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Derivation of acoustical streaming equations for nonlinear and dispersive fluids

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ABSTRACT

The equations of streaming generated by an acoustic mod propagating in a nonlinear dispersive medium (exhibiting absorption and dispersion of phase sound speed) are derived with an arbitrarily shaped incident acoustical field assumed. This field may be periodic or non-periodic. A general dispersion model represented by a convolution operator taking into account relaxation effects was taken into account. Making the assumption of a periodic acoustic field from the general streaming equation.

The quasi-stationary flow is driven by a force given by the average value of the dispersion operator with respect to the velocity and acoustic pressure fields. In the spectral representation, it is given by the weighted spectral power density distribution of the acoustic field. The weight of the distribution is the dispersion coefficient - the eigenvalue of the dispersion operator. A new result also reveals the effect of the refractive index deviation on the driving force of streaming. The possibility of generalizing the description of streaming in the simplest case of a non-Newtonian fluid was analyzed. The Reiner-Revlin model of a simple liquid was assumed. It was also noted that the streaming model in the Maxwell liquid is analytically solvable. It was found that asymptotic states of streaming in this model and the Navier-Stokes model are identical.

The derivations use new methods different from those used so far. They are based on the separation of nonlinear modes in the momentum transport equation and on the properties of the Gauss-Weierstrass function for the Fick diffusion operator. So far, the method of successive approximations has been used. The consistency of the obtained equations with the assumptions was checked. The obtained formulas generalize the known descriptions of the form of forces driving streaming and extend their application to the case of nonlinear propagation.

1. Introduction

Acoustic streaming is a steady or a quasi steady flow in a fluid driven by stress accompanying the propagation of the acoustic field. Scientific and practical importance phenomena in the range of Acoustical Radiation Force (ARF) and streaming results from the fact that enable testing and influences on the thermo-mechanical properties of matter, in particular liquid and non-contact mass and energy flow control. An extensive list of possible technical and biotechnological applications of ARF and streaming can be compiled from papers [1–5]. Of those specific to medical applications, we will only mention here research into the possibility of influencing thrombus movement and controlling drug transport in blood vessels to increase the efficiency of thrombolysis methods [6]. In addition to streaming, ARF can induce shear waves. Their use in the diagnosis of tissue pathology allows increasing the sensitivity of the examination by many orders compared to conventional diagnostics using longitudinal waves.

In the literature, the term ARF has no unambiguous assignment in terms of the definition of an appropriate force. Typically this term means forces defined on the basis of acoustic components either of the Cauchy stress tensor or tensor of the density of momentum flux (balance Cauchy and Reynolds tensors).

In the case of the force driving streaming, the term Acoustical Driving Force (ADF) will be used throughout the paper instead of ARF. In this work, we determine the force generated by an arbitrarily shaped acoustic beam propagating in a non-linear and dispersive medium. To determine the ADF we use method equivalent to the second of the mentioned approaches (Cauchy and Reynolds tensors).

It is believed that in lossy homogeneous fluid streaming is the result of non elastic momentum transfer through acoustic mode to a fluid and

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is of the first order with respect to acoustical Mach number. This is true, however, not all of it. We show this in this work. The sound intensity and dispersion (absorption and refractive index dispersion) of the medium are the factors that drive streaming. Shear viscosity resists the flow. The influence of classically viscous absorption was shown by Eckart [7]. It is believed to be the first theoretical analysis of streaming in a free sound field, or with weak influence of spatial constraints, on a spatial scale much larger than the acoustic wavelength.

This work focuses on Eckart's type of streaming. His model of sound absorption considered only viscosities. He suggested that the ratio of the bulk and shear viscosities can be determined by streaming measurements. Merkham [8] extended the absorption model by introducing a so-called relaxation term with an appropriate coefficient into the equation of state. However, this term was local in time and space and mathematically equivalent to the volume viscosity [9]. Nevertheless, Merkham's modification allowed for a better description of the experimental results [10]. It drew attention, to the suggestion already made in the work of Fox and Herzfeld [11], that a correct description of streaming requires the search for and consideration of all the mechanisms that build sound absorption. In this work we show this explicitly by assuming a suitably general model of dissipative losses (viscous stresses and relaxations) for a nonlinearly propagating disturbance.

In the derivations, we use the modal distribution of the general flow field. We use, arguments derived from various characteristic time–space scales of variation. These conditions are discussed in Section 2. The formulas obtained for the ADF are valid for nonlinear sound propagation. They result from the potential description and acoustic approximation rather than from the method of successive approximations. They are of second order with respect to a nonlinearly propagating field and not of second order with respect to a linearly propagating field. Nevertheless, they can be reduced to those obtained by the method of successive approximations.

The ADF results from the balance of forces generated by the Cauchy stress tensor and the inertial reaction of the medium, represented by the substantial derivative of the flow (in other notations of the momentum equation by the Reynolds tensor). We derive the ADF for the general case, also for non-periodic waves or single pulses. However, we focus on the form of ADF induced by periodic, continuous or impulse disturbances. In Section 3 we show that ADF for quasi stationary flow is determined by average value of the dispersion (absorption) operator, which is, in generally, integro-differential operator of the convolution type. This operator describes the dispersion, particularly heat generation due to acoustic energy loss. Using general equations and relationships for the acoustic field we express ADF (mean value of the dispersion operator) as weighted average of the dispersion coefficient considering Fourier spectrum of the sound intensity vector. This is of great importance for the description of streaming in non-linearly propagating beams, when intense harmonic generation occurs (see observations of Starritt et all [12]). In this way, we maintain the independence of the description from the specific form of the acoustic beam. However, it can be easily reduced to known results [8,9,13–17]. On the other hand, a completely new result is the reveal of the effect on the ADF of the deviation of the real part of the refractive index. It is determined by the imaginary part of the dispersion coefficient. We show that there is also a fine-scale, fast-varying flow generated by beam power fluctuations. In an asymptotic timescale it is carried by the stationary, "Green" function.

Most of the derivations were made in Apendix A and B, then rewriting the results into the main body of the paper in Section 3. The methodology used can be briefly described as follows. In general, the equations of flow in the acoustic field are formed by the sum of coupled operators describing potential (in particular, acoustic) and non-potential flow.

The streaming equations are obtained using the consequences of the assumption that the potential flow equations are satisfied. Based on the potential operator, the momentum transport equation was transformed into the Poison equation. The source term in the resulting equation is the operator describing the non-potential flow. The solving Poisson's equation allowed determining the density as a function of the remaining fields. It has been shown formally that the effect of the vortex mode on density is the quantity of the second order of magnitude to the effect of potential (acoustic) fields. In this way, the acoustic approximation of the potential flow, described with precision to the first-order nonlinear terms with respect to the Mach number, is free from couplings with non-potential terms. Thus, satisfying the potential flow equation or even satisfying this equation in acoustic approximation implies zeroing out the operator describing the non-potential flow. We verify this by substituting the formulas for density and pressure obtained in the acoustic approximation into the momentum transport equation. We find that the potential part (potential operator) zeros out to the required accuracy.

The fundamental evolution of the vortex mode is described by the Fick's diffusion operator. The force driving the vortex flow is defined by second order quantities with respect to the acoustic field. It has a constant and a fast time-varying component. Therefore, a separation of the vortex moduli, the slow-variable (streaming) and the fast-variable, was made by referring to the properties of the Gauss-Weierstrass function for Fick's diffusion equation.

Constitutive models of non-Newtonian fluids are complex even in the simplest of cases. Perhaps that is why no suitable example of a theoretical description of streaming in such a fluid has been found so far. In this paper, such a generalization is considered. The Reiner-Rewlin model [18] of a simple fluid was used. This is the simplest generalization of the Navier-Stokes model describing shear flows for the case of velocity-dependent viscosity. A model of streaming in a linear, elastic-viscous fluid described by the Maxwell model is also presented. Some conclusions from the comparison of the description of streaming in Navier-Stokes and Maxwell fluids are presented.

In a number of works from many years ago, attempts were made to construct simple formulas estimating the speed of the axial component of streaming [17]. In this work, in Appendix B, we presented an example of the solution of the Dirichlet boundary value problem for the equations obtained. In particular, they take a simple form for the streaming axial components. They are presented in Section 4 using the method from work [19]. With the current PC power, these formulas are easy to numerically evaluate, deriving a solution, not just an estimation.

2. Basic equations and assumptions

(a) Normalizations

In this work we use normalized system of independent and dependent variables. The dimensional variables and operators are accentuated. The normalized coordinates in space and time are $\mathbf{x} := K_0^{'}\mathbf{x}^{'}; t := \Omega_0^{'} t^{'}$, whereas $\nabla := \nabla^{'}/K_0^{'}$ is the normalized Nabla vector operator and $\partial_t := \partial_{t^{'}}/\Omega_0^{'}$ is the derivation operator with respect to time. The characteristic pulsation $\Omega_0^{'} := 2\pi/T_0^{'}$ and wave number $K_0^{'} = 2\pi/\lambda_0^{'}$ are restricted by the relation:$K_0^{'}c_0^{'} = \Omega_0^{'}, \lambda_0^{'} = c_0^{'}T_0^{'}$, $T_0^{'}$ is the reference time (e.g. time window, repetition time or period for pulse or singular sine wave stimulation of the medium). In normalized units $T_0^{'} \rightarrow T_0 = 2\pi, \lambda_0^{'} \rightarrow \lambda_0 = 2\pi$. The normalized pulsation (frequency) and the wave number in dispersion-less, homogenous media are $\omega := \omega^{'}/\Omega_0^{'}$ and $k(\omega) = k_0(\omega) = \pm\omega/c_0 = \pm\omega$ respectively. For Fourier series representations of the disturbances ω are discrete variables (integer) whose numerate the components of the series. For $\omega = 1, \omega^{'} = \Omega_0^{'}$. The normalized density and speed of the sound are $g(\boldsymbol{x}, t) := g^{'}/g_0^{'}$, $c(\boldsymbol{x}, t) := c^{'}/c_0^{'}$ respectively. Where $g_0^{'}$ and $c_0^{'}$ are density and speed of sound in reference homogenous medium respectively. This means that $g = g_0 = 1$ and $c = c_0 = 1$ for reference medium in equilibrium. The pressure and vector of the velocity field are normalized as follows $\widetilde{P} := \widetilde{P}^{'}/P_0^{'}, \mathbf{v} =$

$\mathbf{v}^{'}/v_0^{'}$. For acoustical disturbances $P_0^{'} = g_0^{'}c_0^{'}v_0^{'}$, $v_0^{'}$ is the maximum of the absolute value of the velocity amplitude of the acoustical disturbance source; $q := v_0^{'}/c_0^{'} = P^{'}_0/g^{'}_0 c_0^{'2}$ is the acoustical Mach number. The so-called dispersion coefficients are of great importance for this work. They are normalized as follows $a(\omega) := a^{'}(\omega^{'}/\Omega_0^{'})/K_0^{'}$.

