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Genuine quantum coherence

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Abstract

Any quantum resource theory is based on free states and free operations, i.e. states and operations which can be created and performed at no cost. In the resource theory of coherence free states are diagonal in some fixed basis, and free operations are those which cannot create coherence for some particular experimental realization. Recently, some problems of this approach have been discussed, and new sets of operations have been proposed to resolve these problems. We propose here the framework of genuine quantum coherence. This approach is based on a simple principle: we demand that a genuinely incoherent operation preserves all incoherent states. This framework captures coherence under additional constraints such as energy preservation and all genuinely incoherent operations are incoherent regardless of their particular experimental realization. We also introduce the full class of operations with this property, which we call fully incoherent. We analyze in detail the mathematical structure of these classes and also study possible state transformations. We show that deterministic manipulation is severely limited, even in the asymptotic settings. In particular, this framework does not have a unique golden unit, i.e. there is no single state from which all other states can be created deterministically with the free operations. This suggests that any reasonably powerful resource theory of coherence must contain free operations which can potentially create coherence in some experimental realization.

Keywords: quantum coherence, resource theories, incoherent transformations and state manipulation

1. Introduction

Quantum mechanics offers a radically different description of reality that collides with the intuition behind that of classical physics. At first this was only regarded from the foundational point of view. However, in recent decades it has been realized that the fundamentally different features of quantum theory can be exploited to realize revolutionary applications [1]. Quantum information theory has taught us that quantum technologies can outperform classical ones in a large variety of tasks such as communication, computation or metrology. This has led to identify and study nonclassical salient properties of quantum theory, like entanglement [2, 3] or nonlocality [4] which stem from the tensor product structure. This is done in order to understand better from a theoretical perspective the full potential of quantum resources and, also, to seek for new paths for applications. Undoubtedly, the superposition principle, which leads to coherence, is another characteristic trait of quantum mechanics; however, a rigorous theoretical study of this phenomenon on the analogy of the aforementioned resources has only been initiated very recently [5–12]. Nevertheless, quantum coherence is the basis of single-particle interferometry [13–15] and it is believed to play a nontrivial role in the outstanding efficiency of several biological processes [16–19]. This grants coherence the status of a resource and makes necessary to develop a solid framework allowing to assess and quantify this phenomenon together with the rules for its manipulation.

Resource theories have proven to be a very successful framework to build a rigorous and systematic study of the possibilities and limitations of distinct features of quantum information theory. Originally developed in the case of entanglement theory [2, 3], the conceptual elegance and applicability of this approach has led to consider in the last years, resource theories for several quantum features such as frame alignment [6], stabilizer computation [20], nonlocality [21] or steering [22]. In such theories one considers a set of free states and of free operations. The latter constitutes the set of transformations that the physical setting allows to implement. Free states must be mapped to free states under all free operations and they are useless in this physical setting (it is usually assumed that they can be prepared at no cost). With this, non-free states can be regarded as resource states: they allow to overcome the limitations imposed by state manipulation under the set of free operations. Furthermore, free operations then provide all possible protocols to manipulate the resource and induce the most natural ordering among states since the resource cannot increase under this set of transformations. This allows to rigorously construct resource measures: these quantifiers must not increase under free operations. For instance, in entanglement theory the set of free operations is local (quantum) operations and classical communication (LOCC) while free states are separable states (entangled states are then the resource states) and the basic principle behind entanglement measures is that they must not increase under LOCC [2, 3]. Recent literature has set the first steps to build a resource theory of coherence [5–12], which has allowed to study the role of coherence in quantum theory [23–45], its dynamics under noisy evolution [46–57], and to obtain new coherence measures [58–75].

However, there is an ongoing intense debate on how this theory should be exactly formulated with several alternatives being considered [5, 7, 11, 12, 39, 73, 76–78]. Notice that in the case of entanglement the physical setting clearly identifies the set of allowed operations: the parties, who might be spatially separated, can only act locally. However, while in the case of coherence it is clear that free states should correspond to incoherent states (see below for definitions), the physical setting does not impose any clear restriction on what free operations should be. Thus, any set of operations that map incoherent states to incoherent states might qualify in principle as a good candidate. It is therefore fundamental to identify reasonable sets of free operations from the physical and/or mathematical perspective and to study the different

features of such resource theories. In this paper, we analyze in detail two such sets and we thoroughly study the possibilities and limitations of the emergent resource theories for state manipulation.

In the framework of coherence, the physical setting identifies a particular set of basis states as classical. A state on $H \simeq \mathbb{C}^d$ is called incoherent if it is diagonal in the fixed aforementioned basis $\{|i\rangle\}$ ($i = 1, \dots, d$) and otherwise coherent. In the standard resource theory of coherence of Baumgratz *et al* [11], free operations are given by the so-called incoherent operations. These correspond to those maps that admit an incoherent Kraus decomposition:

$$\Lambda_i[\rho] = \sum_i K_i \rho K_i^\dagger, \quad (1)$$

where, besides the normalization condition $\sum_i K_i^\dagger K_i = 1$, the Kraus operators fulfill that the unnormalized states $K_i \rho K_i^\dagger$ remain incoherent for every i if ρ is. On an experimental level, this definition means that a quantum operation *can* be implemented in an incoherent way. By performing the measurements given by the aforementioned Kraus operators, no coherence can be created from an incoherent state even if we allow for postselection. Thus, this set of operations appears to be a very sensible choice and it leads to a reasonably rich resource theory [12]. However, as mentioned above, no physical reason is known why these operations should be regarded as free in this setting. This naturally leads to consider other possibilities either because they seem physically justified or because they have a convenient structure or properties in a given setting [5, 7, 12, 39, 73, 76–78]. In particular, one can think of scenarios where the particular forms of implementing quantum operations are restricted. Moreover, one can consider resource theories of speakable or unspeakable information [78]. The former are theories where the quality of the resource is independent of the physical encoding while in the latter it depends on the underlying degrees of freedom. It turns out that in the resource theory of incoherent operations, coherence is a speakable resource and it might be desirable to consider theories where coherence is unspeakable [78]. In this paper we introduce and study the sets of *genuinely incoherent operations* (GIO) and *fully incoherent operations* (FIO). GIO leads to a resource theory of unspeakable coherence and is derived from one simple condition, namely that the operation preserves *all* incoherent states. We will show in this work that these operations are incoherent irrespectively of the implementation (i.e. of the Kraus decomposition), and exhibit several interesting properties which are not present for the set introduced in [11]. On the other hand the set FIO is the most general set of operations which are incoherent for all Kraus decompositions and leads to a theory of speakable coherence.

This paper is structured as follows. In section 2 we consider GIO. We define and motivate this set of operations and we analyze in detail its mathematical structure. This allows us to derive genuine coherence measures and to study extensively the possibilities and limitations of GIO for state manipulation. We show that deterministic transformations are very constrained in this framework. However, we show that stochastic transformations have a much richer structure. We also consider the asymptotic setting, which plays a key role in resource theories, and show that any general form of distillation and dilution is impossible. Motivated by these limitations in section 3 we study the set of FIO. We characterize mathematically this kind of operations, which allows us to prove that this set is a strict superset of GIO. Although this is reflected in a strictly more powerful capability for state manipulation, we provide several results suggesting that FIO are still rather limited. Finally, in section 4 we provide a discussion on the relation of the operations presented in this work to other alternative incoherent operations presented in the literature.

2. Genuinely incoherent operations

2.1. Definition and motivation

The underlying principle in every resource theory of coherence is that incoherent states should not be a resource. Thus, lacking a clear intuition on the operations the physical setting allows to implement, any set of operations that maps incoherent states to incoherent states is a good candidate to be regarded as the free operations from the mathematical point of view. The largest class of maps with this property has been studied in previous literature and is referred to as maximally incoherent operations [5]. It is natural to consider the opposite extreme case, i.e. those operations for which all incoherent states are fixed points. We define genuinely incoherent (GI) operations to be quantum operations which have this property of preserving all incoherent states:

$$\Lambda_{\text{gi}}[\rho_i] = \rho_i \quad (2)$$

for any incoherent state ρ_i . Thus, in the GI formalism, incoherent states are not resourceful in an extreme way: if we are provided with such a state, we are bound to it and no protocol is possible. In particular, incoherent states cannot even be transformed to other incoherent states. From this point of view, in this setting incoherent states are not free states, i.e. they cannot be prepared at no cost with the allowed set of operations. What grants coherent states the status of a resource in this case is the fact that these are the only states for which non-trivial protocols are in principle possible.

One reason to study the resource theory of genuine coherence is that this is arguably the most contrived theory one can think of at the level of allowed operations. In this sense, GI manipulation can be regarded as a building block for protocols in other resource theories in which the free operations have a richer structure. More importantly, genuine coherence is an extreme form of a resource theory of *unspeakable* coherence. Another example of such a theory is the resource theory of asymmetry [7, 76, 77], while the framework of Baumgratz *et al* [11] is a representative of *speakable* coherence. As discussed in [78], resource theories of speakable coherence are those where the means of encoding the information is irrelevant. On the other hand, unspeakable information can only be encoded in certain degrees of freedom. In this case, coherent states given by a particular superposition of classical states need not be equally useful as the same superposition of a different set of classical states. The resource theory of genuine coherence is an extreme form of unspeakable coherence in the sense that no coherent state is interchangeable with any other. This might be particularly meaningful from the physical point of view in scenarios where the classical states are constrained by e.g. energy preservation rules. Indeed, if the states $|i\rangle$ are the eigenstates of some nondegenerate Hamiltonian, GI operations correspond to energy-preserving operations as defined in [79]. Another example would be noiseless excitation transport. If the classical states (i.e. the position of the excitation) can be freely mapped to each other, in the absence of dissipation the process of transport is trivial.

Another interesting feature of GI operations, as we will see in the next section, is that GI maps are incoherent independently of the Kraus decomposition (see equation (1)). This is not the case in the standard framework of Baumgratz *et al* [11]. To see this, notice that a quantum operation admits different Kraus representations as characterized in the following well-known theorem (see e.g. [80, 81]).

Theorem 1. *Two sets of Kraus operators $\{K_j\}$ and $\{L_i\}$ correspond to Kraus representations of the same map if and only if there exists a partial isometry matrix V such that*

$$L_i = \sum_j V_{ij} K_j. \quad (3)$$

Thus, as an example for the above claim take the single-qubit operation given by the following Kraus operators:

$$K_0 = |0\rangle\langle +| \quad \text{and} \quad K_1 = |1\rangle\langle -| \tag{4}$$

with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. These operators are incoherent as can be seen by noting that $K_i|\psi\rangle \sim |i\rangle$ holds true for any pure state $|\psi\rangle$. If we now apply theorem 1 to the channel defined above with $V = H/\sqrt{2}$ with the 2×2 Hadamard matrix H , we get the Kraus operators

$$L_{\pm} = \frac{1}{\sqrt{2}}(|0\rangle\langle +| \pm |1\rangle\langle -|). \tag{5}$$

Note that the Kraus operators L_{\pm} are not incoherent, which can be directly checked by applying them to the state $|0\rangle$: $L_{\pm}|0\rangle = (|0\rangle \pm |1\rangle)/2$. Thus, these Kraus operators convert the incoherent state $|0\rangle$ into one of the maximally coherent states $|+\rangle$ or $|-\rangle$.

The above example shows that a quantum operation which is incoherent in one Kraus decomposition is not necessarily incoherent in another Kraus decomposition. This observation is not surprising, a similar effect appears also in entanglement theory. There, the set of LOCC operations is also defined via a certain structure of Kraus operators which is lost when other Kraus decompositions are considered. However, this feature is clearly justified by the physical setting: as long as a map has a Kraus representation of the LOCC form, there are physical means for the spatially separated parties to implement the corresponding protocol. On the other hand, the definition of incoherent operations is rather abstract and we do not have a physical reason that guarantees that a certain map is implementable. It is certainly admissible to take the analogy of LOCC and consider maps which have one Kraus representation that does not create coherence. However, it also seems reasonable *a priori* to explore the alternative case in which maps must not create coherence independently of the Kraus decomposition. This would be relevant in resource theories of coherence where one is not granted with the power to choose different particular experimental implementations of a map.

2.2. Mathematical characterization

In order to understand the potential of a resource theory based on GI operations, it will be useful to have a more detailed mathematical description of these operations beyond its mere definition given in equation (2). The framework of *Schur operations* plays a key role here. A quantum operation Λ acting on a Hilbert space of dimension d is called a Schur operation if there exists a $d \times d$ matrix A such that

$$\Lambda[\rho] = A \odot \rho. \tag{6}$$

Here, \odot denotes the Schur or Hadamard product, i.e. entry-wise product for two matrices of the same dimension:

$$(X \odot Y)_{ij} = X_{ij}Y_{ij}. \tag{7}$$

The fact that Λ is a quantum channel—and thus a trace preserving completely positive map—adds additional constraints on the matrix A [80]:

- A must be positive semidefinite (PSD),
- the diagonal elements of A must be $A_{ii} = 1$.

