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RESEARCH PAPER

C. Graczykowski · T. Lewiński

Michell cantilevers constructed within trapezoidal domains—Part III: force fields

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Abstract This paper complements the analysis of geometric properties of the Hencky nets within the Michell cantilevers constructed in the trapezoidal domains by providing the analytical formulae for the force fields. The force field analysis introduces a new division of the cantilever domain and enables an alternative method for computing the optimal weights.

1 Introduction

Geometric and kinematic analyses of the Michell cantilevers designed within the trapezoidal domains, given in the previous parts of the paper, do not exhaust the problem. The concentrated force applied introduces force fields within the cantilevers and in the reinforcing bars. Our aim now is to find these fields, prove that they fulfil all equilibrium requirements—both differential and algebraic—and provide the analytical formulae for the function h representing the density of fibres, see (I.2.13). By integration in (I.2.7) one can find the total weight of the lightest cantilevers. On the other hand, the same result should be provided by (I.2.12) or by the dual formula involving the trial displacement fields. Only upon proving the equivalence of both the formulae can one be sure that the solution is correct. Appropriate checks will be reported in part IV. Thus, the formulae for stress fields (or, rather, force fields, since the quantities T_1, T_2 considered here are of dimension of force) are indispensable for a thorough verification of the final formulae for the optimal weights.

The concentrated force applied introduces a new division of the optimal design into subdomains of a static division. The aim of the present paper is to show the interfaces between these subdomains and disclose possible jumps of the force fields. The force fields are found either by Riemann's method or by solving appropriate Volterra-like integral equations. Thus, the mathematical methods used are characteristic for the problems of mathematical physics governed by the hyperbolic equations.

The Riemann method leads to integral formulae. One of the main tools to make them explicit is the integral formula (B.1); its non-trivial proof is reported in Appendix B.

We adopt here the conventions already used in the previous parts of the paper. For instance, (B.1) of Appendix B in part I or in Graczykowski and Lewiński (2006a) will be referred to as (I.B.1). (10) and (166) of the papers Lewiński et al. (1994a,b), will be labelled (a.10) and (b.166), respectively.

C. Graczykowski (✉)
Institute of Fundamental Technological Research,
Polish Academy of Sciences,
Świętokrzyska 21, 00-049, Warsaw, Poland
e-mail: Cezary.Graczykowski@ippt.gov.pl

T. Lewiński
Faculty of Civil Engineering, Institute of Structural Mechanics,
Warsaw University of Technology,
al.Armii Ludowej 16, 00-637, Warsaw, Poland
e-mail: T.Lewinski@il.pw.edu.pl

2 Equilibrium equations of Michell's cantilevers

The state of stress in the Michell cantilevers considered is described by the tensor field $\mathbf{N} = \left(N^{\gamma\delta}\right)$, $\gamma, \delta = 1, 2$, within the interior of the cantilever, by the longitudinal force $F_C{=}F_C(s)$ in the compression reinforcing bars and by the longitudinal force $F_T{=}F_T(s)$ in the tension bars; here, s is a natural parameter of the neutral axes of the reinforcing bars.

The variational equation (I.2.5) comprises all the conditions of equilibrium: of the node of application of the force $\mathbf{P}$, of the reinforcing bars and of the fibrous interior of the cantilevers [see the conditions (a, b, and c) in Section I.2].

The Hencky net (α, β) forms the trajectories of virtual strains, and the same net forms the trajectories of principal stress resultants N_{I}, N_{II}. Because, by convention, $N_{\mathrm{I}}{\geq}N_{\mathrm{II}}$, we note that $N_{\mathrm{I}}{\geq}0$ are stress resultants in the tension fibres, while $N_{\mathrm{II}}{\leq}0$ refer to the compressed fibres.

The explicit form of the abstract equation div $\mathbf{N{=}0}$ (see I.2.3) is as follows (see Novozhilov 1962, chap. II, (2.2)):

$$-\frac{\partial (BN_{\mathrm{I}})}{\partial\alpha}+\frac{\partial B}{\partial\alpha}N_{\mathrm{II}}=0,\quad -\frac{\partial (AN_{\mathrm{II}})}{\partial\beta}+\frac{\partial A}{\partial\beta}N_{\mathrm{I}}=0. \quad (2.1)$$

Following Hemp (1973) we introduce now the fields

$$T_1 = BN_{\mathrm{I}}, \quad T_2 = AN_{\mathrm{II}} \tag{2.2}$$

of force dimension $[T_1]=[T_2]=N$. Thus, the fields T_1, T_2 will be called the force fields or internal force fields and not the stress fields. They satisfy the equilibrium equations in the form

$$-\frac{\partial T_1}{\partial \alpha} + \frac{1}{A}\frac{\partial B}{\partial \alpha}T_2 = 0, \quad -\frac{\partial T_2}{\partial \beta} + \frac{1}{B}\frac{\partial A}{\partial \beta}T_1 = 0. \tag{2.3}$$

In the regions where $\phi(\alpha, \beta)=\beta-\alpha$, these equations assume a remarkably simple form:

$$T_2 = \frac{\partial T_1}{\partial \alpha}, \quad T_1 = \frac{\partial T_2}{\partial \beta}, \tag{2.4}$$

which follows from (I.6.2). Thus, the equilibrium equations (2.4) do not involve Lamé coefficients. Moreover, both the fields T_1, T_2 satisfy the hyperbolic equations $LT_1=0$, $LT_2=0$, with L given by (I.6.3). This property paves the way for the Riemann method of integrating the given system (2.4) of differential equations.

To perform the static analysis of a cantilever, we start with the point of application of the concentrated force and find the values of the longitudinal forces in the reinforcing bars at this node. Because the bars do not resist to bending and transverse shearing, one can say that they are cables, yet they are capable of transmitting compression. Then, by using the boundary conditions, one can find one of the unknown force fields T_1 or T_2. Not in all the cases can this be done directly; in some cases, an auxiliary integral equation of Volterra type has to be solved to make further progress. Fortunately, this equation can always be analytically solved, as will be shown later. The second force field can be found by one of the (2.4). Having the formulae for both the force fields, one can compute the effective thickness h within the cantilever by using (I.2.13) and then compute the volume of the cantilever by appropriate integration. This process is postponed to part IV. The present paper is confined to the force fields analysis, the emphasis being put on casting all the fields in all the subdomains into the analytical formulae expressed in terms of Lommel–Chan functions G_n, F_n (see Chan 1967 and Lewiński et al. 1994a,b).

3 The force applied within RAN

If the concentrated force **P** acts within the domain RAN, the optimal cantilever consists of two orthogonal bars with an empty interior (see Section I.4). The values of the longitudinal forces in the bars are

$$F_{\mathrm{T}} = P\cos(\varphi - \gamma_2) \text{ - the upper bar ;}$$
$$F_{\mathrm{C}} = P\sin(\varphi - \gamma_2) \text{ - the lower bar,} \tag{3.1}$$

where $P=|\mathbf{P}|$ and

$$\gamma_2 = \arctan\left(\kappa^{-1/2}\right), \quad \gamma_1 = \pi/2 - \gamma_2, \tag{3.2}$$

φ being an angle, measured in a counterclockwise direction, between the force and the vertical line of the support (see Fig. I.4).

4 The force applied within the fan domains

4.1 Fan NAC

Assume that the point load is applied at P=C′ within the domain NAC. Then the optimal cantilever consists of the straight tension member R′A′, curved tension bar A′C′, straight compression member NC′ and the compressed circular fan (see Fig. 1).

By using the equilibrium conditions of the node P of application of the force **P**, one can find the forces in both the reinforcing bars

$$F_{\mathrm{T}} = P\sin\left(\gamma_1 + \alpha_p + \varphi\right),$$
$$F_{\mathrm{C}} = -P\cos\left(\gamma_1 + \alpha_p + \varphi\right), \tag{4.1}$$

where γ_1 is an angle between the compression bar and the support; the angle is given by (I.4.2); α_p, β_p are coordinates of point P=C′.

The fibres go orthogonally to the reinforcing bars; the tangent loading is absent. Thus, according to (I.2.2), the longitudinal force in the bar is constant along the bar. To find the internal force T_2 within the compressed fan we make use of (I.2.2) and take into account that the normal load $N_n(s)$ is linked with the internal force T_2 by

$$T_2(s) = N_n(s)R(s), \tag{4.2}$$

where $R(s)=r_1$. We find

$$T_2(s) = -F_{\mathrm{T}} = -P\sin\left(\gamma_1 + \alpha_p + \varphi\right), \tag{4.3}$$

This value is independent of $s=r_1\alpha$. The force does not vary also in the β_1 direction because there are no bars in the

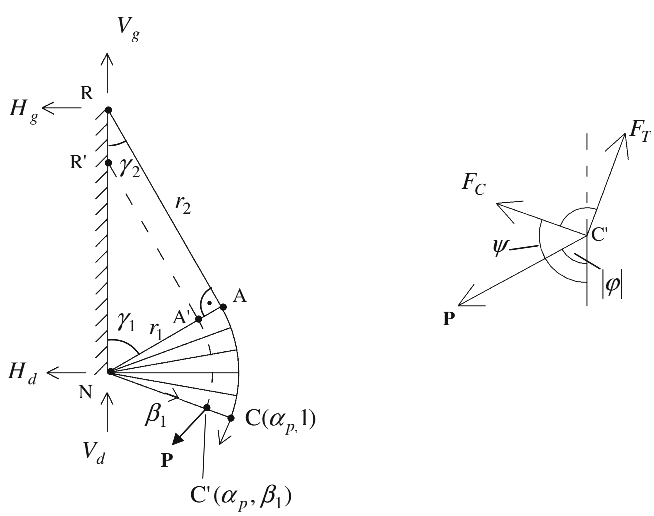

Fig. 1 Optimal cantilever composed of straight members and a circular fan. $\psi=\gamma_1+\alpha_p$; here, $\varphi<0$

circumferential direction. To make the analysis complete we check now the global conditions of equilibrium. The reactions at the upper node are

$$H_g = F_T \sin\gamma_2, \quad V_g = F_T \cos\gamma_2. \tag{4.4}$$

The horizontal reaction is directed opposite to axis x_0, and the vertical reaction is directed along y_0. This convention holds in this and in the next part of the present paper; index g means "upper," and index d means "lower." The reactions at the lower node are computed as follows:

$$\begin{aligned} H_d &= \int_0^{\alpha_p} T_2(s(\alpha)) \sin(\alpha+\gamma_1) d\alpha + F_C \sin(\alpha_p+\gamma_1) \\ &= F_T \cos(\alpha_p+\gamma_1) - F_T\cos\gamma_1 + F_C \sin(\alpha_p+\gamma_1) \\ V_d &= -\int_0^{\alpha_p} T_2(s(\alpha)) \cos(\alpha+\gamma_1) d\alpha - F_C \cos(\alpha_p+\gamma_1) \\ &= F_T \sin(\alpha_p+\gamma_1) - F_T \sin\gamma_1 - F_C\cos(\alpha_p+\gamma_1). \end{aligned} \tag{4.5}$$

It is easy to check that the global equilibrium conditions

$$V_g + V_d = P\cos\varphi, \quad H_g + H_d = P\sin\varphi \tag{4.6}$$

and the condition of the total moment at N being zero are fulfilled identically.

4.2 Fan RBA

Let the point load be applied within RBA at P=B′(α_1, β_p). The optimal structure consists of the straight tension bar RB′, curved compression bar B′A′ , the straight compression bar A′N′ and the circular fan in tension (see Fig. 2).

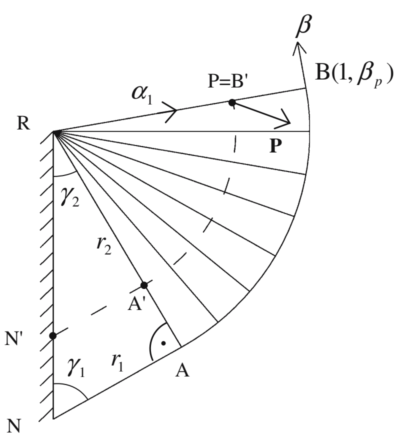

Fig. 2 Optimal cantilever composed of straight members and a circular fan

The longitudinal forces in the reinforcing bars do not vary; they can be found by equilibrium conditions of point P=B′. Their values are

$$\begin{aligned} F_T &= P\sin(\pi/2 - \gamma_2 - \beta_p + \varphi), \\ F_C &= -P\cos(\pi/2 - \gamma_2 - \beta_p + \varphi). \end{aligned} \tag{4.7}$$

We shall assume that the point load is directed such that the upper bar is in tension and the lower is compressed. Within the circular fan the only internal force present is T_1 going in the radial direction. The value of this force can be found similarly as for the lower fan by (I.2.2). The force found this way occurs to be equal (up to a sign) to the force in the reinforcing bar and does not depend on the coordinates parameterizing the fan:

$$T_1 = -F_C = P\cos(\pi/2 - \gamma_2 - \beta_p + \varphi). \tag{4.8}$$

The reader can check now that the force field found satisfies the global conditions of equilibrium.

5 Prager–Hill domain ABDC

Let the point load be applied within domain ABDC. The optimal cantilever consists of the upper tension bar RD, lower compression bar ND, the upper tension fan RBA, the lower compression fan NAC and the fibrous domain ABDC, in which one family of fibres is in compression and one in tension (see Fig. 3). We assume that $\gamma_2+\beta_p\le\gamma_R$, $\gamma_1+\alpha_p\le\gamma_N$ (see Fig. I.1).