Although we use normalized quantities, in the formulas we have preserved the symbolism of quantities which, after normalization, take the value 1, as for example $g_0 = 1, c_0 = 1$. In this way, the formulas preserve the structure of the non-normalized quantities which makes it easier to give them a physical dimension and to identify them with the non-normalized quantities used in the literature.

(**b**) Basic equations

In Eulerian coordinates evolution in time of the primary variables (fields) $g(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{x},t)$ is given as follows:

$$\partial_t g + q\nabla\cdot g\mathbf{v} = 0 \tag{1}$$

$$g\partial_t\mathbf{v} + qg\frac{1}{2}\nabla v^2 - qg\mathbf{v}\times\nabla\times\mathbf{v} + \nabla\widetilde{P}(g,\ \mathbf{v}) + \overline{\eta}\nabla\times\nabla\times\mathbf{v} + \mathbf{N}(\mathbf{v}) + o\left(q^2\right) = 0 \tag{2}$$

$$\widetilde{P}(g,\mathbf{v}) := \frac{g_0 c_0^2}{q\gamma}\left[\left(\frac{g}{g_0}\right)^{\gamma} - 1\right] + 2g_0 c_0 \mathscr{A}^{v}\mathbf{v}, \quad v = |\mathbf{v}| \tag{3}$$

Where $\widetilde{P}(g,\mathbf{v})$ is the pressure disturbance in dissipative medium (viscous, relaxing, heat conducting) in respect to the equilibrium state $g = g_0, \mathbf{v} = \mathbf{0}$. Because acoustic disturbance is applied as primarily then the nearly adiabatic transition of the medium was assumed; γ is the exponent of the adiabate. In more general case γ is the polytrophic exponent. We assume that, $\mathscr{A}^{v}$ is the convolution type operator describes the viscous stress or relaxation processes. They lead to dispersion, that is, of the sound energy dissipation and dispersion of phase speed of sound [20–23]. It is a generalization of the equation of state (for volumetric disturbances) mentioned in the introduction, proposed in [8] and used in [9,10]. In particular, the experiments carried out in organic media show that the trace of the viscous stress tensor for a classically viscous medium is insufficient for a correct description of the absorption (dispersion) of acoustic disturbances. From this point of view, it is more convenient to use the dispersion (absorption) operator $\mathscr{A}^{x,t} = \mathscr{A}^{v}\nabla$. Its analytical properties are described in more detail in [20–23]. Then $A^{x,t}$ is the integro-differential operator of the convolution-type with kernel $A(\mathbf{x},t)$ $\mathscr{A}^{x,t}P := A\underset{x,t}{\otimes}P$. Here P denotes the trial function. Later it will be either scalar or vector functions describing the sound field. Generalized Fourier transform of the kernel A, $a^{\varsigma,\omega}(\varsigma,\widehat{\omega}) = \mathrm{F}^{x,t}[A]$ is a function of dispersion, where ς is the complex wave number, $\widehat{\omega}$ is the complex pulsation. That is $a^{\varsigma,\omega}$ is the eigenvalue of $A^{x,t}$ corresponding to the eigenfunction $f = \exp(-i\varsigma\mathbf{e}_K\cdot\mathbf{x} + i\widehat{\omega}t), \mathscr{A}^{x,t}f = a^{\varsigma,\omega}f$; $\mathbf{K} = \varsigma\mathbf{e}_K$ is the complex wave vector; i is the imaginary unit. Using methods from [22,23] we obtain the analytical form of the dispersion coefficient $a^{\omega}(\widehat{\omega}) = a^{\varsigma,\omega}(\varsigma(\widehat{\omega}),\widehat{\omega})$, where $\varsigma(\widehat{\omega})$ is the solution to the dispersion equation. On the real axis $\omega = Re(\widehat{\omega})$ we get $a^{\omega}(\omega) =: a(\omega) = a_0(\omega) + ih_0(\omega)$, $a(-\omega) = a(\omega)^*$, where $a_0(\omega)$ is the weak-signal absorption coefficient, $k_0 n_I = a_0$, $k_0\delta n_r = k_0(1 - n_r) = h_0$; n_I, $n_r := c_0/c_f(\omega)$ imaginary and real part of the refraction coefficient, $c_f(\omega)$ is the phase velocity. As is known a_0 i h_0 related by Kramers-Kronig relations (Hilbert transforms). We note that the procedure presented here in brief allows us to replace the time-dependent (mixed) model of dispersion by a time-dependent (homogeneous) model since the transition from $a^{\varsigma,\omega}\to a^{\omega}$ corresponds to $\mathscr{A}^{x,t}\to\mathscr{A}^{t}\equiv\mathscr{A}$ (see details in [22]). We also have $\mathscr{A}^{t}f(\widehat{\omega},t) = a^{\omega}f(\widehat{\omega},t)$, $f = \exp(i\widehat{\omega}t)$.

$$\mathscr{A}e^{i\omega t} = a(\omega)e^{i\omega t}, \quad A(t) = F^{-1}[a(\omega)] \tag{4}$$

Because of the relatively easy measurement of the absorption coefficient, it is usually the basis for the determination of the full dispersion, i.e.h_0. Similarly model $\mathscr{A}^{x,t}$ can be converted into space dependent homogeneous model $\mathscr{A}^{x}$. In [23], two dispersion models, specific to acoustics, are presented. The first represents the Maxwell medium. In the zero relaxation time limit, the dispersion operator of this medium corresponds to the Navier-Stokes viscous stress model which is the source of the classical absorption model. In the second model the absorption (dispersion) coefficient is characteristic for many organic media.

For classic viscous media $\mathscr{A}^{v} := -(\eta_h/c_0)\nabla\cdot, \mathscr{A}^{x,t}\equiv\mathscr{A}^{x} = -(\eta_h/2c_0)\Delta, \eta_h := \eta_h^{'}\cdot K_0/2c_0^{'}$ $\eta_h^{'} := (4\eta_s^{'}/3 + \eta_b^{'} + (\gamma-1)\mu^{'}/g_0^{'}c_{sp}^{'})$; $\eta_h^{'}$ is the hybrid viscosity,$\eta_s^{'},\eta_b^{'},\mu^{'}$, $c_{sp}^{'}$ are, kinematic coefficients of shear and bulk viscosity, heat conduction and specific heat, respectively. A corresponding operator $\mathscr{A}$ in the time domain is not local in time. Its eigenvalue (dispersion coefficient) is [23].

$$a(\omega) = a_0(\omega) + ih_0(\omega) = (2\cdot\pi)^2\frac{\alpha_2\omega^2 + i\left(2c_0(2\pi)^2\alpha_2 + t_r\right)\alpha_2\omega^3}{1 + \omega^2\left(2c_0(2\pi)^2\alpha_2 + t_r\right)^2} \tag{5}$$

where, $\alpha_2 := \eta_h/2c_0^3$ (in non normalized units $\alpha^{'}_2 := \eta^{'}_h/2c_0^{'3}$). However, it should be remembered that the values $\alpha_2^{'}$ ($\eta_h^{'}$), determined on the basis of the theory may significantly differ from the measured one. For classical viscous media $t_r = 0$, for Maxwell media $t_r \neq 0$ is the relaxation time. In the literature, we most often encounter the "non-analytical dispersion model" for a classically viscous medium (for details see [23]). The value for water $\alpha_2^{'} = 2.5\cdot10^{-14}\mathrm{Np/(m\cdot Hz^2)}$, and for glycerin $\alpha_2^{'} \cong (70 \div 700)\cdot10^{-14}\mathrm{Np/(m\cdot Hz^2)}$ in the temperature range from 39°C to 5°C [24]. In normalized units and for $\Omega_0^{'}/2\pi = 3\mathrm{MHz}$, $\alpha_2 = 1.79\cdot10^{-5}$ for water and $\alpha_2 = (6.4 \div 64)\cdot10^{-4}$ for glycerin. Thus, for water in the band up to 100 MHz ($\omega = 33$) $h_0/a_0 = 8\pi^2 c_0\alpha_2\omega \leqslant 0.047$ ($c_0 = 1$) and the amplitude modulation effects of nonlinear interactions far outweigh those of phase modulation. However, for glycerin $h_0/a_0 = 8\pi^2 c_0\alpha_2\omega \leqslant (1.67 \div 16.67)$ in the same frequency range. For many biological substances it is assumed that $a_0(\omega) = \alpha_\kappa|\omega|^\kappa$, For $\kappa \neq 2$ we heave non-classical absorption. For example for $\kappa = 1$, $a(\omega) = \alpha_1|\omega| + i(2\alpha_1/\pi)\omega\ln(|\omega|)$ [23]. For blood $a_0(\omega) = \alpha_{1.2}|\omega|^{1.2}$, $\alpha_1^{'} \cong 11\cdot10^{-8}$ 10^{-8} $\mathrm{Np/mHz^{1.2}}$ [25], in normalized variables $\alpha_{1.2} \cong 5.5\cdot10^{-4}$.

Analytical derivation of $h(\omega)$ using the Kramers-Kronig formulas for fractional cases is impossible. Likewise for experimental data, since they are available only from a limited frequency band and not over the required range $\omega \in [0,\infty)$(e.g. blood, etc.). In [26], a good, approximate solution to this problem is presented. Here, however, we will use the estimation resulting from the assumption that for blood $\alpha_{1.2} \approx\sim 1$.. We get $h_0/a_0 \approx (2/\pi)\ln(\omega) < 2.2$ in the frequency range up to 100 MHz, $h_0/a_0 \leqslant 1$ in the range up to 15 MHz.

