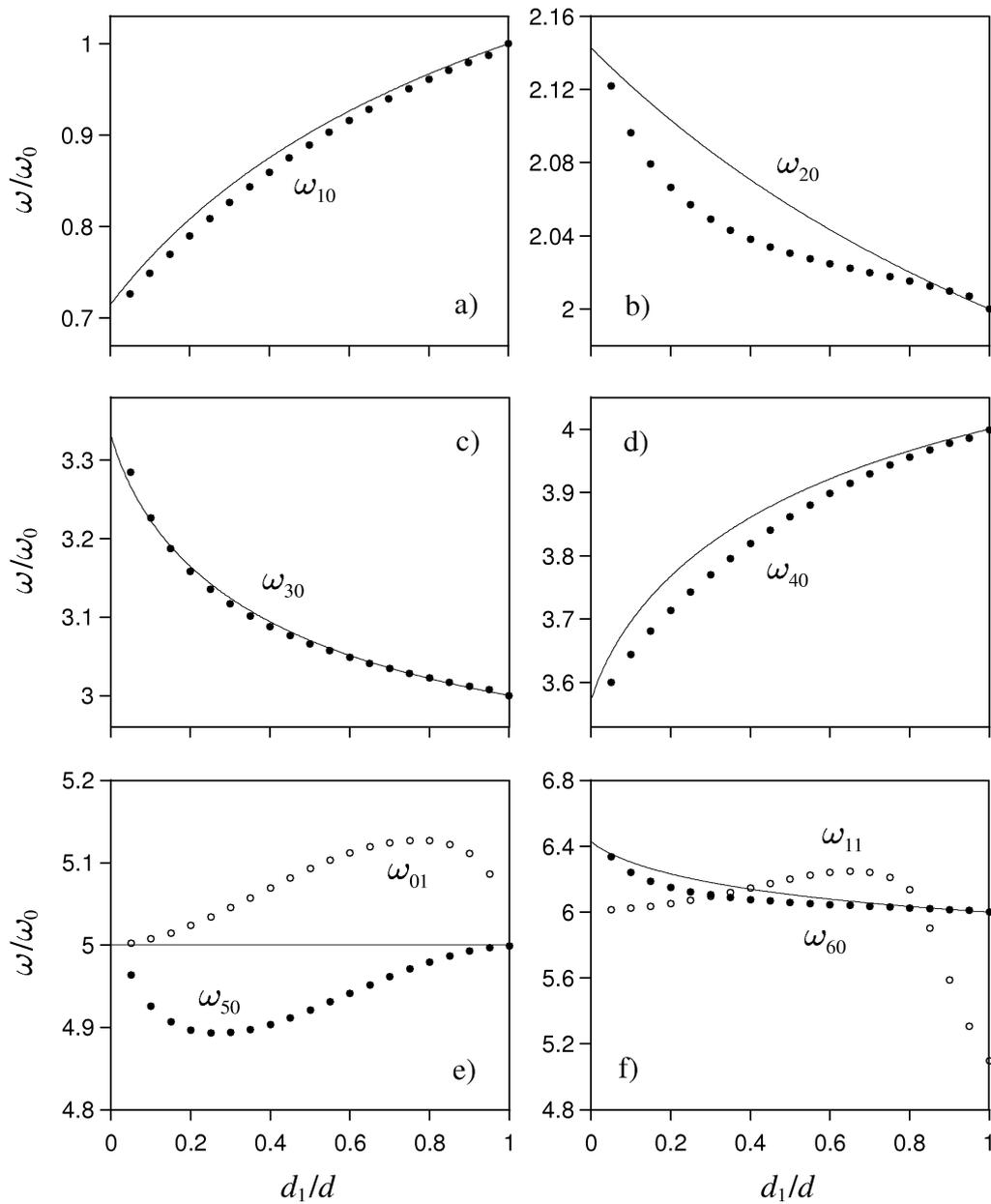


Fig. 3. Non-dimensional resonant frequencies versus contraction ratio  $d_1/d$  for duct with one cross-sectional discontinuity. Solid lines: results obtained from Eq. (5). Symbols: numerical data.