First, let us compute the forces in reinforcing bars from the equilibrium condition of the node P=D(α_p, β_p). We find F_C=F_C(P), F_T=F_T(P), where

$$F_C(P) = -P\cdot\cos(\psi+\varphi), \quad F_T(P) = P\cdot\sin(\psi+\varphi), \tag{5.1}$$

where ψ is an angle between the compression bar and the vertical direction

$$\psi = \gamma_1 + \alpha_p - \beta_p. \tag{5.2a}$$

The conditions $F_C<0$ and $F_T>0$ imply

$$0 \le \psi+\varphi \le \frac{\pi}{2}, \quad 0 \le \gamma_1+\alpha_p-\beta_p+\varphi \le \frac{\pi}{2} \tag{5.2b}$$

and only then further analysis applies.

The longitudinal forces in the reinforcing bars are constant because the fibres go to the bars orthogonally or are absent. The static boundary conditions for the domain ABDC follow from (I.2.2). Thus, we find the value of T_1 along the lower line β (α=α_p) and the value of T_2 along the upper line α(β=β_p):

$$T_1(\alpha_p, \beta) = -F_C, \quad T_2(\alpha, \beta_p) = -F_T. \tag{5.3}$$

Let us define new functions

$$\overline{T}_\gamma(x, y) = T_\gamma(\alpha_p - x, \beta_p - y), \quad \gamma = 1, 2.$$

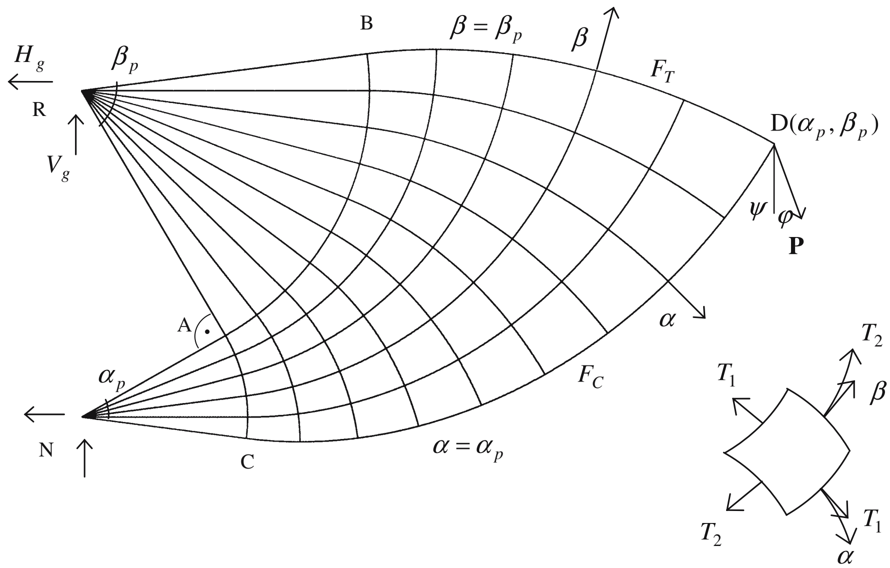

Fig. 3 Cantilever consisting of two circular fans and fibrous Prager–Hill domain. The coordinates of depicted points are: A=(0,0), B=(0, β_p), C=(α_p, 0), D=(α_p, β_p)

They are linked by

$$\overline{T}_2(x,y) = -\frac{\partial \overline{T}_1(x,y)}{\partial x}, \quad \overline{T}_1(x,y) = -\frac{\partial \overline{T}_2(x,y)}{\partial y}$$

and satisfy the equation $L\overline{T}_\gamma = 0$, with L given here by $L = \frac{\partial^2}{\partial x \partial y} - 1$. Let us rewrite (5.3) in the form

$$\overline{T}_1\left(0, \overline{\beta}\right) = -F_\mathrm{C}, \qquad \overline{T}_2(\overline{\alpha}, 0) = -F_\mathrm{T},$$

where $\overline{\beta} = \beta_p - \beta$, $\overline{\alpha} = \alpha_p - \alpha$. Thus, we have

$$\overline{T}_1(0,0) = -F_c, \quad \frac{\partial \overline{T}_1(0,\overline{\beta})}{\partial \overline{\beta}} = 0$$

$$\frac{\partial \overline{T}_1(\overline{\alpha}, 0)}{\partial \overline{\alpha}} = -\overline{T}_2(\overline{\alpha}, 0) = F_\mathrm{T}. \tag{5.4}$$

This makes it possible to apply Riemann's formula (see a.23) for finding $\overline{T}_1\left(\overline{\alpha}, \overline{\beta}\right)$ within ABDC:

$$\overline{T}_1\left(\overline{\lambda}, \overline{\mu}\right) = -F_\mathrm{C} G_0\left(\overline{\lambda}, \overline{\mu}\right) + \int_0^{\overline{\lambda}} G_0\left(\overline{\lambda} - \overline{\alpha}, \overline{\mu}\right) F_\mathrm{T} d\overline{\alpha} \tag{5.5a}$$

or

$$\overline{T}_1\left(\overline{\lambda}, \overline{\mu}\right) = -F_\mathrm{C} G_0\left(\overline{\lambda}, \overline{\mu}\right) + F_\mathrm{T} G_1\left(\overline{\lambda}, \overline{\mu}\right),$$

the final result being found by using the property (a.4) for H=G, n=1. Coming back to the original parameterization we find

$$T_1(\alpha,\beta) = -F_\mathrm{C} G_0\left(\alpha_p - \alpha, \beta_p - \beta\right) + F_\mathrm{T} G_1\left(\alpha_p - \alpha, \beta_p - \beta\right). \tag{5.5b}$$

The Riemann formula can also be applied for finding T_2 (α, β). However, it is easier to use the differential rule (2.4) to get

$$T_2(\alpha,\beta) = -F_T G_0\left(\alpha_p - \alpha, \beta_p - \beta\right) + F_C G_1\left(\beta_p - \beta, \alpha_p - \alpha\right). \tag{5.6}$$

In the circular fans the hoop internal force vanishes. This follows from the equilibrium (I.2.2) under the condition of the reinforcing bar being straight. Thus, the internal radial force is constant along the radial direction and equal to that on the line adjacent to the domain ABDC. In the lower circular domain we have

$$T_2(\alpha, \beta_1) = T_2^{ABDC}(\alpha, 0). \tag{5.7}$$

In the upper circular domain,

$$T_1(\alpha_1, \beta) = T_1^{ABDC}(0, \beta). \tag{5.8}$$

The formulae (5.5b) and (5.6) are common for all Michell cantilevers in which the Hencky net is characterized by ϕ= ϕ_0+β−α. They describe force field distribution caused by a concentrated load. In particular, these formulae are valid if the cantilever is supported around the circular boundary (see Graczykowski and Lewiński 2005, (26), cited in part I).

6 The upper Chan's domain BDH

Assume that γ_2+θ_2=γ_R and γ_1+α_p≤γ_N (see Fig. I.1) and Fig. 4. Moreover, assume that the point P of application of the force **P** lies within Chan's domain BDH. Two cases should

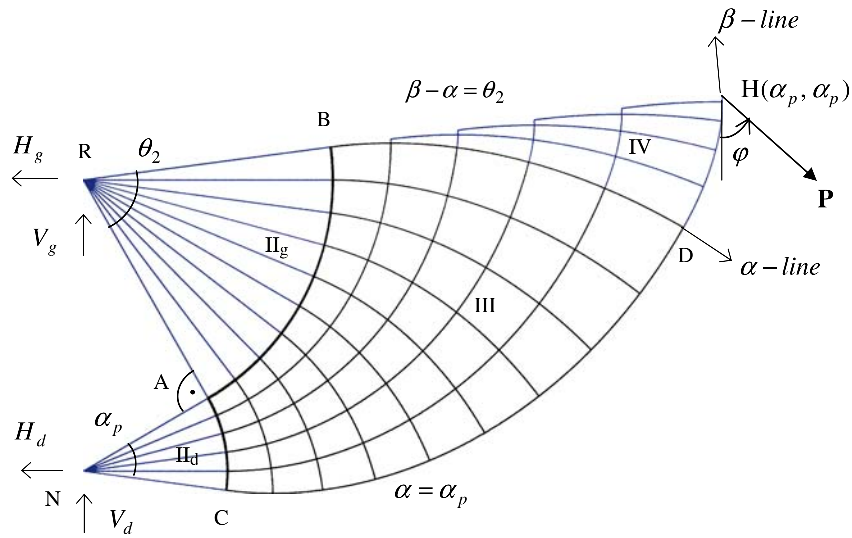

Fig. 4 Michell cantilever with the force applied on the straight boundary. Here the α, β coordinates of depicted points are A=(0,0), D=(α_p, θ_2), B=(0, θ_2) and C=(α_p, 0)

be considered separately. First, we consider the case of point P lying on the straight boundary BH and then the case of P within the domain. These two cases result in different divisions of the cantilever called static divisions.

Consider now the case of P lying on BH, which corresponds to $\alpha_p \leq \theta_1$, $\theta_1 = \gamma_N - \gamma_1$, and $\beta_p = \alpha_p + \theta_2$. The feasible domain will be shrunk appropriately, the notation of points C,D,H being kept (see Fig. 4). Within the cantilever one can indicate three subdomains of the static division: one comprising the geometric division-based domains III = BDH and IV = ABDC, and two subdomains on the upper and lower circular domains: II_g = RBA and II_d = NAC. The force in the compressed bar is constant; it can be found from the equilibrium equation of the node P. The tension bar is straight, and the characteristic lines of the fibrous domain go tangentially to this bar. Thus, the longitudinal force in the tension bar varies along the bar and cannot be determined from purely algebraic analysis of equilibrium.

The force field T_2 satisfies the hyperbolic equation LT_2=0, with L given by (I.6.3). Thus, this field can be found by Riemann formula (A.6) referred to the domain QFE (see Fig. 5) for T=T_2:

$$T_2(\lambda, \mu) = \frac{1}{2}\left(T_{2|F} + T_{2|E}\right) + \frac{1}{2}\int_{\mu-\theta_2}^{\lambda} \left[T_2(\alpha, \alpha+\theta_2)\left(\frac{\partial G}{\partial \beta} - \frac{\partial G}{\partial \alpha}\right)_{|\beta=\alpha+\theta_2} + G(\alpha, \alpha+\theta_2)\left(\frac{\partial T_2}{\partial \alpha} - \frac{\partial T_2}{\partial \beta}\right)_{|\beta=\alpha+\theta_2} \right] d\alpha. \tag{6.1}$$

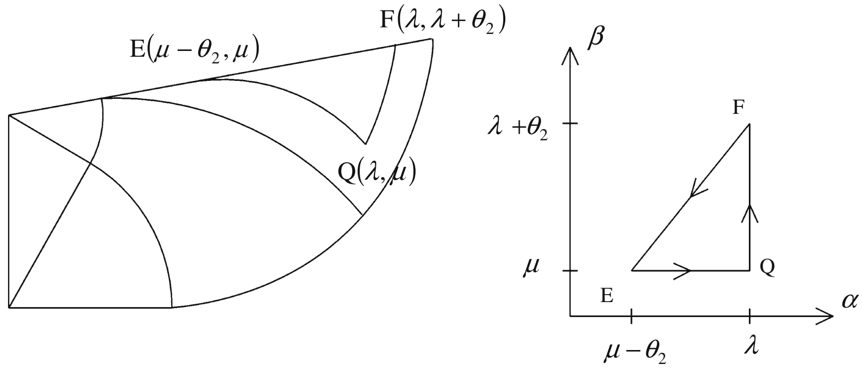

Fig. 5 Triangular domain for which Riemann formula (A.6) is applied

Within QFE we have $\alpha \le \lambda$ and $\mu \le \beta$, and we see that the arguments of the function $G(\alpha, \beta) = D_0(\lambda - \alpha, \beta - \mu)$ are non-negative. The force T_2 vanishes on the straight segment EF lying within the boundary BH

$$T_2(\alpha, \alpha + \theta_2) = 0. \tag{6.2}$$

Thus, the formula (6.1) reduces to

$$T_2(\lambda, \mu) = \frac{1}{2} \int_{\mu-\theta_2}^{\lambda} D_0(\lambda - \alpha, \alpha + \theta_2 - \mu)\chi(\alpha)d\alpha, \tag{6.3}$$

where the function $\chi(\alpha)$ is defined on the straight segment BH as follows:

$$\chi(\alpha) = \left(\frac{\partial T_2}{\partial \alpha} - \frac{\partial T_2}{\partial \beta}\right)\Big|_{\beta=\alpha+\theta_2} . \tag{6.4}$$

This function will be an unknown in the integral equation (6.3) specified for the line $\beta(\alpha{=}\alpha_p)$. The equilibrium equations of the boundary provide the values of the force T_1 along the line $\beta(\alpha{=}\alpha_p)$

$$T_1(\alpha_p, \beta) = -F_C, \tag{6.5}$$

where $F_C{=}F_C(\mathrm{P})$ is given by (5.1). The argument P is omitted for brevity. By using the equilibrium equation (2.4) one finds

$$T_2(\alpha_p, \beta) = -F_C\beta + C. \tag{6.6}$$

The constant C will be determined by the condition of the force T_2 being zero at point H, where the point load is applied, or for $\alpha_p{=}\beta_p{-}\theta_2$. Thus, the force T_2 on the line $\alpha{=}\alpha_p$ equals

$$T_2(\alpha_p, \beta) = -F_C(\beta - \alpha_p - \theta_2). \tag{6.7}$$

Now we can write (6.3) for $\lambda{=}\alpha_p$:

$$T_2(\alpha_p, \mu) = \frac{1}{2} \int_{\mu-\theta_2}^{\alpha_p} D_0(\alpha_p - \alpha, \alpha + \theta_2 - \mu)\chi(\alpha)d\alpha; \tag{6.8a}$$

hence, we find the integral equation

$$-F_C(\mu - \alpha_p - \theta_2) = \frac{1}{2} \int_{\mu-\theta_2}^{\alpha_p} D_0(\alpha_p - \alpha, \alpha + \theta_2 - \mu)\chi(\alpha)d\alpha \tag{6.8b}$$

with the function $\chi(\alpha)$ as an unknown. We concentrate now our attention on solving this equation. First, we change the variables

$$\tilde{\alpha} = \alpha - \mu + \theta_2, \quad \theta_2 + \alpha_p - \mu = t,$$
$$\chi(\alpha) = \tilde{\chi}(\alpha - \mu + \theta_2) \tag{6.9}$$

to arrive at the integral equation of the form

$$F_C t = \frac{1}{2} \int_0^t D_0(t - \tilde{\alpha}, \tilde{\alpha})\tilde{\chi}(\tilde{\alpha})d\tilde{\alpha}. \tag{6.10}$$