For the classic viscous fluid $\mathbf{N}(\mathbf{v}) + o(q^2) \equiv \mathbf{0}$ and $\overline{\eta}(\mathbf{v}) = \eta$, where $\eta := \eta^{'}_s K_0^{'2}/\Omega^{'}_0 = \eta_s^{'}K_0^{'}/c_0^{'} = \eta^{'}_s\Omega^{'}_0/c_0^{'2}$ is the normalized shear viscosity. The following quantities play an important role in the description of flow phenomena, $\delta_{0s}^{'} := \sqrt{2\eta_s^{'}/\Omega_0^{'}}$ is the viscous wave range (or boundary layer thickness) for pulsation $\Omega_0^{'}$,$\kappa_{0s}^{'} := (1+i)k_{0s}^{'} = (1+i)/\delta_{0s}^{'}$, $\kappa^{'}_{0s}$ is the complex wave number. For pulsation ω, $\kappa_s := (1+i)k_s = (1+i)/\delta_s$ and $\delta_s := \sqrt{2\eta/\omega}$ are normalized counterparts of $k_s^{'}$ and $\delta_s^{'}$ given for pulsation $\omega^{'}$ [27],$Re := Re_0(\lambda^{'}/\lambda_0^{'}) = Re_0\lambda$, $Re_0 := 2\pi v_0^{'}/K_0^{'}\eta_s^{'}$ is the Reynolds number respect scale $\lambda_0^{'}$, while Re in the scale $\lambda^{'}$. Particularly, for flows in viscous waves $Re_s := Re_0/k_s$ is the Reynolds number. We also have a relationship $\eta = K_0^{'2}/k_{0s}^{'2} = \left(K_0^{'}\delta_{0s}^{'}\right)^2/2 = 2\pi q/Re_0 = 2\pi\lambda q/Re$. $\eta_s^{'} \cong (11.7 \div 2.6)\cdot10^{-4}\ \mathrm{m^2/s}$, in the temperature range from 20°C to 40°C [28] and the flow scale $\lambda_0^{'} \sim 0.1\,\mathrm{m}$,$\eta \sim (7.65\cdot10^{-5} \div 1.7)\cdot10^{-5}$. Vector function.

$$\mathbf{N}(\mathbf{v}) + o\left(q^2\right) := \nabla\overline{\eta}(v)\times\nabla\times\mathbf{v} - 2\nabla\overline{\eta}(v)\nabla\cdot\mathbf{v} + 2\nabla\overline{\eta}(v)\cdot\nabla\mathbf{v} + o\left(q^2\right) \tag{6}$$

follows from the constitutive equation $\boldsymbol{\sigma} := \overline{\eta}\boldsymbol{\varepsilon} + \overline{\eta}_2\boldsymbol{\varepsilon}^2$, $\boldsymbol{\varepsilon} = (\nabla^\circ\boldsymbol{\upsilon} + \nabla^\circ\boldsymbol{\upsilon}^T)/2$ for a simple liquid [18],$(\nabla^\circ\boldsymbol{\upsilon})_{i,j} := \partial\upsilon_i/\partial x_j$, $(.)^T$ is the transposed tensor. It was proposed by Reiner [29] and Rivlin [30] and is the simplest generalization of the classical Stokes equation. In the simplest case, scalar functions $\overline{\eta}$, $\overline{\eta}_2$ are polynomials with respect to invariants $\mathrm{tr}(\boldsymbol{\varepsilon}) = \nabla\cdot\boldsymbol{\upsilon}$,$\mathrm{tr}(\boldsymbol{\varepsilon}^2)$ e.t.c.. Empirical models are also used [31,32]. For simplicity, the explicit representation of the terms at least $o(q^2)$ generated by the square of the strain rate tensor $\overline{\eta}_2$ has been omitted. Even a rough analysis of the shear stress model generalized in Eq.(2) seems necessary due to the presentation of a generalized model of dispersion (dissipation) associated with volume (acoustic) disturbances, the effects of which are different from the classical ones. Moreover, a number of media, including the blood mentioned, have properties different from Newtonian fluid, corresponding to velocity-dependent viscosity. Of course, blood is a complex suspension, but in the range of suitably low rates, it can be considered a continuous medium. Assuming a simple case $\overline{\eta} := \eta(1 + q\tau\nabla\cdot\boldsymbol{\upsilon})$ (in non-normalized variables $\overline{\eta}' := \eta_s'(1 + \tau'\nabla'\cdot\boldsymbol{\upsilon}')$), Eq.(6) takes the form $N(\boldsymbol{\upsilon}) = -\eta\tau q(\nabla\times\ \boldsymbol{\upsilon}\times\ +\ 2\nabla\cdot\boldsymbol{\upsilon} - \nabla^\circ\boldsymbol{\upsilon})\nabla\nabla\cdot\boldsymbol{\upsilon} + o(q^2)$,$\tau\ = \tau'\ \Omega_0'$, $\eta_s'\tau'$ is a material constant.

We assume the following decomposition of the velocity field $\boldsymbol{\upsilon}\ = \mathbf{v} + \boldsymbol{\xi}_1 + \widetilde{\boldsymbol{\xi}}$,$\mathbf{v}\ = \nabla\Phi$,$\widetilde{\boldsymbol{\xi}}\ = \mathbf{w} + \boldsymbol{\xi}$,$\nabla\cdot(\widetilde{\boldsymbol{\xi}} + \boldsymbol{\xi}_1)\ = 0$. Where Φ is the potential of the irrotational flow, $\boldsymbol{\xi}_1$ is the vortex field generated in the first order by the acoustic field. In a heterogeneous medium or on fluid constraints, the $\boldsymbol{\xi}_1$ describes the vortex motion in the boundary layer adjacent to the interfacial boundary. In Eckart type of streaming in the homogeneous fluid when acoustic beam not interact with vessel walls or boundary layer thickness is small $\boldsymbol{\xi}_1$ may be neglected. The field $\widetilde{\boldsymbol{\xi}}$ is second order respect acoustical field. In this sense, it is counterpart for the second-order components in the method of successive approximations from the works, e.g.[8,9,13–15,17]. But, in our case it is a complete mode representing non-potential flow. It is determined by quantities described by nonlinear acoustical equations. The field $\mathbf{w}(\mathbf{x}, t)$ is a slowly varying in time respect acoustical field changes and tends to the stationary state velocity $\mathbf{w}^s(\mathbf{x})$, while $\boldsymbol{\xi}$ is periodic with dominant frequency doublet the acoustical one (carrier) and describe small scale volumetric flow changes characterized by δ_s(like $\boldsymbol{\xi}_1$), which may be relevant in the case of micro streaming. According to above $\boldsymbol{\upsilon} = \mathbf{v} + \widetilde{\boldsymbol{\xi}} = \mathbf{v} + \mathbf{w} + \boldsymbol{\xi}$ and mean value definition.

$$\langle\cdot\rangle := \int_t^{t+T_0} (\cdot)dt \Big/ T_0\ , \tag{7}$$

$\langle\mathbf{v}\rangle\ = \mathbf{0}$. In transient states, the average calculated according to Eq. (7) may depend on time. In long scale observations $t \gg T_0$,$\langle\partial_t\mathbf{w}\rangle = (\mathbf{w}(\mathbf{x}, t + T_0) - \mathbf{w}(\mathbf{x}, t))/T_0 \underset{t\to\infty}{\rightarrow} 0$ and in stationary state $\langle\mathbf{w}\rangle \rightarrow \mathbf{w}^s(\mathbf{x}) :=\ \langle\boldsymbol{\upsilon}\rangle$,$\langle\boldsymbol{\xi}\rangle\ =\ \mathbf{0}$. For the above velocity distribution $N(\boldsymbol{\upsilon}) = -\eta\tau q(\nabla\times\widetilde{\boldsymbol{\xi}}\times + 2\nabla\cdot\mathbf{v}\quad -\nabla^\circ\boldsymbol{\upsilon})\cdot\nabla\nabla\cdot\mathbf{v}$. Because $\eta = q2\pi/Re_0$ then $N(\boldsymbol{\upsilon}) \sim o(q^2/Re_0)2\pi$. Note that the above relationship may still apply even when $Re \ll 1$. Only when $(2\pi\eta_s'/\lambda_0') \sim c_0'$ we have $\eta\ =\ 2\pi q/Re_0 \sim 1$.

As can be seen from the $N(\boldsymbol{\upsilon})$ and $\overline{\eta}(\boldsymbol{\upsilon})$ forms in the assumed model, the non-Newtonian flow is supported by the presence of the acoustic field. It disappears in the absence of normal (volume) stresses. Moreover, it is revealed in an important way for micro flows. Additionally, in the asymptotic timescale some of the N components disappear. Although we will omit these terms, we will mention some of them in the further analysis, pointing to their meaning and positions in the obtained equations - if they had been included. This is especially true for the term $-\eta\tau q\nabla(\nabla\cdot\mathbf{v})^2$ contained in N. Its inclusion in the equations leads to an additional ADF component (Apendix A). The first non-vanishing component $\overline{\eta}$, with the "off" sound field, that would generate the velocity-dependent viscosity in this model, is proportional to $\mathrm{tr}(\boldsymbol{\varepsilon}^2) \sim o(q^2)$.

For linear fluids showing shear stress relaxation (deformation memory), which can be described by the Maxwell model in Eq. (3) the $-\overline{\eta}\nabla\times\nabla\times\boldsymbol{\upsilon} = \eta\Delta\widetilde{\boldsymbol{\xi}}$ term should be replaced by.

$$\eta\Delta\widetilde{\boldsymbol{\xi}} \rightarrow (\eta/\mathrm{t}_r)\Delta\int_{-\infty}^{t} e^{-(t-t')/\mathrm{t}_r}\widetilde{\boldsymbol{\xi}}(\mathbf{x}, t')dt' \tag{8}$$

Where t_r is different from the relaxation time in Eq. (5). As for the normal components of the stress tensor leading to the dispersion operator $\mathscr{A}$, for the tangent components and $\mathrm{t}_r \rightarrow 0$ we obtain a classic viscous fluid.

(c) Auto Coriolis Force rating

We discuss conditions under which, components of the "Auto Coriolis Force" (ACF) $qg\boldsymbol{\upsilon}\times\nabla\times\boldsymbol{\upsilon}$ may or may not be neglected compared to the viscous drag force or other forces generated by the inertial response of the medium (Reynolds tensor). The general condition for neglecting this force is a small Reynolds number for stationary flow [33,34]. In a fixed medium, this leads to limitations on the amplitudes of the acoustic field (intensity) generating streaming. The question arises whether this would exclude situations where, for consistency, it would be required to neglect also the other components of the substance derivative (or Reynolds tensor), which are fundamental for the description of streaming.

To derive the streaming equations, the method of successive approximation is most commonly used. It eliminates the ACF from the equations. However, it assumes a priori that the streaming velocity is a small quantity of the higher order (second) respect magnitude of acoustical velocity, which satisfies the linear propagation equation. As in the paper [15], we believe that this is a significant and redundant restriction on the applicability of the streaming equations. As is the use of the argument $Re_0 < 1$. Instead, in addition to amplitude relations, we primarily use differences in time–space scales of mode variation.

In the common notation of both forces $(q\boldsymbol{\upsilon} - \eta\nabla)\times\ \nabla\times\ (\boldsymbol{\xi} +\ \mathbf{w})$, $(g \sim 1)$, the corresponding operators act on the same field. If their estimates satisfy the condition $q\|\boldsymbol{\upsilon}\times\| \ll \eta\|\nabla\times\|$, where $\|\bullet\|$ is the appropriate norm, then ACF can be omitted.

In relation to $\boldsymbol{\xi}$, $\|\eta\nabla\times\| \sim \eta k_s = \sqrt{2\eta\omega}$ and conditions $1 \ll \sqrt{2\eta\omega}/q = 1/Re_s = c_S/q$ is met in a wide range of parameters and in the whole range of pulsations of the effective Fourier spectrum of the acoustic disturbance, $c_S = c_S'/c_0'$ is normalized speed of the shear (viscous) waves propagation ($\upsilon_0' \ll \sqrt{2\eta'\omega'}\ = c_S'$),$\eta k_S/\upsilon_0 = q/Re_S$,$Re_S = Re_0/k_s$, $\upsilon_0 = 1$ (see section b). Re_S is the Reynolds number for the flow $\boldsymbol{\xi}$ (or $\boldsymbol{\xi}_1$)).