For the duct without a cross-sectional discontinuity, the eigenfunctions  $\Psi_{m0}$  of longitudinal modes have simple cosinusoidal shapes. Thus, as indicated by Eq. (16), variations of modal frequencies in the duct with a discontinuity in a cross-section are connected directly with changes in distributions of these eigen-

functions on the surface  $S$ . The plots in Fig. 4 show longitudinal mode shapes along the axis of symmetry of eigenfunctions  $\Psi_{m0}$  in a duct with one cross-sectional discontinuity for the contraction ratio  $d_1/d$  of 0.1 (solid lines) and in a duct without discontinuity (dashed lines). As may be noted, a duct discontinuity causes a visible deformation of the cosinusoidal shape of eigenfunction and an increase in the eigenfunction amplitude in a range of  $x$  where there is a duct con-

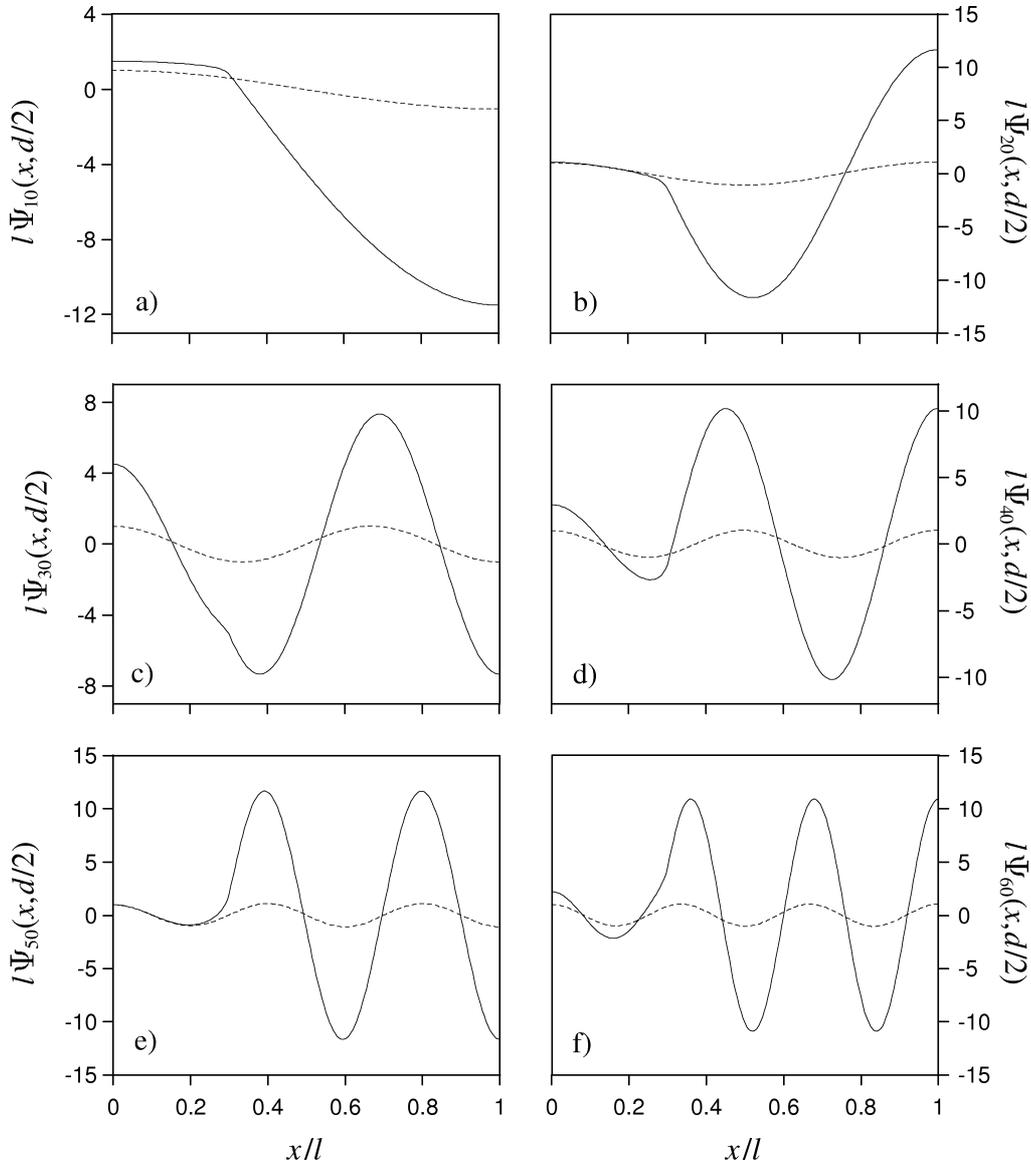


Fig. 4. Longitudinal mode shapes along axis of symmetry of eigenfunctions  $\Psi_{m0}$  in duct with one cross-sectional discontinuity for contraction ratio  $d_1/d = 0.1$  (solid lines) and in duct without discontinuity (dashed lines).

traction. Since eigenfunctions are orthonormal, the last mentioned effect is a result of a reduction of the surface  $S$  with a decrease in the contraction ratio  $d_1/d$ .

A geometry of the duct with two discontinuities in a cross-section is depicted in Fig. 5. In this system a part of duct with smaller area of cross-section is placed symmetrically with respect to closed ends and furthermore, a size of a duct contraction is assumed to be very small ( $l_2/l = 0.05$ ). The results of frequency calculations, obtained both analytically and numerically, are presented in Fig. 6. It can be seen from this figure