We perform the Laplace transform (see a.25) of both the sides:

$$\frac{F_C}{p^2} = L_t\left\{\frac{1}{2}\int_0^t D_0\left(t - \tilde{\alpha}, \hat{\alpha}\right) \cdot \tilde{\chi}(\tilde{\alpha})d\hat{\alpha}\right\}. \tag{6.11}$$

We rearrange the left-hand side

$$\frac{F_C}{p^2} = \frac{F_C}{p^2 - 1} - \frac{F_C}{p^2(p^2 - 1)} \tag{6.12}$$

and recall the result (see b.167, b.171 and b.176) for $n \ge 0$:

$$L_t\left\{\int_0^t D_0(x, t - x)G_n(x, x)dx\right\} = \frac{1}{p^n(p^2 - 1)} \tag{6.13a}$$

or

$$L_t\left\{\int_0^t D_0(x, t - x)G_n(t - x, t - x)dx\right\} = \frac{1}{p^n(p^2 - 1)}. \tag{6.13b}$$

Thus, the left-hand side of (6.11) can be expressed as Laplace transform

$$\frac{F_C}{p^2} = L_t\left\{F_C \cdot \int_0^t D_0(\tilde{\alpha}, t - \tilde{\alpha})[G_0(t - \tilde{\alpha}, t - \tilde{\alpha}) - G_2(t - \tilde{\alpha}, t - \tilde{\alpha})]d\tilde{\alpha}\right\} \tag{6.14a}$$

or

$$\frac{F_C}{p^2} = L_t\left\{F_C \cdot \int_0^t D_0(\tilde{\alpha}, t - \tilde{\alpha})[G_0(\tilde{\alpha}, \tilde{\alpha}) - G_2(\tilde{\alpha}, \tilde{\alpha})]d\tilde{\alpha}\right\}. \tag{6.14b}$$

By equating the right-hand sides of (6.11) and (6.14a, b), one obtains two possible solutions:

$$\frac{1}{2}\tilde{\chi}(\tilde{\alpha}) = F_C \cdot G_0(t - \tilde{\alpha}, t - \tilde{\alpha}) - F_C G_2(t - \tilde{\alpha}, t - \tilde{\alpha}) \tag{6.15a}$$

and

$$\frac{1}{2}\widetilde{\chi}(\widetilde{\alpha}) = F_C G_0(\widetilde{\alpha},\widetilde{\alpha}) - F_C G_2(\widetilde{\alpha},\widetilde{\alpha}). \tag{6.15b}$$

Let us discuss now the solution (6.15a). Taking into account that $t-\widetilde{\alpha}=\alpha_p-\alpha$ we find

$$\chi(\alpha) = \widetilde{\chi}(\alpha-\mu+\theta_2) = 2F_C G_0\big(\alpha_p-\alpha,\alpha_p-\alpha\big) - 2F_C G_2\big(\alpha_p-\alpha,\alpha_p-\alpha\big). \tag{6.16}$$

To have $T_2(\lambda,\mu)$ within BDH we insert (6.16) into (6.3) and obtain

$$T_2(\lambda,\mu) = F_C \int_{\mu-\theta_2}^{\lambda} D_0(\lambda-\alpha,\alpha+\theta_2-\mu)\cdot\big[G_0\big(\alpha_p-\alpha,\alpha_p-\alpha\big) - G_2\big(\alpha_p-\alpha,\alpha_p-\alpha\big)\big]d\alpha. \tag{6.17}$$

We change the variables

$$\widetilde{\alpha}=\alpha-\mu+\theta_2,\qquad t=\alpha_p-\mu+\theta_2,$$
$$t_\lambda=\lambda-\mu+\theta_2; \tag{6.18}$$

hence, $t-\widetilde{\alpha}=\alpha_p-\alpha$, $t_\lambda-\widetilde{\alpha}=\lambda-\alpha$, and we rewrite (6.17) as

$$T_2(\lambda,\mu) = F_C \int_0^{t_\lambda} D_0(t_\lambda-\widetilde{\alpha},\widetilde{\alpha})\cdot[G_0(t-\widetilde{\alpha},t-\widetilde{\alpha}) - G_2(t-\widetilde{\alpha},t-\widetilde{\alpha})]d\widetilde{\alpha}. \tag{6.19}$$

Now we are ready to use the result (B.1) for θ=0 and n=0, which gives

$$T_2(\lambda,\mu) = F_C[G_1(t,t-t_\lambda) - G_1(t-t_\lambda,t)]. \tag{6.20}$$

We substitute the original variable

$$t-t_\lambda=\alpha_p-\lambda \tag{6.21}$$

and find

$$T_2(\lambda,\mu) = F_C\big[G_1\big(\alpha_p+\theta_2-\mu,\alpha_p-\lambda\big) - G_1\big(\alpha_p-\lambda,\alpha_p+\theta_2-\mu\big)\big] \tag{6.22}$$

Recalling that α_p=$\beta_p-\theta_2$ we rearrange the above result to the form

$$T_2(\lambda,\mu) = F_C\big[G_1\big(\beta_p-\mu,\alpha_p-\lambda\big) - G_1\big(\alpha_p-\lambda,\beta_p-\mu\big)\big]. \tag{6.23}$$

By using (2.4) one finds

$$T_1(\lambda,\mu) = F_C\big[-G_0\big(\beta_p-\mu,\alpha_p-\lambda\big) + G_2\big(\alpha_p-\lambda,\beta_p-\mu\big)\big]. \tag{6.24}$$

Let us look more closely at the solution (6.15b). Changing the variables by (6.9) one finds the function

$$\chi(\alpha) = \widetilde{\chi}(\alpha-\mu+\theta_2) = 2F_C G_0(\alpha-\mu+\theta_2,\alpha-\mu+\theta_2) - 2F_C G_2(\alpha-\mu+\theta_2,\alpha-\mu+\theta_2) \tag{6.25}$$

depending on μ, which contradicts the definition of function $\chi(\alpha)$ defined on the line BH, where β=α+θ_2. If one inserts this value into (6.25) one obtains the wrong result $\chi(\alpha)$=0.

Let us consider this solution further. One can insert now the function (6.25) into (6.3) to find the field $T_2(\lambda,\mu)$. We apply the change of variables

$$\widetilde{\alpha}=\alpha+\theta_2-\mu,\quad t_\lambda=\lambda-\mu+\theta_2 \tag{6.26}$$

and find $T_2(\lambda,\mu)=f(t_\lambda)$, with

$$f(t_\lambda) = \int_0^{t_\lambda} D_0(t_\lambda-\widetilde{\alpha})\cdot[G_0(\widetilde{\alpha},\widetilde{\alpha}) - G_2(\widetilde{\alpha},\widetilde{\alpha})]d\widetilde{\alpha}.$$

By performing the Laplace transform one finds

$$L_T[f(t_\lambda)] = \frac{1}{p^2-1} - \frac{1}{p^2\big(p^2-1\big)} = \frac{1}{p^2}; \tag{6.27}$$

hence, $f(t_\lambda)=t_\lambda$, which gives

$$T_2(\lambda,\mu)=\lambda-\mu+\theta_2,\quad T_1(\lambda,\mu)=\frac{\partial T_2}{\partial\mu}=-1. \tag{6.28}$$

This solution is not statically admissible because the second equilibrium equation is not satisfied; the solution (6.25) must be rejected. Thus, the correct solution for the ABHC domain is given by the formulae (6.23) and (6.24).

In the circular domains only the radial internal forces are present: T_2 in the lower domain and T_1 in the upper circular domain. These forces are equal to their boundary values adjacent to ABHC.

For the lower circular domain we have

$$T_1(\alpha, \beta_1) = 0$$
$$T_2(\alpha, \beta_1) = F_C G_1(\beta_p, \alpha_p - \alpha) - F_C G_1(\alpha_p - \alpha, \beta_p), \tag{6.29a}$$

while in the upper circular region,

$$T_1(\alpha_1, \beta) = -F_C G_0(\beta_p - \beta, \alpha_p) + F_C G_2(\alpha_p, \beta - \beta_p)$$
$$T_2(\alpha_1, \beta) = 0. \tag{6.29b}$$

The case of the point load being applied within the domain BDH (see Fig. 6) is more complicated.

This position of the point load is described by the inequalities $0<\alpha_p<\theta_1$, $\theta_2<\beta_p<\alpha_p+\theta_2$. The compression bar lies entirely along the parametric line, while the tension bar lies partly on a parametric line and is partly tangent to other parametric lines of the same family. Thus, we have two kinds of boundary conditions along the tension bar: the condition $T_2=0$ along the straight segment and the condition $T_2=-F_T$ along the curved part. Thus, the domain ABPC is divided into two domains of static division, separated by the line ZM along which the force field T_2 suffers a jump. Let us emphasize that the static division does not coincide here with the geometric division. Along the line BD the geometry of the net changes, while the force fields do not change. On the contrary, along the line ZM perpendicular to the previous line, the force fields change while the geometrical characteristics remain unchanged. The following domains occur: $\text{III}^2=\text{ABQM}$, $\text{III}^1=\text{MQDC}$, $\text{IV}^2=\text{BQZ}$ and $\text{IV}^1=\text{ZPDQ}$.

The longitudinal forces in the compression bar NCP and in the tension bar on the segment PZ are computed from the equilibrium equations of the node P

$$F_C^{PN} = F_C(P) = -P\cos(\psi+\varphi),$$
$$F_T^{PZ} = F_T(P) = P\sin(\psi+\varphi), \tag{6.30}$$

where $\psi=\gamma_1+\alpha_p-\beta_p$.

The force fields in the domain MZPC are determined directly from Riemann's formula by using the values of T_1, T_2 being known along appropriate curved boundaries. We find the results (5.5b and 5.6) in the same manner as for the ABDC domain.

Now we shall find the force fields in the domain ABZM. We start from finding T_2 along the line ZQ, by using the known values of T_1 on this line, using the boundary condition of vanishing of T_2 at point Z and by using (2.4). We find

$$T_2(\beta_p - \theta_2, \beta) = F_C G_1(\beta_p - \beta, \alpha_p - \beta_p + \theta_2)$$
$$-F_T G_0(\alpha_p - \beta_p + \theta_2, \beta_p - \beta) + F_T; \tag{6.31}$$

here, $F_C=F_C(P)$, $F_T=F_T(P)$. We omit the argument P for brevity. We write the integral (6.3) on the line $\lambda=\beta_p-\theta_2$

$$\frac{1}{2}\int_{\mu-\theta_2}^{\beta_p-\theta_2} D_0(\beta_p - \theta_2 - \alpha, \alpha + \theta_2 - \mu)\,\chi(\alpha)d\alpha$$
$$= -F_T G_0(\alpha_p - \beta_p + \theta_2, \beta_p - \mu)$$
$$+F_C G_1(\beta_p - \mu, \alpha_p - \beta_p + \theta_2) + F_T \tag{6.32}$$

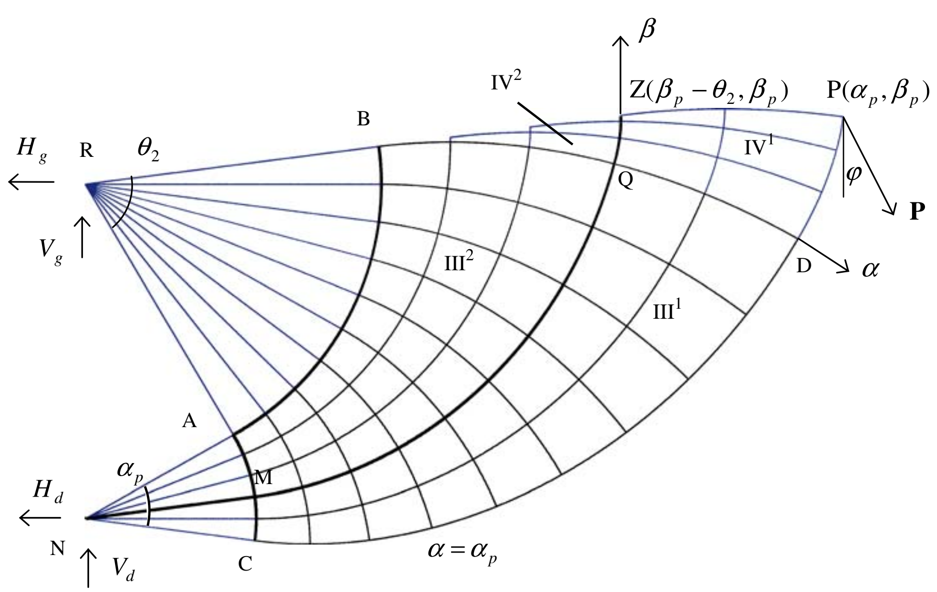

Fig. 6 Michell cantilever with the force applied within Chan's domain. Here $M=(\beta_p-\theta_2, 0)$, $Q=(\beta_p-\theta_2, \theta_2)$ and $D=(\alpha_p, \theta_2)$

and change the variables

$$\widetilde{\alpha}=\alpha-\mu+\theta_2\,,\ t=\beta_p-\mu,\ \theta=\alpha_p-\beta_p+\theta_2; \tag{6.33}$$

hence, $t-\widetilde{\alpha}=\beta_p-\alpha-\theta_2\,,\ t-\widetilde{\alpha}+\theta=\alpha_p-\alpha$, which rearranges (6.32) to the form

$$\frac{1}{2}\int_0^t D_0(t-\widetilde{\alpha},\widetilde{\alpha})\widetilde{\chi}(t-\widetilde{\alpha})d\widetilde{\alpha}=F_{\mathrm{T}}-F_{\mathrm{T}}G_0(\theta,t)+F_{\mathrm{C}}G_1(t,\theta), \tag{6.34}$$

where $\chi(\alpha)=\widetilde{\chi}(t-\widetilde{\alpha})$. The alternative substitution $\chi(\alpha)=\widetilde{\chi}(\widetilde{\alpha})$ leads to incorrect results; hence, it is now rejected. We perform the Laplace transform with respect to t on both sides of (6.34); by using (b.174), we find

$$\frac{1}{2}\frac{1}{p}\cdot\widetilde{\chi}^*\left(p+\frac{1}{p}\right)=\frac{F_{\mathrm{T}}}{p}-\frac{F_{\mathrm{T}}}{p}e^{\theta/p}+\frac{F_{\mathrm{C}}}{p^2}e^{\theta/p}. \tag{6.35}$$