In relation to $\mathbf{w}$ we apply the following estimation of the norm $\|\eta\nabla\times\| \sim \eta k_{be} := \eta 2\pi/d_{be}$. Where $d_{be} = 2\pi\cdot 0.61\cdot z/(k_0(\omega)r_t)$ is the radius of the first dark circle of the cross section of intensity profile of the acoustical beam, z is the distance along beam axis from the sound source of the radius r_t. This estimation is based on intuition and experimental observations that the transverse profile of the streaming and the intensity of the acoustic beam are similar and that the transverse gradients of the leading (axial) component $\mathbf{w}$ in the flow core can be estimated by the parameters of the cross section of the leading component of the acoustic beam intensity (see [12]).For the plane transducers, for $z = z_{fr} := k_0(\omega)r_t^2/2\pi$ corresponding to a minimum beam width and a maximum of field, and for strongly focused transducer in focus $z\ = z_{foc} \sim 2r_t$, we obtain $k_{be} \approx 4\pi/r_t$, $k_{be}\widetilde{k(\omega)}$ respectively. Adopt last estimations we obtain $1 \ll \eta k_0(\omega)/q\ = \eta\omega/q\ = \eta_s'\omega'/\upsilon_0'c_0' = g_0'\eta_s'\omega'/P_0' = k_0(\omega)/Re_0$. For water ($\eta_s' \cong 10^{-6}$ m^2/s), the condition $Re_0 < 0.1$ is fulfilled for asymptotically high frequencies $\omega'/2\pi > 200$ MHz for $P_0' =\ 0.1$ MPa ($q = 4.4\cdot 10^{-5}$), either for a reasonable frequency range $\omega'/2\pi > 0.2$ MHz but asymptotically low pressures $P_0 =\ \ 0.0001$ MPa. For glycerin $\eta_s' \cong (11.7 \div 2.6)\cdot 10^{-4}$ m^2/s, in the temperature range from 20°C to 40°C, $g_0' = 1250$ Kg/m^3, $c_0' = 1920$ m/s [28] and $P_0' = 0.1$ MPa ($q = 3.6\cdot 10^{-5}$) for $\omega'/2\pi > 0.4$ MHz.

Thus, despite the adoption of extreme estimates for k_{be} the omission $\mathbf{\upsilon} \times \nabla \times \mathbf{w}$ components of ACF on the basis of comparison with the force of viscous resistance can only occur for very viscous liquids. It should also be remembered that the Mach number for disturbances in the focus is from several to several dozen times higher (nonlinear propagation) than the a priori established on the basis of boundary conditions. The remaining, acoustic components of the inertial reaction contained in the substance derivative (or in Reynolds stress) are of the same order (q) as ACF. Therefore their use for estimating ACF significance is more appropriate.

Instantaneous and average ACF can be estimate, respectively as follows $\|\mathbf{\upsilon} \times \nabla \times \mathbf{w}\| \leqslant \upsilon \|\nabla\| \mathrm{w} \sim \upsilon k_{be} w, \|\mathbf{w}^s \times \nabla \times \mathbf{w}^s\| \leqslant \|\nabla\| w^{s2} \sim k_{be} w^{s2}, w := |\mathbf{w}|$. The magnitude of the remaining forces in Eq.(2) primarily depends on the variability of the acoustic field, which is characterized by the acoustical wave number $k(\omega)$, then we obtain $\|\nabla \upsilon^2\| \leqslant \upsilon \|\nabla\| \upsilon \sim k(\omega)\upsilon^2$. If $k_{be} \ll k(\omega)$ then from above estimations $k_{be} w \ll k(\omega)\upsilon$, even if $w \sim \upsilon$ and $k_{be} w^{s2} \ll k(\omega)\langle \upsilon^2 \rangle$ in instantaneous and averaging cases respectively and ACF may be neglected.

The terms majoring ACF, especially $\nabla \upsilon^2$, are significant only in the area of the acoustic beam. Thus, apart from the core of the vortex flow $\mathbf{\upsilon} \cong \mathbf{w}^s$, when the back flow is taken into account, the value of the Reynolds number becomes the determining factor. Also in the vicinity of foci of highly concentrated beams $k_{be} \sim k(\omega)$, and because of their high intensity, the above strong inequality ($k_{be} \ll k(\omega)$) may not occur, and although $w < \upsilon$ then is $w \sim \mathrm{v}$ or $w > \mathrm{v}$. See also Starritt et al. [12] comment on streaming models for focused fields. An additional argument that can be used to omit ACF is that it is perpendicular to the general velocity and vortex fields and do not affect on the kinetic energy balance equation in the instantaneous and averaging cases. It means that it does not affect directly on the modulus of the velocity. However it affect on formation of the distribution of velocity field directions. Since we are interested in the flow only in the beam area in which the energy transfer takes place, the formation of the value of speed and direction of streaming under the influence of the acoustic intensity vector. Therefore, we may omit it in comparison with other generated by inertial reaction (or Reynolds tensor) or viscous part of the stress tensor.

(d) Acoustic mod

Based on the considerations in Apendix A and Eqs.(A.5-A.7), it can be seen that the potential mod determines the form of the density and pressure fields. In potential approximation from Eq.(A.7) we obtain.

$$\frac{g}{g_0} = \left[1 - q\frac{\gamma - 1}{c_0^2}\left(\partial_t \Phi + q\frac{1}{2}\upsilon^2 + 2c_0 \mathscr{A}\Phi\right)\right]^{\frac{1}{\gamma-1}} = 1 - \frac{q}{c_0^2}(\partial_t \Phi + 2c_0 \mathscr{A}\Phi) + o(q^2) \tag{9}$$

As can be seen, Eq.(9) is a solution of the equation $\mathrm{B}[\Phi, \upsilon, g] = 0$ with respect to g. The solution of the $\nabla \mathrm{B} = \mathbf{0}$ equation is $\mathrm{B} = \mathrm{f(t)}$. However, $\mathrm{f(t)} = 0$ can be assumed without losing generality. This is also the justification why in Eq.(A.6) zero Boundary conditions are assumed. After substituting Eq.(9) into Eq.(2) and expanding with accuracy to $o(q)$, the pressure is obtained.

$$\widetilde{P} = -g_0 \partial_t \Phi - q\frac{g_0}{2}\left[\upsilon^2 - \left(\frac{\partial_t \Phi}{c_0}\right)^2\right] + q2\frac{g_0}{c_0}\partial_t \Phi \mathscr{A}\Phi + o^2 \tag{10}$$

The description of the potential mode evolution using Φ is obtained by substituting Eq.(9) into the continuity equation Eq.(1). The obtained equation is long-term notation and is of the order $o(q^2)$ see Eq.(A.11) and (A.13). For our purposes or the determination of Eq.(10), an acoustic approximation of the evolution of the potential mode is sufficient. There are several such approximations of the order $o(q)$ e.g. Kuznetsov equation [35]. Sufficiently accurate and simple, numerically and experimentally proven, the description of the non-linear propagation of acoustic disturbances is given by [21,23,36].

$$\Delta\Phi - \frac{1}{c_0^2}\partial_{tt}\Phi - \frac{2}{c_0}\partial_t \mathscr{A}\Phi - \widehat{q}\partial_t(\partial_t\Phi)^2 = 0 + \mathrm{o}^2, \qquad \widehat{q} := q\frac{\gamma+1}{2c_0^4} \tag{11}$$

The acoustic pressure is given by $P = -g_0 \partial_t \Phi, \langle P \rangle = 0$.

$$\Delta P - \frac{1}{c_0^2}\partial_{tt}P - \frac{2}{c_0}\partial_t AP + \frac{\widehat{q}}{g_0}\partial_{tt}P^2 = 0 + o^2 \tag{12}$$

A characteristic feature of the propagation described by Eq.(12) is the nonlinear widening of the spectrum, which is manifested by the generation of harmonics of the boundary (initial) disturbance. Therefore the solution of Eq.(12) is presented in the form of Fourier series,

$$P(\mathbf{x}, t) = \sum_{\omega \geqslant 1} P_\omega = (1/2)\sum_{\omega \geqslant 1} C_\omega(\mathbf{x})e^{-i\omega t} + c.c \tag{13}$$

$$\mathbf{v}(\mathbf{x}, t) = \frac{1}{2}\sum_{\omega \geqslant 1} \mathbf{v}_\omega(\mathbf{x})\mathbf{e}^{-i\omega t} + c.c \quad , \quad \mathbf{v}_\omega(\mathbf{x}) = \frac{\nabla C_\omega(\mathbf{x})}{i\omega g_0} \tag{14}$$

Calculating the Fourier transform Eq.(12) or substituting Eq.(13) to Eq.(12) we get.

$$\Delta C_\omega + k^2 C_\omega + \widehat{q} NL[C_\omega] = 0 + o(q^2) \tag{15}$$

$$k^2 := k_0^2(1 + 2ia(\omega)/k_0) \tag{16}$$

Where,$k(\omega) \cong k_0(\omega)\mathrm{n}_r + ia_0(\omega)$, (see paragraf **b**), $qNL[C_\omega] := (\widehat{q}/g_0) F[\partial_{tt}P^2]$ [22,23].

Further calculations will require the calculation of the mean value $-g_0\langle \partial_t \Phi \mathscr{A} \nabla \Phi \rangle = \langle P \mathscr{A} \mathbf{v} \rangle$. Using Eq.(4), Eqs. (13), (14) we obtain.

$$\begin{aligned}\langle P\mathscr{A}\mathbf{v}\rangle &= \sum_{\omega \geqslant 1} a_0(\omega)\mathbf{I}_\omega(\mathbf{x}) - \sum_{\omega \geqslant 1} h_0(\omega)\nabla|C_\omega|^2 \Big/ 4\omega g_0 \\ &= \sum_{\omega \geqslant 1}\left(a_0(\omega)\mathbf{I}_\omega(\mathbf{x}) - \delta \mathrm{n}_r(\omega)\nabla|\mathrm{C}_\omega|^2/4g_0 c_0\right)\end{aligned} \tag{17}$$

In acoustics, vector $\mathbf{I} := \left\langle \widetilde{\mathbf{I}} \right\rangle$ is called sound intensity. $\widetilde{\mathbf{I}} := P\mathbf{v} + o(q)$ is the density of the acoustical energy current. However in use is also term instantaneous value of the power intensity vector,

$$\mathbf{I}(\mathbf{x}) = \sum_{\omega \geqslant 1}\mathbf{I}_\omega(\mathbf{x}) = \frac{1}{4}\sum_{\omega \geqslant 1} C_\omega^* \mathbf{v}_\omega + C_\omega \mathbf{v}_\omega^* = \sum_{\omega \geqslant 1}\left(C_\omega^* \nabla C_\omega - C_\omega \nabla C_\omega^*\right) \Big/ 4g_0 i\omega \tag{18}$$

The Fourier components of acoustic beam may by presented in the form.

$$C_\omega = e^{ik_0(\omega)|\mathbf{x}|}\overline{C}_\omega(\mathbf{x}), \qquad \overline{C} = |\overline{C}|e^{i\varphi}, \qquad |C| = |\overline{C}| \tag{19}$$