that for all modes Eq. (8) quite precisely describes the relation between the modal frequency and the contraction ratio. Therefore, this simple formula can be successfully used for approximate predictions of the frequency of lowest-order longitudinal modes.

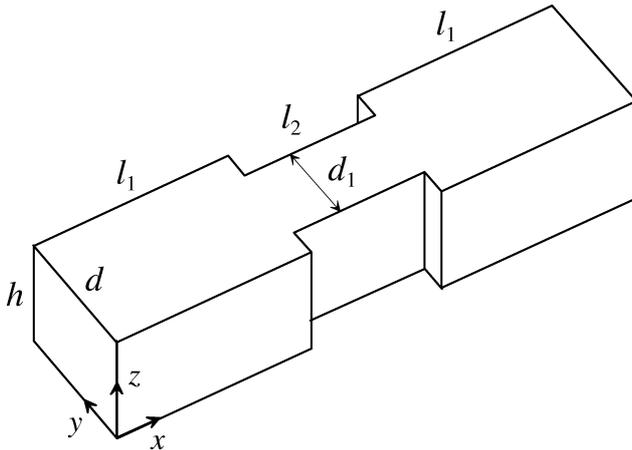


Fig. 5. Rectangular duct under consideration having two cross-sectional discontinuities. Proportion of duct dimensions:  $d/l = 0.2$ ,  $h/l = 0.15$ ,  $l_2/l = 0.05$ , where  $l = 2l_1 + l_2$ .

In Fig. 6 such a regularity can be observed that for modes with odd mode numbers the resonant frequency increases with a growing value of  $d_1/d$ , whereas for even mode numbers this frequency steadily decreases. For some modes, a dependence between the frequency and the contraction ratio seems to be almost precisely linear (Fig. 6b,d). A substantial variation of a resonant frequency is noted for the fundamental longitudinal mode (Fig. 6a), for which the frequency becomes nearly twice smaller with a decrease in  $d_1/d$  from 1 to 0.05 (numerical results). Moreover, the analytical data show that a frequency of this mode decreases to zero when ratio  $d_1/d$  approaches a zero value. This would mean that an eigenfunction for the fundamental mode continuously varies from the cosinusoidal form:  $\sqrt{2} \cos(\pi x/l)/\sqrt{l}$  for  $d_1/d$  equal to unity, to a shape resembling the Helmholtz mode for  $d_1/d$  close to zero. In order to check that, in Fig. 7 the graphs illustrating a modification of fundamental mode shape with a decrease in  $d_1/d$  are depicted. They confirm that for diminishing value of  $d_1/d$  a distribution of eigenfunction outside a duct contraction becomes more and more uniform.