The following theorem provides a simple characterization of GIO that will be used throughout this paper.

Theorem 2. *The following statements are equivalent:*

1. Λ is a genuinely incoherent quantum operation, i.e. $\Lambda[\rho] = \rho$ for every incoherent state ρ .
2. Any Kraus representation of Λ as in equation (1) has all Kraus operators $\{K_i\}$ diagonal.
3. Λ can be written as

$$\Lambda[\rho] = A \odot \rho \tag{8}$$

with a PSD matrix A such that $A_{ii} = 1$.

Proof. Let us start by showing that 1 implies 2. For any GI operation Λ and any pure incoherent state $|j\rangle$ it must hold that

$$\Lambda[|j\rangle\langle j|] = \sum_i K_i |j\rangle\langle j| K_i^\dagger = |j\rangle\langle j|, \tag{9}$$

where $\{K_i\}$ are Kraus operators of Λ . However, this equality can only hold true if $|j\rangle$ is an eigenstate of every Kraus operator:

$$K_i |j\rangle \propto |j\rangle. \tag{10}$$

This shows that every Kraus operator is diagonal. The fact that 3 \Rightarrow 1 is straightforward. Any operation defined as in equation (8) is indeed genuinely incoherent, i.e. it preserves all incoherent states. It remains to prove that 2 implies 3. This is a direct consequence of theorem 4.19 in [80]. \square

Part 2 of this theorem clearly shows that the set of GI operations is a strict subset of incoherent operations introduced by Baumgratz *et al* [11]. Furthermore, this also immediately proves our claim of the previous section that every Kraus decomposition of a GI map does not create coherence. By definition, every GI operation is *unital*, i.e. it preserves the maximally mixed state: $\Lambda(\mathbb{1}/d) = \mathbb{1}/d$. An important example for a GIO is a convex combination of unitaries diagonal in the incoherent basis:

$$\Lambda_{\text{gi}}[\rho] = \sum_k p_k U_k \rho U_k^\dagger \tag{11}$$

with probabilities p_k and unitaries U_k defined as $U_k = \sum_l e^{i\phi_k} |l\rangle\langle l|$. It is now natural to ask if any GIO can be written in this form. The characterization provided in theorem 2 allows to study in great detail the mathematical structure of the set of GIOs and, in particular, to answer the above question.

Theorem 3. *For qubits and qutrits any GIO is of the form (11). This is no longer true for dimension 4 and above.*

For the proof of the theorem we refer to appendix A.

In the context of resource theories, one can also be interested in tasks that, although impossible deterministically, might be implemented with a certain non-zero probability of success. Interestingly, theorem 2 can also be generalized to the stochastic scenario. In this case we need to consider trace non-increasing genuinely incoherent operations, i.e. transformations of the form

$$\Lambda_{\text{sgi}}[\rho] = \sum_i K_i \rho K_i^\dagger, \tag{12}$$

where the Kraus operators K_i are all diagonal in the incoherent basis but do not need to form a complete set, i.e. $\sum_i K_i^\dagger K_i \leq \mathbb{1}$. This means that the process is not deterministic, but occurs with probability given by $p = \text{Tr}[\sum_i K_i \rho K_i^\dagger]$. We will call such a map *stochastic genuinely incoherent (SGI)* operation. It is important to note that any SGI operation can be completed to a genuinely incoherent

operation by another SGI operation. Thus, a transformation among two states can be implemented with some non-zero probability of success if and only if there exists an SGI operation connecting them (up to normalization). The following theorem generalizes theorem 2 to the stochastic scenario.

Theorem 4. *A quantum operation Λ is SGI if and only if it can be written as*

$$\Lambda[\rho] = A \odot \rho$$

with a PSD matrix A such that $0 \leq A_{ii} \leq 1$.

Proof. Proposition 4.17 and theorem 4.19 in [80] establish the equivalence of Schur maps with a PSD matrix A and maps with diagonal Kraus operators independently of whether the maps are trace-preserving or not. SGI maps correspond to the case of trace non-increasing maps. Thus, it only remains to check that this condition is fulfilled if and only if $0 \leq A_{ii} \leq 1 \forall i$. First of all it must hold that $A_{ii} \geq 0 \forall i$ in order for the matrix to be PSD. Then, on the one hand, $A_{ii} \leq 1 \forall i$ is clearly sufficient for the Schur map to be trace non-increasing. On the other hand, looking at the action of the map on the states $\{|i\rangle\langle i|\}$ the bound is also found to be necessary. \square

The power of the above theorem lies in the fact that it gives a simple characterization of all SGI operations, which will be very useful when we study stochastic state transformations in section 2.5.

2.3. Coherence rank and coherence set

In entanglement theory [3], local unitaries are invertible local operations, and states related by local unitaries have the same amount of entanglement. Thus, for most problems concerning bipartite pure-state entanglement, it is sufficient to consider the Schmidt coefficients of the corresponding states.

In direct analogy, we notice that diagonal unitaries are invertible GI operations. Hence, for any measure of genuine coherence, states related by diagonal unitaries are equally coherent. Thus, without loss of generality, we can restrict our considerations to pure states

$$|\psi\rangle = \sum_i \psi_i |i\rangle \quad (13)$$

such that $\psi_i \geq 0$. Obviously, a pure state is incoherent if and only if $\psi_i = 1$ for some i .

In analogy to the Schmidt rank in entanglement theory [3], one can define the *coherence rank* of a pure state $r(|\psi\rangle)$ as the number of basis elements for which $\psi_i \neq 0$ [61]. The coherence rank, like its analogous in entanglement theory, provides useful information about the coherence content of a state and constrains the possible transformations among resource states. For instance, the coherence rank cannot increase under incoherent operations [12]. As we will see later, one particularity of genuine coherence is the following. It is not only relevant the coherence rank but also for which basis elements a state has zero components. We encode this information in the *coherence set*.

Definition 5. The *coherence set* $R(\psi)$ of $|\psi\rangle = \sum_{i=1}^d \psi_i |i\rangle$ denotes the subset of $\{1, 2, \dots, d\}$ for which $\psi_i \neq 0$.

The coherence set captures one of the crucial differences between the formalism of GI operations and that of incoherent operations of [11]. In the latter, incoherent states are exchangeable. However, diagonal unitaries do not allow to permute basis elements and by its very definition an incoherent state cannot be transformed by GI operations into a different incoherent state. Thus, the relevance of the coherence set arises from the fact that we are dealing with

a resource theory of unspeakable coherence. Unless otherwise stated, in the following when we start with a state $|\psi\rangle = \sum_i \psi_i |i\rangle$ it should be assumed that all sums go over the elements of $R(\psi)$. Finally, we will always use the notation ρ_ψ for the density matrix corresponding to the pure state $|\psi\rangle$ (i.e. $\rho_\psi = |\psi\rangle\langle\psi|$).

2.4. Quantifying genuine coherence

Having introduced the framework of genuine coherence, we will provide methods to quantify the amount of genuine coherence in a given state. For this, we will follow established notions for entanglement and coherence quantifiers [3, 11, 82, 83].

A measure of genuine coherence G should have at least the following two properties.

- (G1) Nonnegativity: G is nonnegative, and zero if and only if the state ρ is incoherent.
- (G2) Monotonicity: G does not increase under GI operations, $G(\Lambda_{\text{gi}}[\rho]) \leq G(\rho)$.

It is instrumental to compare the above conditions to the corresponding conditions in entanglement theory [3, 82, 83]. There, the condition corresponding to G2 implies that an entanglement measure does not increase under LOCC. This condition and nonnegativity are regarded as the most fundamental conditions for an entanglement measure [3].

The following two conditions will be regarded as desirable but less fundamental.

- (G2') Strong monotonicity: G does not increase on average under the action of GI operations for any set of Kraus operators $\{K_i\}$, i.e. $\sum_i q_i G(\sigma_i) \leq G(\rho)$ with probabilities $q_i = \text{Tr}[K_i \rho K_i^\dagger]$ and states $\sigma_i = K_i \rho K_i^\dagger / q_i$.
- (G3) Convexity: G is a convex function of the state, $G(\sum_i p_i \rho_i) \leq \sum_i p_i G(\rho_i)$.

The entanglement equivalent of G2' states that the entanglement measure does not increase on average under selective LOCC. This condition as well as convexity are not mandatory for a good entanglement measure [3]. Following the notion from entanglement theory, we will consider conditions G1 and G2 to be more fundamental than G2' and G3. A measure which fulfills conditions G1, G2, and G2' will be called *genuine coherence monotone*. Additionally, the corresponding measure (monotone) will be called *convex* if condition G3 is fulfilled as well. Note that G2' and G3 in combination imply G2.

Since the set of GI operations is a subset of general incoherent operations, any coherence monotone in the sense of Baumgratz *et al* is also a genuine coherence monotone. Examples for such monotones are the relative entropy of coherence, the l_1 -norm of coherence, and the geometric coherence [11, 60]. However, it is possible that some quantities which do not give rise to a good coherence measure in the framework of Baumgratz *et al* are still good measures of genuine coherence. This is indeed the case for the Wigner–Yanase skew information [84]: $S_H(\rho) = -\frac{1}{2} \text{Tr}([H, \sqrt{\rho}]^2)$ with the commutator $[X, Y] = XY - YX$, and H is some nondegenerate Hermitian operator diagonal in the incoherent basis. As is shown in appendix B, S_H is a convex measure of genuine coherence, it fulfills conditions G1, G2, and G3. It remains open if it also fulfills the condition G2'.

Alternatively, a very general measure of genuine coherence can be defined as follows:

$$G_D(\rho) = \min_{\sigma \in \mathcal{I}} D(\rho, \sigma), \quad (14)$$

where \mathcal{I} is the set of incoherent states and D is an arbitrary distance which does not increase under unital operations Λ_u , i.e. $D(\Lambda_u[\rho], \Lambda_u[\sigma]) \leq D(\rho, \sigma)$ for any operation which preserves the maximally mixed state $\Lambda_u(\mathbb{1}/d) = \mathbb{1}/d$. The following theorem shows that G_D fulfills the corresponding conditions.

Theorem 6. G_D is a measure of genuine coherence, it satisfies conditions G1 and G2. If D is jointly convex, G_D also satisfies G3.

Proof. The proof that G_D satisfies condition G1 follows from the fact that any distance is nonnegative and zero if and only if $\rho = \sigma$. For proving G2, let τ be the closest incoherent state to ρ , i.e. $G_D(\rho) = D(\rho, \tau)$. The fact that any genuinely incoherent operation is unital together with the requirement that D does not increase under unital maps implies:

$$G_D(\rho) = D(\rho, \tau) \geq D(\Lambda_{\text{gi}}[\rho], \Lambda_{\text{gi}}[\tau]) \geq G_D(\Lambda_{\text{gi}}[\rho]), \quad (15)$$

where in the last inequality we used the fact that $\Lambda_{\text{gi}}[\tau]$ is incoherent.

We will now show that G_D is also convex if the distance D is jointly convex, i.e. if it satisfies

$$D\left(\sum_i p_i \rho_i, \sum_j p_j \sigma_j\right) \leq \sum_i p_i D(\rho_i, \sigma_i). \quad (16)$$

For this, let τ_i be the closest incoherent state to ρ_i : $G_D(\rho_i) = D(\rho_i, \tau_i)$. Then we have

$$\begin{aligned} \sum_i p_i G_D(\rho_i) &= \sum_i p_i D(\rho_i, \tau_i) \geq D\left(\sum_i p_i \rho_i, \sum_j p_j \tau_j\right) \\ &\geq G_D\left(\sum_i p_i \rho_i\right), \end{aligned} \quad (17)$$

which is the desired statement. \square

An example for such a distance is the quantum relative entropy $S(\rho||\sigma) = \text{Tr}[\rho \log_2 \rho] - \text{Tr}[\rho \log_2 \sigma]$, and the corresponding measure is known as the relative entropy of coherence [11]. As mentioned above, this measure also satisfies strong monotonicity G2' [11].

Remarkably, theorem 6 also holds for all distances based on Schatten p -norms

$$D(\rho, \sigma) = \|\rho - \sigma\|_p \quad (18)$$

with the Schatten p -norm $\|M\|_p = (\text{Tr}[(M^\dagger M)^{p/2}])^{1/p}$ and $p \geq 1$. This follows from the fact that Schatten p -norms do not increase under unital operations [85]. This result is surprising since Schatten p -norms are generally problematic in quantum information theory. In particular, the attempt to quantify entanglement via these norms leads to quantities which can increase under local operations for $p > 1$ [86, 87]. Similar problems arise for other types of quantum correlations such as quantum discord [88, 89].