We multiply both sides by p and change the variables

$$p+\frac{1}{p}=s\,;\qquad \frac{1}{p}=\frac{2}{s+R}\,;$$
$$R=\sqrt{s^2-4}\,;\qquad p=\frac{1}{2}(s+R) \tag{6.36}$$

to obtain

$$\frac{1}{2}\widetilde{\chi}^*(s)=F_{\mathrm{T}}-F_{\mathrm{T}}\exp\left(\frac{2\theta}{s+R}\right)+\frac{2F_C}{s+R}\exp\left(\frac{2\theta}{s+R}\right). \tag{6.37}$$

We introduce a new variable as follows:

$$h=\frac{1}{s+R};\qquad \frac{dh}{ds}=-\frac{h}{R}, \tag{6.38}$$

which rearranges (6.37) to the form

$$\frac{1}{2}\widetilde{\chi}^*(s)=F_{\mathrm{T}}-F_{\mathrm{T}}\exp(2\theta\cdot h)+F_{\mathrm{C}}2h\exp(2\theta\cdot h) \tag{6.39}$$

Now we differentiate both sides with respect to s:

$$\frac{1}{2}\frac{d\widetilde{\chi}^*(s)}{ds}=2F_{\mathrm{C}}\left(\frac{dh}{ds}e^{2\theta h}+h\cdot 2\theta\frac{dh}{ds}e^{2\theta h}\right)-F_{\mathrm{T}}\cdot 2\theta\frac{dh}{ds}e^{2\theta h}, \tag{6.40}$$

and by using (6.38), we get

$$\frac{1}{2}\frac{d\widetilde{\chi}^*(s)}{ds}=-F_{\mathrm{C}}\frac{2^1}{R(s+R)}\exp\left[\frac{\theta}{2}(s-R)\right]-F_{\mathrm{C}}\theta\frac{2^2}{R(s+R)}\exp\left[\frac{\theta}{2}(s-R)\right]+F_{\mathrm{T}}\theta\frac{2^1}{R(s+R)}\exp\left[\frac{\theta}{2}(s-R)\right]. \tag{6.41}$$

Now we perform the Laplace transform with the use of (b.164) and of the known rule for the transform of a derivative to have

$$-\frac{1}{2}L_t[t\cdot\widetilde{\chi}(t)]=-F_{\mathrm{C}}\cdot L_t[G_1(t,t+\theta)]-F_{\mathrm{C}}\theta\cdot L_t[G_2(t,t+\theta)]+F_{\mathrm{T}}\theta\cdot L_t[G_1(t,t+\theta)], \tag{6.42}$$

which implies

$$-\frac{1}{2}\widetilde{\chi}(t)=-\frac{F_{\mathrm{C}}G_1(t,t+\theta)}{t}-\frac{F_{\mathrm{C}}\theta\cdot G_2(t,t+\theta)}{t}+\frac{F_{\mathrm{T}}\theta\cdot G_1(t,t+\theta)}{t}. \tag{6.43}$$

By using $\chi(\alpha)=\widetilde{\chi}(t-\widetilde{\alpha})$ and coming back to the original variables we get

$$\frac{1}{2}\chi(\alpha)=\frac{F_{\mathrm{C}}G_1\big(\beta_p-\theta_2-\alpha,\alpha_p-\alpha\big)}{\beta_p-\theta_2-\alpha}+\frac{F_{\mathrm{C}}\big(\alpha_p-\beta_p+\theta_2\big)G_2\big(\beta_p-\theta_2-\alpha,\alpha_p-\alpha\big)}{\beta_p-\theta_2-\alpha}-\frac{F_{\mathrm{T}}\big(\alpha_p-\beta_p+\theta_2\big)G_1\big(\beta_p-\theta_2-\alpha,\alpha_p-\alpha\big)}{\beta_p-\theta_2-\alpha}. \tag{6.44}$$

The expression above can be simplified by using the identity (a.10).

We omit the derivation and report the final result

$$\begin{aligned}\chi(\alpha)=\,&2F_{\mathrm{C}}G_0\left(\beta_p-\theta_2-\alpha,\alpha_p-\alpha\right)\\&-2F_{\mathrm{C}}G_2\left(\beta_p-\theta_2-\alpha,\alpha_p-\alpha\right)\\&-2F_{\mathrm{T}}G_1\left(\alpha_p-\alpha,\beta_p-\theta_2-\alpha\right)\\&+2F_{\mathrm{T}}G_1\left(\beta_p-\theta_2-\alpha,\alpha_p-\alpha\right).\end{aligned} \tag{6.45}$$

To find now the force field $T_2(\lambda,\mu)$ we substitute (6.45) into (6.3), use (a.12$_1$) and obtain a complicated integral expression:

$$\begin{aligned}T_2(\lambda,\mu)=\frac{1}{2}\int_{\mu-\theta_2}^{\lambda}&D_0(\lambda-\alpha,\alpha+\theta_2-\mu)\\&\cdot\big[2F_{\mathrm{C}}G_0\left(\beta_p-\theta_2-\alpha,\alpha_p-\alpha\right)\\&-2F_{\mathrm{C}}G_2\left(\beta_p-\theta_2-\alpha,\alpha_p-\alpha\right)\\&-2F_{\mathrm{T}}G_{-1}\left(\beta_p-\theta_2-\alpha,\alpha_p-\alpha\right)\\&+2F_{\mathrm{T}}G_1\left(\beta_p-\theta_2-\alpha,\alpha_p-\alpha\right)\big]\,d\alpha.\end{aligned} \tag{6.46}$$

This integral can be put in terms of Lommel functions. To show this we change the variables

$$\widetilde{\alpha} = \alpha - \mu + \theta_2\,;$$
$$\theta = \alpha_p - \beta_p + \theta_2,$$
$$t = \beta_p - \mu\,,$$
$$t_\lambda = \lambda - \mu + \theta_2; \tag{6.47}$$

hence,

$$t_\lambda - \widetilde{\alpha} = \lambda - \alpha\,,$$
$$t - \widetilde{\alpha} = \beta_p - \alpha - \theta_2,$$
$$t - \widetilde{\alpha} + \theta = \alpha_p - \alpha,$$

and rewrite (6.46) in the form

$$\begin{aligned} T_2(\lambda,\mu) = \int_0^{t_\lambda} D_0(t_\lambda - \widetilde{\alpha}, \widetilde{\alpha}) \cdot [F_C G_0(t-\widetilde{\alpha}, t-\widetilde{\alpha}+\theta) \\ - F_C G_2(t-\widetilde{\alpha}, t-\widetilde{\alpha}+\theta) \\ - F_T G_{-1}(t-\widetilde{\alpha}, t-\widetilde{\alpha}+\theta) \\ + F_T G_1(t-\widetilde{\alpha}, t-\widetilde{\alpha}+\theta)]\, d\widetilde{\alpha}. \end{aligned} \tag{6.48}$$

The rule (B.1) is crucial here; it gives directly the following result:

$$\begin{aligned} T_2(\lambda,\mu) &= F_C G_1(t, t-t_\lambda+\theta) - F_C G_1(t-t_\lambda, t+\theta) \\ &\quad - F_T G_0(t, t-t_\lambda+\theta) + F_T G_0(t-t_\lambda, t+\theta)\,. \end{aligned} \tag{6.49}$$

Now we recall the formulae

$$t - t_\lambda = \beta_p - \lambda - \theta_2,$$
$$t - t_\lambda + \theta = \alpha_p - \lambda,$$
$$t + \theta = \alpha_p - \mu + \theta_2 \tag{6.50}$$

and arrive at

$$\begin{aligned} T_2(\lambda,\mu) &= F_C G_1\left(\beta_p - \mu, \alpha_p - \lambda\right) \\ &\quad - F_T G_0\left(\beta_p - \mu, \alpha_p - \lambda\right) \\ &\quad - F_C G_1\left(\beta_p - \theta_2 - \lambda, \alpha_p + \theta_2 - \mu\right) \\ &\quad + F_T G_0\left(\beta_p - \theta_2 - \lambda, \alpha_p + \theta_2 - \mu\right), \\ T_1(\lambda,\mu) &= F_T G_1\left(\alpha_p - \lambda, \beta_p - \mu\right) \\ &\quad - F_C G_0\left(\beta_p - \mu, \alpha_p - \lambda\right) \\ &\quad + F_C G_2\left(\beta_p - \theta_2 - \lambda, \alpha_p + \theta_2 - \mu\right) \\ &\quad - F_T G_1\left(\beta_p - \theta_2 - \lambda, \alpha_p + \theta_2 - \mu\right), \end{aligned} \tag{6.51}$$

the latter formula being obtained by (2.4). Above formulae determine force fields within the domain ABZM.

In the case of $\beta_p = \alpha_p + \theta_2$ the integral equation assumes the form (6.8b), whilst (6.51) assumes the forms (6.23) and (6.24). Then the subdomain ZNP of the static division disappears.

The force fields within the circular domains can be found by continuity conditions of the force field T_1 along the arc AB and of the force field T_2 along the arc AC. The final results are as follows

The lower circular domain NMC

$$T_1(\alpha, \beta_1) = 0$$
$$T_2(\alpha, \beta_1) = F_C G_1\left(\beta_p, \alpha_p - \alpha\right) - F_T G_0\left(\beta_p, \alpha_p - \alpha\right); \tag{6.52}$$

The lower circular domain NAM

$$T_1(\alpha, \beta_1) = 0$$
$$\begin{aligned} T_2(\alpha, \beta_1) &= -F_C G_1\left(\beta_p, \alpha_p - \alpha\right) \\ &\quad - F_C G_1\left(\beta_p - \theta_2 - \alpha, \alpha_p + \theta_2\right) \\ &\quad - F_T G_0\left(\beta_p, \alpha_p - \alpha\right) \\ &\quad + F_T G_0\left(\beta_p - \theta_2 - \alpha, \alpha_p + \theta_2\right); \end{aligned} \tag{6.53}$$

The upper circular domain RBA

$$\begin{aligned} T_1(\alpha_1, \beta) &= -F_C G_0\left(\beta_p - \beta, \alpha_p\right) \\ &\quad + F_C G_2\left(\beta_p - \theta_2, \alpha_p + \theta_2 - \beta\right) \\ &\quad + F_T G_1\left(\alpha_p, \beta_p - \beta\right) \\ &\quad - F_T G_1\left(\beta_p - \theta_2, \alpha_p + \theta_2 - \beta\right) \\ T_2(\alpha_1, \beta) &= 0. \end{aligned} \tag{6.54}$$

Within the compression bar BZ, where the fibres α are joined tangentially with the reinforcing bar (not shown in Fig. 6), the longitudinal force varies according to the rule

$$F_T^{BZ} = F_T(\mathrm{P}) + \int_\alpha^{\beta_p - \theta_2} T_1^{BQZ}(\overline{\alpha}, \overline{\alpha} + \theta_2)\, d\overline{\alpha}. \tag{6.55}$$

The tension force in the segment RB is constant:

$$F_T^{RB} = F_T^{BZ}(\mathrm{B}). \tag{6.56}$$

The compression force F_C is constant in the reinforcing bar NCP and equals $F_C(\mathrm{P})$ (see 6.30).