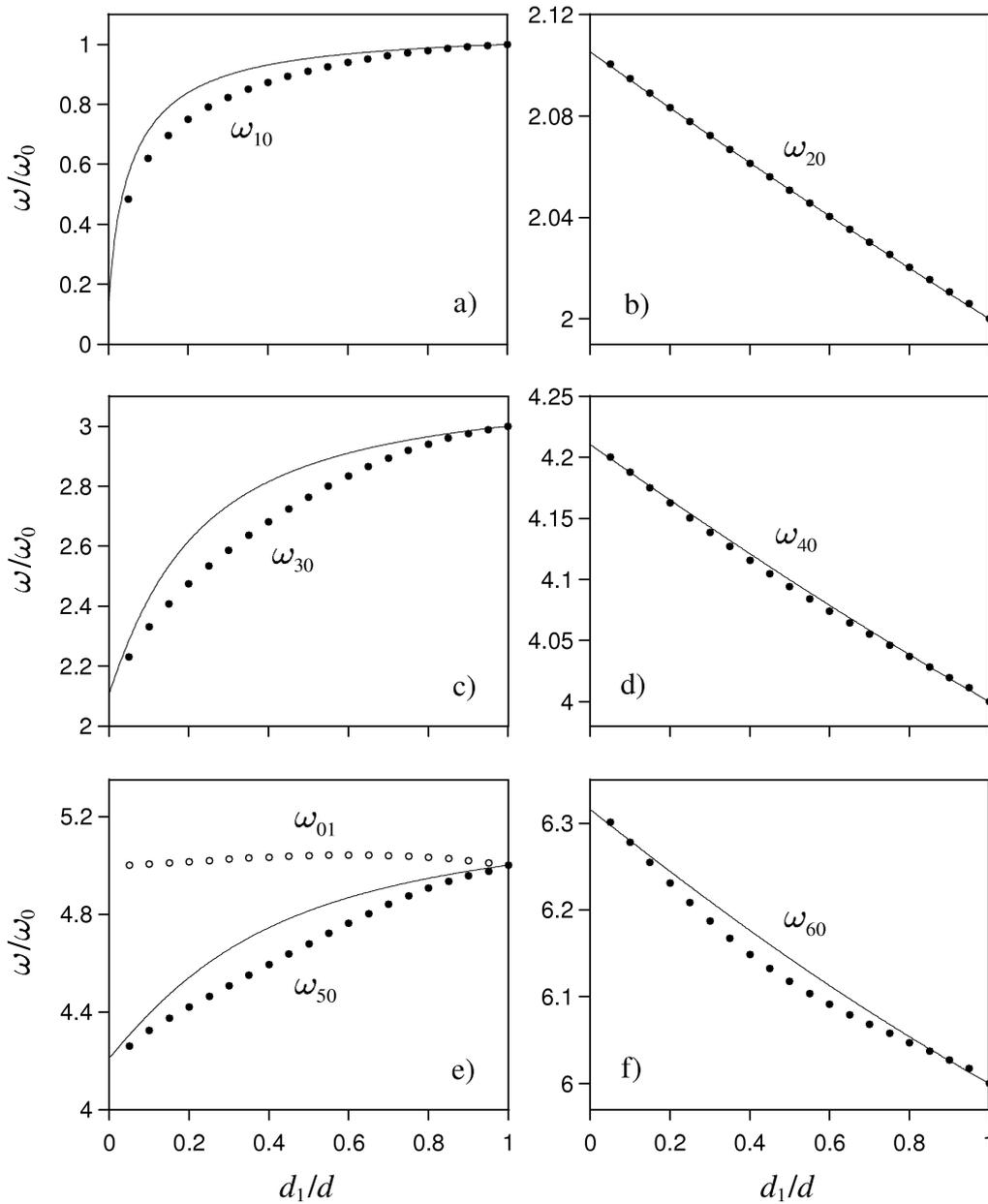


Fig. 6. Non-dimensional resonant frequencies versus contraction ratio  $d_1/d$  for duct with two cross-sectional discontinuities. Solid lines: results obtained from Eq. (8). Symbols: numerical data.