For $p = 1$ and $p = 2$ the corresponding distances are also known as trace distance and Hilbert–Schmidt distance. In the case of the Hilbert–Schmidt distance the coherence measure can be evaluated explicitly [11]:

$$C_{\text{HS}}(\rho) = \|\rho - \Delta[\rho]\|_2, \quad (19)$$

where $\Delta[\rho] = \sum_i \langle i|\rho|i\rangle |i\rangle\langle i|$ denotes complete dephasing in the incoherent basis. However, for general Schatten norms $\Delta[\rho]$ is not the closest incoherent state to ρ [11].

While the distance-based measure of coherence defined in equation (14) does not admit a closed expression for a general distance D , we prove now that the following simple quantity is also a valid measure of coherence:

$$\tilde{G}_D(\rho) = D(\rho, \Delta[\rho]). \quad (20)$$

In particular, \tilde{G}_D satisfies conditions G1 and G2 if the distance D is contractive under unital operations. To see this, notice that the distance $D(\rho, \Delta[\rho])$ is nonnegative and zero if and only if $\rho = \Delta[\rho]$. Hence, G1 is fulfilled. For condition G2, recall that all Kraus operators of a GI operation are diagonal in the incoherent basis, and thus any GI operation commutes with the dephasing operation Δ :

$$\Lambda_{\text{gi}}[\Delta[\rho]] = \Delta[\Lambda_{\text{gi}}[\rho]]. \quad (21)$$

It follows that

$$\begin{aligned} \tilde{G}_D(\Lambda_{\text{gi}}[\rho]) &= D(\Lambda_{\text{gi}}[\rho], \Delta[\Lambda_{\text{gi}}[\rho]]) \\ &= D(\Lambda_{\text{gi}}[\rho], \Lambda_{\text{gi}}[\Delta[\rho]]) \\ &\leq D(\rho, \Delta[\rho]) = \tilde{G}_D(\rho), \end{aligned} \quad (22)$$

where the inequality follows from the fact that Λ_{gi} is unital. Additionally, \tilde{G}_D is convex if the distance D is jointly convex, which is true for all distances based on Schatten p -norms [11]. The above claim can be proven directly via the following calculation:

$$\begin{aligned} \sum_i p_i \tilde{G}_D(\rho_i) &= \sum_i p_i D(\rho_i, \Delta[\rho_i]) \geq D\left(\sum_i p_i \rho_i, \sum_j p_j \Delta[\rho_j]\right) \\ &= D\left(\sum_i p_i \rho_i, \Delta\left[\sum_j p_j \rho_j\right]\right) = \tilde{G}_D\left(\sum_i p_i \rho_i\right). \end{aligned} \quad (23)$$

It is worth noticing that it remains unclear if the measures \tilde{G}_D are also genuine coherence monotones, i.e. if they satisfy G2'⁴.

2.5. State manipulation under GI operations

In any resource theory the free operations play a key role regarding the ordering of states relative to their usefulness. If ρ can be transformed to σ , then ρ cannot be less useful than σ . This is because any task that can be achieved by σ can also be implemented by ρ since the latter can be transformed at no cost to the former but not necessarily the other way around. Thus, as we have seen in the previous section, any measure of genuine coherence should be non-increasing under GI operations (property G2). Therefore, in order to understand the power of the resource theory of genuine coherence it is important to clarify the possibilities and limitations of GI operations for state transformation. In this section we carry out a thorough analysis considering both pure and mixed-state conversions, deterministic and stochastic manipulation and single and many-copy scenarios. As we will see, it turns out that state manipulation under GI operations is rather contrived.

2.5.1. Single-state transformations. In the standard resource theory of coherence the state

$$|+_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle \quad (24)$$

⁴Note that quantifiers based on Schatten p -norms as defined in equation (14) do not give rise to coherence monotones in the sense of Baumgratz *et al* for $p > 1$ [69]. However, this does not exclude the possibility of genuine coherence monotones based on Schatten norms.

represents the golden unit: it can be transformed into any other state on \mathbb{C}^d via incoherent operations [11] and, therefore, it can be considered as the maximally coherent state. We will see now that this is no longer the case for genuine coherence. As we will show in the following theorem, the situation for state manipulation under GI operations is much more drastic.

Theorem 7. *A pure state $|\psi\rangle$ can be deterministically transformed into another pure state $|\phi\rangle$ via GI operations if and only if $|\phi\rangle = U|\psi\rangle$ with a genuinely incoherent unitary U .*

Proof. From theorem 2 it follows that for any GI operation Λ the states ρ and $\Lambda[\rho]$ have the same diagonal elements, i.e. $\langle i|\rho|i\rangle = \langle i|\Lambda[\rho]|i\rangle$. For pure states ρ_ψ and $\rho_\phi = \Lambda[\rho_\psi]$ this means that $|\langle i|\psi\rangle|^2 = |\langle i|\phi\rangle|^2$, and thus the states are the same up to a unitary U which is diagonal in the incoherent basis. \square

This theorem shows that deterministic GI transformations among pure states are trivial. We can only use invertible operations to transform states within the classes of equally coherent states. This resembles the case of multipartite entangled states where almost no pure state can be transformed to any other state outside its respective equivalence class [90].

We have therefore seen that the framework of genuine coherence does not have a golden unit, i.e. there is no unique state from which all other states can be prepared via GI operations. However, it is still possible that for every mixed state ρ there exists some pure state $|\psi\rangle$ from which ρ can potentially be created via GI operations. In the following theorem we will show that this is indeed the case.

Theorem 8. *For every mixed state ρ there exists a pure state ρ_ψ and a GI operation Λ such that*

$$\rho = \Lambda[\rho_\psi]. \tag{25}$$

Proof. Let $\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j|$ be an arbitrary mixed state. Since ρ is PSD, we can assume that $\rho_{ii} \neq 0$ for all i . We will now provide a pure state $|\psi\rangle$ and a GI operation Λ such that $\Lambda(\rho_\psi) = \rho$. As we will prove in the following, the desired state and GI operation are given as

$$|\psi\rangle = \sum_i \sqrt{\rho_{ii}} |i\rangle, \tag{26}$$

$$\Lambda[X] = \rho \odot \rho_{\tilde{\psi}} \odot X \tag{27}$$

with the matrix $\rho_{\tilde{\psi}} = \sum_{ij} (\sqrt{\rho_{ii}\rho_{jj}})^{-1} |i\rangle\langle j|$ ⁵.

For proving this, we first note that the matrix $A = \rho \odot \rho_{\tilde{\psi}}$ is PSD, since it is the Schur product of two PSD matrices [91]. Notice moreover that $A_{ii} = 1 \forall i$; hence, using theorem 2, $\Lambda[X] = A \odot X$ is a GI operation. Since $\rho_{\tilde{\psi}}$ is the Schur inverse of ρ_ψ we finally obtain that

$$\Lambda[\rho_\psi] = \rho \odot \rho_{\tilde{\psi}} \odot \rho_\psi = \rho, \tag{28}$$

which is the desired result. \square

This theorem shows that in the framework of genuine quantum coherence the set of all pure states can be regarded as a resource: all mixed states can be obtained from some pure states via GI operations. Thus, although there is no maximally genuinely coherent state, there is a maximal genuinely coherent set in the terminology of [90]. Moreover, noticing that transformations under GI operations require that the diagonal entries of the density matrices are preserved, the theorem further implies the following corollary, which characterizes all conversions from pure states to mixed states.

⁵Note that $\rho_{\tilde{\psi}}$ is not normalized, i.e. $\text{Tr}[\rho_{\tilde{\psi}}] \neq 1$ in general.

Corollary 9. *A pure state $|\psi\rangle$ can be deterministically transformed by GI operations into the mixed state ρ if and only if $\langle i|\rho_\psi|i\rangle = \langle i|\rho|i\rangle$.*

At this point we also note that mixed states cannot be deterministically transformed to a pure state. Indeed, let a mixed state have spectral decomposition $\rho = \sum_i \lambda_i \rho_{\psi_i}$ with $0 < \lambda_i < 1$. Then, if there existed a GI map Λ such that $\Lambda(\rho) = \rho_\phi$ for some pure state ρ_ϕ , we would need that $\Lambda(\rho_{\psi_i}) = \rho_\phi \forall i$, which is forbidden by theorem 7 (unless $|\psi_i\rangle = U|\phi\rangle$ for all i for some genuinely incoherent unitary, which would imply that ρ is pure).

The impossibility of deterministic GI conversions among pure states calls for the analysis of probabilistic transformations. As explained above this amounts to the use of SGI operations. In the following theorem we evaluate the optimal probability for pure state conversion via SGI operations. In this theorem we will also explicitly use definition 5 for the coherence set R .

Theorem 10. *A probabilistic transformation by GI operations from $|\psi\rangle$ to $|\phi\rangle$ is possible if and only if $R(\phi) \subseteq R(\psi)$. The optimal probability of conversion is*

$$P(\rho_\psi \rightarrow \rho_\phi) = \min_{i \in R(\phi)} \frac{\langle i|\rho_\psi|i\rangle}{\langle i|\rho_\phi|i\rangle}. \quad (29)$$

Proof. Without loss of generality we can write

$$|\psi\rangle = \sum_{i \in R(\psi)} \psi_i |i\rangle, \quad |\phi\rangle = \sum_{j \in R(\phi)} \phi_j |j\rangle. \quad (30)$$

We will first show that the condition $R(\phi) \subseteq R(\psi)$ is necessary for a probabilistic transformation. Let $\Lambda(\cdot) = \sum_{i=1}^n K_i \cdot K_i^\dagger$ be an SGI map such that $\Lambda(\rho_\psi) \propto \rho_\phi$. Then, it must be that $K_i |\psi\rangle \propto |\phi\rangle \forall i$. Hence, since the Kraus operators are diagonal, if for some index k we have $\psi_k = 0$ then $\phi_k = 0$ must be true as well. This proves that $R(\phi) \subseteq R(\psi)$ is a necessary condition for probabilistic transformation.

We will now show that for $R(\phi) \subseteq R(\psi)$ there exists a protocol implementing the transformation with the aforementioned probability. For this, we additionally define the matrix

$$\rho_{\tilde{\psi}} = \sum_{i,j \in R(\psi)} (\psi_i \psi_j)^{-1} |i\rangle \langle j| \quad (31)$$

and the number $c = \max_i (\phi_i / \psi_i)^2$. Using again the Schur product theorem, we have that the matrix $A = c^{-1} \rho_\phi \odot \rho_{\tilde{\psi}}$ is PSD with $A_{ii} \leq 1$. Hence, there exists an SGI map Λ such that $\Lambda(X) = A \odot X$. Moreover, $\Lambda(\rho_\psi) = c^{-1} \rho_\phi$ and $\text{tr} \Lambda(\rho_\psi) = c^{-1}$. Thus, $|\psi\rangle$ can be transformed to $|\phi\rangle$ with probability c^{-1} (notice that $1 < c < \infty$).

In the final step, we show that there cannot exist a protocol with larger probability of success. We do this by contradiction. Suppose that there exists an SGI map Λ with Schur representation given by the PSD matrix A such that $A \odot \rho_\psi = k \rho_\phi$ with $k > c^{-1}$. Let i be the index for which $c = (\phi_i / \psi_i)^2$. Since $A_{ii} \psi_i^2 = k \phi_i^2$, we would have that $A_{ii} = kc > 1$, which is in contradiction with the fact that Λ is a trace non-increasing map. \square

Notice that the state $|+_d\rangle$ is not maximally genuinely coherent even under the stochastic point of view. Indeed, let $|\chi\rangle$ be the state for which $\{\chi_i^2\}$ give rise to $\{1/2, (2(d-1))^{-1}, \dots, (2(d-1))^{-1}\}$. Then, for $d > 2$, $P(\rho_\chi \rightarrow \rho_+) > P(\rho_+ \rightarrow \rho_\chi)$. Given the impossibility to relate pure states by deterministic GI operations, it would be tempting to order the set of pure coherent states by $|\psi\rangle > |\phi\rangle$ if $P(\rho_\psi \rightarrow \rho_\phi) > P(\rho_\phi \rightarrow \rho_\psi)$. Unfortunately, it turns out that such an order would be not well defined. To see that, consider the state $|\psi\rangle$ with squared components $\{1/4, 5/8, 1/8\}$.

For $d = 3$, we have that $P(\rho_\chi \rightarrow \rho_+) > P(\rho_+ \rightarrow \rho_\chi)$ and $P(\rho_+ \rightarrow \rho_\psi) > P(\rho_\psi \rightarrow \rho_+)$ but $P(\rho_\psi \rightarrow \rho_\chi) > P(\rho_\chi \rightarrow \rho_\psi)$. On the other hand, by arguing as in [92], we have that, fixing a target state, the function $f_{\rho_\phi}(\rho_\psi) = P(\rho_\psi \rightarrow \rho_\phi)$ gives a computable genuine coherence monotone for all pure states. Furthermore, $f_{\rho_\phi}(\rho_\psi) = 1$ iff $|\phi\rangle$ and $|\psi\rangle$ are related via diagonal unitaries. Hence, by changing the target state we obtain different monotones, each of them being maximal for a different state in the maximal genuinely coherent set.