7 The lower Chan's domain CDG

Two different cases of the point load applied at the boundary and in the interior of the domain CDG (see Fig. I.19 and Fig. 7) will be considered simultaneously. The application point is parameterized by $0 < \beta_p \leq \theta_2$, $\theta_1 < \alpha_p \leq \beta_p + \theta_1$. Along the tension bar NP two types of the boundary conditions are present: $T_1 = 0$ along the straight boundary NZ′ and $T_1 = -F_C$ along the curved boundary Z′P. On the line RM′Z′ the force field T_1 undergoes a jump. Thus, this line divides the domain

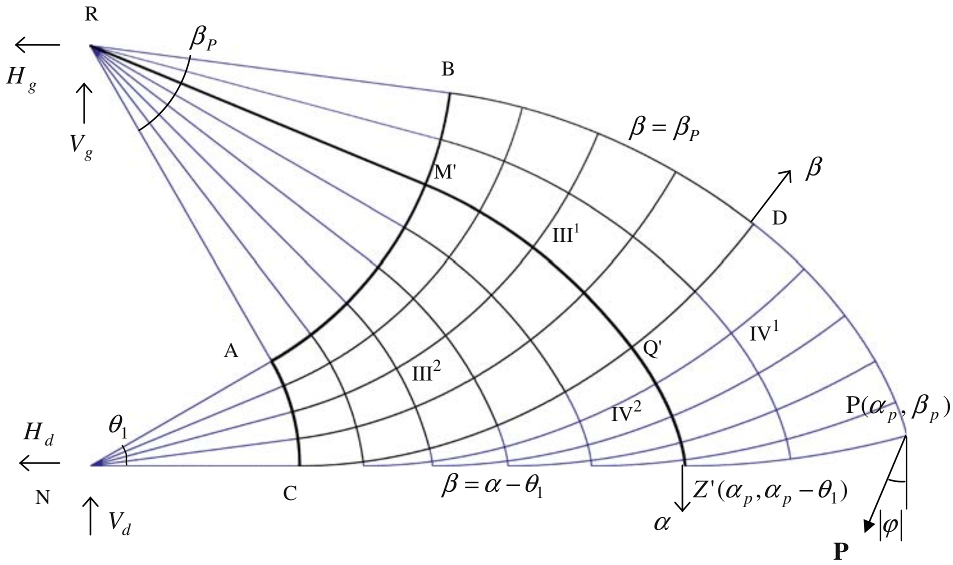

Fig. 7 Optimal cantilever with the force applied within lower Chan domain. Here $\varphi<0$

into two subdomains of the static division. Along the line RM′Z′ the force field T_2 is continuous.

The vertices of the domains have curvilinear coordinates (α, β):

$$\mathrm{B}\big(0, \beta_p\big), \mathrm{M}'(0, \alpha_p - \theta_1), \mathrm{D}\big(\theta_1, \beta_p\big), \mathrm{Q}'\big(\theta_1, \alpha_p - \theta_1\big),$$

$$\mathrm{Z}'\big(\alpha_p, \alpha_p - \theta_1\big), \mathrm{C}(\theta_1, 0).$$

The forces in the tension bar RBDP and in the compression bar along PZ′ are determined by

$$F_{\mathrm{T}} = P \sin(\psi + \varphi), \quad F_{\mathrm{C}}^{PZ'} = -P \cos(\psi + \varphi) \tag{7.1}$$

with

$$\psi = \gamma_1 + \alpha_p - \beta_p. \tag{7.2}$$

The force fields inside M′BPZ′ are computed directly from the Riemann formula, which gives the results (5.5b) and (5.6). Further computation needs the formula for T_1 along the line Q′Z′, where $\beta{=}\alpha_p{-}\theta_1$. First, note that the force field T_2 is known along this line. By using the equilibrium (2.4) and the condition of vanishing of T_1 at point Z′, one finds

$$T_1\left(\alpha, \alpha_p - \theta_1\right) = F_{\mathrm{T}} G_1\left(\alpha_p - \alpha, \beta_p - \alpha_p + \theta_1\right) - F_{\mathrm{C}} G_0\left(\beta_p - \alpha_p + \theta_1, \alpha_p - \alpha\right) + F_{\mathrm{C}}, \tag{7.3}$$

where $F_{\mathrm{C}}{=}F_{\mathrm{C}}(\mathrm{P})$, $F_{\mathrm{T}}{=}F_{\mathrm{T}}(\mathrm{P})$ (see 5.1) and argument P is omitted for brevity.

The following subdomains occur: III^1 = BM′Q′D, III^2 = AM′Q′C, IV^1 = Q′DPZ′ and IV^2 = CQ′Z′.

The distribution of the force field T_1 within AM′Z′C will be found by Riemann formula in the form (see Appendix I.B, (I.B.5) and Fig. 8)

$$T_1(\xi, \eta) = \frac{1}{2}\left(T_{1|_F} + T_{1|_E}\right) - \frac{1}{2}\int_{\xi}^{\eta+\theta_1}\left[T_1(\alpha, \alpha-\theta_1)\left(\frac{\partial G}{\partial \beta} - \frac{\partial G}{\partial \alpha}\right)_{|\beta=\alpha-\theta_1} + G(\alpha, \alpha-\theta_1)\left(\frac{\partial T_1}{\partial \alpha} - \frac{\partial T_1}{\partial \beta}\right)_{|\beta=\alpha-\theta_1}\right] d\alpha \tag{7.4}$$

for the domain QFE adjacent to the straight line CZ′. The force T_1 vanishes on the straight boundary CZ′, where $\beta{=}\alpha{-}\theta_1$. Thus, the formula (7.4) reduces to

$$T_1(\xi, \eta) = -\frac{1}{2}\int_{\xi}^{\eta+\theta_1} G(\alpha, \alpha-\theta_1)\left(\frac{\partial T_1}{\partial \alpha} - \frac{\partial T_1}{\partial \beta}\right)_{|\beta=\alpha-\theta_1} d\alpha \tag{7.5a}$$

or

$$T_1(\xi, \eta) = \frac{1}{2}\int_{\xi-\theta_1}^{\eta} G(\beta+\theta_1, \beta)\left(\frac{\partial T_1}{\partial \beta} - \frac{\partial T_1}{\partial \alpha}\right)_{|\alpha=\beta+\theta_1} d\beta \tag{7.5b}$$

or

$$T_1(\xi, \eta) = \frac{1}{2}\int_{\xi-\theta_1}^{\eta} D_0(\eta-\beta, \beta+\theta_1-\xi)\chi(\beta) d\beta. \tag{7.5c}$$

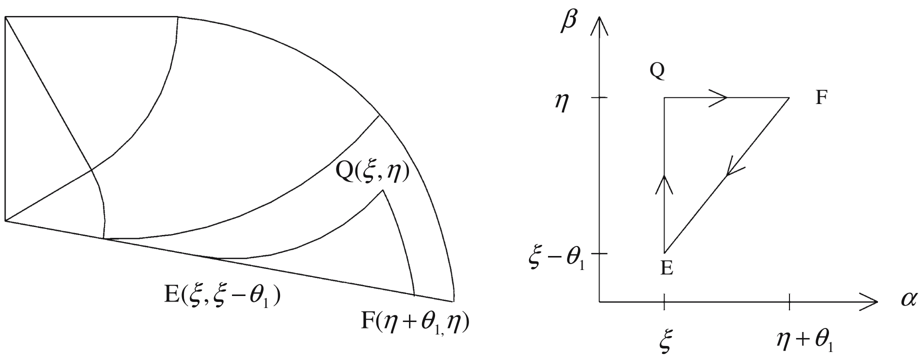

Fig. 8 Application of Riemann formulae for deriving distribution of internal force within CDG domain

The function $\chi(\beta)$ determined on the straight edge CZ′ is defined by

$$\chi(\beta)=\left(\frac{\partial T_1}{\partial\beta}-\frac{\partial T_1}{\partial\alpha}\right)\Big|_{\alpha=\beta+\theta_1}\,. \tag{7.6}$$

The formula (7.5c) will be written for the line Q′Z′ or for $\eta{=}\alpha_p{-}\theta_1$

$$T_1\left(\xi,\alpha_p-\theta_1\right)=\frac{1}{2}\int_{\xi-\theta_1}^{\alpha_p-\theta_1}D_0\left(\alpha_p-\theta_1-\beta,\,\beta+\theta_1-\xi\right)\chi(\beta)d\beta. \tag{7.7}$$

By equating the right-hand sides of (7.3) (for $\alpha{=}\xi$) and (7.7), one finds

$$\begin{aligned}\frac{1}{2}\int_{\xi-\theta_1}^{\alpha_p-\theta_1}D_0\ \left(\alpha_p-\theta_1-\beta,\,\beta+\theta_1-\xi\right)\chi(\beta)\,d\beta\\ =-F_{\mathrm{C}}G_0\left(\beta_p-\alpha_p+\theta_1,\,\alpha_p-\xi\right)\\ +F_{\mathrm{T}}G_1\left(\alpha_p-\xi,\,\beta_p-\alpha_p+\theta_1\right)+F_{\mathrm{C}}.\end{aligned} \tag{7.8}$$

The integral equation obtained above is equivalent to (6.32), which can be shown upon substitution:

$$\begin{array}{lll}\alpha\to\beta,\ \mu\to\xi, & \alpha_p\to\beta_p,\,\theta_2\to\theta_1, & F_{\mathrm{T}}\to F_{\mathrm{C}}\\ \quad\ \lambda\to\eta & \beta_p\to\alpha_p & F_{\mathrm{C}}\to F_{\mathrm{T}}\end{array}\,. \tag{7.9}$$

The physically correct solution of (7.8) is similar to (6.45) and reads

$$\begin{aligned}\chi(\beta)=\,&2F_{\mathrm{T}}G_0\left(\alpha_p-\theta_1-\beta,\,\beta_p-\beta\right)\\ &-2F_{\mathrm{T}}G_2\left(\alpha_p-\theta_1-\beta,\,\beta_p-\beta\right)\\ &-2F_{\mathrm{C}}G_1\left(\beta_p-\beta,\,\alpha_p-\theta_1-\beta\right)\\ &+2F_{\mathrm{C}}G_1\left(\alpha_p-\theta_1-\beta,\,\beta_p-\beta\right).\end{aligned} \tag{7.10}$$

The function $\chi(\beta)$ thus found should now be inserted into (7.5c). By virtue of (B.1) the integration can be performed analytically to find

$$\begin{aligned}T_1(\xi,\eta)=\,&F_{\mathrm{T}}G_1\left(\alpha_p-\xi,\,\beta_p-\eta\right)\\ &-F_{\mathrm{C}}G_0\left(\alpha_p-\xi,\,\beta_p-\eta\right)\\ &-F_{\mathrm{T}}G_1\left(\alpha_p-\theta_1-\eta,\,\beta_p+\theta_1-\xi\right)\\ &+F_{\mathrm{C}}G_0\left(\alpha_p-\theta_1-\eta,\,\beta_p+\theta_1-\xi\right).\end{aligned} \tag{7.11a}$$

By (2.4) one can find now the second force field

$$\begin{aligned}T_2(\xi,\eta)=\,&F_{\mathrm{C}}G_1\left(\beta_p-\eta,\,\alpha_p-\xi\right)\\ &-F_{\mathrm{T}}G_0\left(\alpha_p-\xi,\,\beta_p-\eta\right)\\ &+F_{\mathrm{T}}G_2\left(\alpha_p-\theta_1-\eta,\,\beta_p+\theta_1-\xi\right)\\ &-F_{\mathrm{C}}G_1\left(\alpha_p-\theta_1-\eta,\,\beta_p+\theta_1-\xi\right).\end{aligned} \tag{7.11b}$$

The formulae (7.11a, b) correspond to the formulae (6.51) concerning the upper Chan's domain upon the change of the variables by (7.9). (7.11a, b) determines values of the force fields within the domain AM′Z′C.

In the circular domains the force fields are constant and are directed radially; they are equal to the forces in the arcs lying on the boundaries between the fan domains and the Hill domain.

Within the compression bar CZ′, where the fibres β are joined tangentially with the reinforcing bar, the longitudinal force varies according to the rule

$$F_{\mathrm{C}}^{CZ'}=F_{\mathrm{C}}(\mathrm{P})+\int_{\beta}^{\alpha_p-\theta_1}T_2^{CQ'Z'}\left(\overline{\beta}+\theta_1,\overline{\beta}\right)d\overline{\beta}. \tag{7.12}$$

The compression force in the segment NC is constant:

$$F_{\mathrm{C}}^{NC}=F_{\mathrm{C}}^{CZ'}(\mathrm{C}). \tag{7.13}$$

The solution above holds for the case of the force being applied at the lower boundary of the feasible domain or for $\beta_p{\le}\theta_2$, $\alpha_p{=}\beta_p{+}\theta_1$. The curve RM′Z′ is then the upper edge of the cantilever domain, whilst the domain RM′Z′PBR does not occur in the cantilever.

8 The force applied within the domain DHJG

The vertices of the domains have the following curvilinear coordinates:

$$\mathrm{A}(0,0), \mathrm{B}(0,\theta_2), \mathrm{M}'(0,\alpha_p-\theta_1), \mathrm{D}(\theta_1,\theta_2),$$
$$\mathrm{Q}'(\theta_1,\alpha_p-\theta_1), \mathrm{Z}'(\alpha_p,\alpha_p-\theta_1), \mathrm{C}(\theta_1,0), \mathrm{S}_g(\theta_1,\beta_p),$$
$$\mathrm{M}(\beta_p-\theta_2,0), \mathrm{W}(\beta_p-\theta_2,\alpha_p-\theta_1), \mathrm{Q}(\beta_p-\theta_2,\theta_2),$$
$$\mathrm{Z}(\beta_p-\theta_2,\beta_p), \mathrm{S}_d(\alpha_p,\theta_2)$$

In the case of the point load being applied within the domain DHJG corresponding to the intervals $\theta_1<\alpha_p\leq\theta_1+\theta_2$, $\theta_2<\beta_p\leq\theta_1+\theta_2$, both along the reinforcing tension bar and the compression bar, two types of the boundary conditions occur. The internal force orthogonal to the bar lying along the parametric line is equal to the force in the reinforcing bar, with the opposite sign, whilst the internal force normal to the straight reinforcing bar vanishes. The lines NZ and RZ′ starting at the points of a discontinuity of the boundary condition are the lines of the static division of the cantilever. On these lines the forces T_2 or T_1 suffer jumps. We note that the cantilever is now divided into eight domains of static division separated by thin lines in Fig. 9. Some of the force fields have already been found while analyzing the force fields in the previously considered domains. The following subdomains occur:

$$\mathrm{III}=\mathrm{AM'WM}\,,\ \mathrm{III}^1=\mathrm{WQDQ'},\ \mathrm{III}^2_{\mathrm{d}}=\mathrm{CMWQ'},$$
$$\mathrm{III}^2_{\mathrm{g}}=\mathrm{BM'WQ},\ \mathrm{IV}^2_{\mathrm{g}}=\mathrm{BQZ},\ \mathrm{IV}^2_{\mathrm{d}}=\mathrm{CQ'Z'},$$
$$\mathrm{IV}^1_{\mathrm{g}}=\mathrm{QZS_gD},\ \mathrm{IV}^1_{\mathrm{d}}=\mathrm{Q'DS_dZ'},\ \mathrm{V}=\mathrm{DS_gPS_d}.$$

The force field within the domain WZPZ′ can be found by applying the standard Riemann formula, making use of the values of the force fields being known along the boundaries, thus arriving at (5.5b) and (5.6).