Thus, it should be expected that for a sufficiently small  $d_1/d$  the eigenfunction will be approximately constant outside the duct contraction providing the modal frequency close to zero.

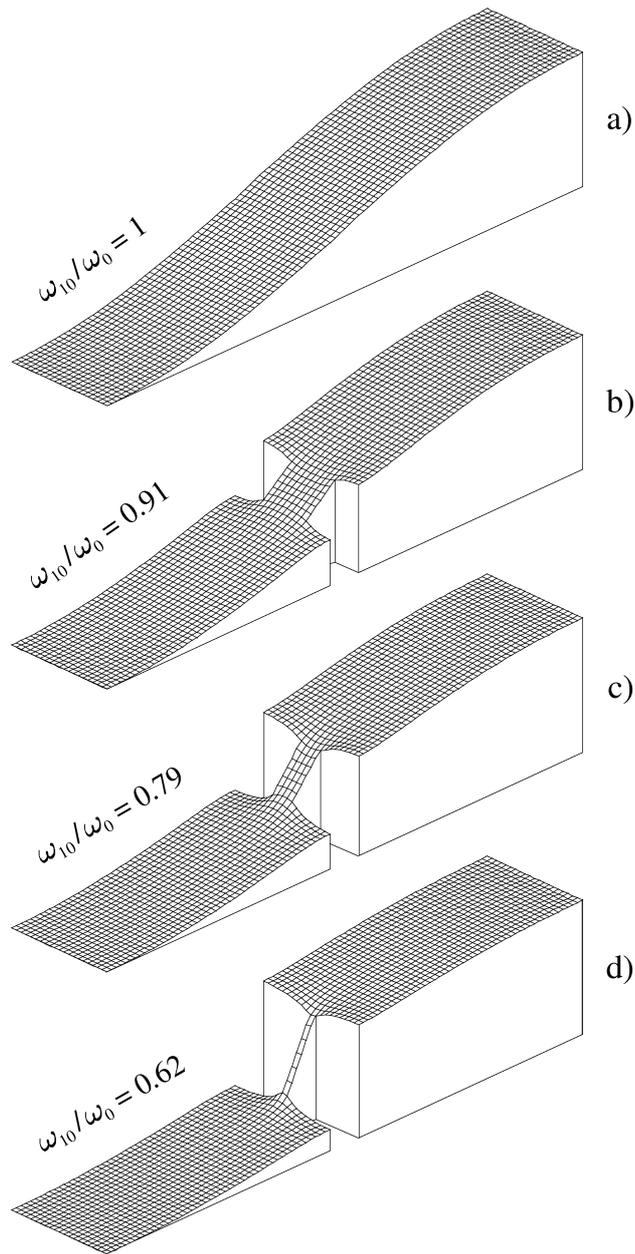


Fig. 7. Shapes of fundamental longitudinal mode in duct with two cross-sectional discontinuities for contraction ratio  $d_1/d$ : a) 1, b) 0.5, c) 0.25 and d) 0.1.

The mode shapes along the axis of symmetry of eigenfunctions  $\Psi_{m0}$  in the duct with two cross-sectional discontinuities are plotted in Fig. 8. These data evince that for modes with odd mode numbers a duct contraction causes a large non-