Finally, one may wonder whether mixed states can be transformed by GI operations with some non-zero probability into a pure coherent state. A simple example is given by $\rho = p|\psi\rangle\langle\psi| + (1-p)|\phi\rangle\langle\phi|$ where the two pure states $|\psi\rangle$ and $|\phi\rangle$ are respectively supported on the orthogonal subspaces $W = \text{span}\{|i\rangle\}_{i=1}^n$ and $W^\perp = \text{span}\{|i\rangle\}_{i=n+1}^d$. If we denote by P_X the projector onto the subspace X , then the GI map with Kraus operators $K_1 = P_W$ and $K_2 = P_{W^\perp}$ transforms ρ into $|\psi\rangle$ with probability p and into $|\phi\rangle$ with probability $1-p$. The next theorem shows that this is essentially the only possibility.

Theorem 11. *Let W denote a subspace spanned by a subset of two elements of the incoherent basis. Then, a non-pure coherent state ρ can be transformed by GI operations with non-zero probability to some pure coherent state if and only if $P_W \rho P_W$ is a pure coherent state for some choice of W .*

Proof. The ‘if’ part of the theorem is immediate since a map with a unique Kraus operator given by P_W is clearly an SGI operation. To prove the ‘only if’ part we will show that if $P_W \rho P_W$ is not pure for any possible choice of W , then ρ cannot be transformed by GI operations with non-zero probability into a pure coherent state. We will proceed by assuming the opposite and arriving at a contradiction. Suppose that there exists an SGI map Λ with Schur representation given by the PSD matrix A such that $A \odot \rho = k\rho_\psi$ with $0 < k < 1$ and $\rho_\psi = \sum_{i,j} \psi_i \psi_j |i\rangle\langle j|$. By our premise, the projection of ρ on every 2-dimensional subspace spanned by two elements of the incoherent basis must be positive definite (not PSD). This, together with the fact that ρ is coherent implies that there must exist i, j such that $\rho_{ii}, \rho_{jj}, \rho_{ij} \neq 0$ and $\rho_{ii}\rho_{jj} > |\rho_{ij}|^2$. The existence of the SGI map imposes the following three equations

$$\begin{aligned} A_{ii}\rho_{ii} &= k\psi_i^2, \\ A_{jj}\rho_{jj} &= k\psi_j^2, \\ A_{ij}\rho_{ij} &= k\psi_i\psi_j, \end{aligned}$$

which altogether yield

$$|A_{ij}|^2 = \frac{A_{ii}A_{jj}\rho_{ii}\rho_{jj}}{|\rho_{ij}|^2} > A_{ii}A_{jj}.$$

However, this implies that A cannot be PSD and, hence, a contradiction. \square

Hence, most mixed states cannot be stochastically transformed to any pure state. Thus, if, as discussed above, we regard pure states as the most resourceful states over mixed states, it turns out that most less resourceful states cannot be transformed, even with small probability, to a resource state in the one-copy regime.

2.5.2. Multiple-state and multiple-copy transformations. So far we have just discussed possible transformations acting on a single copy of a state. However, in quantum information theory it is standard to find that multiple-state transformations broaden the possibilities for resource manipulation. An important example of this are activation phenomena. This means

that the transformation $\rho \otimes \sigma \rightarrow \rho' \otimes \text{junk}$ (or, more generally, $\rho \otimes \sigma \rightarrow \tau$ with $\text{tr}_{\text{junk}} \tau = \rho'$) is possible even though it is impossible to implement the conversion $\rho \rightarrow \rho'$. In this case the state σ is called an activator. In the particular case when the activator can be returned, i.e. $\rho \otimes \sigma \rightarrow \rho' \otimes \sigma$, the process is known as catalysis.

Another example, and probably the most paradigmatic one, is distillation. In these protocols one aims at transforming many copies of a less useful state into less copies of a maximally useful state in the asymptotic limit of infinitely many available copies. For instance, in entanglement theory this target state that acts as a golden standard to measure the usefulness of the resource is the maximally entangled two-qubit state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.

In general, we say that a state σ can be distilled from the state ρ at rate $0 < R \leq 1$ if $\rho^{\otimes n} \rightarrow \tau$ and $\text{tr}_{\text{junk}} \tau$ is ε -close to $\sigma^{\otimes nR}$ and $\varepsilon \rightarrow 0$ as $n \rightarrow \infty$. As a measure of closeness we will use the trace norm; that is, it must hold that $\|\text{tr}_{\text{junk}} \tau - \sigma^{\otimes nR}\| \leq \varepsilon$ where $\|M\| = \text{tr} \sqrt{M^\dagger M}$. The optimal rate at which distillation is possible, i.e. the supremum of R over all protocols fulfilling the aforementioned conditions, is a very relevant figure of merit known as distillable resource and plays a key role for the quantification of usefulness in resource theories. The reversed protocol, which is known as dilution, is also an interesting object of study. In this case one seeks for the optimal rate at which less copies of maximally useful state can be converted into more copies of a less useful state. This leads to another figure of merit: the resource-cost. In more detail, the cost of ρ is the infimum of the rate R over all protocols with $0 < R \leq 1$ such that $\sigma^{\otimes nR} \otimes \text{junk}$ (where σ is a golden unit maximally resourceful state) transforms ε -close to $\rho^{\otimes n}$ and $\varepsilon \rightarrow 0$ as $n \rightarrow \infty$. The distillable entanglement and entanglement-cost have been widely studied in entanglement theory [3] and allow to establish the phenomenon of irreversibility. More recently, the distillable coherence and coherence-cost have been characterized and irreversibility has also been identified in this setting [12].

In order to discuss multiple-state and multiple-copy manipulation under GI operations, it should be made clear what the set of allowed maps is in this setting. If we are allowed to act jointly on n different states each of them acting on the Hilbert space $H \simeq \mathbb{C}^d$, we define the incoherent basis in the total Hilbert space $H^{\otimes n}$ as $\{|i_1 i_2 \dots i_n\rangle\}$ ($i_j = 1, \dots, d \forall j$), where $\{|i_j\rangle\}$ is the incoherent basis in each Hilbert space [46, 60]. This can be further justified by the no superactivation postulate (see [73]). Thus, joint GI operations should preserve incoherent states in this basis and they will be characterized by having Kraus operators diagonal in the joint incoherent basis. By the same reasons as in section 2.2, these GI maps will also admit a Schur representation in the joint incoherent basis.

We are now in the position to state our results on multiple-state and multiple-copy manipulation under joint GI operations. It turns out that these protocols are out of reach: activation and any non-trivial form of distillation and dilution are impossible. This claim is a consequence of the following lemma.

Lemma 12. *For every two states $\rho, \sigma \in \mathcal{L}(\mathbb{C}^d)$ and every GI map Λ acting on $\mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d)$ such that $\Lambda(\rho \otimes \sigma) = \tau$, there exists another GI map $\tilde{\Lambda}$ acting on $\mathcal{L}(\mathbb{C}^d)$ such that $\tilde{\Lambda}(\rho) = \tau_1 := \text{tr}_2(\tau)$.*

Proof. By assumption together with theorem 2, there exists a PSD matrix A with diagonal entries equal to 1,

$$A = \sum_{ijkl} A_{ik,jl} |ik\rangle\langle jl| \quad (A_{ik,ik} = 1 \forall i, k), \quad (32)$$

which induces the GI operation Λ such that

$$\tau = \Lambda(\rho \otimes \sigma) = A \odot (\rho \otimes \sigma) = \sum_{ijkl} \rho_{ij} \sigma_{kl} A_{ik,jl} |ik\rangle\langle jl|. \quad (33)$$

The state τ_1 is obtained by taking the partial trace over the second subsystem:

$$\tau_1 = \text{tr}_2(\tau) = \sum_{ij} \left(\sum_k \sigma_{kk} A_{ik,jk} \right) \rho_{ij} |i\rangle\langle j| = \tilde{A} \odot \rho \quad (34)$$

with the matrix \tilde{A} with entries $\tilde{A}_{ij} = \sum_k \sigma_{kk} A_{ik,jk}$. The proof is complete if we can show that the operator \tilde{A} is PSD and that $\tilde{A}_{ii} = 1$ holds for all i , since in this case by theorem 2 there must exist a GI operation $\tilde{\Lambda}$ such that $\tilde{\Lambda}(\rho) = \tilde{A} \odot \rho$. It is straightforward to verify that $\tilde{A}_{ii} = 1$ holds true for all i . To see that \tilde{A} is PSD, notice that $\tilde{A} = \text{tr}_2(X \odot A)$, where the operator X can be written as

$$X = \sum_{ijkl} \sigma_{kl} |ik\rangle\langle jl| = |v\rangle\langle v| \otimes \sigma \quad (35)$$

with the (unnormalized) vector $|v\rangle = \sum_i |i\rangle$. Thus, X is PSD (and so is A by assumption). Hence, by the Schur product theorem, \tilde{A} is the partial trace of a PSD matrix and it must be PSD too. \square

The above lemma shows that there cannot be any activation phenomena in the resource theory of genuine coherence and it will be very useful in our study of coherence distillation and dilution with GI operations. In order to analyze the possibility of distillation one first needs to discuss what the target state is going to be. However, this is not at all clear under GI operations since, as we pointed out in the previous section, there is no unique state which would allow to create all other states via GI operations. In the following, we will show that in general it is not possible to distill the state σ from ρ via GI operations if σ has more coherence than ρ . Here, we measure the coherence by the relative entropy of coherence [11]

$$C_r(\rho) = \min_{\sigma \in \mathcal{I}} S(\rho || \sigma) \quad (36)$$

with the quantum relative entropy $S(\rho || \sigma) = \text{tr}(\rho \log_2 \rho) - \text{tr}(\rho \log_2 \sigma)$ and \mathcal{I} denotes the set of all incoherent states. The relative entropy of coherence is known to be equal to the distillable coherence [12], and C_r is also a faithful genuine coherence monotone (see section 2.4). We are now in the position to prove the following theorem.

Theorem 13. *Given two states ρ and σ with*

$$C_r(\rho) < C_r(\sigma), \quad (37)$$

it is not possible to distill σ from ρ at any rate $R > 0$ via GI operations.

Proof. We will prove the statement by contradiction, assuming that distillation is possible for some state ρ with $C_r(\rho) < C_r(\sigma)$. In particular, this would imply that for n large enough it is possible to approximate one copy of the state σ . To be more precise, for any $\varepsilon > 0$ there exists an integer n and a GI operation Λ such that

$$\|\text{tr}_{n-1}(\Lambda[\rho^{\otimes n}]) - \sigma\| \leq \varepsilon, \quad (38)$$

where the partial trace is taken over some subset of $n - 1$ copies.

In the next step we use lemma 12 to note that the map $\text{tr}_{n-1}(\Lambda[\rho^{\otimes n}])$ can always be written as a GI operation $\tilde{\Lambda}$ acting on just one copy of ρ :

$$\tilde{\Lambda}[\rho] = \text{tr}_{n-1}(\Lambda[\rho^{\otimes n}]). \quad (39)$$

Combining the aforementioned arguments, we conclude that for any $\varepsilon > 0$ there exists a GI operation $\tilde{\Lambda}$ such that

$$\|\tilde{\Lambda}[\rho] - \sigma\| \leq \varepsilon. \quad (40)$$

In the final step we will use the asymptotic continuity of the relative entropy of coherence (see lemma 12 in [12]). It implies that

$$\left| C_r(\tilde{\Lambda}[\rho]) - C_r(\sigma) \right| \leq \varepsilon \log_2 d + 2h\left(\frac{\varepsilon}{2}\right) \quad (41)$$

with the binary entropy $h(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ and d the fixed dimension of the Hilbert space. Using the fact that the bound on the right-hand side of this inequality is continuous for $\varepsilon \in (0, 1)$ and that it goes to zero as $\varepsilon \rightarrow 0$, we can say that for any $\delta > 0$ there exists some GI operation $\tilde{\Lambda}$ such that

$$\left| C_r(\tilde{\Lambda}[\rho]) - C_r(\sigma) \right| < \delta. \quad (42)$$

On the other hand, the assumption $C_r(\rho) < C_r(\sigma)$ implies that there exists some $\delta > 0$ such that

$$C_r(\sigma) - C_r(\rho) \geq \delta. \quad (43)$$

Recalling that the relative entropy of coherence is a genuine coherence monotone, i.e. $C_r(\tilde{\Lambda}[\rho]) \leq C_r(\rho)$, we arrive at the following result:

$$C_r(\sigma) - C_r(\tilde{\Lambda}[\rho]) \geq \delta \quad (44)$$

for some $\delta > 0$ and any GI operation $\tilde{\Lambda}$. This is a contradiction to equation (42), and the proof of the theorem is complete. \square

From the above theorem it follows that it is not possible to distill the state $|+\rangle$ from any non-equivalent single-qubit state ρ , since $C_r(\rho) < 1 = C_r(|+\rangle)$.