In the upper Chan's domain $\mathrm{IV}^2_{\mathrm{g}}$ the force field T_2 can be determined from the integral equation using the known value of this force along the straight edge BZ and on the arc ZQ. Computations shown in Section 6 lead to the result (6.51). This result is valid in the domain $\mathrm{III}^2_{\mathrm{g}}$. In the lower Chan's domain $\mathrm{IV}^2_{\mathrm{d}}$ the force field T_1 can be found from the integral equation by using the known value of this force along the straight segment CZ′ and on the arc Z′Q′. This analysis has been performed in Section 7, leading to the result (7.11a, b), which can be applied also in the domain $\mathrm{III}^2_{\mathrm{d}}$. Hence, the force fields within the domain BZWM′ are determined by formula (6.51) and within the domain CMWZ′ by formula (7.11a, b).

As can be easily noted the formulae (6.51) and (7.11a, b) have the same first two terms, the difference between them lying in the last two terms. If the solution referring to the upper Chan's domain is augmented with two additional terms referring to the lower domain, then the boundary conditions concerning the force field T_1 along the line M′W and the boundary conditions concerning the force field T_2 along MW will be satisfied. The formula for the force fields within AM′WM becomes

$$\begin{aligned}T_1(\alpha,\beta)=&-F_{\mathrm{C}}G_0(\beta_p-\beta,\alpha_p-\alpha)\\&+F_{\mathrm{T}}G_1(\alpha_p-\alpha,\beta_p-\beta)\\&+F_{\mathrm{C}}G_2(\beta_p-\theta_2-\alpha,\alpha_p+\theta_2-\beta)\\&-F_{\mathrm{T}}G_1(\beta_p-\theta_2-\alpha,\alpha_p+\theta_2-\beta)\\&+F_{\mathrm{C}}G_0(\alpha_p-\theta_1-\beta,\beta_p+\theta_1-\alpha)\\&-F_{\mathrm{T}}G_1(\alpha_p-\theta_1-\beta,\beta_p+\theta_1-\alpha)\end{aligned}$$

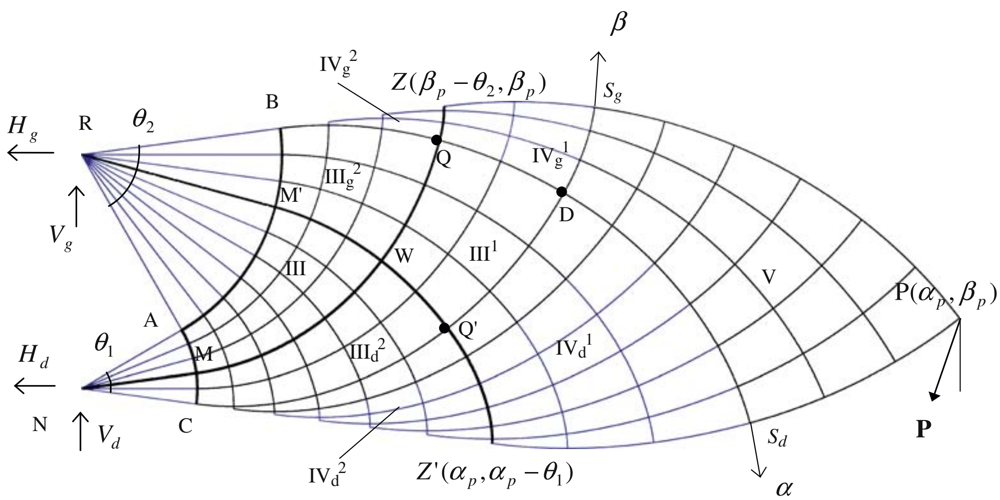

Fig. 9 Static division of the optimal cantilever while load is applied within DHJG domain (see Fig. I.19)

$$
\begin{aligned}
T_2(\alpha,\beta) = {} & F_C G_1(\beta_p-\beta,\alpha_p-\alpha) \\
& -F_T G_0(\beta_p-\beta,\alpha_p-\alpha) \\
& -F_C G_1(\beta_p-\theta_2-\alpha,\alpha_p+\theta_2-\beta) \\
& +F_T G_0(\beta_p-\theta_2-\alpha,\alpha_p+\theta_2-\beta) \\
& -F_C G_1(\alpha_p-\theta_1-\beta,\beta_p+\theta_1-\alpha) \\
& +F_T G_2(\alpha_p-\theta_1-\beta,\beta_p+\theta_1-\alpha), \qquad (8.1)
\end{aligned}
$$

where $F_C = F_C(P)$, $F_T = F_T(P)$. The equations above can be found by using Riemann formula, knowing the values of the corresponding functions along the lines MW and M′W.

In both the circular domains we note two domains of static division separated by the lines RM′ and NM. The force in the tension bar is constant along ZP and RB and increases on BZ according to (6.55). The force in the compression bar is constant along NC and Z′P and varies on CZ′ according to (7.12).

9 The force applied within HJH_2, GJG_2 and $JH_2J_2G_2$

Let us consider the point load applied within the domain HJH_2 (see Fig. I.19). The position of the force is determined by the inequalities $\theta_1 < \alpha_p < \theta_1 + \theta_2$, $\theta_1 + \theta_2 < \beta_p < \alpha_p + \theta_2$. The whole cantilever is divided into eight domains of static division, four of them lying within the circular fans. The force fields within the domains of upper indices 1–4 have been found in Section 8; thus, we confine our analysis to the domain IV_d^5 (see Fig. 10).

The vertices of the domains have the following coordinates:

$A(0,0)$, $B(0,\theta_2)$, $M'(0,\alpha_p-\theta_1)$, $D(\theta_1,\theta_2)$,
$M_1(\theta_1,\beta_p-\theta_1-\theta_2)$, $C(\theta_1,0)$, $Z(\beta_p-\theta_2,\beta_p)$,
$A_1(0,\beta_p-\theta_1-\theta_2)$, $Z_1(\beta_p-\theta_2,\beta_p-\theta_1-\theta_2)$,
$D_3(\theta_1,\alpha_p-\theta_1)$, $D_4(\beta_p-\theta_2,\alpha_p-\theta_1)$, $H(\theta_1,\theta_1+\theta_2)$,
$D_2(\beta_p-\theta_2,\theta_2)$, $B_1(\alpha_p,\theta_2)$, $D_5(\alpha_p,\theta_1+\theta_2)$,
$D_1(\beta_p-\theta_2,\theta_1+\theta_2)$.

The following subdomains occur:

$III^5 = AA_1M_1C$, $III^4 = A_1M'D_3M_1$, $III^2 = M'BDD_3$,
$IV_g^2 = BDH$, $IV_d^2 = D_3DD_2D_4$, $IV_d^4 = M_1D_3D_4Z_1$,
$IV_d^3 = Z_1D_4Z'$, $IV_d^1 = D_4D_2B_1Z'$, $IV_d^5 = CM_1Z_1$,
$V^1 = D_2D_1D_5B_1$, $V^2 = DHD_1D_2$,
$VI_g^2 = HD_1Z$, $VI_g^1 = ZD_1D_5P$.

We shall find the field T_1 within CM_1Z_1 by setting the integral equation on the line $\beta = \beta_p - \theta_1 - \theta_2$ or $A_1M_1Z_1$; along this line, the function T_2 is continuous. By (2.4) we have

$$
\begin{aligned}
T_1^{(5)}(\alpha,\beta_p-\theta_2-\theta_1) &= \int T_2(\alpha,\beta_p-\theta_2-\theta_1)\,d\alpha \\
&= T_1^{(4)}(\alpha,\beta_p-\theta_2-\theta_1) + C, \qquad (9.1)
\end{aligned}
$$

where the upper index (5) indicates the subdomain IV_d^5 of static division. The constant C is computed from the conditions

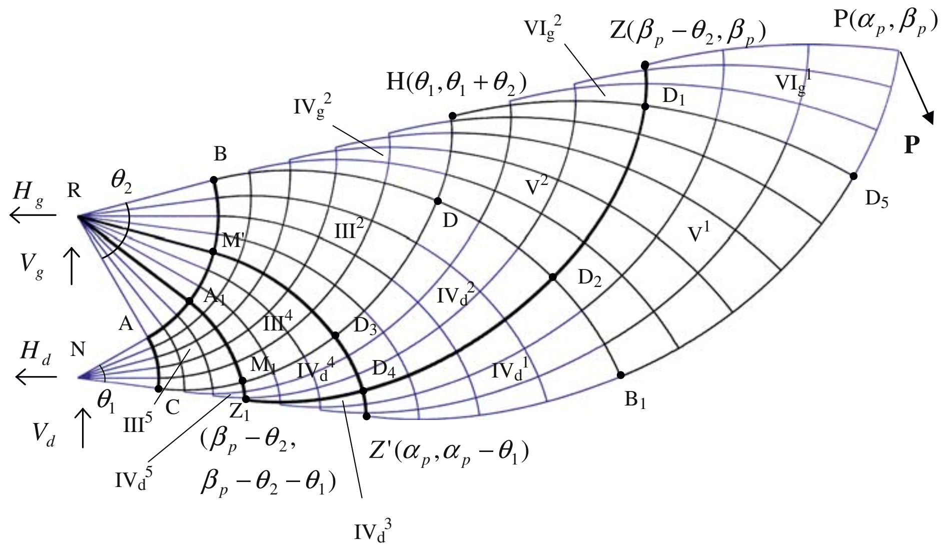

Fig. 10 Michell cantilever with the force applied within upper Chan domain of the second rank

$T_1^i(\beta_p-\theta_2, \beta_p-\theta_1-\theta_2)=0$, for i=4,5. Hence, C=0. The terms corresponding to the domain no $\mathrm{IV}_{\mathrm{d}}^3$ satisfy the condition of the force T_1 being zero along the straight boundary. Consequently, these terms will be still valid in domain no IV_d^5. The integral equation (7.5c) will be written only for the terms appearing at the passage from domain $\mathrm{IV}_{\mathrm{d}}^3$ to domain $\mathrm{IV}_{\mathrm{d}}^4$ (see 6.51):

$$\frac{1}{2}\int_{\xi-\theta_1}^{\beta_p-\theta_2-\theta_1} D_0\big(\beta_p-\theta_2-\theta_1-\beta, \beta+\theta_1-\xi\big)\chi(\beta)d\beta$$
$$= -F_{\mathrm{T}}G_1\big(\beta_p-\xi-\theta_2, \alpha_p-\beta_p+\theta_1+2\theta_2\big)$$
$$+F_{\mathrm{C}}G_2\big(\beta_p-\xi-\theta_2, \alpha_p-\beta_p+\theta_1+2\theta_2\big), \tag{9.2}$$

where F_{T}=F_{T}(P), F_{C}=F_{C}(P). Function $\chi(\beta)$ determines the solution T_1 as follows:

$$T_1(\alpha,\beta) = T_1^{IVd4}(\alpha,\beta) + \widehat{T}_1(\alpha,\beta), \tag{9.3a}$$

where ${T_1}^{IVd4}$ is given by (7.11a) and

$$\widehat{T}_1(\xi,\eta) = \frac{1}{2}\int_{\xi-\theta_1}^{\eta} D_0(\eta-\beta, \beta+\theta_1-\xi)\chi(\beta)d\beta \tag{9.3b}$$

To solve (9.2) we change the variables

$$\widetilde{\beta} = \beta-\xi+\theta_1, t=\beta_p-\theta_2-\xi\ , \theta=\alpha_p-\beta_p+\theta_1+2\theta_2;$$

hence, $t-\widetilde{\beta} = \beta_p-\theta_2-\theta_1-\beta,\ t-\widetilde{\beta}+\theta = \beta_p+\theta_2$. The integral equation assumes the form

$$\frac{1}{2}\int_0^t D_0\big(t-\widetilde{\beta},\widetilde{\beta}\big)\widetilde{\chi}\big(t-\widetilde{\beta}\big)d\widetilde{\beta} = -F_{\mathrm{T}}G_1(t,\theta)+F_{\mathrm{C}}G_2(t,\theta), \tag{9.4}$$

where $\chi(\beta) = \widetilde{\chi}\big(t-\widetilde{\beta}\big)$. By the result (B.21) one finds

$$\frac{1}{2}\widetilde{\chi}\left(t-\widetilde{\beta}\right) = -F_{\mathrm{T}}G_0\left(t-\widetilde{\beta}, t-\widetilde{\beta}+\theta\right)$$
$$+F_{\mathrm{T}}G_2\left(t-\widetilde{\beta}, t-\widetilde{\beta}+\theta\right)$$
$$+F_{\mathrm{C}}G_1\left(t-\widetilde{\beta}, t-\widetilde{\beta}+\theta\right)$$
$$-F_{\mathrm{C}}G_3\big(t-\widetilde{\beta}, t-\widetilde{\beta}+\theta\big) \tag{9.5}$$