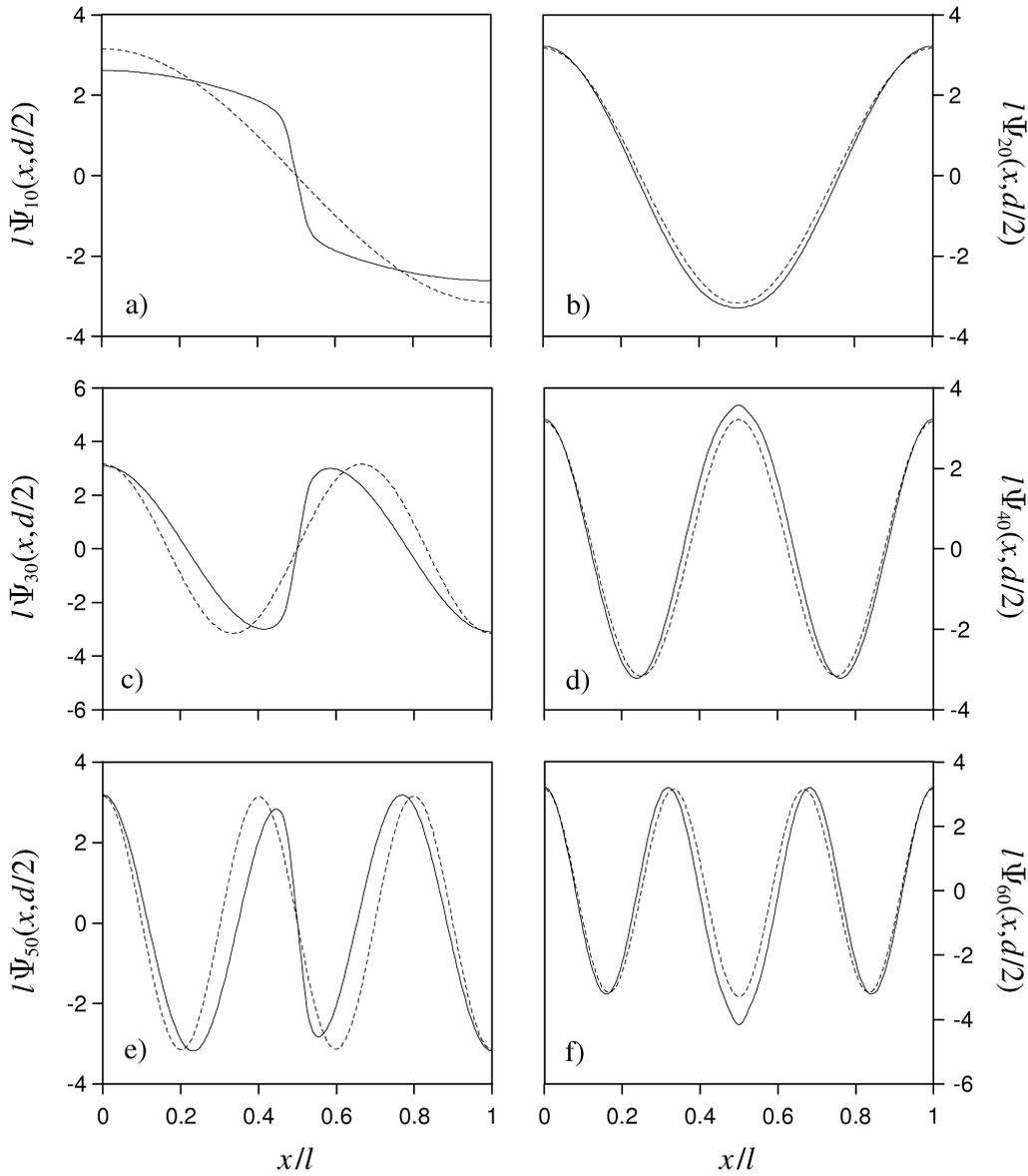


Fig. 8. Longitudinal mode shapes along axis of symmetry of eigenfunctions  $\Psi_{m_0}$  in duct with two cross-sectional discontinuities for contraction ratio  $d_1/d = 0.1$  (solid lines) and in duct without discontinuity (dashed lines).

linear deformation of an initially cosinusoidal shape of modes. On the contrary, for even mode numbers slight changes of eigenfunctions are observed and it is related to relatively small variations of resonant frequencies versus  $d_1/d$  for these modes. Because of small size of a duct contraction, a previously noted increase in the eigenfunction amplitude is not observed.

## 5. Summary and conclusions

The aim of this study was to demonstrate how cross-sectional discontinuities in closed hard-walled ducts influenced resonant conditions for lowest-order longitudinal modes. In the first part of the paper, the analytical method based on plane wave assumption was developed to obtain expressions for the frequency of longitudinal modes in the simplest duct configurations, i.e. a duct with one discontinuity in a cross-section and ducts with two and three discontinuities. These expressions have a form of transcendental equations where a degree of each duct discontinuity is characterised by the appropriate contraction ratio. Using the same approach, the analytical formulae for ducts having more cross-sectional discontinuities can be found. Moreover, by changing impedance conditions on duct terminations, the analytical method presented here can be extended to other resonant systems with discontinuities in a cross-section such as an open duct or a resonator with one end closed and one end open.