In the final part of this section we address the impossibility of dilution. Interestingly, it turns out that diluting less copies of a state into more copies of another is generically impossible independently of which state is picked as a golden unit.

Theorem 14. *Given any two coherent states ρ and σ of the same dimensionality it is not possible to dilute σ to ρ at any rate $R < 1$ via GI operations.*

Proof. If dilution at rate $R < 1$ (i.e. leaving aside one-copy deterministic transformations when possible) was possible, this would require that $\forall \varepsilon > 0$ there existed integers $m < n$ such that

$$\|\Lambda(\sigma^{\otimes m} \otimes \text{junk}) - \rho^{\otimes n}\| = \|\tau - \rho^{\otimes n}\| \leq \varepsilon. \quad (45)$$

Notice that the presence of some junk incoherent part is indispensable as GI operations cannot increase the dimensionality. Since the trace distance cannot increase by quantum operations, by tracing out the m particles in the system the above equation requires in particular that

$$\|\text{tr}_{\text{system}} \tau - \rho^{\otimes(n-m)}\| \leq \varepsilon. \quad (46)$$

Now, lemma 12 implies that $\text{tr}_{\text{system}} \tau = \tilde{\Lambda}(\text{junk})$ for some GI map $\tilde{\Lambda}$. Hence, it must hold that

$$\|\tilde{\Lambda}(\text{junk}) - \rho^{\otimes(n-m)}\| \leq \varepsilon. \quad (47)$$

However, the junk part is incoherent and, therefore, so must be $\tilde{\Lambda}(\text{junk})$. Any coherent state is bounded away from the set of incoherent states and, thus, the above inequality cannot hold $\forall \varepsilon > 0$ for any coherent state ρ . \square

Thus, dilution with rate $R < 1$ from a more useful qudit state into a less useful qudit state is impossible by GI operations independently of the measure of coherence used.

It is known that quantum resource theories where the free operations are maximal (resource-non-generating maps in the asymptotic limit) are asymptotically reversible and the optimal rate is given by the regularized relative entropy [93]. GI operations are more contrived and represent the opposite extreme: non-trivial forms of distillation and dilution are impossible.

2.6. Relation to entanglement theory of maximally correlated states

Recently, several authors conjectured [12, 42, 60] that the resource theory of coherence as introduced by Baumgratz *et al* is equivalent to the resource theory of entanglement, if the latter is restricted to the set of maximally correlated states. Given a state $\rho = \sum_{i,j} \rho_{ij} |i\rangle\langle j|$, we can always associate with it a bipartite maximally correlated state $\rho_{\text{mc}} = \sum_{i,j} \rho_{ij} |ii\rangle\langle jj|$. This connection has led to several important results, e.g. the distillable coherence, the coherence cost, and the coherence of assistance of ρ are all equal to the corresponding entanglement equivalent of ρ_{mc} [12, 42]. Moreover, it has been conjectured [42] that for any two states ρ and σ related via some incoherent operation Λ_i such that $\sigma = \Lambda_i[\rho]$, the maximally correlated states ρ_{mc} and σ_{mc} are related via some LOCC operation: $\sigma_{\text{mc}} = \Lambda_{\text{LOCC}}[\rho_{\text{mc}}]$. As we will see in the following theorem, this conjecture is true if genuinely incoherent operations are considered.

Theorem 15. *Given two states ρ and σ related via a genuinely incoherent operation such that $\sigma = \Lambda_{\text{gi}}[\rho]$, there always exists a LOCC operation relating the corresponding maximally correlated states: $\sigma_{\text{mc}} = \Lambda_{\text{LOCC}}[\rho_{\text{mc}}]$.*

Proof. We will prove this statement by showing that Λ_{LOCC} can be chosen as Λ_{gi}^A , i.e. the genuinely incoherent operation acting on Alice’s subsystem. For this, we first write σ explicitly:

$$\begin{aligned} \sigma &= \Lambda_{\text{gi}}[\rho] = \sum_i K_i \rho K_i^\dagger \\ &= \sum_{i,m,l} \rho_{ml} K_i |m\rangle\langle l| K_i^\dagger = \sum_{i,m,l} \rho_{ml} a_{im} a_{il}^* |m\rangle\langle l| \end{aligned} \tag{48}$$

with $\rho = \sum_{m,l} \rho_{ml} |m\rangle\langle l|$ and $K_i |m\rangle = a_{im} |m\rangle$.

We will now apply the same operation Λ_{gi} on Alice’s subsystem of the maximally correlated state $\rho_{\text{mc}} = \sum_{m,l} \rho_{ml} |mm\rangle\langle ll|$:

$$\begin{aligned} \Lambda_{\text{gi}}^A[\rho_{\text{mc}}] &= \sum_{i,m,l} \rho_{ml} K_i |m\rangle\langle l| K_i^\dagger \otimes |m\rangle\langle l| \\ &= \sum_{i,m,l} \rho_{ml} a_{im} a_{il}^* |mm\rangle\langle ll| = \sigma_{\text{mc}}. \end{aligned} \tag{49}$$

This completes the proof of the theorem. \square

This theorem lifts the relation between coherence and entanglement to a new level. For an arbitrary coherence-based task involving genuinely incoherent operations, we can immediately make statements about the corresponding entanglement-based task involving LOCC operations.

3. Fully incoherent operations

3.1. General concept

As discussed in section 2.1, one of the reasons to consider GI operations was to rule out any form of hidden coherence in the free operations of a resource theory of coherence. However, we have seen in section 2.5 that state manipulation under GI operations might be too limited. This leads to think whether one could consider a larger set of allowed operations still having the property that every Kraus representation is incoherent but that could allow for a richer structure for state manipulation.

We recall that the incoherent operations considered in the resource theory of coherence introduced by Baumgratz *et al* in [11] are given by Kraus operators $\{K_i\}$ such that for every incoherent state ρ , $K_i\rho K_i^\dagger$ is (up to normalization) an incoherent state as well $\forall i$. It will be relevant in the following to notice that such $\{K_i\}$ are characterized by having at most one non-zero entry in every column [28]. As we used already in section 2.1, the fact that the Kraus operators of a different Kraus representation of some incoherent operation might not be incoherent can be easily seen using theorem 1. On the contrary, with this theorem it is straightforward to check that GI maps are incoherent (in fact, even diagonal) in every Kraus representation. As discussed above, it comes as a natural question whether GI maps constitute the most general class of operations having this property. Interestingly, as we will show in the following, the answer is no: there exist operations which are not GI but still incoherent in every Kraus representation. A simple example for such an operation is the erasing map, which puts every input state onto the state $|0\rangle\langle 0|$. This operation is incoherent in every Kraus representation because it must hold that $K_i\rho K_i^\dagger \propto |0\rangle\langle 0|$ for every Kraus operator. On the other hand, the operation is clearly not GI because any incoherent state that is not $|0\rangle\langle 0|$ does not remain invariant under this map. We will call this class of operations which are incoherent in every Kraus representation *fully incoherent (FI)*.

One might wonder then what are the properties of the class of FI maps and which differences it has with the class of GI maps. In particular, we want to compare these sets of operations in the task of state transformation to see whether FI induces a richer structure. For this, we will first provide a full characterization of FI operations in the following theorem.

Theorem 16. *A quantum operation is FI if and only if all Kraus operators are incoherent and have the same form.*

Before we prove the theorem some remarks are in place. The requirement that all Kraus operators have the same form means that their nonzero entries are all at the same position (i.e. whenever there is a non-zero entry in a given column, it must occur at the same row for every Kraus operator). As an example, according to the theorem any single-qubit quantum operation defined by the Kraus operators

$$K_1 = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \quad (50)$$

is fully incoherent, since both Kraus operators are incoherent and have the same form. The completeness condition $\sum_i K_i^\dagger K_i = \mathbb{1}$ puts constraints on the complex parameters: $|a|^2 + |c|^2 = |b|^2 + |d|^2 = 1$ and $a^*b + c^*d = 0$. Note that—according to the theorem—this map is FI, but it is not GI since the Kraus operators are not diagonal. Indeed, this is exactly the erasing map which maps every state onto $|0\rangle\langle 0|$. We will now provide the proof of theorem 16.

Proof. That maps with this property are FI is immediate. If the Kraus operators in one representation are all incoherent and have a particular given form, then, by theorem 1, so does every Kraus representation since equation (3) preserves this structure. Hence, every Kraus representation is incoherent.

Thus, to complete the proof we only need to see that any map which has one (incoherent) Kraus representation in which not all operators are of the same form cannot be FI, i.e. it must then admit another Kraus representation which is not incoherent. For such, we can take without loss of generality that

$$\text{col}_i(K_1) = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{col}_i(K_2) = \begin{pmatrix} 0 \\ * \\ \vdots \\ 0 \end{pmatrix}, \quad (51)$$

where $\text{col}_i(A)$ denotes the i th column of the matrix A and $*$ an arbitrary non-zero number. Moreover, we define two unitary matrices V and U :

$$V = \begin{pmatrix} U & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (52)$$

Since V is unitary the set of Kraus operators $\{L_i\}$ constructed using equation (3) is another Kraus representation of the map given by $\{K_i\}$. However, we find that

$$\text{col}_i(L_i) = \begin{pmatrix} * \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (53)$$

and the representation given by $\{L_i\}$ is therefore not incoherent. □

From theorem 16 it is clear that GI maps are contained in the class of FI maps since every Kraus operator in every Kraus representation is diagonal, hence fulfilling the condition of theorem 16. As also noted above theorem 16, there exist operations which are FI but not GI, and the erasing map $\Lambda[\rho] = |0\rangle\langle 0|$ is one example. Another instance of FI maps, taking for example maps on $\mathcal{L}(\mathbb{C}^3)$, are those for which the Kraus operators are given by

$$K_i = \begin{pmatrix} a_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_i & c_i \end{pmatrix}. \quad (54)$$

A property fulfilled by FI maps is that these operations allow to prepare any pure incoherent state from an arbitrary input state. For that one would use the corresponding erasure map with the property $\Lambda[\rho] = |i\rangle\langle i|$ for every state ρ . It can be easily seen that any such map is always FI for any choice of incoherent state $|i\rangle\langle i|$. This means that, contrary to the case of GI operations, in this setting pure incoherent states are indeed free states, i.e. they can be prepared with the free operations.

Another feature of FI maps is that any incoherent unitary transformation can be implemented. Thus, elements of the incoherent basis can be permuted and the coherence set is no longer meaningful. Hence, the coherence rank r takes the relevant role instead, similarly to state manipulation under incoherent operations. Thus, in this case coherence is regarded as a speakable resource.

On the other hand, a striking feature of FI maps is that they constitute a non-convex set. More precisely, given two FI operations Λ_1 and Λ_2 , their convex combination

$$\Lambda[\rho] = p\Lambda_1[\rho] + (1-p)\Lambda_2[\rho] \quad (55)$$

is not always fully incoherent. This can be demonstrated with the following single-qubit operations:

$$\Lambda_1[\rho] = \sigma_x \rho \sigma_x, \quad (56)$$

$$\Lambda_2[\rho] = \sigma_z \rho \sigma_z \quad (57)$$

with the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (58)$$

While both operations Λ_1 and Λ_2 are fully incoherent, their convex combination Λ in equation (55) is not fully incoherent for $0 < p < 1$. This can be seen by noting that the Kraus operators of Λ are given by $K_1 = \sqrt{p}\sigma_x$ and $K_2 = \sqrt{1-p}\sigma_z$. Since these Kraus operators do not have the same form for $0 < p < 1$, by theorem 16 the operation cannot be fully incoherent.

This non-convexity is, thus, a consequence of the fact that FI maps are characterized by a property of the set of implemented Kraus operators and not of each individual operator as it is the case for incoherent or GI operations. In practice, this means that if one considers state manipulation under FI maps, it turns out that two particular operations might be free but not to implement each of them with a given probability. In particular, this implies that, although pure incoherent states can be prepared with the free operations as pointed out above, this result does not need to extend to mixed incoherent states despite the fact that they constitute free states as well. This is because it is not allowed to mix different FI fixed-output maps. Actually, it can be checked that it is not the case that every state can be transformed to any mixed incoherent state by some FI map. The interested reader is referred to appendix C for a particular example. This construction relies on an observation on the structure of FI maps used in theorem 17 below.

3.2. Permutations as basic FI operations

As mentioned above, any genuinely incoherent operation is also fully incoherent. Here, we will consider another important class of FI operations, which we call *permutations*. A permutation P_{ij} with $i \neq j$ is a unitary which interchanges the states $|i\rangle$ and $|j\rangle$, and preserves all states $|k\rangle$ for $k \neq i, j$:

$$P_{ij}|i\rangle = |j\rangle, \quad (59)$$

$$P_{ij}|j\rangle = |i\rangle, \quad (60)$$

$$P_{ij}|k\rangle = |k\rangle \text{ for all } k \neq i, j. \quad (61)$$

The above definition involves the permutation of only two states $|i\rangle$ and $|j\rangle$. In the following, we will also consider more general permutations with more than two elements. We will denote an arbitrary general permutation by P . Any such permutation can be decomposed as a product of permutations of only two states. Notice that any such P corresponds to an FI unitary transformation.