The solution $\widetilde{\chi}\big(t-\widetilde{\beta}\big)$ will now be substituted into (9.3b). We introduce the variable $t_\eta = \eta-\xi+\theta_1$ and write

$$\widehat{T}_1(\xi,\eta) = -F_{\mathrm{T}}\int_0^{t_\eta} D_0\big(t_\eta-\widetilde{\beta},\widetilde{\beta}\big)\cdot\big[G_0\big(t-\widetilde{\beta}, t-\widetilde{\beta}+\theta\big)$$
$$-G_2\big(t-\widetilde{\beta}, t-\widetilde{\beta}+\theta\big)\big]$$
$$+F_{\mathrm{C}}\int_0^{t_\eta} D_0\big(t_\eta-\widetilde{\beta},\widetilde{\beta}\big)\cdot\big[G_1\big(t-\widetilde{\beta}, t-\widetilde{\beta}+\theta\big)$$
$$-G_3\big(t-\widetilde{\beta}, t-\widetilde{\beta}+\theta\big)\big]\,d\widetilde{\beta} \tag{9.6}$$

The above integrals are computed now by (B.1) to get

$$\widehat{T}_1\left(\xi,\eta\right) = -F_{\mathrm{T}}G_1\left(t, t-t_\eta+\theta\right) + F_{\mathrm{T}}G_1\left(t-t_\eta, t+\theta\right)$$
$$+F_{\mathrm{C}}G_2\left(t, t-t_\eta+\theta\right) - F_{\mathrm{C}}G_2\left(t-t_\eta, t+\theta\right). \tag{9.7}$$

We make use of the relations

$$t-t_\eta = \beta_p-\theta_2-\theta_1-\eta,\ t-t_\eta+\theta = \alpha_p+\theta_2-\eta,$$
$$t+\theta = \alpha_p-\xi+\theta_1+\theta_2 \tag{9.8}$$

and arrive at the final form of the solution

$$\widehat{T}_1(\xi,\eta) = -F_{\mathrm{T}}G_1\big(\beta_p-\theta_2-\xi, \alpha_p+\theta_2-\eta\big)$$
$$+F_{\mathrm{T}}G_1\big(\beta_p-\theta_2-\theta_1-\eta, \alpha_p-\xi+\theta_1+\theta_2\big)$$
$$+F_{\mathrm{C}}G_2\big(\beta_p-\theta_2-\xi, \alpha_p+\theta_2-\eta\big)$$
$$-F_{\mathrm{C}}G_2\big(\beta_p-\theta_2-\theta_1-\eta, \alpha_p-\xi+\theta_1+\theta_2\big). \tag{9.9}$$

The solution found consists of terms corresponding to the domain IV_d^2 and of two new terms. The complete solution for the domain IV_d^5 consists of the terms giving function $\widehat{T}_1(\xi,\eta)$ and the terms of the domain IV_d^3 of the static division (see 9.3a):

$$T_1(\alpha,\beta) = -F_{\mathrm{C}}G_0\big(\beta_p-\beta, \alpha_p-\alpha\big)$$
$$+F_{\mathrm{T}}G_1\big(\alpha_p-\alpha, \beta_p-\beta\big)$$
$$+F_{\mathrm{C}}G_0\big(\alpha_p-\beta-\theta_1, \beta_p-\alpha+\theta_1\big)$$
$$-F_{\mathrm{T}}G_1\big(\alpha_p-\beta-\theta_1, \beta_p-\alpha+\theta_1\big)$$
$$-F_{\mathrm{T}}G_1\big(\beta_p-\alpha-\theta_2, \alpha_p-\beta+\theta_2\big)$$
$$+F_{\mathrm{C}}G_2\big(\beta_p-\alpha-\theta_2, \alpha_p-\beta+\theta_2\big)$$
$$+F_{\mathrm{T}}G_1\big(\beta_p-\theta_2-\theta_1-\beta, \alpha_p+\theta_1+\theta_2-\alpha\big)$$
$$-F_{\mathrm{C}}G_2\big(\beta_p-\theta_2-\theta_1-\beta, \alpha_p+\theta_1+\theta_2-\alpha\big) \tag{9.10a}$$

and by (2.4)

$$\begin{aligned}T_2(\alpha,\beta) = & -F_T G_0(\alpha_p-\alpha,\beta_p-\beta)\\ & +F_C G_1(\beta_p-\beta,\alpha_p-\alpha)\\ & -F_C G_1(\alpha_p-\beta-\theta_1,\beta_p-\alpha+\theta_1)\\ & +F_T G_2(\alpha_p-\beta-\theta_1,\beta_p-\alpha+\theta_1)\\ & +F_T G_0(\beta_p-\alpha-\theta_2,\alpha_p-\beta+\theta_2)\\ & -F_C G_1(\beta_p-\alpha-\theta_2,\alpha_p-\beta+\theta_2)\\ & -F_T G_2(\beta_p-\theta_2-\theta_1-\beta,\alpha_p+\theta_1+\theta_2-\alpha)\\ & +F_C G_3(\beta_p-\theta_2-\theta_1-\beta,\alpha_p+\theta_1+\theta_2-\alpha).\end{aligned} \tag{9.10b}$$

In the upper circular domain we note three domains of the static division in which the radial force is constant, and thus, equal to the force measured along the boundary with Hill's domain. In the lower fan domain we note only one domain of static division. The force in the tension bar is constant along the curved boundary, whilst its values along the straight boundary vary according to the rule

$$F_T^{ZH} = F_T(P) + \int_{\alpha}^{\beta_p-\theta_2} T_1^{(2)}(\overline{\alpha},\overline{\alpha}+\theta_2)\,d\overline{\alpha}, \tag{9.11}$$

where $T_1^{(2)}$ is given by (6.51).

The force in the compression bar is constant along the curvilinear boundary. Along the boundary of domain IV_d^3 this force varies by the rule

$$F_C^{Z_1Z'} = F_C(P) + \int_{\beta}^{\alpha_p-\theta_1} T_2^{(3)}(\overline{\beta}+\theta_1,\overline{\beta})\,d\overline{\beta}, \tag{9.12}$$

where $T_2^{(3)}$ is given by (7.11b). Along the boundary of domain IV_d^5 it can be computed by

$$F_C^{CZ_1} = F_C(P) + \int_{\beta_p-\theta_2-\theta_1}^{\alpha_p-\theta_1} T_2^{(3)}(\overline{\beta}+\theta_1,\overline{\beta})\,d\overline{\beta} + \int_{\beta}^{\beta_p-\theta_2-\theta_1} T_2^{(5)}(\overline{\beta}+\theta_1,\overline{\beta})\,d\overline{\beta}, \tag{9.13}$$

where $T_2^{(5)}$ is given by (9.10b).

The formulae for the force fields within the fans and in the reinforcing bars hold for the case of the point load being applied at the upper straight segment of the boundary of the feasible domain or for $\theta_1<\alpha_p<\theta_1+\theta_2$, $\beta_p=\alpha_p+\theta_2$. Then the domains VI_g^1, V^1, IV_d^1, IV_d^3 and IV_d^4 disappear (see Fig. 10). (6.51) is replaced with (6.23), whilst (9.10a, b) reduce to the form

$$\begin{aligned}T_1(\xi,\eta) = & -F_C G_0(\alpha_p-\xi,\beta_p-\eta)\\ & +F_C G_0(\alpha_p-\theta_1-\eta,\beta_p+\theta_1-\xi)\\ & +F_C G_2(\alpha_p-\xi,\beta_p-\eta)\\ & -F_C G_2(\alpha_p-\theta_1-\eta,\beta_p+\theta_1-\xi)\\ T_2(\xi,\eta) = & F_C G_1(\beta_p-\eta,\alpha_p-\xi)\\ & -F_C G_1(\alpha_p-\theta_1-\eta,\beta_p+\theta_1-\xi)\\ & -F_C G_1(\alpha_p-\xi,\beta_p-\eta)\\ & +F_C G_3(\alpha_p-\theta_1-\eta,\beta_p+\theta_1-\xi),\end{aligned} \tag{9.14}$$

where $F_C=F_C(P)$.

The force fields for the case of the point load applied within GJG_2 can be found from (9.14) by interchanging T_1 with T_2 and using the change of variables (7.9).

Consider now the case of the force **P** applied within domain $JH_2J_2G_2$ (Fig. 11).

Position of the point load in the domain $JH_2J_2G_2$ is described by $\theta_1+\theta_2<\alpha_p<2\theta_1+\theta_2$, $\theta_1+\theta_2<\beta_p<\theta_1+2\theta_2$. The

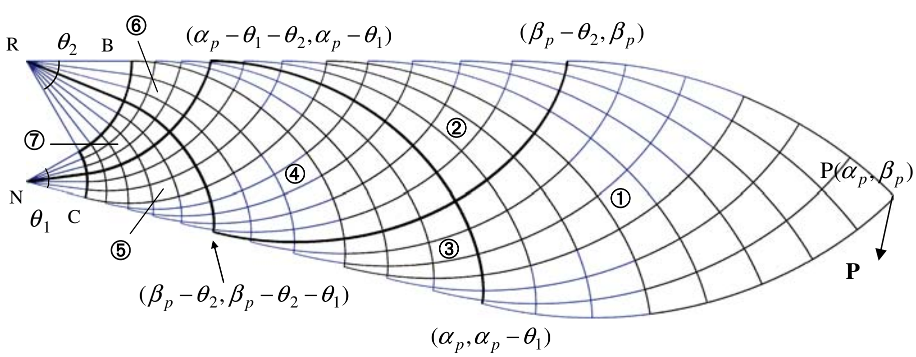

Fig. 11 Static division of the optimal cantilever while load **P** is applied within $JH_2J_2G_2$ domain

cantilever is divided into 11 subdomains of static division. The equations describing domains 1–5 have been derived previously. To find the field T_1 in domain 6 we use the formula (9.10b) for the force T_2 and change the variables by (7.9)

$$\begin{aligned}T_1^{(6)}(\alpha,\beta) = &-F_C G_0(\beta_p-\beta,\alpha_p-\alpha)\\ &+F_T G_1(\alpha_p-\alpha,\beta_p-\beta)\\ &-F_T G_1(\beta_p-\alpha-\theta_2,\alpha_p-\beta+\theta_2)\\ &+F_C G_2(\beta_p-\alpha-\theta_2,\alpha_p-\beta+\theta_2)\\ &+F_C G_0(\alpha_p-\beta-\theta_1,\beta_p-\alpha+\theta_1)\\ &-F_T G_1(\alpha_p-\beta-\theta_1,\beta_p-\alpha+\theta_1)\\ &-F_C G_2(\alpha_p-\theta_1-\theta_2-\alpha,\beta_p+\theta_1+\theta_2-\beta)\\ &+F_T G_3(\alpha_p-\theta_1-\theta_2-\alpha,\beta_p+\theta_1+\theta_2-\beta).\end{aligned} \tag{9.15a}$$

The second force is found by (2.4):

$$\begin{aligned}T_2^{(6)}(\alpha,\beta) = &-F_T G_0(\alpha_p-\alpha,\beta_p-\beta)\\ &+F_C G_1(\beta_p-\beta,\alpha_p-\alpha)\\ &+F_T G_0(\beta_p-\alpha-\theta_2,\alpha_p-\beta+\theta_2)\\ &-F_C G_1(\beta_p-\alpha-\theta_2,\alpha_p-\beta+\theta_2)\\ &-F_C G_1(\alpha_p-\beta-\theta_1,\beta_p-\alpha+\theta_1)\\ &+F_T G_2(\alpha_p-\beta-\theta_1,\beta_p-\alpha+\theta_1)\\ &-F_C G_1(\alpha_p-\theta_1-\theta_2-\alpha,\beta_p+\theta_1+\theta_2-\beta)\\ &-F_T G_2(\alpha_p-\theta_1-\theta_2-\alpha,\beta_p+\theta_1+\theta_2-\beta),\end{aligned} \tag{9.15b}$$

where $F_C=F_C(P)$, $F_T=F_T(P)$.

The formulae for the force fields within domain 7 will be guessed in the manner similar to that concerning domain III, corresponding to the case of the point load applied within DHJG. The formulae for $T_1^{(7)}$, $T_2^{(7)}$ include the terms corresponding to the domain 4 and new terms concerning domains 5 and 6:

$$\begin{aligned}T_1^{(7)}(\alpha,\beta) = T_1^{(4)}(\alpha,\beta)\\ &+F_T G_1(\beta_p-\theta_2-\theta_1-\beta,\alpha_p+\theta_1+\theta_2-\alpha)\\ &-F_C G_2(\beta_p-\theta_2-\theta_1-\beta,\alpha_p+\theta_1+\theta_2-\alpha)\\ &-F_C G_2(\alpha_p-\theta_1-\theta_2-\alpha,\beta_p+\theta_1+\theta_2-\beta)\\ &+F_T G_3(\alpha_p-\theta_1-\theta_2-\alpha,\beta_p+\theta_1+\theta_2-\beta)\end{aligned} \tag{9.16a}$$

$$\begin{aligned}T_2^{(7)}(\alpha,\beta) = T_2^{(4)}(\alpha,\beta) +\\ &-F_T G_2(\beta_p-\theta_2-\theta_1-\beta,\alpha_p+\theta_1+\theta_2-\alpha)\\ &+F_C G_3(\beta_p-\theta_2-\theta_1-\beta,\alpha_p+\theta_1+\theta_2-\alpha)\\ &+F_C G_1(\alpha_p-\theta_1-\theta_2-\alpha,\beta_p+\theta_1+\theta_2-\beta)\\ &-F_T G_2(\alpha_p-\theta_1-\theta_2-\alpha,\beta_p+\theta_1+\theta_2-\beta),\end{aligned} \tag{9.16b}$$

where $T_1^{(2)}$ is given by (6.51) and where $T_1^{(4)}$, $T_2^{(4)}$ are given by (8.1), where $F_C=F_C(P)$, $F_T=F_T(P)$.