For the two simplest duct geometries, the analytical predictions were compared to numerical data obtained by a wave equation solver employing the force oscillator method. Computations have been done for the first six longitudinal modes and they confirmed a reasonably good agreement between analytical and numerical results. Calculation data obtained for the duct with two discontinuities have shown that for modes with odd mode numbers the resonant frequency increases with the growing contraction ratio, whereas for even mode numbers this frequency steadily decreases. For this duct a substantial variation of a frequency of the fundamental longitudinal mode was also noted. An inspection of eigenfunction shapes has proved that a large decrease in a modal frequency for the contraction ratio close to zero is a result of eigenfunction smoothing outside a duct contraction.

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## References

1. KANG S.W., LEE J.M. (2000), *Eigenmode analysis of arbitrarily shaped two-dimensional cavities by the method of point-matching*, Journal of the Acoustical Society of America, **107**, 3, 1153–1160.
2. KARAL F.C. (1953), *The analogous acoustical impedance for discontinuities and constrictions of circular cross section*, Journal of the Acoustical Society of America, **25**, 2, 327–334.
3. KERGMARD J., GARCIA A. (1987), *Simple discontinuities in acoustic waveguides at low frequencies: critical analysis and formulae*, Journal of Sound and Vibration, **114**, 3, 465–479.

4. MATSUI K. (2010), *Linear acoustic field in a closed pipe with an abrupt change in cross-sectional area* [in German], *Acta Acustica united with Acustica*, **96**, 1, 14–25.
5. MEISSNER M. (1999), *The influence of acoustic nonlinearity on absorption properties of Helmholtz resonators. Part I: theory*, *Archives of Acoustics*, **24**, 2, 179–190.
6. MEISSNER M. (2000), *The influence of acoustic nonlinearity on absorption properties of Helmholtz resonators. Part II: experiment*, *Archives of Acoustics*, **25**, 2, 175–190.
7. MEISSNER M. (2007), *Computational studies of steady-state sound field and reverberant sound decay in a system of two coupled rooms*, *Central European Journal of Physics*, **5**, 3, 293–312.
8. MEISSNER M. (2009), *Computer modelling of coupled spaces: variations of eigenmodes frequency due to a change in coupling area*, *Archives of Acoustics*, **34**, 2, 157–168.
9. MILES J.W. (1946a), *The analysis of plane discontinuities in cylindrical tubes. Part I*, *Journal of the Acoustical Society of America*, **17**, 3, 259–271.
10. MILES J.W. (1946b), *The analysis of plane discontinuities in cylindrical tubes. Part II*, *Journal of the Acoustical Society of America*, **17**, 3, 272–284.
11. MUEHLEISEN R.T., SWANSON D.C. (2002), *Modal coupling in acoustic waveguides: planar discontinuities*, *Applied Acoustics*, **63**, 12, 1375–1392.
12. NAKAYAMA T., YAKUBO K. (2001), *The forced oscillator method: eigenvalue analysis and computing linear response functions*, *Physics Reports*, **349**, 3, 239–299.
13. PAGNEUX V., AMIR N., KERGOMARD J. (1996), *A study of wave propagation in varying cross-section waveguides by modal decomposition. Part I. Theory and validation*, *Journal of the Acoustical Society of America*, **100**, 4, 2034–2048.
14. SAHASRABUDHE A.D., MUNJAL M.L. (1995), *Analysis of inertance due to the higher order mode effects in a sudden area discontinuity*, *Journal of Sound and Vibration*, **185**, 3, 515–529.
15. SAPOVAL B., HAEBERLÉ O., RUSS S. (1997), *Acoustical properties of irregular and fractal cavities*, *Journal of the Acoustical Society of America*, **102**, 4, 2014–2019.