Where $\overline{C}$ is the envelope, φ is the phase of envelope. Substitute Eq.(19) in to Eq.(18) we obtain.

$$\mathbf{I}_\omega(\mathbf{x}) = \frac{1}{2g_0 c_0}\left(|\overline{C}_\omega(\mathbf{x})|^2(\mathbf{e} + \nabla\varphi/k_0(\omega))\right), \quad \mathbf{e} := \nabla|\mathbf{x}| \tag{20}$$

$I = \sum_\omega I_\omega, I_\omega = |\mathbf{I}_\omega|$. Phase φ is slowly varying function $\nabla\varphi/k_0(\omega) \ll 1$ in the almost all area of the propagation (except area near the source, a size of the sound source). When the z axis is the beam propagation and symmetry axis, it is better to take $k_0 z$, $z \geqslant 0$ instead of $k_0|\mathbf{x}|$ in Eq.(19). Then in Eq.(20)$\mathbf{e} = \mathbf{e}_z := (1; 0; 0), \nabla := (\partial_z; \nabla_\perp), \nabla_\perp$ denotes the transverse components of the gradient. In Fraunhofer approximation $\nabla\varphi = 0$. In the case of nonlinear beam propagation the condition of slow variability is getting better and better fulfilled, for harmonic components of fundamental beam [19 20]. In approximation of the slowly varying phase,$\mathbf{I}_\omega = I_\omega \mathbf{e}, I_\omega \cong |\overline{C}_\omega|^2/2g_0 c_0$, or $\mathbf{I}_\omega \cong (I_{z,\omega}; \mathbf{0}) = (I_\omega, \mathbf{0})$ when $\mathbf{e} = \mathbf{e}_z$.

3. Streaming equations and the acoustical driving force in homogeneous medium

We derive the equations of streaming generated by the acoustic field propagating in a nonlinear and lossy medium. The equations will be more general and, at the same time, of a simpler form of ADF than previously presented in the literature.

Substituting Eqs.(9,10) into Eq.(2) we get (see Appendix A).

$$g\partial_t \mathfrak{v} + qg\frac{1}{2}\nabla \mathfrak{v}^2 - qg\mathfrak{v}\times\nabla\times\mathfrak{v} + \nabla\tilde{P}(g,\mathfrak{v}) + \eta\nabla\times\nabla\times\mathfrak{v} \equiv$$

$$\equiv g_0\partial_t\tilde{\boldsymbol{\xi}} - \eta\Delta\tilde{\boldsymbol{\xi}} - qg_0\mathfrak{v}\times\nabla\times\tilde{\boldsymbol{\xi}} - q(2/c_0)P\mathscr{A}\mathbf{v} = \mathbf{0} + o(q^2) \tag{21}$$

Equation Eq.(21) is the general equation of streaming driven by the acoustic field generating ADF Eq.(22), we are looking for,

$$d_\vartheta\mathbf{F}_{ADF} := q(2/c_0)P\mathscr{A}\mathbf{v} \tag{22}$$

In Eq.(21) the ADF may be non-periodic. Despite discussions in Section 2 we have retained the ACF in Eq.(21).

Further we deal with the case of the periodic ADF. This allows for a clear separation of mods, quasi stationary $\mathbf{w}$ and fast $\boldsymbol{\xi}$. Substituting to Eq.(21) the distributions, $\tilde{\boldsymbol{\xi}} = \mathbf{w} + \boldsymbol{\xi}, P\mathscr{A}\mathbf{v} = \langle P\mathscr{A}\mathbf{v}\rangle + \delta(P\mathscr{A}\mathbf{v})$, where $\delta(\cdot)$ is the deviation from the mean value, and using the properties of the Gauss-Weierstrass function for the Fick operator, based on the results in Appendix B and Section 2 we get.

$$g_0\partial_t\mathbf{w} - \eta\Delta\mathbf{w} = \langle d_\vartheta\mathbf{F}_{ADF}\rangle = 2(q/c_0)\langle P\mathscr{A}\mathbf{v}\rangle \tag{23}$$

$$= \frac{q}{c_0}\sum_{\omega\geqslant 1}(2a_0(\omega)\mathbf{I}_\omega - \delta\mathrm{n}_\mathrm{r}(\omega)\nabla\mathrm{I}_\omega) + o(q^2)$$

$$g_0\partial_t\boldsymbol{\xi} - \eta\Delta\boldsymbol{\xi} = q2/c_0\delta(P\mathscr{A}\mathbf{v}) + o(q^2) \tag{24}$$

In Eq.(23) approximation $|C_\omega|^2/2c_0g_0 \cong I_\omega$ was used, so.

$$2h_0(\omega)\nabla|C_\omega|^2/4\omega g_0 = \delta\mathrm{n}_\mathrm{r}(\omega)\nabla|C_\omega|^2/2c_0g_0 \cong \delta\mathrm{n}_\mathrm{r}(\omega)\nabla I_\omega \tag{25}$$

Let us note that for Eq.(11) the conservation law is valid [21].

$$\nabla\cdot\mathbf{I} = -(2/g_0c_0)\langle P\mathscr{A}P\rangle = -2\sum_{\omega\geqslant 1}a_0(\omega)|C_\omega|^2\Big/2g_0c_0 \tag{26}$$

In the case of a non-linear description of the propagation, for the component $\boldsymbol{I}_\omega$.

$$\nabla\cdot\mathbf{I}_\omega = -2a_0(\omega)|C_\omega|^2/2g_0c_0 + (\hat{q}/g_0)\langle P_\omega\partial_t P^2\rangle \tag{27}$$

In slowly varying phase approximation Eq.(23) takes the shape.

$$g_0\partial_t\mathbf{w} - \eta\Delta\mathbf{w} = -(q\Big/c_0)\left(\mathbf{e}\nabla\cdot\mathbf{I} + \sum_{\omega\geqslant 1}\delta\mathrm{n}_\mathrm{r}(\omega)\nabla I_\omega\right) + o(q^2) \tag{28}$$

$$= (q\Big/c_0)\left(\mathbf{e}Q - \sum_{\omega\geqslant 1}\delta\mathrm{n}_\mathrm{r}(\omega)\nabla I_\omega\right) + o(q^2)$$

$$Q := \sum_{\omega\geqslant 1}Q_\omega = \sum_{\omega\geqslant 1}2a_0(\omega)I_\omega(\mathbf{x}) \tag{29}$$

Quantity $Q = -\nabla\cdot\mathbf{I}$ is the power loss density of the sound or equivalently, power density of the heat sources generated by acoustical mode in the medium.

On the basis of Eqs.(19,20) the mean value of the density of the driving force has the form.

$$\langle d_\vartheta\mathbf{F}_{ADF}\rangle = (q/g_0c_0^2)\sum_\omega|C_{0\omega}|^2(2a_0(\omega)\mathbf{e} - \delta\mathrm{n}_\mathrm{r}(\omega)\nabla)|\overline{S}_\omega(\mathbf{x})|^2 + o(q^2) \tag{30}$$

Where, $C_{0\omega}$ is the transmitted pulse Fourier spectrum, $\overline{S}_\omega(\mathbf{x})$ is the envelope of the spatial characteristic of the ω component of the Fourier spectrum. In the case of linear propagation, $\overline{S}_\omega(\mathbf{x})$ is given in Appendix C. For a non-linearly propagating Gaussian beam see paper [23].

Due to the form of $d_\vartheta\mathbf{F}_{ADF}$, $\langle d_\vartheta\mathbf{F}_{ADF}\rangle$ of the ADF densities forcing the general flow and quasi-stationary flow, and the inertial evolutionary term $g_0\partial_t\mathbf{w}$, Eqs.(22,23,28) and Eq.(30) are generalization of the result obtained in [8,9,13–15,17].

The streaming kinetic energy equation has the form.

$$\frac{g_0}{2}\partial_t w^2 = \eta\nabla\cdot(\mathbf{w}\times\nabla\times\mathbf{w}) - \eta|\nabla\times\mathbf{w}|^2 \tag{31}$$

$$+\frac{q}{c_0}\mathbf{w}\cdot\sum_{\omega\geqslant 1}(2a_0(\omega)\mathbf{I}_\omega(\mathbf{x}) - \delta\mathrm{n}_\mathrm{r}(\omega)\nabla I_\omega) + o^2$$

The coupling between modes were neglected, $-\Delta\mathbf{w} = \nabla\times\nabla\times\mathbf{w}, \nabla\cdot\mathbf{w} = 0$. The dissipation of the kinetic energy of streaming in the differential area $d\vartheta$ is caused by the transported and existing vorticity of the flow. ACF does not contribute to the kinetic energy balance.

4. Estimates. Additional conditions

So far, we have not characterized the size of the dispersion operator. For this purpose, it is convenient to introduce the parameter α_A and the (energy) norm of the operator A[21],

$$\|\mathscr{A}\| := \alpha_A = \max_\mathbf{x}\frac{|\langle P\mathscr{A}\mathbf{v}\rangle|}{I} \tag{32}$$

On the basis of the formulas from Section 2.b) and 2.d) we have $|\langle P\mathscr{A}\mathbf{v}\rangle| \leqslant \sum_\omega a_0 I_\omega + |\delta\mathrm{n}_\mathrm{r}(\omega)\nabla I_\omega| \leqslant \sum_\omega a_0 I_\omega + |\delta\mathrm{n}_\mathrm{r}(\omega)k_0(\omega)|I_\omega \leqslant \sum_\omega(a_0 + |h_0|)I_\omega$ and.

$$\alpha_A \leqslant \max_\mathbf{x}\sum_{\omega\geqslant 1}(a_0 + |h_0|)I_\omega\Big/I \tag{33}$$

It can be shown that $\alpha_{A(NL)} > \alpha_{A(L)}$. Subscripts (NL) and (L) correspond to nonlinear and linear solutions of the propagation equations for the same boundary conditions. In the case of developed nonlinearity of the focus sources α_A may not be small and the distribution $d_\vartheta\mathbf{F}_{ADF}$ is highly concentrated around the focal point. Since the physical effects associated with sound propagation are determined by the spatial distribution of the Q function then the physical focus size should be determined by the spatial distribution of this function. In the case of nonlinear propagation, the cross-sectional size of the physical focus determined on this basis may be even several times smaller than that determined from the pressure (or intensity) distribution. On the axis of an axisymmetric beam, the steady-state velocity distribution based on Eqs.(B.5), (B.7) is given by [19].

$$\mathbf{w}^S(z,0) = \frac{1}{2g_0\eta}\int_0^\infty dz'\left(\int_{|z-z'|}^\infty d_\vartheta\mathbf{F}_{ADF}\left(z',\sqrt{s^2-(z-z')^2}\right)ds\right.$$
$$\left.-\int_{z+z'}^\infty d_\vartheta\mathbf{F}_{ADF}\left(z',\sqrt{s^2-(z+z')^2}\right)ds\right) \tag{34}$$