3.3. State manipulation under FI operations

3.3.1. Pure state deterministic transformations. Given the limitations of GI operations, in this section we study the potential for deterministic state manipulation if the allowed set of operations is given by FI maps. Interestingly, we will see in the following that—contrary to the case of GI maps—transformations among pure states are possible. However, these are rather limited as shown in the following theorem.

Theorem 17. *A deterministic FI transformation from $|\psi\rangle$ to $|\phi\rangle$ is possible only if $r(\phi) \leq r(\psi)$. Moreover, for $r(\phi) = r(\psi)$ a transformation is possible if and only if $|\phi\rangle = U|\psi\rangle$ with a fully incoherent unitary U .*

Proof. The first part of the theorem is true due to the more general result stating that the coherence rank cannot increase under incoherent operations as we already mentioned in section 2.3. Since FI operations are a subclass of general incoherent operations, the coherence rank also cannot increase under FI operations.

We will now prove the second part of the theorem, assuming that $|\psi\rangle$ and $|\phi\rangle$ are two states with the same coherence rank, and that

$$\rho_\phi = \Lambda[\rho_\psi] \quad (62)$$

with some FI operation Λ . Since both states have the same coherence rank, the FI operation Λ must contain at least one Kraus operator K which has one nonzero element in each row and column. Moreover, all remaining Kraus operators must have the same shape as K , i.e. their nonzero elements must be at the same positions as the nonzero elements of K . It is now crucial to note that any such map must be a composition of a GI operation Λ_{gi} and a permutation P :

$$\Lambda[\rho] = P\Lambda_{\text{gi}}[\rho]P^\dagger. \quad (63)$$

Now, since we assume that $\Lambda[\rho_\psi]$ is pure, it must be that $\Lambda_{\text{gi}}[\rho_\psi]$ is also pure, since the permutation P is a unitary operation. By theorem 7, the fact that $\Lambda_{\text{gi}}[\rho_\psi]$ is pure implies that $\Lambda_{\text{gi}}[\rho_\psi] = U_{\text{gi}}\rho_\psi U_{\text{gi}}^\dagger$ with some diagonal unitary U_{gi} . Combining these results we arrive at the following expression:

$$\rho_\phi = \Lambda[\rho_\psi] = P\Lambda_{\text{gi}}[\rho_\psi]P^\dagger = PU_{\text{gi}}\rho_\psi U_{\text{gi}}^\dagger P^\dagger. \quad (64)$$

The proof of the theorem is complete by noting that PU_{gi} is a fully incoherent unitary. \square

If we apply now the above theorem to study single-qubit FI operations from an arbitrary single-qubit state $|\psi\rangle$, we see that the only possible output for transformations to a pure state is either an incoherent state or any state which is equivalent to $|\psi\rangle$ under FI unitaries. This shows that FI operations are strictly less powerful than general incoherent operations in their ability to convert pure quantum states into each other. In particular, general incoherent operations can convert the state $|+2\rangle$ into any other single-qubit state [11]. This is not possible via FI operations, as follows from the above discussion by taking $|\psi\rangle = |+2\rangle$.

Having seen that FI operations are less powerful than general incoherent operations, we will now show that FI operations are still more powerful than GI operations. For this, we can use a fixed-output map to see that by FI operations it is possible to transform the state $|+2\rangle$ into the state $|0\rangle$. By theorem 7 this process is not possible with GI operations. However, this is a transformation to a non-resource state. One might wonder whether FI operations allow for nontrivial transformations to a pure coherent state. In the following, we provide such an example. Consider a FI map with Kraus operators given by

$$K_1 = \begin{pmatrix} a_1 & 0 & c_1 \\ 0 & b_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} a_2 & 0 & c_2 \\ 0 & b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (65)$$

The normalization condition $\sum_i K_i^\dagger K_i = \mathbb{1}$ imposes that

$$\begin{aligned} a_1 c_1^* + a_2 c_2^* &= 0, \\ |x_1|^2 + |x_2|^2 &= 1, \quad x = a, b, c. \end{aligned} \quad (66)$$

In order for the (normalized) state

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \in \mathbb{R}^3 \quad (67)$$

to be convertible with this map into another pure state, the only requirement is that $K_1|\psi\rangle = kK_2|\psi\rangle$ for some constant $k \in \mathbb{C}$. As can be verified by inspection, this condition is equivalent to

$$\psi_3 = \frac{a_2 b_1 - a_1 b_2}{b_2 c_1 - b_1 c_2} \psi_1. \quad (68)$$

Now, a proper choice of the free parameters can be made so that the conditions (66) and (68) are met. For instance, we can take $b_1 = b_2 = 1/\sqrt{2}$, $a_1 = c_2 = \sqrt{3}/2$ and $a_2 = -c_1 = 1/2$, which leads to

$$\psi_3 = \frac{(\sqrt{3} - 1)^2}{2} \psi_1. \quad (69)$$

With a real choice of ψ_1 small enough the above equation can be fulfilled with a properly normalized state. Indeed, if we further choose $\psi_1 = 1/2$, condition (69) implies that $\psi_3 = 1 - \sqrt{3}/2$. The remaining parameter ψ_2 is restricted by the normalization of the state $|\psi\rangle$, and can be chosen as $\psi_2 = (1 - \psi_1^2 - \psi_3^2)^{1/2} = (\sqrt{3} - 1)^{1/2}$. The pure final state $\Lambda[\rho_\psi] = \rho_\phi$ is then given by

$$|\phi\rangle = \frac{\sqrt{6} - \sqrt{2}}{2} |0\rangle + (\sqrt{3} - 1)^{1/2} |1\rangle. \quad (70)$$

This example shows that FI operations allow for nontrivial transformations between pure states which are not possible with genuinely incoherent operations. Still, as we will see in the next theorem with the particular example of the state $|+_3\rangle$, it seems that FI transformations among pure coherent states are nevertheless very constrained.

Theorem 18. *Via deterministic FI operations, the state $|+_3\rangle$ can be transformed into one of the following pure states:*

$$\Lambda[|+_3\rangle\langle+_3|] = \begin{cases} |0\rangle\langle 0| \\ |\psi\rangle\langle\psi| \\ |+_3\rangle\langle+_3| \end{cases} \quad (71)$$

with $|\psi\rangle = \sqrt{2/3} |0\rangle + \sqrt{1/3} |1\rangle$. Moreover, up to FI unitaries, this is the full set of pure states which can be obtained from $|+_3\rangle$ via deterministic FI operations.

Proof. We will prove the statement by studying states with a fixed coherence rank that can be obtained from the state $|+_3\rangle$ via FI operations. We start with states of coherence rank 1, i.e. incoherent states $|i\rangle$. The state $|0\rangle$ can be obtained from the state $|+_3\rangle$ via the erasing operation which maps every state onto $|0\rangle$. As mentioned earlier, this operation is fully incoherent. Any incoherent state $|i\rangle$ can be obtained from $|+_3\rangle$ by first performing the erasing operation, and then applying a fully incoherent unitary which transforms $|0\rangle$ to $|i\rangle$.

Before we study states of coherence rank 2, we first consider the case of coherence rank 3 for simplicity. Since the initial state $|+_3\rangle$ has coherence rank 3, we can apply theorem 17, stating that FI transformation between states with the same coherence rank must be necessarily unitary, i.e. the final state must be $|+_3\rangle$ or FI unitary equivalent of it.

We now come to the most difficult part of the proof, where we will show that for final states of coherence rank 2, the only possible pure state is given by

$$|\psi\rangle = \sqrt{2/3}|0\rangle + \sqrt{1/3}|1\rangle \quad (72)$$

and FI unitary equivalents of it. In particular, we will show that no other pure state of coherence rank 2 can be obtained from $|+_3\rangle$ via FI operations.

According to theorem 16, the only FI maps that are able to produce states of coherence rank 2 must have Kraus operators of the form

$$K_i = P \begin{pmatrix} a_i & 0 & c_i \\ 0 & b_i & 0 \\ 0 & 0 & 0 \end{pmatrix} P', \quad (73)$$

where P and P' are arbitrary (but fixed $\forall i$) permutations. Since permutations are FI unitaries and the state $|+_3\rangle$ is invariant under any permutation it is thus sufficient to study Kraus operators of the form

$$K_i = \begin{pmatrix} a_i & 0 & c_i \\ 0 & b_i & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (74)$$

The completeness condition $\sum_i K_i^\dagger K_i = \mathbb{1}$ imposes that

$$\begin{aligned} \sum_i a_i c_i^* &= 0, \\ \sum_i |z_i|^2 &= 1, \quad z = a, b, c. \end{aligned} \quad (75)$$

Each outcome of such FI maps leads to the following unnormalized state

$$K_i |+_3\rangle \propto \begin{pmatrix} a_i + c_i \\ b_i \\ 0 \end{pmatrix}. \quad (76)$$

Assume now that a deterministic transformation to the state $|\phi\rangle$ is possible. This means that $K_i |+_3\rangle \propto |\phi\rangle \forall i$. In particular, this implies that $|\phi\rangle \propto \sum_i K_i |+_3\rangle$ and, thus

$$|\phi\rangle \propto \begin{pmatrix} \sum_i x_i \\ \sum_i b_i \\ 0 \end{pmatrix} \quad (x_i = a_i + c_i), \quad (77)$$

with the above normalization conditions imposing that $\sum_i |x_i|^2 = 2$ and $\sum_i |b_i|^2 = 1$. Moreover, the fact that $K_i |+_3\rangle \propto K_j |+_3\rangle \forall i \neq j$ demands that $x_i = k_i x$ and $b_i = k_i b$ for some complex numbers $\{k_i\}$, x and b . Thus,

$$|\phi\rangle \propto \sum_i k_i \begin{pmatrix} x \\ b \\ 0 \end{pmatrix} \propto \begin{pmatrix} x \\ b \\ 0 \end{pmatrix}. \tag{78}$$

Furthermore, the normalization conditions then lead to $|x|^2 = 2|b|^2$. This means that it must hold that

$$|\phi\rangle \propto \begin{pmatrix} \sqrt{2} e^{i\alpha} \\ 1 \\ 0 \end{pmatrix}. \tag{79}$$

After normalization this state is FI unitary equivalent to the state $|\psi\rangle$ given in equation (72). These arguments prove that apart from $|\psi\rangle$ and FI unitary equivalents no other pure state with coherence rank 2 can be obtained from $|+_3\rangle$ via FI operations.

To see that the transformation to $|\psi\rangle$ is indeed possible, we choose an FI operation given by the following two Kraus operators:

$$K_1 = \begin{pmatrix} i/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{80}$$

$$K_2 = \begin{pmatrix} 1/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This map acting on $|+_3\rangle$ leads to the state $\sqrt{2/3} e^{i\pi/4} |0\rangle + \sqrt{1/3} |1\rangle$, which can be then transformed to the aforementioned state with an FI unitary. \square

Theorem 18 provides severe constraints on the pure state conversion via FI operations. In particular, it shows that via FI operations the state $|+_3\rangle$ can only be converted into three different pure states, and their FI unitary equivalents. Being FI operations a particular class of general incoherent operations, it seems natural to compare their power. In fact, the latter class of transformations shows a much richer structure. The question whether two pure states can be converted into each other via incoherent operations has been recently addressed in [12, 28]. The answer is closely related to the corresponding problem in entanglement theory which was solved by Nielsen in [94] relying on the theory of majorization. For two density matrices ρ and σ with corresponding eigenvalues $\{p_i\}_{i=1}^d$ and $\{q_j\}_{j=1}^d$, the majorization relation $\rho \succ \sigma$ means that the inequality

$$\sum_{i=1}^t p_i \geq \sum_{j=1}^t q_j \tag{81}$$

holds true for all $t \leq d$.

It has been shown in [12] that given two pure states $|\psi\rangle$ and $|\phi\rangle$ with the same coherence rank, there exists an incoherent operation transforming $|\psi\rangle$ into $|\phi\rangle$ if and only if $\Delta[\rho_\phi] \succ \Delta[\rho_\psi]$. Here, $\Delta[\rho] = \sum_i \langle i | \rho | i \rangle |i\rangle \langle i|$ denotes full dephasing in the incoherent basis. The earlier reference [28] states this same result without the assumption that both states have

the same coherence rank. However, as pointed out recently by Winter and Yang [12] and by Chitambar and Gour [73], the part of the proof showing the necessity of the majorization condition for a transformation to be possible has serious flaws. Notwithstanding, its sufficiency is clearly established. With this we can compare the power of incoherent and FI transformations. Theorem 17 already shows a limitation of the latter. While for any pair of incoherent-unitary inequivalent states of the same coherence rank, the transformation $|\psi\rangle \rightarrow |\phi\rangle$ is possible by incoherent operations whenever $\Delta[\rho_\psi] \succ \Delta[\rho_\phi]$ is fulfilled, this is not possible with FI operations. Moreover, $\Delta[\rho_\psi] \succ \Delta[\rho_{+3}]$ for any qutrit-state $|\phi\rangle$ and, therefore, $|+3\rangle$ can be transformed to any qutrit-state by incoherent operations. However, we have seen in theorem 18 that the only coherence-rank-two state obtainable from this state by deterministic FI manipulation is $\sqrt{2/3}|0\rangle + \sqrt{1/3}|1\rangle$ (and its FI unitary equivalents). Thus, this shows that even under the constraint that the coherence rank decreases the majorization condition is far from being a sufficient condition for FI transformations.