The longitudinal force in the tension bar is constant along the boundary of domain 1 and along the fan, while it varies along the edges of the domains 2 and 6. We have

$$F_T^{(2)} = F_T(P) + \int_{\alpha}^{\beta_p-\theta_2} T_1^{(2)}(\overline{\alpha},\overline{\alpha}+\theta_2)\,d\overline{\alpha} \tag{9.17a}$$

$$F_T^{(6)} = F_T(P) + \int_{\alpha_p-\theta_1-\theta_2}^{\beta_p-\theta_2} T_1^{(2)}(\overline{\alpha},\overline{\alpha}+\theta_2)\,d\overline{\alpha} + \int_{\alpha}^{\alpha_p-\theta_1-\theta_2} T_1^{(6)}(\overline{\alpha},\overline{\alpha}+\theta_2)\,d\overline{\alpha}. \tag{9.17b}$$

The force in the compression bar is constant along the boundary of domain 1 and along the fan, while it varies along the boundary of domains 3 and 5:

$$F_C^{(3)} = F_C(P) + \int_{\beta}^{\alpha_p+\theta_1} T_2^{(3)}(\overline{\beta}+\theta_1,\overline{\beta})\,d\overline{\beta} \tag{9.18a}$$

$$F_C^{(5)} = F_C(P) + \int_{\beta_p-\theta_1-\theta_2}^{\alpha_p+\theta_1} T_2^{(3)}(\overline{\beta}+\theta_1,\overline{\beta})\,d\overline{\beta} + \int_{\beta}^{\beta_p-\theta_1-\theta_2} T_2^{(5)}(\overline{\beta}+\theta_1,\overline{\beta})\,d\overline{\beta}, \tag{9.18b}$$

where $T_2^{(3)}$ is given by (7.11b) and $T_2^{(5)}$ by (9.10b).

10 Final remarks

The derivation presented can be continued and all the results put in terms of Lommel-like functions. The graphs of the fields T_1, T_2 will be shown in part IV for selected examples. Although the cantilevers are composed of infinite number of members, the problems turn out to be statically determinate. In all the cases the boundary conditions suffice to solve the equilibrium problem directly by equilibrium equations. It is noteworthy to recall that in solid mechanics there are only two commonly known two-dimensional statically determinate problems: axisymmetric torsion of annular plates and selected static problems of membrane shells. Other known statically determinate problems are one-dimensional; they concern frames, grillages and trusses. Now we see that Michell structures, being neither continuum nor discrete, constitute the third large class of two-dimensional statically determinate structures.

Appendix A. Formula of Riemann referred to the domain QFE

Assume that the function T satisfies: LT=f within a domain Ω, parameterized by (α, β), with L defined by (I.6.3) and with an arbitrary function f given in this domain. Let G be defined by $G(\alpha, \beta)$=$D_0(\lambda-\alpha, \beta-\mu)$ (see I.7.7). Then, according to Appendix (I.B), (I.B.2), the following identity holds:

$$2\int_{\Omega} fGd\alpha d\beta = \int_{\partial\Omega}\left(-G\frac{\partial T}{\partial\alpha}+T\frac{\partial G}{\partial\alpha}\right)d\alpha + \int_{\partial\Omega}\left(G\frac{\partial T}{\partial\beta}-T\frac{\partial G}{\partial\beta}\right)d\beta. \tag{A.1}$$

Assume now that Ω=QFE (see Fig. 5). By integration by parts, using the property $G(\alpha, \mu)$=1, we compute the boundary integral

$$\begin{aligned}
&\int_{EQ}\left(-G\frac{\partial T}{\partial\alpha}+T\frac{\partial G}{\partial\alpha}\right)d\alpha \\
&= -\int_{\mu-\theta_2}^{\lambda} G(\alpha,\mu)\frac{\partial T(\alpha,\mu)}{\partial\alpha}d\alpha + T(\alpha,\mu)G(\alpha,\mu)\,\Big|_{\mu-\theta_2}^{\lambda} \\
&\quad -\int_{\mu-\theta_2}^{\lambda} G(\alpha,\mu)\frac{\partial T(\alpha,\mu)}{\partial\alpha}d\alpha \\
&= -T(\alpha,\mu)\,\Big|_{\mu-\theta_2}^{\lambda} + T(\alpha,\mu)\,\Big|_{\mu-\theta_2}^{\lambda} - T(\alpha,\mu)\,\Big|_{\mu-\theta_2}^{\lambda} \\
&= -T(\lambda,\mu) + T(\mu-\theta_2,\mu) = T_E - T_Q.
\end{aligned} \tag{A.2}$$

Taking into account that $G(\lambda, \beta)$=1, one finds

$$\begin{aligned}
&\int_{QF}\left(G\frac{\partial T}{\partial\beta}-T\frac{\partial G}{\partial\beta}\right)d\beta \\
&= \int_{\mu}^{\lambda+\theta_2} G(\lambda,\beta)\frac{\partial T(\lambda,\beta)}{\partial\beta}d\beta - T(\lambda,\beta)G(\lambda,\beta)\,\Big|_{\mu}^{\lambda+\theta_2} \\
&\quad + \int_{\mu}^{\lambda+\theta_2} G(\lambda,\beta)\frac{\partial T(\lambda,\beta)}{\partial\beta}d\beta \\
&= T(\alpha,\mu)\,\Big|_{\mu-\theta_2}^{\lambda} - T(\alpha,\mu)\,\Big|_{\mu-\theta_2}^{\lambda} + T(\alpha,\mu)\,\Big|_{\mu-\theta_2}^{\lambda} \\
&= T(\lambda,\lambda+\theta_2) - T(\lambda,\mu) = T_F - T_Q.
\end{aligned} \tag{A.3}$$

The integral along FE is computed as follows:

$$\begin{aligned}
&\int_{FE}\left(-G\frac{\partial T}{\partial\alpha}+T\frac{\partial G}{\partial\alpha}\right)d\alpha + \int_{FE}\left(G\frac{\partial T}{\partial\beta}-T\frac{\partial G}{\partial\beta}\right)d\alpha \\
&= \int_{\lambda}^{\mu-\theta_2}\Bigg[T(\alpha,\alpha+\theta_2)\left(\frac{\partial G}{\partial\alpha}-\frac{\partial G}{\partial\beta}\right)_{|\beta=\alpha+\theta_2} \\
&\qquad + G(\alpha,\alpha+\theta_2)\left(\frac{\partial T}{\partial\beta}-\frac{\partial T}{\partial\alpha}\right)_{|\beta=\alpha+\theta_2}\Bigg]d\alpha.
\end{aligned} \tag{A.4}$$

We rewrite (A.1) in the form

$$\begin{aligned}
2\int_{\Omega} fGd\alpha d\beta &= -2T(\lambda,\mu) + T_F + T_E \\
&\quad + \int_{\mu-\theta_2}^{\lambda}\Bigg[T(\alpha,\alpha+\theta_2)\left(\frac{\partial G}{\partial\beta}-\frac{\partial G}{\partial\alpha}\right) \\
&\quad + G(\alpha,\alpha+\theta_2)\left(\frac{\partial T}{\partial\alpha}-\frac{\partial T}{\partial\beta}\right)\Bigg]d\alpha.
\end{aligned} \tag{A.5}$$

Assuming that f=0 we find the formula sought

$$\begin{aligned}
T(\lambda,\mu) &= \frac{1}{2}(T_F+T_E) \\
&\quad + \frac{1}{2}\int_{\mu-\theta_2}^{\lambda}\Bigg[T(\alpha,\alpha+\theta_2)\left(\frac{\partial G}{\partial\beta}-\frac{\partial G}{\partial\alpha}\right)_{|\beta=\alpha+\theta_2} \\
&\quad + G(\alpha,\alpha+\theta_2)\left(\frac{\partial T}{\partial\alpha}-\frac{\partial T}{\partial\beta}\right)_{|\beta=\alpha+\theta_2}\Bigg]d\alpha.
\end{aligned} \tag{A.6}$$

Appendix B. An important integral formula

The following equality will be proven ($n \geq -1$):

$$\begin{aligned}
&\int_0^{t_\lambda} D_0(t_\lambda-\widetilde{\alpha},\widetilde{\alpha})\big[G_n(t-\widetilde{\alpha},t-\widetilde{\alpha}+\theta) - G_{n+2}(t-\widetilde{\alpha},t-\widetilde{\alpha}+\theta)\big]d\widetilde{\alpha} \\
&\quad = G_{n+1}(t,t-t_\lambda+\theta) - G_{n+1}(t-t_\lambda,t+\theta)
\end{aligned} \tag{B.1}$$

This will be proven not by a direct integration but by showing that the function

$$\chi(w,\alpha) = G_{m-1}(w-\alpha,w-\alpha+\theta) - G_{m+1}(w-\alpha,w-\alpha+\theta) \tag{B.2}$$

is a solution of the integral equation:

$$\int_0^t D_0(t-\alpha,\alpha)\chi(w,\alpha)d\alpha = G_m(w,w-t+\theta) - G_m(w-t,w+\theta). \tag{B.3}$$

Let us introduce the Laplace transform in two forms:

$$L_w\{f(w)\} = \int_0^\infty e^{-wr}f(w)dw \tag{B.4}$$

$$L_t\{f(t)\} = \int_0^\infty e^{-tp}f(t)dt. \tag{B.5}$$

According to (a.164) we recall here the result:

$$L_w\{G_m(w,w-\xi_1)\} = \frac{2^m\exp\left(-\frac{1}{2}\left(r-\sqrt{r^2-4}\right)\xi_1\right)}{\sqrt{r^2-4}\left(r+\sqrt{r^2-4}\right)^m}, \tag{B.6}$$

and consequently,

$$L_w\{G_m(w-t,w-t-\xi_2)\}=\frac{2^m\exp\left(-\frac{1}{2}\left(r-\sqrt{r^2-4}\right)\xi_2-rt\right)}{\sqrt{r^2-4}\left(r+\sqrt{r^2-4}\right)^m}. \tag{B.7}$$

We put $\xi_1{=}t{-}\theta$, $\xi_2{=}{-}t{-}\theta$, $S=\sqrt{r^2-4}$ and find

$$L_w\{G_m(w,w-t+\theta)-G_m(w-t,w+\theta)\} = \frac{2^m\exp\left(\frac{\theta}{2}(r-S)\right)}{S(r+S)^m}\left(e^{-a_1t}-e^{-a_2t}\right), \tag{B.8}$$

where

$$a_1=\frac{1}{2}(r-S),\quad a_2=\frac{1}{2}(r+S). \tag{B.9}$$

Let us perform the Laplace transform of both sides of (B.8) using (B.5):

$$L_tL_w\{G_m(w,w-t+\theta)-G_m(w-t,w+\theta)\} = \frac{2^m\exp\left(\frac{\theta}{2}(r-S)\right)}{S(r+S)^m}\left(\frac{1}{p+a_1}-\frac{1}{p+a_2}\right). \tag{B.10}$$

We transform both sides of (B.3) using (B.4) to find

$$\int_0^t D_0(t-\alpha,\alpha)\chi^*(r,\alpha)d\alpha = L_w\{G_m(w,w-t+\theta)-G_m(w-t,w+\theta)\} \tag{B.11}$$

with

$$\chi^*(r,\alpha)=L_w\{\chi(w,\alpha)\}. \tag{B.12}$$

Then, making use of (b.174), we transform both the sides of (B.11) according to (B.5):

$$\frac{1}{p}\chi^{**}\left(r,p+\frac{1}{p}\right)=\text{r.h.s. of (B.10)}, \tag{B.13}$$

where

$$\chi^{**}(r,p)=L_t\left\{\chi^*(r,t)\right\}. \tag{B.14}$$

We take into account notation (6.36) and use the identity

$$\frac{1}{p+a_1}-\frac{1}{p+a_2}=\frac{S}{p(r+s)}. \tag{B.15}$$

The equality (B.13) is equivalent to

$$\chi^{**}(r,s)=\frac{2^m\exp\left(\frac{\theta}{2}(r-S)\right)}{(r+S)^m}\frac{1}{s+r}. \tag{B.16}$$

We replace s by p and rewrite (B.16) in the form

$$L_t\left\{\chi^*(r,t)\right\}=\frac{2^m\exp\left(\frac{\theta}{2}(r-S)\right)}{(r+S)^m}L_t\left(e^{-rt}\right); \tag{B.17}$$

hence,

$$\chi^*(r,t)=\frac{2^m\exp(-rt)}{(r+S)^{m-1}}\frac{\exp\left(-\frac{1}{2}(r-S)(-\theta)\right)}{r+S}. \tag{B.18}$$

We use the decomposition

$$\frac{1}{r+S}=\frac{1}{2S}-\frac{2}{S(r+S)^2} \tag{B.19}$$

and write

$$\chi^*(r,t)=2^{m-1}\frac{\exp\left(-\frac{1}{2}(r-S)(-\theta)-rt\right)}{S(r+S)^{m-1}} -2^{m+1}\frac{\exp\left(-\frac{1}{2}(r-S)(-\theta)-rt\right)}{S(r+S)^{m+1}}. \tag{B.20}$$

The formula (B.7) gives the desired result (B.2), which ends the derivation.

Substitution of $t_\lambda=t$ into (B.1) gives

$$\int_0^t D_0(t-\alpha,\alpha)\left[G_n(t-\alpha,t-\alpha+\theta)-G_{n+2}(t-\alpha,t-\alpha+\theta)\right]d\alpha = \begin{cases} G_0(t,\theta)-1 & \text{for } n=-1 \\ G_{n+1}(t,\theta) & \text{for } n\geq 0 \end{cases}. \tag{B.21}$$

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