Where, $d_\vartheta\mathbf{F}_{ADF}(\mathbf{x}) = d_\vartheta\mathbf{F}_{ADF}(z,r)$ is given by Eq.(30). For slowly variable phase approximation and the case $\mathbf{e} = \mathbf{e}_z$ considered above, we get $\mathbf{w}^S(z,0) = (w_z(z,0);0)$.

$$w_z = \frac{q}{\eta g_0c_0}\sum_{\omega\geqslant 1}a_0(\omega)\int_0^\infty dz'\left(\int_{|z-z'|}^\infty I_\omega\left(z',\sqrt{s^2-(z-z')^2}\right)ds\right.$$
$$\left.-\int_{z+z'}^\infty I_\omega\left(z',\sqrt{s^2-(z+z')^2}\right)ds\right) \tag{35}$$

$$-\frac{q}{\eta g_0 c_0}\sum_{\omega\geqslant 1}\delta n_r(\omega)\int_0^\infty dz'\left(\int_{|z-z'|}^\infty \partial_{z'} I_\omega\left(z', \sqrt{s^2-(z-z')^2}\right)ds - \int_{z+z'}^\infty \partial_{z'} I_\omega\left(z', \sqrt{s^2-(z+z')^2}\right)ds\right)$$

Let us check the consistency of Eq.(23). When we deriving them, we suppose $\nabla\cdot\mathbf{w} = 0$. However, Eq.(23) contains a gradient hence $\mathbf{w}$ is not a "pure" vortex field. Let us determine the conditions required for the divergence of the right side of Eq.(23) to be zero with the required accuracy. Using Eqs.(13–16) it can be shown that,

$$\Delta I_\omega = \left(k_0^2/g_0 c_0\right)\left((g_0 c_0|\mathbf{v}_\omega|)^2 - (1-2\delta n_r)|P_\omega|^2\right) + o(q) \tag{36}$$

For a one-dimensional plane wave, when $\delta n_r \equiv 0$, the expression in brackets of Eq.(36) is equal to zero. This is due to the basic impedance relationships. In work [21] it was shown that it is of the order of in the general case $o(q)$. In that case $o(\Delta I_\omega) = o(q) + o(\delta n_r)$.

Calculating the divergences of both sides of Eq.(23), using Eq.(26) and the above analysis of the order Eq.(36), we obtain.

$$\mathbf{0} = \mathbf{0} + o(q^2) + o(q\alpha_A^2) + o(q^2\alpha_A) + o(q\delta n_r^2) + o(q^2\delta n_r) \tag{37}$$

The last two terms concern the characterization of the divergence orders of the term containing δn_r in Eq.(23). So it is required that $\alpha_A \sim \sqrt{q}$ and $\delta n_r \sim \sqrt{q}$. Therefore $\nabla\cdot\mathbf{w} = 0 + o(q^2)$ (because of the potential term $\sim \nabla I_\omega$). On the basis Eq.(5) given for classically viscous fluids and relation $k_0\delta n_r = h_0$ from the section **2.b,** we obtain $\delta n_r = h_0(\omega)/k_0(\omega)$. Then for the water $\delta n_r \in (0\,; 1.1\cdot 10^{-3})$, for glycerin $\delta n_r \in (0;\ 0.37 \div 0.5)$ (in the temperature range from 39°C to 5°C) in the range up to $\omega'/2\pi \leqslant 100\,\text{MHz}$, while $\delta n_r \in (0;\ 0.03 \div 0.43)$ in the range up to $\omega'/2\pi \leqslant 15\,\text{MHz}$. For blood (not classically viscous medium) $\delta n_r \leqslant 3\cdot 10^{-3}$ in the same frequency range.

5. Discussion and conclusions

The equation of streaming generated in a homogeneous lossy fluid by a non-linearly propagating acoustic field was derived. It is a generalization of the result from works [8,9,13,14,17] in which the ADF was obtained using the method of successive approximations and solutions of the linear equation of sound propagation. Let us compute $\nabla\times$ both sides of Eq.(23).

$$g_0\partial_t\boldsymbol{\Omega} - \eta\Delta\boldsymbol{\Omega} = \langle\nabla\times d_\vartheta\mathbf{F}_{ADF}\rangle = 2(q/c_0)\langle\nabla P\times\mathscr{A}\mathbf{v}\rangle, \quad \boldsymbol{\Omega} := \nabla\times\mathbf{w} \tag{38}$$

$$\langle\nabla\times d_\vartheta\mathbf{F}_{ADF}\rangle = 2(q/c_0)\langle\nabla P\times\mathscr{A}\mathbf{v}\rangle = \frac{q}{c_0}\sum_{\omega\geqslant 1}2a_0(\omega)\nabla\times\mathbf{I}_\omega + o(q^2)$$

$$= -2(q/c_0)g_0\langle\partial_t\mathbf{v}\times\mathscr{A}\mathbf{v}\rangle = \frac{qg_0}{c_0}\sum_{\omega\geqslant 1}4i\omega a_0(\omega)\mathbf{v}_\omega\times\mathbf{v}_\omega^* + o(q^2) \tag{39}$$

For a classically viscous medium (classical absorption) $\mathscr{A} \approx -(\eta_h/2c_0^3)\partial_{tt}$. For solutions of the wave equation $\mathscr{A} = \mathscr{A}^x := -(\eta_h/2c_0)\Delta + o(q)$ (see [23]). Because $\Delta\mathbf{v} = \nabla\nabla\cdot\mathbf{v}$ then after substituting to Eq.(39) we get $\langle\nabla\times d_\vartheta\mathbf{F}_{ADF}\rangle = -(q\eta_h/c_0^2)\langle\nabla P\times\nabla\nabla\cdot\mathbf{v}\rangle$. In the first order of approximation from the Navier-Stokes equation, we obtain the linear equation of sound propagation where [7,8,9,14,17] $\nabla P_{(1)} = c_0^2\nabla g_{(1)}, \nabla\cdot\mathbf{v}_{(1)} = -\partial_t g_{(1)}/qg_0, \mathbf{v} = \mathbf{v}_{(1)} + \mathbf{v}_{(2)} + \dots, g = g_0 + g_{(1)} + \dots$. After substituting into "force" term we get $\langle\nabla\times d_\vartheta\mathbf{F}_{ADF}\rangle = (\eta_h/g_0)\cdot\left\langle\nabla g_{(1)}\times\nabla\partial_t g_{(1)}\right\rangle$. It is exactly the source term in the equation derived in [7,8,9,14,17] (here η_h is the kinematic shear viscosity). The circulation in Eq.(38) corresponds to $\boldsymbol{\Omega} := \nabla\times\mathbf{v}_2$. We have shown above that the equations in these works can be integrated with respect to the $\nabla\times$ operator. The formal result of integration is given with the accuracy of the gradient of a certain expression. However, for these equations, logic requires that it be the gradient that has been eliminated by the rotation operator. However, it has not been shown to be equal to zero. Moreover, regardless of the method, the introduction of vortex field $\boldsymbol{\Omega}$ to the description, eliminates the possibility of revealing the influence of dispersion from the description.

The force density driving the streaming in the general case was determined. In the case of periodic disturbances, it has been shown that it is proportional to the spectral loss distribution of the acoustic power density and the gradients of the intensity distributions of harmonics. The derivation shows that acoustic streaming is a consequence of medium dispersion. In general, the importance of absorption (dissipation) for streaming was known from the very beginning of the research. Although absorption and dispersion are related to Hilbert transformations (Kramers-Kronig relationships) and are components of the same complex function, it is a surprise and a new result that the impact of the deviation of the real part of the refractive index (sound phase velocity dispersion) has an impact on the streaming.

In classically viscous media, when the gradient of the intensity modulus is small, this effect is rather weak - much smaller than the effect of absorption.

Note that the estimate given after Eq.(32) shows that the relationship between the absorption and refractive components of the force driving streaming is given by the components of the dispersion coefficient $|h_0|/a_0$. For water and glycerin, the value of this ratio, in the frequency range up to $\omega = 33(100\text{ MHz})$ is given after the Eq.(5). For glycerin, in the temperature range up to 20 °C, this is no longer a small quantity.

However, in a media (like) type blood, at the higher frequency range (however, wavelengths greater than 0.04 mm) may be comparable to or greater than the effect of absorption. In general, along with the refractive index deviation, the spatial variation in the gradient sign of the intensity modulus can be a spatially segmenting factor for streaming.

Dispersions can be considered as a feature of a quasi particle, the scattering potential of which has an inelastic component - the absorption coefficient, and an elastic component - the deviation of the real part of the refractive index. The formation of ADF can be considered a special case of ARP - the result of the interaction of the acoustic field with this potential.

An interesting experimental work [37] presents the average ADF value for a classically viscous medium in the form of a spectral distribution. It formally corresponds to the one derived in this work for this medium. However, as we can guess (no details), it only relates to the axial component of the full force and depends solely on absorption.

A nonlinear generalization of the description of streaming with the non-Newtonian model of shear stresses "adjacent" to the Navier-Stokes model was considered. It turned out to be of too high order in relation to the Mach number (perhaps with the exception of the micro-flows). The evolution of the velocity-dependent viscosity, in the lowest order of the Mach number with respect to only normal (volume, acoustic) disturbances, may appear non-physical. Nevertheless, it shows how important the acoustic disturbance in the fluid is from the point of view of testing the material properties and the mode "content" of the medium. Maxwell fluid model is shown. This model for the equations Eqs.(23,24) with shear stresses described by Eq. (8) is analytically solvable. Using Laplace transforms, it is easy to show that for $t\to\infty$ (stationary states) the $\mathbf{w} = \mathbf{w}^S(\mathbf{x})$ solutions ($\mathbf{w}^1$ in Appendix B) of the Navier-Stokes and Maxwell models are identical. However, $\boldsymbol{\xi}^1\to\mathbf{0}$, $\boldsymbol{\xi}(\boldsymbol{x},t)\to\mathbf{0}$ in the Maxwell model. In the case of a Maxwell fluid, ACF rating can be carried out by estimating the integral Eq.(8). It can also be conducted by application of the Laplace transform to the ACF and to the convolution Eq.(8) and using the properties of the Laplace spectra of the slowly and quickly varying components of the flow. The results are similar to these presented in Section 2. c).