3.3.2. Pure state stochastic transformations. One may ask for the potential of FI operations for stochastic manipulation of states. Being GI operations a subset of FI operations, the corresponding optimal protocols of theorem 10 provide lower bounds on the maximal probability of conversion under SFI operations. In general, these protocols are suboptimal as, for example, from the previous section we know that $|+3\rangle \rightarrow \sqrt{2/3}|0\rangle + \sqrt{1/3}|1\rangle$ can be realized in this case with probability one. Notice, however, that these bounds can be strengthened since the FI setting allows to apply a permutation to the input state before implementing the protocol. With this observation and the insight of theorem 10 we can characterize the optimal probability for SFI transformations among states with the same coherence rank and establish non-trivial lower bounds for the case of coherence-rank-decreasing operations (obviously, a transformation to a state with a higher coherence rank cannot be accomplished with nonzero probability).

Theorem 19. *The optimal probability of transforming a state into another by FI operations fulfills*

$$P(\rho_\psi \rightarrow \rho_\phi) \geq \max_P \min_i \frac{\langle i|P\rho_\psi P^\dagger|i\rangle}{\langle i|\rho_\phi|i\rangle}, \tag{82}$$

where P is any permutation (and the minimization goes over all i such that $\langle i|\rho_\phi|i\rangle \neq 0$ if $r(\psi) > r(\phi)$). Moreover, the inequality becomes an equality if $r(\psi) = r(\phi)$.

Proof. As discussed above, the bound is obvious from theorem 10 as we can use the protocol of the GI setting optimized over all FI unitary equivalents of the input state. To see that this gives the actual optimal probability when the input and output state have the same coherence rank, notice that in this case at least one Kraus operator of the corresponding SFI map must be full-rank. Since all Kraus operators must be of the same form, this means that they all must take the form $K_i = D_i P$ with some permutation P and diagonal matrices D_i . Thus, the most general SFI maps to achieve such transformation are given by $\Lambda_{\text{sgi}}(P\rho_\psi P^\dagger)$ where Λ_{sgi} is any SGI map and P is any permutation. \square

3.3.3. Many-copy transformations. The most striking limitation of state manipulation under GI operations is the generic impossibility of distillation and dilution. These are a consequence of the obstruction to any form of activation proven in lemma 12. It would be very interesting

to see whether, although deterministic one-copy transformations are rather limited as well for FI operations, distillation and dilution protocols are possible in this case. Interestingly, we show in the following that lemma 12 is no longer true for FI operations. This leaves open the possibility that many-copy transformations in this case might have a rich structure.

To prove our claim, it suffices to consider a counterexample. Take the obvious extension to \mathbb{C}^4 of the protocol implementing $|+_3\rangle \rightarrow \sqrt{2/3}|0\rangle + \sqrt{1/3}|1\rangle$ to see that $|+_4\rangle \rightarrow (\sqrt{2}|0\rangle + |1\rangle + |3\rangle)/2$ is possible by FI. For this, a map with Kraus operators given by

$$\begin{aligned} K_1 &= \begin{pmatrix} i/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} \end{pmatrix}, \\ K_2 &= \begin{pmatrix} 1/\sqrt{2} & 0 & i/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} \end{pmatrix}, \end{aligned} \quad (83)$$

together with an FI unitary transformation does the job. This shows that $|+_2\rangle^{\otimes 2} \rightarrow (\sqrt{2}|00\rangle + |01\rangle + |11\rangle)/2$ can be done with two copies. If we now trace out the second qubit we see that this activates the transformation $|+_2\rangle \rightarrow \rho$ where

$$\rho = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}. \quad (84)$$

This transformation is clearly impossible with one copy because of similar arguments used before. Since ρ is full-rank, the transformation $|+_2\rangle \rightarrow \rho$ could only be implemented by an FI operation with full-rank Kraus operators. This means that we can only use compositions of permutations and GI maps but corollary 9 tells us that this would require the diagonal entries of $|+_2\rangle\langle+_2|$ and ρ to be equal (up to permutations), which is not the case.

4. Relation to other concepts of quantum coherence

In this section we will discuss the relation of genuinely incoherent operations and fully incoherent operations to other concepts of coherence. We start with a list of alternative incoherent operations recently proposed in the literature:

- Maximally incoherent operations (MIO) [5]: operations which preserve the set of incoherent states, i.e. $\Lambda(\rho) \in \mathcal{I}$ for all $\rho \in \mathcal{I}$.
- Dephasing-covariant incoherent operations (DIO) [73, 78]: operations which commute with dephasing, i.e. $\Delta[\Lambda(\rho)] = \Lambda(\Delta[\rho])$.
- Translationally invariant operations (TIO) [7, 76, 77]: operations which commute with the unitary translation e^{-itH} for some nondegenerate Hermitian operator H diagonal in the incoherent basis, i.e. $\Lambda[e^{-itH}\rho e^{itH}] = e^{-itH}\Lambda[\rho]e^{itH}$ ⁶.

⁶Some references (see [73, 78]) allow for degenerate operators H . This is a legitimate choice which is relevant in certain scenarios. However, this induces a different set of free states and leads to a different resource theory in which coherence is measured relative to the eigenspaces of H . This is why we do not consider here this possibility.

- Strictly incoherent operations (SIO) [12, 39]: operations with a Kraus decomposition $\{K_i\}$ such that each Kraus operator commutes with dephasing, i.e. $\Delta[K_i\rho K_i^\dagger] = K_i\Delta[\rho]K_i^\dagger$.
- Physical incoherent operations (PIO) [73]: maps with Kraus operators of the form $K_j = \sum_x e^{i\theta_j} |\pi_j(x)\rangle\langle x| P_j$ and their convex combinations. Here, π_j are permutations and P_j is an orthogonal and complete set of incoherent projectors.

The following inclusions have been proven in the aforementioned references (see e.g. [73])

$$\text{PIO} \subset \text{SIO} \subset \text{DIO} \subset \text{MIO}, \quad (85)$$

$$\text{PIO} \subset \text{SIO} \subset \text{IO} \subset \text{MIO}. \quad (86)$$

In the following we will show that FIO and SIO are not subsets of each other, i.e.

$$\text{FIO} \not\subset \text{SIO}, \quad \text{SIO} \not\subset \text{FIO}. \quad (87)$$

For proving $\text{FIO} \not\subset \text{SIO}$, note that SI operations with Kraus operators $\{K_i\}$ are characterized by the operators K_i^\dagger being also incoherent $\forall i$ [12]. This amounts to the fact that the Kraus operators not only have at most one non-zero entry per column but also at most one non-zero entry per row. FI operations with non-diagonal rank-deficient Kraus operators clearly fail to fulfill this requirement. Notice that this also implies that $\text{FIO} \not\subset \text{PIO}$. Finally, it is easy to see that $\text{SIO} \not\subset \text{FIO}$, since the Kraus operators of SIO need not have the same form.

We proceed to show that FI operations are a subset of DI operations, i.e.

$$\text{FIO} \subset \text{DIO}. \quad (88)$$

Lemma 17 of [73] states that Λ is DIO if and only if $\Lambda(|x\rangle\langle x|)$ is incoherent and $\Delta(\Lambda(|x\rangle\langle x'|)) = 0 \forall x \neq x'$. The first condition is clearly fulfilled by FI maps. To see that the second condition also holds, notice that the (j,j) entry of $\Lambda(|x\rangle\langle x'|)$ is given by $\sum_i K_i(j,x)K_i^*(j,x')$ if Λ has Kraus operators $\{K_i\}$. For FI maps, either $K_i(j,x)K_i^*(j,x') = 0 \forall i$ (and the second condition is then trivially fulfilled for the (j,j) entry) or $K_i(m,x), K_i(m,x') = 0 \forall i$ and $\forall m \neq j$. In this last case the (x,x') entry of the normalization condition $\sum_i K_i^\dagger K_i = \mathbb{1}$ reads then precisely $\sum_i K_i^*(j,x)K_i(j,x') = 0$, as we wanted to prove.

Moreover, it is easy to see the following relations:

$$\text{FIO} \subset \text{IO}, \quad (89)$$

$$\text{GIO} \subset \text{SIO}, \quad (90)$$

$$\text{PIO} \not\subset \text{GIO}. \quad (91)$$

While $\text{FIO} \subset \text{IO}$ is obvious, $\text{GIO} \subset \text{SIO}$ follows from the fact that genuinely incoherent Kraus operators are all diagonal. Finally, $\text{PIO} \not\subset \text{GIO}$ holds true because the Kraus operators of PIO need not be diagonal, and that PI operations allow for distillation [73]. For qubit and qutrit maps it further holds that $\text{GIO} \subset \text{PIO}$, because in this situation GI operations are mixed-unitary, see theorem 3. However, this inclusion breaks down in higher dimensions and, in general, we have that $\text{GIO} \not\subset \text{PIO}$ as well. To see this, consider the GI map with Kraus operators given by

$$K_1 = \text{diag}(1, 0, \cos \theta, \cos \theta), \quad K_2 = \text{diag}(0, 1, \sin \theta, i \sin \theta) \quad (92)$$

for some value of $\theta \in (0, \pi/2)$. It has been shown in [73] that PIO correspond to convex combinations of maps with Kraus operators of the form $K_j = U_j P_j \forall j$, where the U_j are incoherent unitaries and the P_j form an orthogonal and complete set of incoherent projections. If a PI map is also GI without loss of generality we can take the U_j to be diagonal. Thus, according to theorem 1, if the above map was also in PIO, it is necessary that there exists a linear combination

of the Kraus operators $L = \alpha K_1 + \beta K_2$ ($\alpha, \beta \neq 0$) such that for every $i \in \{1, 2, 3, 4\}$ we either have $L_{ii} = 0$ or $|L_{ii}| = \sqrt{p}$ for some fixed $p \in (0, 1]$. Since

$$L = \text{diag}(\alpha, \beta, \alpha \cos \theta + \beta \sin \theta, \alpha \cos \theta + i\beta \sin \theta), \quad (93)$$

the above condition can only hold for $i = 1, 2$ if $|\alpha| = |\beta| = \sqrt{p}$. This furthermore imposes that both $|L_{33}| = \sqrt{p}$ and $|L_{44}| = \sqrt{p}$ since we cannot have then that these entries vanish. However, the first condition then leads to $\text{Re}(\alpha^* \beta) = 0$ and the second to $\text{Im}(\alpha^* \beta) = 0$. Hence, we would need that $\alpha = \beta = 0$, which cannot be. Thus, the given GI map is not in PIO.

Last, it holds that FIO and TIO are not subsets of each other. To see that $\text{FIO} \not\subset \text{TIO}$, it suffices to note that permutations are not TIO since the only unitary operations in TIO are diagonal [78]. In order to prove that $\text{TIO} \not\subset \text{FIO}$, one can consider the fully depolarizing qubit map, which maps every qubit state to $\mathbb{1}/2$. This map is clearly in TIO and has Kraus operators $K_i = \sigma_i/2$ with $i = 0, 1, 2, 3$ where σ_0 is the identity and the others are the standard Pauli matrices. Since the Kraus operators do not have the same form the fully depolarizing map cannot be an FI operation. On the other hand, we have that $\text{GIO} \subset \text{TIO}$, where the inclusion is strict. This is because the Kraus operators $\{K_i\}$ of a GI operation are all diagonal and, hence, $[K_i, H] = 0 \forall i$. This ensures that every GI operation is TI. The erasing operation $\Lambda[\rho] = |0\rangle\langle 0| \forall \rho$ is an example of a TI operation, which is clearly not GI.

5. Conclusions

The physical setting in other resource theories such as entanglement clearly defines the set of free operations. However, in the current efforts to develop a resource theory of coherence, although it is clear that free states should correspond to incoherent states, the notion of free operations is not completely clear from the physical context. Actually, it has been recently discussed [73, 78] that the incoherent operations considered in the standard resource theory of coherence of Baumgratz *et al* [11], although mathematically consistent, have not been provided with a clear physical interpretation. On the other hand, it has also been found that physically more plausible operations can lead to theories with very limited protocols for resource manipulation [73]. In this article we have considered two alternative frameworks of coherence that stem from very clear and simple principles and have thoroughly analyzed the power of the resulting resource theories.