From the required consistency of the left and right sides of Eq.(23), after applying the divergence operation, the applicability conditions of the obtained equations were determined. Moreover, due to dispersion, there is a gradient in the ADF. Hence the streaming field, in particular, is not purely vortex. For periodic perturbations, the general streaming equation Eq.(21) was decomposed into the quasi-stationary flow equation Eq.(23) and the fast-variable fine-scale flow equation Eq.(24). Note, however, that the ADF decomposition based on $P\mathscr{A}\mathbf{v}=\langle P\mathscr{A}\mathbf{v}\rangle+\delta(P\mathscr{A}\mathbf{v})$ is also applicable to continuous non-periodic disturbances.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A

After dividing by g, using $1/g\equiv(1/g-1/g_0)+1/g_0$, Eq. (2), we can write as follows.

$$\nabla \mathrm{B}+\mathbf{S}=\mathbf{0} \tag{A.1}$$

$$\mathrm{B}[\Phi;g]:=\partial_t\Phi+q\frac{1}{2}\upsilon^2+\frac{c_0^2}{q(\gamma-1)}\left(\left(\frac{g}{g_0}\right)^{\gamma-1}-1\right)+2c_0\mathscr{A}\Phi \tag{A.2}$$

$$\mathbf{S}\left[\tilde{\boldsymbol{\xi}}\right]:=\partial_t\tilde{\boldsymbol{\xi}}+\frac{\overline{\eta}}{g}\nabla\times\nabla\times\tilde{\boldsymbol{\xi}}-q\left(\mathbf{v}+\tilde{\boldsymbol{\xi}}\right)\times\nabla\times\tilde{\boldsymbol{\xi}}+2c_0\left(\frac{g_0-g}{g}\right)\mathscr{A}\nabla\Phi \tag{A.3}$$

Where, $\boldsymbol{\upsilon}=\mathbf{v}+\tilde{\boldsymbol{\xi}},\mathbf{v}:=\nabla\Phi,\nabla\cdot\tilde{\boldsymbol{\xi}}=0$. In a spin less motion in the lossless medium, $\mathrm{B}[\cdot]$ is the Cauchy integral. Applying $\nabla\cdot$ to Eq. (A.1) we get the Poisson equation.

$$\Delta\mathrm{B}\;=\;-\nabla\cdot\mathbf{S} \tag{A.4}$$

$$\nabla\cdot\mathbf{S}=\left(\left(q\boldsymbol{\upsilon}-\frac{\nabla\overline{\eta}g}{g^2}\right)\cdot\nabla\times\nabla\times\tilde{\boldsymbol{\xi}}+2c_0\nabla\cdot\left(\frac{g_0-g}{g}\mathscr{A}\mathbf{v}\right)-q\left(\nabla\times\tilde{\boldsymbol{\xi}}\right)^2\right) \tag{A.5}$$

Solution of Eq.(A.4) (also Eq.(A.1) has the form.

$$\mathrm{B}\;=\int_{\mathbf{x}'\in\vartheta}\frac{\nabla\cdot\mathbf{S}'}{4\pi|\mathbf{x}-\mathbf{x}'|}d\vartheta\;\left(=\int_{l(\mathbf{x}_0,\mathbf{x})}\mathbf{S}'\cdot d\mathbf{l}\right),\;\;\mathbf{S}':=\mathbf{S}\left[\tilde{\boldsymbol{\xi}}\left(\mathbf{x}',t\right)\right] \tag{A.6}$$

where, $l(\mathbf{x}_b,\mathbf{x})$ is an integration path with a beginning in $\mathbf{x}_b$ and the end in $\mathbf{x}$,$\mathrm{B}\left[\tilde{\boldsymbol{\xi}}(\mathbf{x}_b,t)\right]=0$. Hence, on the basis of Eq.(A.2) we obtain the functional equation for g,

$$\frac{g}{g_0}=\left[1-q\frac{\gamma-1}{c_0^2}\left(\partial_t\Phi+q\frac{1}{2}\upsilon^2+2c_0\mathscr{A}\Phi-\int_{\mathbf{x}'\in\vartheta}\frac{\nabla\cdot\mathbf{S}'}{4\pi|\mathbf{x}-\mathbf{x}'|}d\vartheta\right)\right]^{\frac{1}{\gamma-1}} \tag{A.7}$$

From Eq.(A.5) it can be seen that the integral term in Eq.(A.7), due to the dependence on g, would generate terms of at least $o(q^2)$ in the iterative process. Including N in the above equations does not extend the set of factors significantly affecting the density disturbance in Eq.(A.7). Including the term $-\eta\tau q\nabla(\nabla\cdot\mathbf{v})^2$ mentioned in section 2.b) leads to the addition of the term $-\eta\tau q(\nabla\cdot\mathbf{v})^2/g_0$ to B in Eq.(A.2) and the term $-\eta\tau q\nabla(\nabla\cdot\mathbf{v})^2(g_0-g)/gg_0$ to S in Eq.(3). Based on Eq.(A.7), it is of the order of $o(q^3/Re_0)$ and has no significance as a contribution to the ADF. The corresponding higher order terms, would be proportional to $\mathrm{tr}(\boldsymbol{\varepsilon}^2)$.

As it can be shown, in the case of an ideal medium, the system of equations Eqs.(1,2) is of the order $o(q^2)$. Thus, the potential mod B determines the form of the density and pressure fields. Substituting the right side of Eq.(9). and Eq.(10) in to Eq.(2), taking into account $\nabla\cdot\tilde{\boldsymbol{\xi}}=0$ we get,

$$\begin{aligned}
&g_0\left(1-\frac{q}{c_0^2}(\partial_t\Phi+2c_0\mathscr{A}\Phi)\right)\partial_t\nabla\Phi+q\frac{g_0}{2}\nabla\upsilon^2\\
&-g_0\nabla\left(\partial_t\Phi+\frac{q}{2}\left[\upsilon^2-\left(\frac{\partial_t\Phi}{c_0}\right)^2\right]-\frac{2q}{c_0}(\partial_t\Phi\mathscr{A}\Phi)\right)\\
&+g_0\left(1-q\frac{1}{c_0^2}(\partial_t\Phi)\right)\partial_t\tilde{\boldsymbol{\xi}}+\eta\nabla\times\nabla\times\tilde{\boldsymbol{\xi}}-qg_0\boldsymbol{\upsilon}\times\nabla\times\tilde{\boldsymbol{\xi}}\\
&\equiv g_0\left(1+qP/g_0c_0^2\right)\partial_t\tilde{\boldsymbol{\xi}}-\eta\Delta\tilde{\boldsymbol{\xi}}-qg_0\boldsymbol{\upsilon}\times\nabla\times\tilde{\boldsymbol{\xi}}+2q\frac{g_0}{c_0}(\partial_t\Phi)\mathscr{A}\nabla\Phi=\mathbf{0}+o\left(q^2\right)
\end{aligned} \tag{A.8}$$

On the other hand, assuming $\mathrm{B}=0$ we also have directly from Eq. (A.1)$\mathbf{S}=\mathbf{0}$, i.e.

$$\partial_t\tilde{\boldsymbol{\xi}}-(\eta/g_0)\Delta\tilde{\boldsymbol{\xi}}-q\left(\mathbf{v}+\tilde{\boldsymbol{\xi}}\right)\times\nabla\times\tilde{\boldsymbol{\xi}}+\frac{2q}{c_0}(\partial_t\Phi)\mathscr{A}\nabla\Phi=\mathbf{0}+o\left(q^2\right) \tag{A.9}$$

Eq.(A.9) was obtained using in Eq.(A.3) the approximation solutions on g Eq.(9), resulting from the equation $\mathrm{B}=0$ and providing the appropriate accuracy. Dividing both sides Eq.(A.8) by $\left(1+qP/g_0c_0^2\right)=(1+\delta g)$ and then expanding with respect to $qP/g_0c_0^2$ and omitting terms higher order than q, we get.

$$g_0\partial_t\tilde{\boldsymbol{\xi}}-\eta\Delta\tilde{\boldsymbol{\xi}}-qg_0\boldsymbol{\upsilon}\times\nabla\times\tilde{\boldsymbol{\xi}}+q2\frac{g_0}{c_0}(\partial_t\Phi)\mathscr{A}\nabla\Phi=\mathbf{0}+o\left(q^2\right)+o(\eta q) \tag{A.10}$$

Based on Section 2.b, we have $o(\eta q)\sim o(q^2/Re_0)$.

Now note that the continuity equation Eq.(1) can be written as follows.

$$D\mathrm{H}(g,\mathbf{S})+(1+q(\gamma-1)\mathrm{H}(g,\mathbf{S}))\nabla\cdot\mathbf{v}=0,\quad \nabla\cdot\boldsymbol{\upsilon}=\nabla\cdot\mathbf{v} \tag{A.11}$$

$$\mathrm{H}(g,\mathbf{S}):=\frac{1}{q(\gamma-1)}\left[\left(\frac{g}{g_0}\right)^{\gamma-1}-1\right]=-\frac{1}{c_0^2}\left(\partial_t\Phi+q\frac{1}{2}\upsilon^2+2c_0\mathscr{A}\Phi-\int_{\mathbf{x}'\in\vartheta}\frac{\nabla\cdot\mathbf{S}'}{4\pi|\mathbf{x}-\mathbf{x}'|}d\vartheta\right)$$

Where $\mathrm{H}(g,\mathbf{S})$ results from the solution of Eq.(A.6) of the equations Eqs.(A.1) and (A.4) and the form $\mathrm{B}[\Phi;g]$ given by Eq.(A2), shortly from the transformed Eq.(A.7). $D:=\partial_t+q\boldsymbol{\upsilon}\cdot\nabla$ is a substantial derivative. So we reduced the system Eqs.(1,2) to Eq.(A.11). A sufficient condition for satisfying Eq.(A.11) or of the system Eqs.(1,2) is satisfying of the equations.

$$\mathbf{S}=\mathbf{0} \tag{A.12}$$

$$D\mathrm{H}(g,\mathbf{0})+(1+q(\gamma-1)\mathrm{H}(g,\mathbf{0}))\nabla\cdot\mathbf{v}=0 \tag{A.13}$$

Of course from Eq.(A.12) we get Eq.(A.9). Equation (A.13) is a description of the potential mode coupled with the vortex through $\upsilon^2=...+\mathbf{v}\cdot\tilde{\boldsymbol{\xi}}$. In acoustical approximation, the modes decouple. Only the acoustic mod affects the vortex generating the ADF. Keeping in Eq.(A.13) the terms of the order $o(q)$, we obtain nonlinear Eq.(11). The described procedure corresponds to a one-time integration of the system Eqs.(1,2). Thus, the forms of the streaming equations Eqs.(A.9), (A.12) and Eq.(A.10) are identical for a fully nonlinear potential description Eq.(A.13) and for the acoustic approximation of Eqs.(11,12).

Appendix B

Since $\tilde{\boldsymbol{\xi}}=\mathbf{w}+\boldsymbol{\xi}$, then Eq.(21) can be rewritten as follows,

$$g_0\partial_t(\mathbf{w}+\boldsymbol{\xi})-\eta\Delta(\mathbf{w}+\boldsymbol{\xi})=qg_0(\boldsymbol{\upsilon}\times\nabla\times\boldsymbol{\xi}+(\mathbf{v}+\boldsymbol{\xi})\times\nabla\times\mathbf{w}+\mathbf{w}\times\nabla\times\mathbf{w}) \tag{B.1}$$

$$+2(q/c_0)(\langle P\mathscr{A}\mathbf{v}\rangle+\delta(P\mathscr{A}\mathbf{v}))$$

Where, $\delta(\cdot)$ denotes deviation from mean value $\langle\delta(\cdot)\rangle=\mathbf{0}$. The right side contains the source.

words $PA\boldsymbol{v}$, and ACF decomposed onto factors that are constant or slowly changing in time, and quickly changing in time. In particular.

$$\delta(P\mathscr{A}\mathbf{v})=\sum_{\omega=1}^{\infty}e^{-i\omega t}\delta_\omega(P\mathscr{A}\mathbf{v})+\ c.c \tag{B.2}$$

Where, based on Eq.(4) and Eqs. (13), (14) $\delta_\omega(P\mathscr{A}\mathbf{v})$ is a combined Fourier component. The sum of all $C_na(m)\mathbf{v}_m$ for combined frequencies such as $\omega=n-m\geqslant1$ or $\omega=n+m\geqslant2$. In the case of linear and non-linear propagation of impulses with a carrier frequency of ω_c, the leading term in Eq.(B.2) are concentrated around the component $\omega=2\omega_c$ ($\omega=l\omega_c$). The above division of sources (and quasi-sources) due to the variability over time was made taking into account the properties of the diffusion operator $g_0\partial_t-\eta\Delta$. The Gauss-Weierstrass ("Green") function of this operator "by itself" would separate the solutions into slow-varying in time, tends to the stationary solutions, dependent on averaged sources, and into time-variables with zero mean value, in particular fast-varying ones that tend to periodic. Therefore, in the distribution $\tilde{\boldsymbol{\xi}}=\mathbf{w}+\boldsymbol{\xi}$, $\mathbf{w}$ depends only on the time averaged sources and $\boldsymbol{\xi}$ the sources determined by the deviations from the mean values, with the variability determined by the acoustic mode. Accordingly, taking into account Eq.(17) and $\nabla\cdot\mathbf{w}=0$, $\nabla\cdot\boldsymbol{\xi}=0$ and omitting the term $\boldsymbol{\upsilon}\times\nabla\times\boldsymbol{\xi}$ discussed in section 2.c) we get.

$$g_0\partial_t\mathbf{w}-\eta\Delta\mathbf{w}=2(q/c_0)\sum_{\omega}\left(a_0(\omega)\mathbf{I}_\omega(\mathbf{x})-h_0(\omega)\nabla|C_\omega|^2/4\omega g_0\right)+qg_0\mathbf{w}\times\nabla\times\mathbf{w} \tag{B3}$$

$$g_0\partial_t\boldsymbol{\xi}-\eta\Delta\boldsymbol{\xi}=2(q/c)\delta(P\mathscr{A}\mathbf{v})+qg_0(\mathbf{v}+\boldsymbol{\xi})\times\nabla\times\mathbf{w} \tag{B.4}$$

Looking for the solution of Eq.(B.3) and Eq.(B.4) in the form of a series of successive approximations to $\mathbf{w}=\mathbf{w}^1+\mathbf{w}^2+...$, $\boldsymbol{\xi}=\boldsymbol{\xi}^1+\boldsymbol{\xi}^2+...$ (for Eq.(B.1) $\tilde{\boldsymbol{\xi}}=\tilde{\boldsymbol{\xi}}^1+\tilde{\boldsymbol{\xi}}^2+...$) it is easy to notice that the sources for $\boldsymbol{w}^m$ and $\boldsymbol{\xi}^m$ are at least $o(q^m)$. Solutions $\mathbf{w}^1$, $\boldsymbol{\xi}^1$ Eq.(B.3) and Eq.(B.4) in the half-space $z\geqslant0$ for the Dirichlet boundary conditions $\mathbf{w}(\mathbf{x},t)|_{z=0}=\mathbf{0}$, $\boldsymbol{\xi}(\mathbf{x},t)|_{z=0}=\mathbf{0}$ (or $\tilde{\boldsymbol{\xi}}(\mathbf{x},t)\big|_{z=0}=\mathbf{0}$) (natural for these fields) and the initial $\mathbf{w}(\mathbf{x},t=0)=\mathbf{0}$, $\boldsymbol{\xi}(\mathbf{x},t=0)=\mathbf{0}$ have the form,

$$\mathbf{w}^1=(2q\Big/g_0c_0)\widehat{G^{\mathbf{w}}}(\mathbf{x},t)\underset{\mathbf{x}}{\otimes}\sum_{\omega\geqslant1}(a_0(\omega)\mathbf{I}_\omega(\mathbf{x})-\delta\mathrm{n}_{\mathrm{r}}(\omega)\nabla\mathrm{I}_\omega/2) \tag{B.5}$$

$$\boldsymbol{\xi}^1=\sum_{\omega\geqslant1}e^{-i\omega t}\boldsymbol{\xi}^1_\omega(\mathbf{x},t)+\ c.c=(2q\Big/g_0c_0)\sum_{\omega\geqslant1}e^{-i\omega t}\overset{\frown}{G^{\boldsymbol{\xi}}_{\omega}}(\mathbf{x},t)\underset{\mathbf{x}}{\otimes}\delta_\omega(P\mathscr{A}\mathbf{v})+\ c.c \tag{B.6}$$

$$\widehat{G^{w;\xi}}(\mathbf{x},t):=G^{w;\xi}(\mathbf{x},t)-G^{w;\xi}(\overline{\mathbf{x}},t),\qquad \overline{\mathbf{x}}:=[-z,x,y]$$

$$G^{w}(\mathbf{x},t) := \mathrm{erfc}\left(|\mathbf{x}|\Big/\sqrt{4\eta t}\right)\Big/4\pi\eta|\mathbf{x}| \tag{B.7}$$

$$G^{\xi}_{\omega}(\mathbf{x},t) := \left[1 - e^{i\omega t}\frac{\exp\left(-|\mathbf{x}|^2/4\eta t\right)}{(4\pi\eta t)^{3/2}}\underset{\mathbf{x}}{\otimes}\right]\frac{\exp\left(-k_S|\mathbf{x}|(1-i)\right)}{4\pi\eta|\mathbf{x}|} \tag{B.8}$$

In Eq.(B.5), Eq.(25) was used. The functions Eqs.(B.7) and (B.8) are the result of convolution respect time of the fundamental solution (Gauss-Weierstrass) with a function constant in time ($\langle P\mathscr{A}\mathbf{v}\rangle$) or $\exp(-i\omega t)$ (see Eqs.(B.1) and (B.2)). The solution of Eq.(B.5) tends to solution of the Poison equation $\partial_t\mathbf{w}^1 \underset{t\to\infty}{\rightarrow} \mathbf{0}, \mathbf{w}^1(\mathbf{x},t) \underset{t\to\infty}{\rightarrow} \mathbf{w}^S(\mathbf{x})$. The solution of Eq.(B.6) tends to the periodic function $\boldsymbol{\xi}^1_\omega(\mathbf{x},t) \underset{t\to\infty}{\rightarrow} \boldsymbol{\xi}^1_\omega(\mathbf{x})$ and its influence is exponentially limited to the regions of the order δ_s within the beam. It follows from the above solutions that the neglected in Eq.(A.10) term $qP\cdot\partial_t(\boldsymbol{\xi}+\mathbf{w}) \sim o(q^2)$. As can be shown by estimating the convolution integrals containing G^{ξ}_{ω} except that they are explicitly $o(q)$ are implicitly $\sim 1/k_s^2 = \delta_s^2 = 2\eta/\omega$ and $\boldsymbol{\xi}_\omega \widetilde{q/\omega}, \omega \geqslant 1$. These properties are induced in the following approximations. By definition, the terms $o(q^2)$ and higher are omitted and $\mathbf{w} = \mathbf{w}^1 + o(q^3), \boldsymbol{\xi} = \boldsymbol{\xi}^1 + o(q^2)$. Eqs.(B.3) and (B.4) takes the form.

$$g_0\partial_t\mathbf{w} - \eta\Delta\mathbf{w} = 2(q/c_0)\sum_{\omega}\left(a_0(\omega)\mathbf{I}_\omega(\mathbf{x}) - h_0(\omega)\nabla|C_\omega|^2/4\omega g_0\right) + qg_0\mathbf{w}\times\nabla\times\mathbf{w} \tag{B.9}$$

$$g_0\partial_t\boldsymbol{\xi} - \eta\Delta\boldsymbol{\xi} = 2(q/c_0)\delta(P\mathscr{A}\mathbf{v}) \tag{B.10}$$

Leaving aside $qg_0\mathbf{w}\times\nabla\times\mathbf{w}$, we rewritten these equations into Section 3 in the form Eqs.(22) and (23). Averaging Eq.(B.3) for $t\to\infty$ gives.

$$\eta\Delta\mathbf{w}^S = -2(q/c_0)\sum_{\omega\geq 1}\left(a_0(\omega)\mathbf{I}_\omega(\mathbf{x}) - h_0(\omega)\nabla|C_\omega|^2/4\omega g_0\right) + o^2 \tag{B.11}$$

As it is know, the solutions of Eq.(B.9) tends to steady state which is described by Eq.(B.11). Typically, the derivation of the streaming equation is based on averaging the power term, ignoring slow-varying transients, i.e. for $t\to\infty$ ($\partial_t w = 0$). The $\boldsymbol{\xi}$ mod is ignored in the considerations, even though its presence in the flow is sometimes noticed [9]. It is in the same order as $\mathbf{w}$. As we have shown, the correct derivation of Eq.(B.11) requires an analysis of the influence of this mode.

Appendix C

For example, if as incident waves we assume the wave satisfies Dirichlet boundary conditions, emitted by plane uniformly apodised transducer perpendicular to z axis, or as the "global" plane wave propagating along z then.

$$\overline{C}_\omega(\mathbf{x}) = C_{0\omega}\overline{S}_\omega(\mathbf{x}) \tag{C.1}$$

$$\overline{S}_\omega(\mathbf{x}) = \begin{cases} 2\partial_z \displaystyle\int_{\mathbf{x}_\sigma\in\sigma} G(\mathbf{x}-\mathbf{x}_\sigma)e^{-ik_0|\mathbf{x}|}d\sigma & \textit{for transducer} \\ e^{-a_0(\omega)z} & \textit{for plane wave} \end{cases} \tag{C.2}$$

Where, $C_{0\omega}$ is transmitted pulse Fourier spectrum, $S_\omega(\cdot)$ is the spatial characteristic of the ω component of the Fourier spectrum, $\overline{S}_\omega(\cdot)$ is the spatial envelope of $S(\mathbf{x}) = \exp(ik_0|\mathbf{x}|)\overline{S}(\mathbf{x}), |S_\omega(\cdot)| = |\overline{S}_\omega(\cdot)|, G(\mathbf{x}-\mathbf{x}_\sigma) = \exp(ik|\mathbf{x}-\mathbf{x}_\sigma|)/4\pi|\mathbf{x}-\mathbf{x}_\sigma|$, σ is the transducer surface. We note that the demodulation transformation Eq.(19) concerns just the function $S_\omega(\cdot)$. The envelope $\overline{S}_\omega(\cdot)$ is not only slow-changing in space but also in relation to ω.

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