In particular, we introduced the concept of genuine quantum coherence. This concept can be reduced to one simple requirement: that the genuinely incoherent quantum operation preserves all incoherent states. The concept obtained in this way captures coherence under additional constraints such as energy preservation, and falls into the class of unspeakable coherence [78]. We provide a general characterization of genuinely incoherent operations via Schur maps, and use this result to prove strong limitations of this framework. In particular, genuinely incoherent operations do not have a golden unit, and also do not allow for asymptotic state transformations in the form of distillation or dilution. On the single-copy level, genuinely incoherent operations only allow for transformations to states with the same diagonal elements. Nevertheless, more general transformations are possible via stochastic GI operations. One of our main results is the optimal probability for this procedure in theorem 10.

We also show that genuinely incoherent operations are incoherent for any Kraus decomposition, i.e. they cannot produce coherence regardless of their particular experimental implementation. Starting from this result, we introduced and studied the class of fully incoherent operations: these are all operations which are incoherent in any Kraus decomposition. We provide a complete characterization of this set of operations, and show that it is strictly larger

than the set of genuinely incoherent operations. On the other hand, we show that state transformations are very limited also in this more general framework: there is still no ‘maximally coherent state’ which would allow for transformations to all other states, and deterministic pure state transformations are only possible between very restricted families of states. It remains open, however, whether distillation and dilution procedures are generally possible in this framework. Since pure incoherent states can be transformed into each other via fully incoherent operations, this concept characterizes speakable coherence [78].

We also analyzed in detail the relation between GI and FI operations with other operations that have been previously considered. Since our operations represent quite restrictive physical conditions, they are usually also implementable in other frameworks. Thus, we hope that the results obtained here regarding coherence manipulation under GI and FI operations could also be used as building blocks for protocols in less-restrictive frameworks.

The results presented in this work lead to significant insights about the limits of any resource theory of quantum coherence. If we demand that a resource theory should have a golden unit, i.e. a unique state from which all other states can be obtained via the free set of operations, it turns out that any such resource theory must contain free operations that can create coherence in some experimental realization. At this point, it is interesting to mention that the set of physically incoherent operations also does not have a golden unit [73]. It is an interesting question if these concepts in combination with the framework proposed in our paper will lead to a resource theory with a golden unit. We leave this question open for future research.

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Appendix A. Mathematical structure of the set of GI maps

A genuinely incoherent operation Λ_{gi} will be called *extremal in the set of GI operations* if it cannot be written as

$$\Lambda_{\text{gi}}[\rho] = p\Lambda'_{\text{gi}}[\rho] + (1-p)\Lambda''_{\text{gi}}[\rho], \quad (\text{A.1})$$

with GI operations $\Lambda'_{\text{gi}} \neq \Lambda''_{\text{gi}}$ and probability $0 < p < 1$. It is straightforward to see that the set of genuinely incoherent operations is compact and convex, which ensures that any GI operation can be written as a convex combination of extremal GI operations [95].

A unital operation Λ_{u} is an operation which preserves the maximally mixed state: $\Lambda_{\text{u}}(\mathbb{1}/d) = \mathbb{1}/d$. It is easy to see that any GI operation is unital, but there exist unital operations which are not GI. We will call a GI operation *extremal in the set of unital operations* if it cannot be written as

$$\Lambda_{\text{gi}}[\rho] = p\Lambda'_{\text{u}}[\rho] + (1-p)\Lambda''_{\text{u}}[\rho], \quad (\text{A.2})$$

with some unital operations $\Lambda'_{\text{u}} \neq \Lambda''_{\text{u}}$ and probability $0 < p < 1$.

Clearly, a GI operation which is extremal in the set of unital operations must also be extremal in the set of GI operations. As we will see in the following theorem, the inverse of this statement is true as well.

Lemma 20. *A GI map is extremal in the set of GI maps if and only if it is extremal in the set of unital maps.*

Proof. As mentioned above the lemma, the implication from right to left is clear. To see the other direction, we show that every map which is not extremal in the set of unital maps is also not extremal in the set of GI maps. Assume that Λ is a GI map which is not extremal in the set of unital maps, i.e. there exist unital maps Φ and Φ' such that $\Lambda = p\Phi + (1-p)\Phi'$ for some $p \in (0, 1)$. This means that if $\{K_i\}$ ($\{K'_j\}$) is a Kraus representation of Φ (Φ'), then $\{\sqrt{p}K_i, \sqrt{1-p}K'_j\}$ is a Kraus representation of Λ . Since every Kraus representation of a GI map is diagonal, the operators $\{K_i\}$ and $\{K'_j\}$ have to be then all diagonal. Thus, Φ and Φ' must be GI maps as well and Λ is therefore not extremal in the set of GI maps. \square

This lemma reveals a close connection between GI operations and unital operations, and will be used to prove several important statements on the structure of these maps in the following.

It has been noted in section 2.2 that mixed-unitary maps, i.e.

$$\Lambda(\rho) = \sum_i p_i U_i \rho U_i^\dagger \tag{A.3}$$

with $\{p_i\}$ probabilities and $\{U_i\}$ unitary matrices, are GI maps when the matrices U_i are all diagonal. It is natural to ask whether the converse is true. Here we show that the answer depends on the dimensions. Every GI operation on $\mathcal{L}(\mathbb{C}^2)$ and $\mathcal{L}(\mathbb{C}^3)$ turns out to be a mixed-unitary map but not otherwise.

Theorem 21. *Every GI operation on $\mathcal{L}(\mathbb{C}^2)$ and $\mathcal{L}(\mathbb{C}^3)$ is mixed-unitary, i.e. can be written as in equation (A.3).*

Proof. The case $d = 2$ follows from the well-known fact that every unital operation on $\mathcal{L}(\mathbb{C}^2)$ is mixed-unitary [1]. Hence, it must be in particular true for GI maps.

The case $d = 3$ requires more work. However, this happens to be equivalent to exercise 4.1 in [80]. For the sake of completeness we provide a full proof here. We recall from above that every GI operation can be written as a convex combination of extremal GI operations. Clearly, diagonal unitaries are extremal GI operations. Thus, the proof is complete if we show that diagonal unitaries are the only extremal GI operations for qutrits.

For this, let $\Lambda(\cdot) = \sum_i K_i \cdot K_i^\dagger$ be a Kraus decomposition of an arbitrary GI map, where without loss of generality the diagonal $\{K_i\}$ are taken to be linearly independent (i. e. a so-called minimal representation [81]). Since a map is then unitary if and only if $|i| = 1$, we have to show that Λ is not extremal whenever $|i| \geq 2$. It is shown in theorem 4.21 in [80] that a unital map on $\mathcal{L}(\mathbb{C}^d)$ given by a set of linearly independent Kraus operators $\{K_i\}$ is extremal in the set of unital maps if and only if the collection of $|i|^2 d^2 \times d^2$ matrices

$$\mathcal{K}_{ij} = \begin{pmatrix} K_i^\dagger K_j & 0 \\ 0 & K_j K_i^\dagger \end{pmatrix} \tag{A.4}$$

is linearly independent. Notice that in the case of GI operations it holds that $[K_i^\dagger, K_j] = 0$ and, therefore, all matrices \mathcal{K}_{ij} are of the form $D_{ij} \oplus D_{ij}$ for diagonal matrices $D_{ij} = K_i^\dagger K_j$. In our case $d = 3$, the matrices \mathcal{K}_{ij} span then at most a 3-dimensional subspace. However, if $|i| \geq 2$ we

have at least four matrices \mathcal{K}_{ij} , and it is hence impossible that all of them are linearly independent. Thus, Λ is not extremal in the set of unital maps whenever $|i| \geq 2$. Lemma 20 ensures that a GI map which is not extremal in the set of unital maps is also not extremal in the subset of GI maps. \square

Theorem 22. *For every $d \geq 4$ there exists a GI map which is not mixed-unitary.*

Proof. We will first show this for $d = 4$, and extend it to all dimensions $d \geq 4$ below. We will prove the statement by presenting a GI map which is extremal but not unitary, and thus cannot be written as a mixture of diagonal unitaries. According to lemma 20, it suffices to check extremality in the set of unital maps.

For this, we define two linearly independent Kraus operators $K_1 = \text{diag}(a_1, a_2, a_3, a_4)$ and $K_2 = \text{diag}(b_1, b_2, b_3, b_4)$ with $a_k = 1/k$ and $b_k = i^k \sqrt{1 - a_k^2}$ for $k = 1, 2, 3, 4$. It is easy to check that these two Kraus operators define a genuinely incoherent quantum operation. We will now show that this operation is extremal, and thus cannot be written as a mixture of unitaries. For this, we will use similar arguments as in the proof of theorem 21. In particular, we now have four 16×16 matrices \mathcal{K}_{ij} given by the Kraus operators K_1 and K_2 via equation (A.4). The proof is complete by proving that these four matrices are linearly independent.

For proving this we note that these matrices are directly related to the following four vectors:

$$t_i = |a_i|^2, \quad u_i = |b_i|^2, \quad v_i = \bar{a}_i b_i, \quad w_i = a_i \bar{b}_i. \quad (\text{A.5})$$

In particular, $\mathcal{K}_{11} = \text{diag}(t) \oplus \text{diag}(t)$, $\mathcal{K}_{22} = \text{diag}(u) \oplus \text{diag}(u)$, $\mathcal{K}_{12} = \text{diag}(v) \oplus \text{diag}(v)$, and $\mathcal{K}_{21} = \text{diag}(w) \oplus \text{diag}(w)$. For completing the proof, it is thus enough to show that these four vectors are linearly independent. This can be done by verifying that the determinant of the matrix (t, u, v, w) is nonzero.

We will now show how the above arguments can also be used for dimensions $d \geq 4$. In this case we define two Kraus operators $K_1 = \text{diag}(a_1, \dots, a_d)$ and $K_2 = \text{diag}(b_1, \dots, b_d)$, where a_i and b_i are defined in the same way as above for $i \leq 4$. For $i > 4$ we define $a_i = 1$ and $b_i = 0$ ⁷. It can be verified by inspection that the previous arguments also apply in this situation, thus leading to a genuinely incoherent operation which is extremal but not unitary. This completes the proof for all dimensions $d \geq 4$. \square

These considerations complete our investigation on the structure of GI operations.

Appendix B. Wigner–Yanase skew information is a convex measure of genuine coherence

Here we will prove that the Wigner–Yanase skew information

$$S_H(\rho) = -\frac{1}{2} \text{Tr} \left([H, \sqrt{\rho}]^2 \right) \quad (\text{B.1})$$

is a convex measure of genuine coherence, i.e. it satisfies G1, G2, and G3 if the Hermitian operator H is nondegenerate and diagonal in the incoherent basis.

The Wigner–Yanase skew information fulfills G1 since it is nonnegative and zero if and only if the commutator $[H, \sqrt{\rho}]$ vanishes. For a nondegenerate Hermitian operator H the

⁷ We chose $a_i = 1$ and $b_i = 0$ for simplicity, but any other choice of parameters a_i and b_i satisfying $|a_i|^2 + |b_i|^2 = 1$ for $i > 4$ will also lead to the desired result.

commutator $[H, \sqrt{\rho}]$ vanishes if and only if ρ and H are diagonal in the same basis, which proves condition G1. Condition G2 follows from the facts that the Wigner–Yanase skew information does not increase under translationally invariant operations [77] and that any GI operation is translationally invariant with respect to a Hamiltonian H diagonal in the incoherent basis, as we have observed in section 4. Since the Wigner–Yanase skew information is convex [84], G3 is also fulfilled.

Appendix C. FI maps do not allow to prepare arbitrary mixed incoherent states from arbitrary input states

As argued in section 3, pure incoherent states can always be obtained from any state by some FI operation. In this section we also mentioned that for mixed incoherent states this is no longer the case even though they are free states as well. To see a particular example, take any d -dimensional state ρ such that it does not hold that $\rho_{ii} = 1/d \forall i$. It turns out that any such state cannot be mapped by FI operations to the d -dimensional maximally mixed state $\mathbb{1}_d/d$, which is incoherent. The reason is the following. Clearly, ρ cannot be mapped by GI operations to $\mathbb{1}_d/d$ (corollary 9). As in the proof of theorem 17, this also excludes any FI map with Kraus operators whose form is any row permutation of a diagonal matrix. This is because these maps can be written as $P\Lambda_{\text{gi}}[\rho]P^\dagger$ for an arbitrary permutation P and GI map Λ_{gi} . However, since the output of the map, $\mathbb{1}_d/d$, is left invariant under permutations, we would need that $\Lambda_{\text{gi}}[\rho] = \mathbb{1}_d/d$, which is impossible as argued above. Thus, the remaining possible FI maps must be such that their Kraus operators are rank-deficient. However, since Kraus operators of FI maps have all the same form, it cannot be that the outputs of such maps are supported in the full d -dimensional space. This shows that the transformation $\rho \rightarrow \mathbb{1}_d/d$ with ρ as defined above cannot be implemented by FI operations.

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